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# On Compatible Normal Odd Partitions in Cubic Graphs

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## Abstract

A *normal odd partition*  $\mathcal{T}$  of the edges of a cubic graph is a partition into *trails* of odd length (no repeated edge) such that each vertex is the end vertex of exactly one trail of the partition and internal in another trail. For each vertex  $v$ , we can distinguish the edge  $e_{\mathcal{T}}(v)$  for which this vertex is pending. Three normal odd partitions are compatible whenever these distinguished edges are distinct for each vertex. We study here this notion and show that a cubic 3-edge colourable graph can always be provided with three compatible normal odd partitions. The Petersen graph have this property and we can construct other cubic graphs with chromatic index 4 with the same property. At last, we give a connexion with the Fan Rapaud [3] conjecture and propose a new conjecture which, if true, would imply this conjecture.

*Key words:* Cubic graph; Edge-partition;

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## 1 Introduction

For basic graph-theoretic terms, we refer the reader to Bondy and Murty [2]. Following Bondy [1], a *walk* in a graph  $G$  is sequence  $W := v_0e_1v_1 \dots e_kv_k$ , where  $v_0, v_1, \dots, v_k$  are vertices of  $G$ , and  $e_1, e_2, \dots, e_k$  are edges of  $G$  and  $v_{i-1}$  and  $v_i$  are the ends of  $e_i$ ,  $1 \leq i \leq k$ .  $v_0$  and  $v_k$  are the *end vertices* and  $e_1$  and  $e_k$  are the *end edges* of this walk, while  $v_1, \dots, v_{k-1}$  are the *internal vertices* and  $e_2, \dots, e_{k-1}$  are the *internal edges*. The *length*  $l(W)$  of  $W$  is the number of edges (namely  $k$ ).  $W$  is *odd* whenever  $k$  is odd and *even* otherwise. The walk  $W$  is a *trail* if its edges  $e_1, e_2, \dots, e_k$  are distincts and a *path* if its vertices  $v_0, v_1, \dots, v_k$  are distincts. If  $W := v_0e_1v_1 \dots e_kv_k$ , is a walk of  $G$   $W' := v_i e_{i+1} \dots e_j v_j$  ( $0 \leq i \leq j \leq k$ ) is a *subwalk* of  $W$  (*subtrails* and *subpaths* are defined analogously). If  $v$  is an internal vertex of a walk  $W$  with ends  $x$  and  $y$ ,  $W(x, v)$  and  $W(v, y)$  are the subwalks of  $W$  obtained in cutting  $W$  in  $v$ . Conversely if  $W_1$  and  $W_2$  have a common end  $v$ , the *concatenation* of these

two walks *on*  $v$  gives rise to a new walk (denoted by  $W_1 + W_2$ ) with  $v$  as an internal vertex. When no confusion, is possible, it will be convenient to omit the edges in the description of a walk, that is  $W := v_0e_1v_1 \dots e_kv_k$  will be shortened in  $W := v_0v_1 \dots v_k$ .

Let  $G = (V, E)$  be a cubic graph (loops and multiple edges are allowed) and let  $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$  be a partition of  $E(G)$  into trails.

**Definition 1** A partition  $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$  of  $E(G)$  into trails is *normal* when every vertex is the end vertex of one trail of  $T_i \in \mathcal{T}$  and internal in  $T_j \in \mathcal{T}$ . The length of a normal partition is the maximum length of the trails in the partition, that is  $\max\{l(T_i) | T_i \in \mathcal{T}\}$ .

When  $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$  is a normal partition, we must have  $k = \frac{n}{2}$ . Moreover, we can associate to each vertex the unique edge with end  $v$  which is the end edge of a trail of  $\mathcal{T}$ . We shall denote this edge by  $e_{\mathcal{T}}(v)$  and it will be convenient to say that  $e_{\mathcal{T}}(v)$  is the *marked* edge associated to  $v$ . When it will be necessary to illustrate our purpose by a figure the marked edge associated to a vertex will be figurate by a  $\vdash$  close to this vertex. Let  $v$  be a vertex,  $v$  is an internal vertex in  $T_i \in \mathcal{T}$  and an end vertex in  $T_j \in \mathcal{T}$  (as an end of  $e_{\mathcal{T}}(v)$ ), we can associate to  $v$  the set  $E_{\mathcal{T}}(v)$  containing the end vertices of  $T_i$ . It must be clear that  $T_i$  and  $T_j$  are not necessarily distinct, in this case we have  $v \in E_{\mathcal{T}}(v)$ . Let  $x$  and  $y$  be the ends of  $T_i$ , one of these two vertices (say  $x$ ) is certainly different from  $v$ . Let  $T'_i = T_i(x, v) + T_j$  and  $T'_j = T_i(v, y)$ . We can thus transform  $\mathcal{T}$  in a new normal partition  $\mathcal{T}'$  by an operation of *switching* defined by:

$$\mathcal{T}' = \mathcal{T} - \{T_i, T_j\} + \{T'_i, T'_j\}$$

**Definition 2** A normal partition  $\mathcal{T} = \{T_1, T_2, \dots, T_{\frac{n}{2}}\}$  of  $E(G)$  into trails is *odd* when every trail in  $\mathcal{T}$  is odd. For each trail of odd length  $T_i \in \mathcal{T}$  let us say that an edge  $e = uv$  of  $T_i$  is *odd* (resp. *even*) whenever the subtrails of  $T_i$  obtained in deleting  $e$  have odd (resp. even) lengths.

Given two normal partitions  $\mathcal{T} = \{T_1, T_2, \dots, T_{\frac{n}{2}}\}$  and  $\mathcal{T}' = \{T'_1, T'_2, \dots, T'_{\frac{n}{2}}\}$ ,  $A_{\mathcal{T}\mathcal{T}'}$  is the set of vertices such that  $e_{\mathcal{T}}(v) = e_{\mathcal{T}'}(v)$ .

**Definition 3** Two normal partitions  $\mathcal{T}$  and  $\mathcal{T}'$  of  $E(G)$  into trails are *compatible* when  $e_{\mathcal{T}}(v) \neq e_{\mathcal{T}'}(v)$  for every vertex  $v \in V(G)$  (in other words  $A_{\mathcal{T}\mathcal{T}'} = \emptyset$ ).

Given three normal partitions  $\mathcal{T} = \{T_1, T_2, \dots, T_{\frac{n}{2}}\}$ ,  $\mathcal{T}' = \{T'_1, T'_2, \dots, T'_{\frac{n}{2}}\}$  and  $\mathcal{T}'' = \{T''_1, T''_2, \dots, T''_{\frac{n}{2}}\}$ , we let  $A = A_{\mathcal{T}\mathcal{T}'} \cup A_{\mathcal{T}'\mathcal{T}''} \cup A_{\mathcal{T}\mathcal{T}''}$ . We shall say that  $G$  has three compatible normal odd partitions  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$  whenever these partitions are pairwise compatible (in other words  $A = \emptyset$ ).

In [4] we showed that a cubic graph without loop can always be provided with 3 compatible normal partitions.

**Theorem 4** [4] *A cubic graph  $G$  has three compatible normal partitions if and only if  $G$  has no loop.*

It turns out that normal odd partitions are directly associated to perfect matchings and it is rather natural to ask whether the problem of finding three compatible normal odd partitions is connected to the problem of the edge colouring. Motivated by this perspective, we show that cubic graphs with chromatic index 3 can be provided with three compatible normal odd partitions. Moreover, there are cubic (non trivial) graphs with chromatic index 4 with that property (in particular the Petersen graph and the Flower Snarks).

## 2 Preliminary results

### 2.1 Switching equivalence

In [4] we have proved that any two normal partitions  $\mathcal{T}$  and  $\mathcal{T}'$  of a cubic graph are *switching equivalent*, that is, we can transform  $\mathcal{T}$  in  $\mathcal{T}'$  by a sequence of at most  $2n$  switchings.

Assume that a normal odd partition  $\mathcal{T}'$  is obtained from the normal odd partition  $\mathcal{T}$  by an operation of switching, this operation of switching will be said an *odd switching*. If we can transform a normal odd partition  $\mathcal{T}$  in  $\mathcal{T}'$  by a sequence of odd switching operations,  $\mathcal{T}$  and  $\mathcal{T}'$  are said *odd switching equivalent*.

**Theorem 5** *Any two normal odd partitions of a cubic graph  $G$  are odd switching equivalent.*

**Proof** Assume that we have chosen two normal odd partitions  $\mathcal{T}$  and  $\mathcal{T}'$  which are not odd switching equivalent and such that  $A_{\mathcal{T}\mathcal{T}'}$  has maximum cardinality.

In particular, we have  $A_{\mathcal{T}\mathcal{T}'} \neq V(G)$ . Let  $v \notin A_{\mathcal{T}\mathcal{T}'}$ ,  $u_1$ ,  $u_2$ , and  $u_3$  being the neighbors of  $v$ . For sake of simplicity  $e_1$ ,  $e_2$  and  $e_3$  respectively denote the edges  $vu_1$ ,  $vu_2$  and  $vu_3$ . Assume without loss of generality that  $e_{\mathcal{T}}(v) = e_2$  and  $e_{\mathcal{T}'}(v) = e_3$ . We denote  $T$  (resp.  $T'$ ) the trail of  $\mathcal{T}$  (resp.  $\mathcal{T}'$ ) which contains  $v$  as an internal vertex.

As a matter of fact the vertex  $v$  is an end point of  $T$ , more precisely

$$T = ve_2u_2 \dots u_1e_1ve_3u_3 \dots x$$

otherwise there would be a normal odd partition  $\mathcal{T}''$  obtained from  $\mathcal{T}$  in using an odd switching operation on  $v$  and such that  $e_{\mathcal{T}''}(v) = e_3$ , a contradiction since  $A_{\mathcal{T}''\mathcal{T}'}$  would have a greater size than  $A_{\mathcal{T}\mathcal{T}'}$ .

Similarly  $T' = ve_3u_3 \dots u_1e_1ve_2u_2 \dots x'$ .

