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# Stability Analysis and Design for Continuous-Time Takagi-Sugeno Control Systems

Mohammed CHADLI, Didier MAQUIN and José RAGOT

## Abstract

This paper discusses conditions on stability and stabilization of continuous Takagi-Sugeno (T-S) systems. Stability analysis is derived via nonquadratic Lyapunov function method and LMIs (Linear Matrix Inequalities) formulation. The nonquadratic Lyapunov function is built by inference of quadratic Lyapunov function on the same basis as the T-S model itself. We show that stability condition of the open-loop T-S systems is assured under certain restrictions on the rate of change of state variables. Following a similar approach, stabilization of the closed-loop continuous T-S systems using the well-known PDC (Parallel Distributed Compensation) technique is investigated. The design methodology is illustrated by numerical examples.

**Keywords:** *Takagi-Sugeno (T-S) models, nonlinear systems, stability analysis, regulators, Lyapunov method, LMIs (Linear Matrix Inequalities), BMIs (Bilinear Matrix Inequalities).*

## 1. Introduction

It is well known that studying the stability of a general non linear model of a process is often very difficult. Thus, we need to develop a simplified model that should capture