But now, in using odd switching operations on  $v$  we obtain normal odd partition  $\mathcal{T}_1$  and  $\mathcal{T}'_1$  respectively odd switching equivalent to  $\mathcal{T}$  and  $\mathcal{T}'$ .

$$\begin{aligned}\mathcal{T}_1 &= \mathcal{T} - \{T\} \cup \{ve_1u_1 \dots u_2e_2ve_3u_3 \dots x\}, \\ \mathcal{T}'_1 &= \mathcal{T}' - \{T'\} \cup \{vve_1u_1 \dots u_3e_3ve_2u_2 \dots x'\}.\end{aligned}$$

It follows that  $|A_{\mathcal{T}_1\mathcal{T}'_1}| > |A_{\mathcal{T}\mathcal{T}'}|$ , a contradiction.  $\square$

**Theorem 6** *Let  $G$  be a cubic graph then  $G$  has an odd normal partition if and only if  $G$  has a perfect matching.*

**Proof** Let  $M$  be a perfect matching in  $G$ . Then  $G - M$  is a 2-factor of  $G$ . let us give any orientation to this 2-factor and for each vertex  $v$  let us denote the outgoing edge  $o(v)$ . For each edge  $e = uv \in M$ , let  $P_{uv}$  be the path of length 3 obtained in concatenating  $o(u)$   $uv$  and  $o(v)$ . Then  $T = \{P_{uv} | uv \in M\}$  is a normal odd partition (of length 3) of  $G$ . Conversely let  $\mathcal{T} = \{T_1, T_2, \dots, T_{\frac{n}{2}}\}$  be a normal odd partition of  $G$ . A vertex  $v \in V(G)$  is internal in exactly one trail of  $\mathcal{T}$ . The edges of this trail being alternatively odd and even,  $v$  is incident to exactly one odd edge. Hence the odd edges so defined induce a perfect matching of  $G$ .  $\square$

Given an odd normal partition  $\mathcal{T}$  of  $G$ , we can define the *associated* perfect matching as the set of odd edges of  $\mathcal{T}$ . Conversely, given a perfect matching  $M$ , we shall say that a normal odd partition  $\mathcal{T}$  is *conformal* to  $M$  whenever  $M$  is the set of odd edges of  $\mathcal{T}$ . Assume that a normal odd partition  $\mathcal{T}'$  is obtained from the normal odd partition  $\mathcal{T}$  by an operation of switching such that the associated perfect matching  $M$  remains unchanged then we shall say that we have proceed to a *conformal (to  $M$ ) switching*. This operation of conformal switching is not always possible on a vertex. Indeed, assume that  $v$  is an internal vertex in  $T \in \mathcal{T}$  and an end vertex of this trail then when  $T(v, v)$  contains the edge of  $M$  incident to  $v$ , the conformal switching is not allowed since we would obtain a cycle in the transformation (see Figure 2.1,

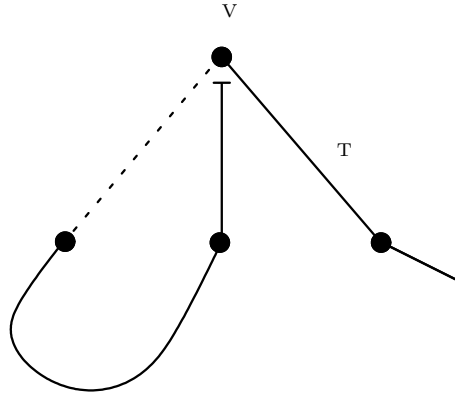


Fig. 1. Conformal switching on  $v$  not allowed

the edge of  $M$  is the dashed edge). In Theorem 7 we investigate the number of equivalence classes with respect to the conformal switching operation for a given perfect matching  $M$ , it turns out that except for the cubic graph on two vertices (see Figure 2.1) the conformal normal partitions are all equivalent.



Fig. 2. Two non equivalent conformal partitions for the cubic graph on 2 vertices

**Theorem 7** *Let  $G$  be a cubic graph of order at least 4 and  $M$  be a perfect matching. Then any two normal partitions  $\mathcal{T}$  and  $\mathcal{T}'$  conformal to  $M$  are conformal switching equivalent.*

**Proof**

Assume that we have chosen two normal odd partitions  $\mathcal{T}$  and  $\mathcal{T}'$  which are not conformal switching equivalent and such that  $A_{\mathcal{T}\mathcal{T}'}$  has maximum cardinality.

In particular, we have  $A_{\mathcal{T}\mathcal{T}'} \neq V(G)$ .

Let  $v \notin A_{\mathcal{T}\mathcal{T}'}$  and let  $u_1, u_2$  and  $u_3$  be its neighbors. W.l.o.g. we can assume that  $vu_1$  is an edge of  $M$  and  $e_{\mathcal{T}}(v) = vu_2$  while  $e_{\mathcal{T}'}(v) = vu_3$ . Since a conformal switch of  $\mathcal{T}$  in  $v$  leads to a conformal normal partition  $\mathcal{T}''$  where  $e_{\mathcal{T}''}(v) = e_{\mathcal{T}'}(v)$  while nothing is changed elsewhere, we can suppose that this conformal switch is not allowed in  $v$ . In the same way a conformal switch of  $\mathcal{T}'$  in  $v$  is not allowed as well. Hence  $v$  is an internal vertex of  $T \in \mathcal{T}$  and an end vertex of this trail. Symmetrically,  $v$  is an internal vertex of  $T' \in \mathcal{T}'$  and an end vertex of this trail (see Figure 2.1). We suppose that  $y$  is the second end vertex of  $T$  and  $y'$  the second end vertex of  $T'$ .

CLAIM 1  $u_2 \neq u_3$

**Proof** If  $u_2 = u_3$  then we have  $e_{\mathcal{T}}(u_3) = u_3v$  (formally we need to still

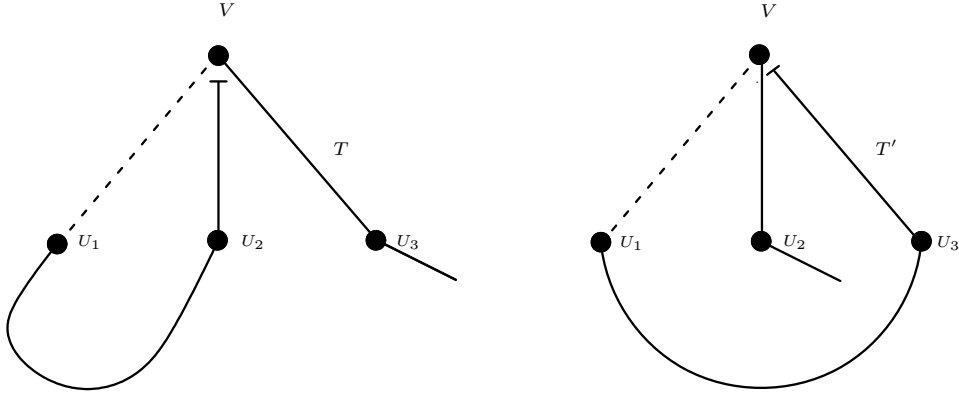


Fig. 3. Conformal switchings on  $v$  are not allowed in  $\mathcal{T}$  and  $\mathcal{T}'$ .

distinguish  $u_2$  and  $u_3$ ) and  $e_{\mathcal{T}'}(u_2) = u_2v$ . Hence  $u_2 \notin A_{\mathcal{T}\mathcal{T}'}$ . Let us denote  $u_2u_4$  the edge of  $M$  incident to  $u_2$  ( $= u_3$ ) being on the subtrail of  $\mathcal{T}$  joining  $u_1$  to  $u_2$  (identical to the the subtrail of  $\mathcal{T}'$  joining these two vertices).

We may assume that  $u_4 \neq u_1$  otherwise  $G$  would be of order 2, a contradiction.

But now, conformal switchings of  $\mathcal{T}$  in  $u_4, u_2$  and  $v$  lead to a normal partition  $\mathcal{T}''$  conformal switching equivalent to  $\mathcal{T}$ . Whatever  $u_4$  belongs or not to  $A_{\mathcal{T}\mathcal{T}'}$ ,  $A_{\mathcal{T}''\mathcal{T}'}$  has more vertices than  $A_{\mathcal{T}\mathcal{T}'}$ , a contradiction. ■

From now on, we can assume that  $u_2 \neq u_3$ .

CLAIM 2  $u_1 \neq u_2$

**Proof** Assume not, thus  $u_1 \notin A_{\mathcal{T}\mathcal{T}'}$ . From Claim 1 we have  $u_1 \neq u_3$ .

We have  $y = u_1$ , otherwise we can transform  $\mathcal{T}$  in  $\mathcal{T}''$  by using a conformal switching on  $u_1$  followed by a conformal switching in  $v$  and we obtain  $|A_{\mathcal{T}''\mathcal{T}'}| > |A_{\mathcal{T}\mathcal{T}'}|$ , a contradiction.

But now, conformal switchings of  $\mathcal{T}$  in  $u_3, u_1$  and  $v$  lead to a normal partition  $\mathcal{T}''$  conformal switching equivalent to  $\mathcal{T}$ . Whatever  $u_3$  belongs or not to  $A_{\mathcal{T}\mathcal{T}'}$ ,  $A_{\mathcal{T}''\mathcal{T}'}$  has more vertices than  $A_{\mathcal{T}\mathcal{T}'}$ , a contradiction. ■

Similarly  $u_1 \neq u_3$ . If we consider the subtrails  $T(u_1, u_2)$  there is a certainly a vertex on that trail for which the associated marked edge  $e_{\mathcal{T}}(w) \neq e_{\mathcal{T}'}(w)$ . Assume that  $w$  is the first vertex with that property when running from  $u_1$  to  $u_2$  on  $T$  (let us remark that  $T(u_1, w) = T'(u_1, w)$ ). Hence  $w \notin A_{\mathcal{T}\mathcal{T}'}$ . Let  $x$  be the neighboring vertex of  $w$  on  $T(v, w)$  (it may happens that  $u_1 = w$ , in that case  $x = v$ ) and let  $Q$  be the trail of  $\mathcal{T}$  ending in  $w$  with the marked edge  $e_{\mathcal{T}}(w)$ .

CLAIM 3  $xw \in M$

**Proof** If  $xw \notin M$  then a conformal switching on  $w$  leads to  $\mathcal{T}''$

$$\mathcal{T}'' = \mathcal{T} - \{T, Q\} + \{T(y, w), Q + T(w, v)\}$$

Whatever is the status of  $w$  relatively to  $A_{\mathcal{T}''\mathcal{T}'}$  we can now perform a conformal switching in  $v$  in order to get a new conformal normal partition  $\mathcal{T}'''$  for which  $A_{\mathcal{T}'''\mathcal{T}'}$  contains at least one vertex more. Indeed let  $R \in \mathcal{T}''$  be the trail containing  $v$  as an internal vertex and  $S \in \mathcal{T}''$  be the trail containing  $v$  as an end vertex then  $R$  ends in  $w$  and  $S$  ends with  $vu_2$  ( $e_{\mathcal{T}''}(v) = vu_2$ ). Hence the conformal switch on  $v$  is now allowed, a contradiction. ■

CLAIM 4  $w = y$

**Proof** Assume that  $w \neq y$ . Since by Claim 3  $xw \in M$ , a conformal switching on  $w$  is possible. This leads to the conformal partition  $\mathcal{T}''$ :

$$\mathcal{T}'' = \mathcal{T} - \{T, Q\} + \{T(y, w) + Q, T(w, v)\}$$

In  $\mathcal{T}''$   $v$  is now on  $T(y, w) + Q$  as an internal vertex and  $T(w, v)$  as an end vertex. A conformal switch is now allowed on  $v$  and we get  $\mathcal{T}'''$  such that  $A_{\mathcal{T}'''\mathcal{T}'}$  contains one vertex more than  $A_{\mathcal{T}\mathcal{T}'}$  (recall that we do not care about  $w$  since  $w \notin A_{\mathcal{T}\mathcal{T}'}$ ), a contradiction. ■

In the same way, we certainly have  $w = y'$  (take  $\mathcal{T}'$  instead of  $\mathcal{T}$ ). Since by Claim 1  $u_2 \neq u_3$ , we must have either  $u_2 \neq w$  or  $u_3 \neq w$ . In considering  $\mathcal{T}$ , we can decide, w.l.o.g. that  $u_2 \neq w$  (if not, we consider  $\mathcal{T}'$  where the roles of  $u_2$  and  $u_3$  are exchanged).

In  $\mathcal{T}$ ,  $u_2$  is an internal vertex of  $T$  and an end vertex of a trail  $S$  with  $S \neq T$ . A conformal switch is allowed in  $u_2$  and this switch leads to the conformal partition:

$$\mathcal{Q} = \mathcal{T} - \{T, S\} + \{T(y, u_2) + S, u_2v\}$$

And we have  $A_{\mathcal{Q}\mathcal{T}'} = A_{\mathcal{T}\mathcal{T}'}$  or  $A_{\mathcal{Q}\mathcal{T}'} = A_{\mathcal{T}\mathcal{T}'} - u_2$ . But now we can perform a conformal switch on  $v$  followed by a conformal switch on  $w$ . The first switch on  $v$  leads to  $\mathcal{R}$

$$\mathcal{R} = \mathcal{Q} - \{T(y, u_2) + S, vu_2\} + \{S + T(u_2, v) + vu_2, T(v, w)\}$$

The second switch on  $w$  leads to  $\mathcal{S}$

$$\mathcal{S} = \mathcal{R} - \{S + T(u_2, v) + vu_2, T(v, w)\} + \{u_2v + T(v, w) + T(w, v), T(w, u_2) + S\}$$



We have now  $v$  and  $w$  in  $A_{S\mathcal{T}'}$ . Since this set has at least one vertex more than  $A_{\mathcal{T}'}$ , we have a contradiction.  $\square$

## 2.2 Miscellaneous

The following proposition will be essential in the next section.

**Proposition 8** *Let  $G$  be a cubic graph having 3 normal partitions  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$  and  $e = xy \in E(G)$  such that  $x$  and  $y$  are not in  $A$  then one of the followings is true:*

- $e$  is an internal edge in exactly one partition
- $e$  is an internal edge in exactly two partitions

*Moreover, in the second case, the edge  $e$  itself is a trail of the third partition.*

**Proof** If  $e$  is not an internal edge in  $\mathcal{T}$ ,  $\mathcal{T}'$  or  $\mathcal{T}''$  then  $e$  is an end edge for a trail of  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$ . In  $x$  or  $y$  we should have two partitions (say  $\mathcal{T}$  and  $\mathcal{T}'$ ) for which  $e_{\mathcal{T}}(x) = e_{\mathcal{T}'}(x)$  ( $e_{\mathcal{T}}(y) = e_{\mathcal{T}'}(y)$  respectively), a contradiction. If  $e$  is an internal edge in  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$ . Let  $a$  and  $b$  the two other neighbors of  $x$ . We should have then

- $e_{\mathcal{T}}(x) = xa$  or  $xb$
- $e_{\mathcal{T}'}(x) = xa$  or  $xb$
- $e_{\mathcal{T}''}(x) = xa$  or  $xb$

which is impossible. Assume now that  $e$  is an internal edge of a trail in  $\mathcal{T}$  and in  $\mathcal{T}'$  and let  $a$  and  $b$  the two other neighbors of  $x$ . Up to the names of vertices we have

- $e_{\mathcal{T}}(x) = xa$
- $e_{\mathcal{T}'}(x) = xb$

From the third partition  $\mathcal{T}''$ , we must have  $e_{\mathcal{T}''}(x) = xy$ . In the same way we should obtain  $e_{\mathcal{T}''}(y) = yx$ . Hence the trail containing  $e = xy$  is reduced to  $e$ , as claimed.  $\square$

Given a normal partition  $\mathcal{T}$   $\mu(\mathcal{T})$  is the mean length of the trails in  $\mathcal{T}$  and  $n_{\mathcal{T}}(i)$  is the number of trails of length  $i$ .

**Proposition 9** [4] *Let  $\mathcal{T}$  be a normal partition of a cubic graph  $G$  on  $n$  vertices. Then*

- $\mu(\mathcal{T}) = 3$
- $\sum_{i=1}^{i=n+1} (3-i)n_{\mathcal{T}}(i) = 0$

Hence a normal partition whose mean length is 3 has all its trail of length 3.

It can be pointed out that cubic graphs with three compatible normal odd partitions are bridgeless.

**Proposition 10** *Let  $G$  be a cubic graph with three compatible normal odd partitions then  $G$  is bridgeless.*

**Proof** Assume that  $xy$  is a bridge of  $G$  and let  $C$  be the connected component of  $G - xy$  containing  $x$ . Since  $G$  has three compatible normal odd partitions, one of these partitions, say  $\mathcal{T}$ , is such that  $e_{\mathcal{T}}(x) = xy$ . The edges of  $C$  are thus partitioned into odd trails (namely the trace of  $\mathcal{T}$  on  $C$ ). We have

$$m = |E(C)| = \frac{3(|C| - 1) + 2}{2}$$

and  $m$  is even whenever  $|C| \equiv 3 \pmod{4}$  while  $m$  is odd whenever  $|C| \equiv 1 \pmod{4}$ . The trace of  $\mathcal{T}$  on  $C$  is a set of  $\frac{|C|-1}{2}$  trails and this number is odd when  $|C| \equiv 3 \pmod{4}$  and even otherwise. Hence, when  $|C| \equiv 3 \pmod{4}$  we must have an odd number of odd trails partitioning  $E(C)$  but, in that case  $m$  is even and when  $|C| \equiv 1 \pmod{4}$  we must have an even number of odd trails partitioning  $E(C)$  but, in that case  $m$  is odd, contradiction.  $\square$

In Figure 2.2, we show  $K_4$  provided with three compatible normal odd partitions. Let us remark that, following Theorem 11, we need to have trails of length 5 in at least one partition.

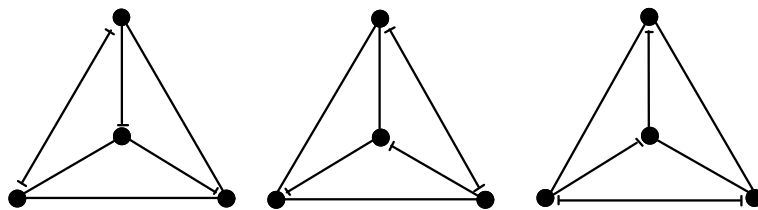


Fig. 4.  $K_4$  with three compatible normal odd partitions

### 3 On cubic graphs with chromatic index 3

In this section the existence of three compatible normal odd partitions in cubic 3-edge colourable graphs is considered.

**Theorem 11** *Let  $G$  be a cubic graph then the followings are equivalent*

- i)  $G$  has three compatible normal odd partitions of length 3
- ii)  $G$  has three compatible normal odd partitions where each edge is an internal edge in exactly one partition
- iii)  $G$  is bipartite

**Proof** Assume first that  $G$  can be provided with three compatible normal odd partitions of length 3  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$ . Since the mean length of each partition is 3 (Proposition 9), each trail of each partition has length exactly 3.  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$  are thus three normal odd partitions and from Proposition 8, each edge is the internal edge of one trail in exactly one partition. Conversely assume that  $G$  can be provided with three compatible normal odd partitions where each edge is an internal edge in exactly one partition, then from Proposition 8 there is no trail of length 1 in any of these partitions. Since the mean length of each partition is 3, that means that each trail in each partition has length exactly 3. Hence  $i) \equiv ii)$ .

We prove now that  $i) \equiv iii)$ . Let  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$  three compatible normal odd partitions of length 3. Following the proof of Theorem 7 the internal edges of trails of  $\mathcal{T}$  ( $\mathcal{T}'$  and  $\mathcal{T}''$  respectively) constitute a perfect matching (say  $M$ ,  $M'$  and  $M''$  respectively).

Let  $a_0a_1a_2a_3$  be a trail of  $\mathcal{T}$  and let  $b_1$  and  $b_2$  the third neighbors of  $a_1$  and  $a_2$  respectively. By definition, we have  $e_{\mathcal{T}}(a_1) = a_1b_1$  and  $e_{\mathcal{T}}(a_2) = a_2b_2$ .

Since  $a_0a_1$  and  $a_2a_3$  must be internal edges in a trail of  $\mathcal{T}'$  or (exclusively)  $\mathcal{T}''$ , assume w.l.o.g. that  $a_0a_1$  is an internal edge of a trail  $T'_1$  of  $\mathcal{T}'$ .  $T'_1$  does not use  $a_1a_2$  otherwise  $e_{\mathcal{T}'}(a_1) = a_1b_1$ , a contradiction with  $e_{\mathcal{T}}(a_1) = a_1b_1$  since  $\mathcal{T}$  and  $\mathcal{T}'$  are compatible. Hence  $T'_1$  uses  $a_1b_1$  and  $e_{\mathcal{T}'}(a_1) = a_1a_2$ .

Assume now that  $a_2a_3$  is an internal edge of a trail  $T'_2$  of  $\mathcal{T}'$ . Reasoning in the same way, we get that  $e_{\mathcal{T}'}(a_2) = a_2a_1$ . These two results lead to the fact that  $a_2a_3$  must be a trail in  $\mathcal{T}'$ , which is impossible since each trail has length exactly 3.

Hence, whenever  $a_0a_1$  is supposed to be an internal edge in a trail of  $\mathcal{T}'$ , we must have  $a_2a_3$  as an internal edge in a trail of  $\mathcal{T}''$ . The two internal vertices of  $a_0a_1a_2a_3$  can be thus distinguished, following the fact that the end edge to whom they are incident is internal in  $\mathcal{T}'$  (say *red* vertices) or  $\mathcal{T}''$  (say *blue* vertices). The same holds for each trail in  $\mathcal{T}$  (and incidently for each partition  $\mathcal{T}'$  and  $\mathcal{T}''$ ). We can remark now that  $a_1b_2$  is an end edge of a trail in  $\mathcal{T}$ . This end edge cannot be an internal edge in  $\mathcal{T}'$  since the trail of length 3 going through  $a_0a_1$  ends with  $a_1b_1$ . Hence  $a_1b_2$  is an internal edge in  $\mathcal{T}''$  and  $b_2$  is a blue vertex. Considering now  $a_0$ , this vertex is the internal vertex of a trail

of length 3 of  $\mathcal{T}$ . Since  $a_0a_1 \in M'$  and  $M'$  is a perfect matching,  $a_0$  cannot be incident to an other internal edge of a trail in  $\mathcal{T}'$  and  $a_0$  must be a blue vertex. Hence  $a_1$  is a red vertex and its neighbors are all blue vertices. Since we can perform this reasoning in each vertex  $G$  is bipartite as claimed.

Conversely, assume that  $G$  is bipartite and let  $V(G) = \{W, B\}$  be the bipartition of its vertex set. In the following, a vertex in  $W$  will be represented by a circle ( $\circ$ ) while a vertex in  $B$  will be represented by a bullet ( $\bullet$ ). From König's theorem [7]  $G$  is a cubic 3-edge colourable graph. Let us consider a coloring of its edge set with three colors  $\{\alpha, \beta, \gamma\}$ . Let us denote by  $\alpha \bullet \beta \circ \gamma$  a trail of length 3 which is obtained in considering an edge  $uv$  ( $u \in B$  and  $v \in W$ ) colored with  $\beta$  together with the edge colored  $\alpha$  incident with  $u$  and the edge colored with  $\gamma$  incident with  $v$ . It can be easily checked that the set  $\mathcal{T}$  of  $\alpha \bullet \beta \circ \gamma$  trails of length 3 is a normal odd partition of length 3. We can define in the same way  $\mathcal{T}'$  as the set of  $\beta \bullet \gamma \circ \alpha$  trails of length 3 and  $\mathcal{T}''$  as the set of  $\gamma \bullet \alpha \circ \beta$  trails of length 3.

Hence  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$  is a set of three normal odd partitions of length 3. We claim that these partitions are compatible. Indeed, let  $v \in W$  be a vertex and  $u_1, u_2$  and  $u_3$  its neighbors. Assume that  $u_1v$  is colored with  $\alpha$ ,  $u_2v$  is colored with  $\beta$  and  $u_3v$  is colored with  $\gamma$ . Hence  $u_1v$  is internal in an  $\gamma \bullet \alpha \circ \beta$  trail of  $\mathcal{T}''$  and  $e_{\mathcal{T}''}(v) = vu_3$ . The edge  $u_2v$  is internal in an  $\alpha \bullet \beta \circ \gamma$  trail of  $\mathcal{T}$  and  $e_{\mathcal{T}}(v) = vu_1$ . The edge  $u_3v$  is internal in an  $\beta \bullet \gamma \circ \alpha$  trail of  $\mathcal{T}'$  and  $e_{\mathcal{T}'}(v) = vu_2$ . Since the same reasoning can be performed in each vertex of  $G$ , the three  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$  partitions are compatible. □

**Theorem 12** *Let  $G$  be a cubic graph with three compatible normal odd partitions  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$  such that*

- $\mathcal{T}$  has length 3
- $\mathcal{T}'$  and  $\mathcal{T}''$  are odd

*Then  $G$  is a cubic 3-edge colourable graph.*

**Proof** Since  $\mathcal{T}$  has length 3, every trail of  $\mathcal{T}$  has length 3 (see Proposition 9). Hence there is no edge which can be an internal edge of a trail of  $\mathcal{T}'$  and a trail of  $\mathcal{T}''$ , since, by Proposition 8 such an edge would be a trail of length 1 in  $\mathcal{T}$ . The perfect matchings associated to  $\mathcal{T}'$  and  $\mathcal{T}''$  (see Theorem 7) are thus disjoint and induce an even 2-factor of  $G$ , which means that  $G$  is a cubic 3-edge colourable graph, as claimed. □

Let  $G$  be a simple cubic 3-edge colourable graph with a proper 3-edge colouring using colours in  $\{\alpha, \beta, \gamma\}$ . According to Theorem 7 above we know that  $G$  has 3 odd normal partitions  $\mathcal{T}_\alpha, \mathcal{T}_\beta, \mathcal{T}_\gamma$  such that  $\mathcal{T}_\alpha$  is conformal to the perfect matching of edges coloured with  $\alpha$ ,  $\mathcal{T}_\beta$  is conformal to  $\beta$  and  $\mathcal{T}_\gamma$  is conformal to  $\gamma$ . Let  $v$  be a vertex, we shall denote by  $v_\alpha, v_\beta$  and  $v_\gamma$  the **edge**, coloured  $\alpha, \beta$  and  $\gamma$  respectively, incident with  $v$ .

**Lemma 13** *Let  $G$  be a cubic 3-edge colourable graph with a proper 3-edge colouring  $\{\alpha, \beta, \gamma\}$ . Assume that  $G$  has three compatible normal odd partitions  $\mathcal{T}_\alpha, \mathcal{T}_\beta, \mathcal{T}_\gamma$  which are, respectively, conformal to  $\alpha, \beta$  and  $\gamma$ . Then  $G'$  obtained from  $G$  in subdividing an edge  $e$  with two vertices  $u$  and  $v$  ( $u$  adjacent to  $x$  and  $v$  adjacent to  $y$ ) and joining these two vertices by an additional edge has also this property.*

**Proof** Assume that  $e = xy$  is coloured with  $\alpha$ . We get a proper 3-edge colouring of  $G'$  as follows : let the edges  $xu$  and  $vy$  be colored  $\alpha$  while the two other edges incident to  $u$  and  $v$  are respectively coloured  $\beta$  and  $\gamma$ . By proposition 9,  $e$  is an internal edge in at least one normal partition (say  $\mathcal{T}_\alpha$ ) and an end edge in at least one other (say  $\mathcal{T}_\beta$ ). We are going to transform the normal odd partition  $\mathcal{T}_\mu$  of  $G$ , conformal to  $\mu$ , into the normal odd partition  $\mathcal{T}'_\mu$  of  $G'$ , conformal to  $\mu \in \{\alpha, \beta, \gamma\}$ .

Let  $T_\alpha \in \mathcal{T}_\alpha$  be the trail containing  $e$ . In order to get the normal odd partition  $\mathcal{T}'_\alpha$  we replace the sub-trail  $xey$  of  $T_\alpha$  with  $xx_\alpha uu_\beta vv_\alpha y$  and we add the trail of length 1,  $uu_\gamma v$ . Obviously all the trails of  $\mathcal{T}'_\alpha$  have odd edges of colour  $\alpha$ .

$e$  is an end-edge of some trail of  $\mathcal{T}_\beta$ . W.l.o.g. we can consider that  $x$  is its end vertex. Let us denote  $T_{\beta,x}$  this trail. We replace the sub-trail  $xey$  of  $T_{\beta,x}$  with  $vv_\gamma uu_\beta vv_\alpha y$  and we consider the trail of length 1,  $xx_\alpha u$ . We get hence a normal odd partition  $\mathcal{T}'_\beta$  of  $G'$  conformal with  $\beta$ .

It remains two cases to consider.

**case 1:**  $e$  is an end-edge of some trail of  $\mathcal{T}_\gamma$

Assume that  $y$  is its end vertex and let us denote this trail  $T_{\gamma,y}$ . We replace the subtrail  $xey$  of  $T_{\gamma,y}$  with  $xx_\alpha uu_\gamma vv_\beta u$  and we consider the trail of length 1  $yy_\alpha v$ .

**case 2:**  $e$  is an internal edge of some trail  $T_\gamma$  of  $\mathcal{T}_\gamma$ . We replace the subtrail  $xey$  of  $T_{\gamma,y}$  with  $xx_\alpha uu_\gamma vv_\beta u$  and we consider the trail of length 1  $yy_\alpha v$ .

In any case, we get a normal odd partition  $\mathcal{T}'_\gamma$  of  $G'$  conformal with  $\gamma$  and we can check that these 3 normal odd partitions  $\mathcal{T}'_\alpha, \mathcal{T}'_\beta$  and  $\mathcal{T}'_\gamma$  of  $G'$  are compatible as expected.  $\square$

**Lemma 14** *Let  $G$  be a cubic 3-edge colourable graph with a proper 3-edge colouring.  $\{\alpha, \beta, \gamma\}$ . Assume that  $G$  has three compatible normal odd partitions  $\mathcal{T}_\alpha, \mathcal{T}_\beta, \mathcal{T}_\gamma$  which are, respectively, conformal to  $\alpha, \beta$  and  $\gamma$ . Then  $G'$  obtained from  $G$  in expanding a vertex by a triangle has also this property.*

**Proof** Let us expand a vertex  $x$  of  $G$  by a triangle say  $abc$ . We colour the edges  $ab, ac$ , and  $bc$  in order to get a proper 3-edge colouring in  $G'$ . Assume w.l.o.g. that the edge  $ab$  (resp.  $bc, ac$ ) is coloured  $\beta$  (resp.  $\alpha, \gamma$ ).

We suppose that  $e_{\mathcal{T}_\alpha}(x) = x_\gamma$ . The vertex  $x$  is an internal vertex of some trail in  $\mathcal{T}_\alpha$ , say  $T_{\alpha,x}$ , we replace  $x$  in  $T_{\alpha,x}$  with  $aa_\beta bb_\alpha c$  and we add the trail of length 1,  $aa_\gamma c$ . Thus we get a normal odd partition of the edge set of  $G'$  whose all trails have odd edges of colour  $\alpha$ .

We proceed in a similar way for  $\mathcal{T}'_\beta$  and  $\mathcal{T}_\gamma$  and we get three compatible normal odd partitions with the desired property.  $\square$

Let  $\mu$  be some colour in  $\{\alpha, \beta, \gamma\}$ , we denote  $T_{\mu,v}$  the trail of  $\mathcal{T}_\mu$  for which  $v$  is an internal vertex while  $T'_{\mu,v}$  denotes the trail of  $\mathcal{T}_\mu$  ending in  $v$ . Since the odd edges of  $T_{\mu,v}$  have colour  $\mu$  we denote the endpoints of this trail as  $a_{T_{\mu,v}}$  and  $b_{T_{\mu,v}}$  in such a way that the sub-trail of  $T_{\mu,v}$  whose endpoints are  $a_{T_{\mu,v}}$  and  $v$  contains  $v_\mu$  as an end-edge. Thus we write  $T_{\mu,v} = a_{T_{\mu,v}} \dots v_\mu v \dots b_{T_{\mu,v}}$ . In addition  $b'_{T_{\mu,v}}$  denotes the endpoint of  $T'_{\mu,v}$  distinct from  $v$ .

**Lemma 15** *Let  $G$  be a simple cubic 3-edge colourable graph with a proper 3-edge colouring using colours in  $\{\alpha, \beta, \gamma\}$ . Let  $\mathcal{T}_\alpha, \mathcal{T}_\beta, \mathcal{T}_\gamma$  three normal odd partitions conformal to  $\alpha, \beta$  and  $\gamma$  and  $A = A_{\mathcal{T}_\alpha \mathcal{T}_\beta} + A_{\mathcal{T}_\beta \mathcal{T}_\gamma} + A_{\mathcal{T}_\gamma \mathcal{T}_\alpha}$  has minimal size. Let  $\mu, \nu, \xi$  be 3 distinct colors of  $\{\alpha, \beta, \gamma\}$ .*

*If  $x \in A_{\mathcal{T}_\mu \mathcal{T}_\nu}$ ,  $x_\mu$  is an end edge of  $T'_{\xi,x}$  and  $y$  is the neighbor of  $x$  such that  $x_\xi = y_\xi$ . Then*

- $a_{T_{\mu,x}} = x$ ,
- $y \notin A$ ,
- $T'_{\nu,x} = xx_\xi y$ ,
- $T_{\xi,x} = xx_\mu \dots y_\nu y x_\xi x x_\nu \dots y_\mu y$ .

**Proof** Assume that  $a_{T_{\mu,x}} \neq x$ . We use an odd switch of  $\mathcal{T}_\mu$  on  $x$  and we get a normal odd partition  $\mathcal{T}'_\mu$  as follows :

$$\mathcal{T}'_\mu = \mathcal{T}_\mu - \{T_{\mu,x}, T'_{\mu,x}\} \cup \{a_{T_{\mu,x}} \dots x_\mu x x_\xi \dots b'_{T_{\mu,x}}, x x_\nu \dots b_{T_{\mu,x}}\}$$

Observe that the trails  $a_{T_{\mu,x}} \dots x_\mu x x_\xi \dots b'_{T_{\mu,x}}$  and  $x x_\nu \dots b_{T_{\mu,x}}$  of  $\mathcal{T}'_\mu$  are odd and their odd edges of colour  $\mu$ . Moreover  $e_{\mathcal{T}'_\mu}(x) = x_\nu$ , consequently we have

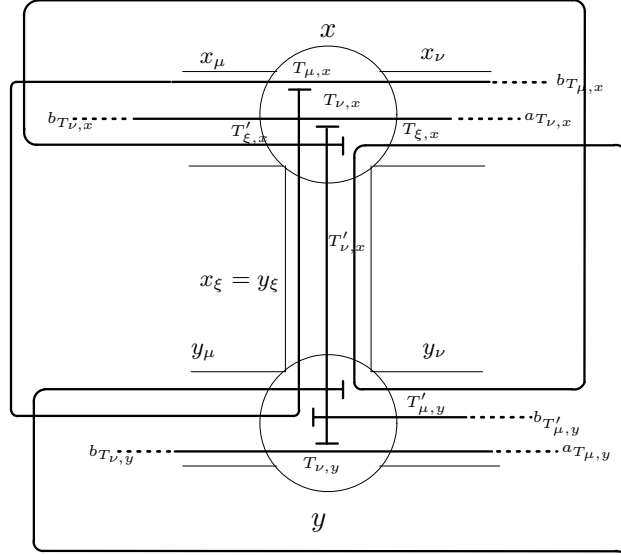
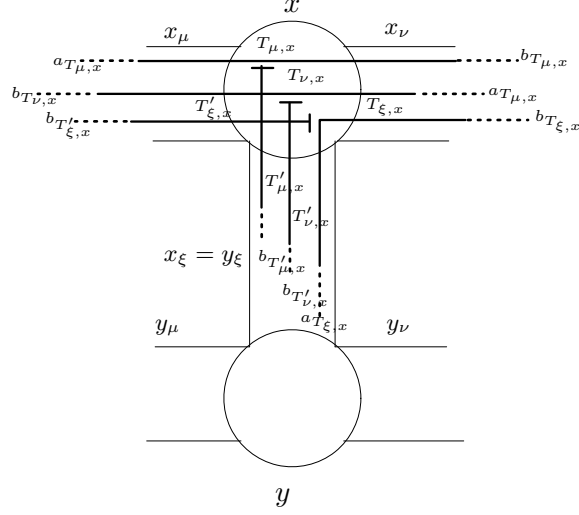


Fig. 5. Situation in Lemma 15

a contradiction with  $A_{T'_\mu T_\nu} \cup A_{T_\nu T_\xi} \cup A_{T'_\mu T_\xi}$  whose size is  $|A| - 1$ . Thus  $a_{T_{\mu, x}} = x$  and  $T_{\mu, x} = T'_{\mu, x}$ .

Assume that  $y \in A$ . Thus whatever the odd switches we will perform on  $y$ , this will not increase the size of  $A$ . We get a normal odd partition  $T'_\mu$  from  $T_\mu$  in using odd switches of  $T_\mu$  on  $y$  and  $x$ . More precisely we write

$$T'_\mu = T_\mu - \{T_{\mu, x}\} \cup \{yy_\mu \dots x_\mu xx_\xi yy_\nu \dots b_{T'_{\nu, y}}, xx_\nu \dots b_{T_{\mu, x}}\}.$$

Once again, when performing those operations we get 3 odd normal partitions which are compatible on  $x$ , a contradiction with the fact that  $A$  has minimal size.

Since  $T_{\mu,x}$  contains the edge  $y_\mu$ , the vertex  $y$  and ends on  $x$  the edge  $y_\nu$  must be an end-edge of  $T'_{\mu,y}$ . moreover  $y$  not being in  $A$ , the trail  $T_{\xi,y}$  must contain the edge  $y_\nu$  and thus  $y_\mu$  is an end-edge of  $T'_{\xi,y}$ , it follows that  $x_\xi$  is an end-edge of  $T'_{\nu,y}$ , that is the trail  $T'_{\nu,x}$  has length 1 namely  $T'_{\nu,x} = xx_\xi y = T'_{\nu,y}$ .

If  $a_{T_{\xi,y}} \neq y$  we can perform 3 odd switches (of  $\mathcal{T}_\mu$ ,  $\mathcal{T}_\nu$  and  $\mathcal{T}_\xi$ ) on  $y$  and the odd switch of  $\mathcal{T}_\mu$  on  $x$ , in other words we write :

$$\begin{aligned} T'_\mu &= \mathcal{T}_\mu - \{T_{\mu,x}\} \cup \{yy_\mu \dots x_\mu xxx_\xi yy_\nu \dots b_{T'_{\mu,y}}\} \\ T'_\nu &= \mathcal{T}_\nu - \{T_{\nu,y}, xx_\xi y\} \cup \{b_{T_{\nu,y}} \dots y_\mu y, xx_\xi yy_\nu \dots a_{T_{\nu,y}}\} \\ T'_\xi &= \mathcal{T}_\xi - \{T_{\xi,y}, T'_{\xi,y}\} \cup \{b_{T'_{\xi,y}} \dots y_\mu yy_\xi xx_\nu \dots a_{T_{\xi,y}}, yy_\nu \dots b_{T_{\xi,y}}\} \end{aligned}$$

But now,  $A_{T'_\mu T'_\nu} \cup A_{T'_\nu T'_\xi} \cup A_{T'_\mu T'_\xi}$  has less vertices than  $A$ , a contradiction. Thus  $T_{\xi,y} = T_{\xi,x} = yy_\mu \dots x_\nu xxx_\xi yy_\nu \dots b_{T_{\xi,y}}$ . When  $b_{T_{\xi,y}} \neq x$  we perform the odd switches of  $\mathcal{T}_\nu$  and  $\mathcal{T}_\xi$  on  $x$  and we get 2 new normal odd partitions, that is :

$$T'_\nu = \mathcal{T}_\nu - \{T_{\nu,x}, xx_\xi y\} \cup \{yx_\xi xx_\nu \dots a_{T_{\nu,x}}, xx_\mu \dots b_{T_{\nu,x}}\}$$

if  $a_{T_{\nu,x}} \neq x$  or

$$T'_\nu = \mathcal{T}_\nu - \{T_{\nu,x}, xx_\xi y\} \cup \{yx_\xi xx_\nu \dots x_\mu x\}$$

otherwise.

In addition we set

$$T'_\xi = \mathcal{T}_\xi - \{T_{\xi,y}, T'_{\xi,x}\} \cup \{b_{T'_{\xi,x}} \dots x_\mu xxx_\xi yy_\nu \dots b_{T_{\xi,y}}\},$$

it follows that  $\mathcal{T}_i, T'_\nu$  and  $T'_\xi$  are compatible on  $x$ , a contradiction. Consequently  $T_{\xi,x} = xx_\mu \dots y_\nu yy_\xi xx_\nu \dots y_\mu y$  which proves the Lemma.  $\square$

**Theorem 16** *Let  $G$  be a cubic 3-edge colourable graph with a proper 3-edge colouring. Then  $G$  has three compatible normal odd partitions  $\mathcal{T}_\alpha$ ,  $\mathcal{T}_\beta$  and  $\mathcal{T}_\gamma$  which are conformal respectively to  $\alpha$ ,  $\beta$  and  $\gamma$ .*

**Proof** Assume that  $G$  has a 3 edge-colouring using colours in  $\{\alpha, \beta, \gamma\}$ . From Lemma 13 and Lemma 14, we can suppose that  $G$  has no multiple edge and no triangle. Let us consider 3 odd normal partitions  $\mathcal{T}_\alpha$ ,  $\mathcal{T}_\beta$ ,  $\mathcal{T}_\gamma$  such that the odd edges are coloured  $\alpha$ ,  $\beta$  and  $\gamma$  respectively. Assume that  $\mathcal{T}_\alpha$ ,  $\mathcal{T}_\beta$  and  $\mathcal{T}_\gamma$  are such that the size of  $A$  is minimum. We suppose that  $A \neq \emptyset$  otherwise  $\mathcal{T}_\alpha$ ,  $\mathcal{T}_\beta$  and  $\mathcal{T}_\gamma$  are compatible and we are done.

Let  $u \in A$ , w.l.o.g we suppose that  $u \in A_{\mathcal{T}_\alpha \mathcal{T}_\beta}$ . Henceforth,  $e_{\mathcal{T}_\alpha}(u) = u_\gamma = e_{\mathcal{T}_\beta}(u)$ . Let  $v$  be the neighbor of  $u$  incident to  $u_\gamma$ .

CLAIM 1  $a_{\mathcal{T}_\alpha, u} = u$ ,  $v \notin A$ ,  $T'_{\beta, u} = uu_\xi v$ , and  $T_{\gamma, u} = uu_\alpha \dots v_\beta vv_\gamma uu_\beta \dots v_\alpha v$ .



**Proof** We just apply Lemma 15 on  $\mathcal{T}_\alpha$ ,  $\mathcal{T}_\beta$  and  $\mathcal{T}_\gamma$  with  $u = x$ ,  $v = y$ ,  $\alpha = \mu$ ,  $\beta = \nu$  and  $\gamma = \xi$ . ■

Let  $v_1$  and  $v_2$  be the neighbors of  $v$  such that  $v_1$  is incident to  $v_\alpha$  and  $v_2$  is incident to  $v_\beta$ .

CLAIM 2  $v_2 \notin A$  and  $b_{T'_{\alpha,v}} = v_2$ .

**Proof** Recall that  $T_{\gamma,u} = uu_\alpha \dots v_\beta v u_\gamma u u_\beta \dots v_\alpha v$  and  $T'_{\beta,u} = uu_\gamma v$ . In using odd switches of  $\mathcal{T}_\beta$  and  $\mathcal{T}_\gamma$  on  $u$  and the odd switch of  $\mathcal{T}_\gamma$  on  $v_2$  we obtain :

$$\mathcal{T}'_\beta = \mathcal{T}_\beta - \{T'_{\beta,u}, T_{\beta,u}\} \cup \{uu_\alpha \dots b_{T_{\beta,u}}, vv_\gamma uu_\beta \dots a_{T_{\beta,u}}\}$$

$$\mathcal{T}'_\gamma = \mathcal{T}_\gamma - \{T_{\gamma,u}, T'_{\gamma,v_2}\} \cup \{uu_\beta \dots v_1 v_\alpha v, u_2 v_\beta v u_\gamma u u_\alpha \dots v_2 \dots b_{T'_{\gamma,v_2}}\}$$

If  $v_2 \in A$ , we have  $A_{\mathcal{T}_\alpha \mathcal{T}'_\beta} \cup A_{\mathcal{T}'_\beta \mathcal{T}'_\gamma} \cup A_{\mathcal{T}_\alpha \mathcal{T}'_\gamma} = A - \{u\}$ , a contradiction with the choice of  $\mathcal{T}_\alpha$ ,  $\mathcal{T}_\beta$  and  $\mathcal{T}_\gamma$ .

Consequently, the edge  $v_\beta$  being an internal edge of  $T_{\beta,v} \in \mathcal{T}_\beta$  and an internal edge of  $T_{\gamma,u} \in \mathcal{T}_\gamma$  and since  $v, v_2 \notin A$ , by Proposition 8, the trail  $T'_{\alpha,v}$  has length 1, we can write  $T'_{\alpha,v} = vv_\beta v_2$ , that is  $b_{T'_{\alpha,v}} = v_2$ . ■

Let us use odd switches of  $\mathcal{T}_\alpha$  on  $u$  and  $v$  as well as an odd switch of  $\mathcal{T}_\beta$  on  $v$ , in other words we get the odd normal partitions :

$$\mathcal{T}'_\alpha = \mathcal{T}_\alpha - \{T_{\alpha,u}, T'_{\alpha,v}\} \cup \{uu_\beta \dots b_{T_{\alpha,u}}, vv_\gamma uu_\alpha \dots v_1 v_\alpha vv_\beta \dots b_{T'_{\alpha,v}}\}$$

and

$$\mathcal{T}'_\beta = \mathcal{T}_\beta - \{T_{\beta,v}, T'_{\beta,v}\} \cup \{uu_\gamma vv_\beta \dots a_{T_{\beta,v}}, vv_\alpha \dots b_{T'_{\beta,v}}\}$$

But now, we apply Lemma 15 with  $x = v$ ,  $y = v_1$ ,  $\mu = \gamma$ ,  $\nu = \beta$ ,  $\xi = \alpha$  and with  $A' = A_{\mathcal{T}'_\alpha \mathcal{T}'_\beta} \cup A_{\mathcal{T}'_\beta \mathcal{T}'_\gamma} \cup A_{\mathcal{T}'_\alpha \mathcal{T}'_\gamma} = A - \{u\} \cup \{v\}$ . It follows that the trail of  $\mathcal{T}'_\alpha$  having  $v_\alpha$  as an internal edge, namely  $vv_\gamma uu_\alpha \dots v_1 v_\alpha vv_\beta \dots b_{T'_{\alpha,v}}$  must have  $v$  and  $v_1$  as endpoints, a contradiction with Claim 2. □

#### 4 On cubic graphs with chromatic index 4

A Snark is a bridgeless cubic graph with edge chromatic number of four. By Proposition 10 a cubic graph with three compatible normal odd partitions must be bridgeless. thus in this section we consider the problem of providing three compatible normal odd partitions for some known Snarks as an extension of the Petersen graph as well as the family of Flower Snarks.

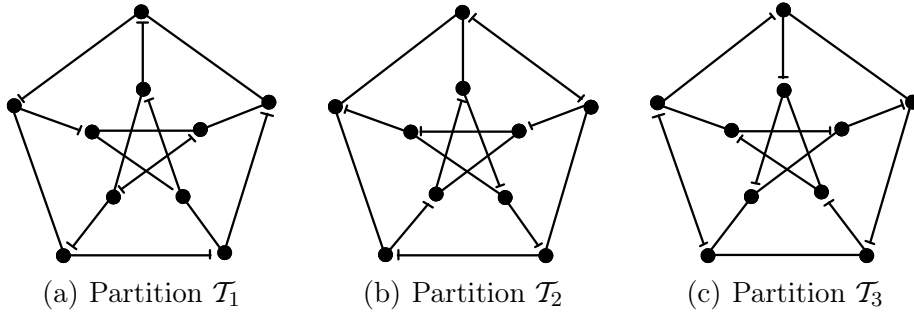


Fig. 6. three compatible normal odd partitions of the Petersen graph

#### 4.1 An extension of the Petersen graph

In Figures 6(a), 6(b) and 6(c) we give three compatible normal odd partitions of the Petersen graph. It can be pointed out that these three compatible normal odd partitions are isomorphic. Indeed, we have, in each partition, a path of length 5, 3 paths of lengths 3 and 1 path of length 1. In some sense this shows that Theorem 12 is best possible.

We have seen that, given a cubic graph with three compatible normal odd partitions, we can obtain a new cubic graph with the same property in subdividing an edge with 2 vertices and adding an edge between these two vertices (see Lemma 13) or in expanding a vertex with a triangle (see Lemma 14). We must say that these Lemmas are proved in a more restrictive context since we deal with conformal odd partitions, but it is an easy task to see that these lemmas remain true when we drop this constraint. We do not know whether the converse operation of the expansion (shrinking a triangle into a single vertex) allow us to preserve three compatible normal odd partitions in the resulting graph.

Our purpose now is to construct an example of a cubic graph with chromatic index 4 and three compatible normal odd partitions starting from two copies of the Petersen graph and using a restricted form of the classical *composition* (see Figure 7) operation. It's a folklore's result that such a composition has chromatic index 4 as soon as one of the two graphs involved has itself chromatic index 4. The interesting point is that we have provided the Petersen graph with three compatible normal odd partitions but two of them are odd while the third one contains 2 even trails.

Assume that  $G$  and  $G'$  are two copies of a cubic graph. In order to be more concise, when  $O$  denote an object (a vertex, a trail ...) of  $G$ ,  $O'$  will denote the corresponding object (using the obvious isomorphism) of  $G'$ . Let  $u \in V(G)$  and assume that  $N(u) = \{a, b, c\}$  (we do not claim that these vertices are distinct since we do not consider only simple graphs). The *composition on*  $\{u, u'\}$  of  $G$  and  $G'$  is obtained in deleting  $u$  in  $G$  and  $u'$  in  $G'$  and joining

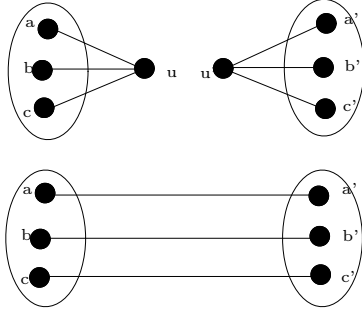
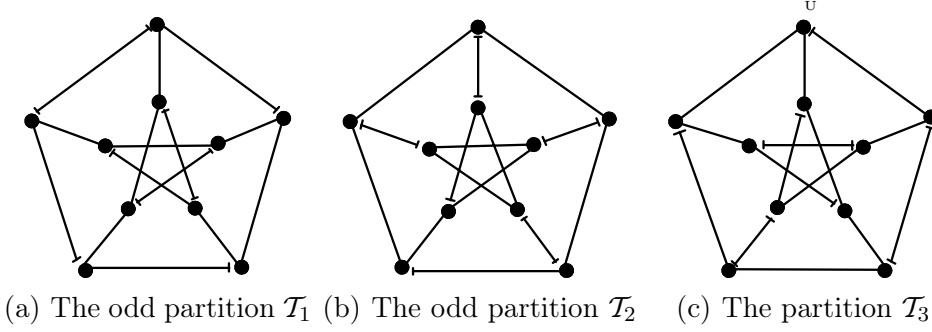


Fig. 7. Composition of two cubic graphs



(a) The odd partition  $\mathcal{T}_1$  (b) The odd partition  $\mathcal{T}_2$  (c) The partition  $\mathcal{T}_3$

Fig. 8. The partitions of the Petersen graph considered in Proposition 18

$a$  to  $a'$ ,  $b$  to  $b'$  and  $c$  to  $c'$ . The restriction of the composition invoked above comes from the fact that we impose to join similar vertices.

**Proposition 17** *Let  $G$  and  $G'$  two copies of a cubic graph with  $u \in V(G)$  and  $u' \in V(G')$ . Let  $H$  be the cubic graph obtained by composition on  $\{u, u'\}$ . Let  $\mathcal{T}_1, \mathcal{T}_2$  and  $\mathcal{T}_3$  be three compatible normal odd partitions of  $G$  and  $\mathcal{T}'_1, \mathcal{T}'_2, \mathcal{T}'_3$  the corresponding normal partitions of  $G'$ .*

*Assume that  $u \notin E_{\mathcal{T}_i}(u)$ , for all  $i \in \{1, 2, 3\}$  then, there exists three compatible normal odd partitions  $\mathcal{Q}_1, \mathcal{Q}_2$  and  $\mathcal{Q}_3$  of  $H$  such that*

*for  $i \in \{1, 2, 3\}$*

- $\forall v \in V(G - u) \ e_{\mathcal{Q}_i}(v) = e_{\mathcal{T}_i}(v)$
- $\forall v' \in V(G' - u') \ e_{\mathcal{Q}_i}(v') = e_{\mathcal{T}'_i}(v')$

**Proof** Let  $i \in \{1, 2, 3\}$ , since  $u \notin E_{\mathcal{T}_i}(u)$ , we have, obviously  $u' \notin E_{\mathcal{T}'_i}(u')$ . Hence, in  $G - u$  the trace of the trails  $T_1$  and  $T_2$  of  $\mathcal{T}_i$  using  $u$  give three distinct trails  $A, B$  and  $C$ . It can be noticed that some of them may have length 0. When  $A$ , by example, has length 0 this means that the marked edge  $e_{\mathcal{T}_1}(a)$  associated to  $a$  is  $au$ . The concatenation of  $A, aa'$  and  $A'$  is a trail  $Q_1$  of  $H$  and we define in the same way  $Q_2$  as the concatenation of  $B, bb'$  and  $B'$  and  $Q_3$  with  $C, cc'$  and  $C'$ . Let  $\mathcal{Q}_i$  be the partition of  $H$

$$\mathcal{Q}_i = \{\mathcal{T}_i - \{T_1, T_2\}\} \cup \{\mathcal{T}'_i - \{T'_1, T'_2\}\} \cup \{Q_1, Q_2, Q_3\}$$

It is a routine matter to check that  $\mathcal{Q}_i$  is a normal partition such that  $\forall v \in V(G - u)$   $e_{\mathcal{Q}_i}(v) = e_{\mathcal{T}_i}(v)$  since we only extend  $A$ ,  $B$  and  $C$  with the edges  $aa'$ ,  $bb'$  and  $cc'$  in order to get  $\mathcal{Q}_1$ ,  $\mathcal{Q}_2$  and  $\mathcal{Q}_3$ . This operation does not affect the marked edges in  $G - u$  (be careful, in the example cited above, when  $A$  has length 0,  $\mathcal{Q}_1$  is reduced to  $aa'$  and the marked edge associated to  $a$  is  $aa'$  as well as the marked edge associated to  $a'$  is  $a'a$ ). The same holds with  $e_{\mathcal{Q}_i}(v')$  when  $v' \in V(G' - u')$ .

Since  $\mathcal{Q}_1$  is the concatenation of  $A$ ,  $aa'$  and  $A'$   $\mathcal{Q}_1$  has obviously odd length since  $A$  and  $A'$  are isomorphic trails. Hence  $\mathcal{Q}_1$ ,  $\mathcal{Q}_2$  and  $\mathcal{Q}_3$  have all odd length.

Thus the partitions  $\mathcal{Q}_1$ ,  $\mathcal{Q}_2$  and  $\mathcal{Q}_3$  are three compatible normal odd partitions of  $H$  as claimed.  $\square$

**Proposition 18** *Let  $H$  be the graph obtained by composition of two Petersen graphs on  $\{u, u'\}$  (where  $u$  is any vertex of the Petersen graph). Then  $H$  can be provided with three compatible normal odd partitions.*

**Proof** Figures 8(a), 8(b) and 8(c) show that the Petersen graph can be provided with three compatible normal odd partitions  $\mathcal{T}_1$ ,  $\mathcal{T}_2$  and  $\mathcal{T}_3$ . In Figures 8(a) and 8(b) we have two odd normal partitions  $\mathcal{T}_1$  and  $\mathcal{T}_2$  while Figure 8(b) shows a normal partition  $\mathcal{T}_3$  with exactly two even trails (of length 8 and 4). These two even trails use  $u$ , the one as an internal vertex, the other as an end vertex, moreover, we can check that  $u \notin E_{\mathcal{T}_3}(u)$

Using Proposition 17, we see that  $\mathcal{T}_1$  and  $\mathcal{T}_1'$  give rise to a normal odd partition  $\mathcal{Q}_1$  of  $H$ ,  $\mathcal{T}_2$  and  $\mathcal{T}_2'$  give a normal odd partition  $\mathcal{Q}_2$  while  $\mathcal{T}_3$  and  $\mathcal{T}_3'$  give a normal partition  $\mathcal{Q}_3$ . Moreover  $\mathcal{Q}_1$ ,  $\mathcal{Q}_2$  and  $\mathcal{Q}_3$  are pairwise compatible.

We claim that  $\mathcal{Q}_3$  is odd. Indeed the two even trails of  $\mathcal{T}_3$  are now transformed into the trails using  $aa'$ ,  $bb'$  and  $cc'$  (where  $N(u) = \{a, bc\}$ ) in the Petersen graph). The last item of Proposition 17 insures us that these trails have odd length. By construction of  $\mathcal{Q}_3$  the other trails are trails of  $\mathcal{T}_3$  or  $\mathcal{T}_3'$  which have odd lengths.  $\square$

## 4.2 Flower Snarks

For an odd  $k \geq 3$  the Flower Snark  $F_k$  (see [6]) is the cubic graph on  $4k$  vertices  $u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_k, w_1, w_2, \dots, w_k, t_1, t_2, \dots, t_k$  such that  $u_1 u_2 \dots u_k$  is an induced cycle of length  $k$ ,  $w_1 w_2 \dots w_k t_1 t_2 \dots t_k$  is an induced cycle of length  $2k$  and for  $i = 1 \dots k$  the vertex  $v_i$  is adjacent to  $u_i$ ,  $w_i$  and  $t_i$ .

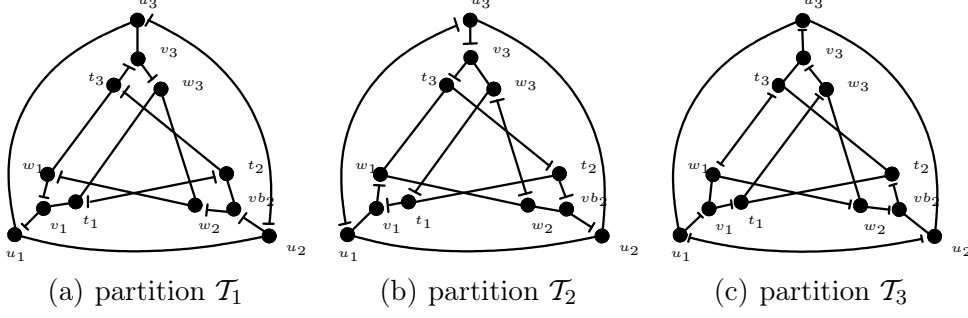


Fig. 9. Three compatible normal odd partitions of the Flower Snark  $F_3$

**Proposition 19** *Let  $F_k$  ( $k \geq 3$ ,  $k$  odd) be a Flower Snark,  $F_k$  can be provided with three compatible normal odd partitions.*

**Proof** In Figure 9 we propose three compatible normal odd partitions, namely  $\mathcal{T}_1$ ,  $\mathcal{T}_2$  and  $\mathcal{T}_3$ , of the Flower Snark  $F_k$  when  $k = 3$ . Moreover, when considering the vertices  $u_1, v_1, w_1, t_1, u_2, v_2, w_2, t_2$  we have the following situation (recall that  $k = 3$ ):

$$e_{\mathcal{T}_1}(u_1) = u_1v_1, e_{\mathcal{T}_1}(v_1) = v_1w_1, e_{\mathcal{T}_1}(w_1) = w_1w_2, e_{\mathcal{T}_1}(t_1) = t_1t_2 \quad (1)$$

$$e_{\mathcal{T}_2}(u_1) = u_1u_2, e_{\mathcal{T}_2}(v_1) = v_1t_1, e_{\mathcal{T}_2}(w_1) = w_1v_1, e_{\mathcal{T}_2}(t_1) = t_1w_2 \quad (2)$$

$$e_{\mathcal{T}_3}(u_1) = u_1u_2, e_{\mathcal{T}_3}(v_1) = v_1u_1, e_{\mathcal{T}_3}(w_1) = w_1t_2, e_{\mathcal{T}_3}(t_1) = t_1v_1 \quad (3)$$

$$e_{\mathcal{T}_1}(u_2) = u_2u_3, e_{\mathcal{T}_1}(v_2) = v_2u_2, e_{\mathcal{T}_1}(w_2) = w_2v_2, e_{\mathcal{T}_1}(t_2) = t_2t_1 \quad (4)$$

$$e_{\mathcal{T}_2}(u_2) = u_2v_2, e_{\mathcal{T}_2}(v_2) = v_2t_2, e_{\mathcal{T}_2}(w_2) = w_2w_3, e_{\mathcal{T}_2}(t_2) = t_2t_3 \quad (5)$$

$$e_{\mathcal{T}_3}(u_2) = u_2u_1, e_{\mathcal{T}_3}(v_2) = v_2w_2, e_{\mathcal{T}_3}(w_2) = w_2w_1, e_{\mathcal{T}_3}(t_2) = t_2v_2 \quad (6)$$

Observe that, among the edges  $u_1u_2, w_1w_2$  and  $t_1t_2$  we have :

$u_1u_2$  is the unique odd edge in  $\mathcal{T}_1$ ,

$t_1t_2$  is the unique odd edge in  $\mathcal{T}_2$  and in  $\mathcal{T}_3$

Assume by induction that for an odd integer  $k$ ,  $k \geq 3$ , the Flower Snark  $F_k$  is provided with three compatible normal odd partitions, namely  $\mathcal{T}_1$ ,  $\mathcal{T}_2$  and  $\mathcal{T}_3$ . Moreover, assume that the above Properties (1) to (6) are verified by  $\mathcal{T}_1$ ,  $\mathcal{T}_2$  and  $\mathcal{T}_3$ .

We derive  $F_{k+2}$  from  $F_k$  as follows :

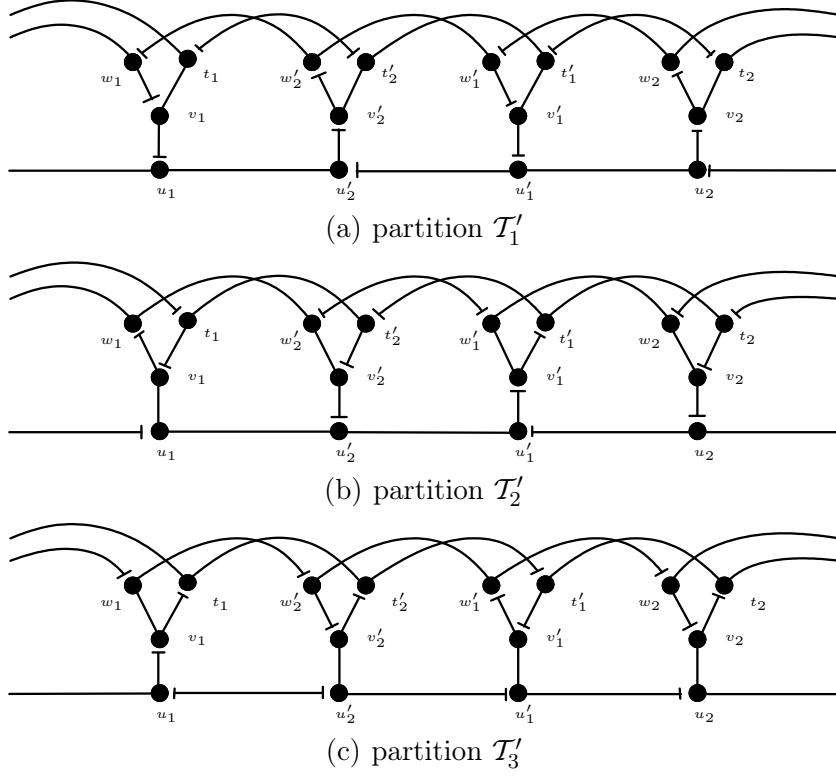


Fig. 10. Extension of three compatible normal odd partitions from the Flower Snark  $F_k$  to the Flower Snark  $F_{k+2}$

$$\begin{aligned}
V(F_{k+2}) &= V(F_k) \cup \{u'_1, v'_1, w'_1, t'_1, u'_2, v'_2, w'_2, t'_2\} \\
E(F_{k+2}) &= E(F_k) - \{u_1u_2, w_1w_2, t_1t_2\} \\
&\quad \cup \{u_1u'_2, w_1w'_2, t_1t'_2\} \\
&\quad \cup \{v'_2u'_2, v'_2w'_2, v'_2t'_2\} \\
&\quad \cup \{u'_2u'_1, w'_2w'_1, t'_2t'_1\} \\
&\quad \cup \{v'_1u'_1, v'_1w'_1, v'_1t'_1\} \\
&\quad \cup \{u'_1u_2, w'_1w_2, t'_1t_2\}
\end{aligned}$$

In other words we add 8 new vertices to  $F_k$ , we delete the edges  $u_1u_2$ ,  $w_1w_2$ ,  $t_1t_2$  and add new edges in order to obtain the Flower Snark  $F_{k+2}$ .

Figure 10(a) 10(b) and 10(c) show three normal partitions of  $F_{k+2}$ ,  $\mathcal{T}'_1$ ,  $\mathcal{T}'_2$  and  $\mathcal{T}'_3$ , respectively obtained from  $\mathcal{T}_1$ ,  $\mathcal{T}_2$  and  $\mathcal{T}_3$ .

But now we rename some vertices of  $F_{k+2}$  as follows : For  $i \geq 2$ ,  $u_i$ ,  $v_i$ ,  $w_i$  and  $t_i$  are respectively renamed  $u_{i+2}$ ,  $v_{i+2}$ ,  $w_{i+2}$ ,  $t_{i+2}$ . The vertices  $u'_1$ ,  $v'_1$ ,  $w'_1$  and  $t'_1$  are respectively renamed  $u_3$ ,  $v_3$ ,  $w_3$  and  $t_3$ . The vertices  $u'_2$ ,  $v'_2$ ,  $w'_2$  and  $t'_2$  are respectively renamed  $u_2$ ,  $v_2$ ,  $w_2$  and  $t_2$ .

It's a routine matter to check that those partitions are odd, compatible and verify Properties (1) to (6).  $\square$

## 5 Conjectures

Fan and Raspaud [3] conjectured that any bridgeless cubic graph can be provided with three perfect matchings with empty intersection.

**Theorem 20** *Let  $G$  be a cubic graph with three compatible normal odd partitions then there exist 3 perfect matching  $M$ ,  $M'$  and  $M''$  such that  $M \cap M' \cap M'' = \emptyset$ .*

**Proof** Let  $M$ ,  $M'$  and  $M''$  be the associated perfect matchings of  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$  respectively. Let  $v$  be any vertex and  $u_1, u_2$  and  $u_3$  its neighbors.  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\mathcal{T}''$  being compatible, we can suppose that  $e_{\mathcal{T}}(v) = vu_1$ ,  $e_{\mathcal{T}'}(v) = vu_2$  and  $e_{\mathcal{T}''}(v) = vu_3$ .  $vu_1$  is an end edge of a trail in  $\mathcal{T}$ , this edge is not an odd edge in  $\mathcal{T}$  and thus  $vu_1 \notin M$ . In the same way  $vu_2 \notin M'$  and  $vu_3 \notin M''$ . Hence, any edge incident to  $v$  is contained in at most two perfect matchings among  $M, M'$  and  $M''$ . Which means that  $M \cap M' \cap M'' = \emptyset$

□

Theorem 20 above implies that the Fan Raspaud Conjecture is true for graphs with three compatible normal odd partitions and we propose as a new conjecture:

**Conjecture 21** *Any bridgeless cubic graph can be provided with three compatible normal odd partitions.*

We do not know whether this new conjecture is equivalent to the Fan Raspaud conjecture or not. By the way, the Fan Raspaud conjecture seems to be originated independently by Jackson. Goddyn [5] indeed mentioned this problem proposed by Jackson for  $r$ -graphs ( $r$ -regular graphs with an even number of vertices such that all odd cuts have size at least  $r$ , as defined by Seymour [8]) in the proceedings of a joint summer research conference on graphs minors which dates back 1991.

**Conjecture 22** [5] *There exists  $k \geq 2$  such that any  $r$ -graph contains  $k + 1$  perfect matchings with empty intersection.*

Seymour [8] conjectured that:

**Conjecture 23** [8] *If  $r \geq 4$  then any  $r$ -graph has a perfect matching whose deletion yields an  $(r-1)$ -graph.*

Hence Seymour's conjecture leads to a specialized form of Jackson's conjecture when dealing with cubic bridgeless graphs and the Fan Raspaud conjecture appears as a refinement of Jackson's conjecture.

Let  $\mathcal{S} = \{\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k\}$  ( $k \geq 3$ ) be a set of odd normal partitions of a cubic graph  $G$ . The set  $\mathcal{S}$  will be said a *complete system of odd normal partitions of order  $k$*  whenever for any vertex  $v$  of  $G$  there is three partitions in  $\mathcal{S}$  which are compatible on  $v$ , that is there is  $\mathcal{T}, \mathcal{T}', \mathcal{T}''$  in  $\mathcal{S}$  such that  $e_{\mathcal{T}}(v)$ ,  $e_{\mathcal{T}'}(v)$  and  $e_{\mathcal{T}''}(v)$  are three distinct edges.

**Conjecture 24** *There exists  $k \geq 3$  such that any bridgeless cubic graph has a complete system of odd normal partitions of order at most  $k$ .*

If a cubic graph has a complete system of normal odd partitions of order  $k$  then it has  $k$  perfect matchings with empty intersection. Thus Conjecture 24 implies Conjecture 22 for bridgeless cubic graphs.

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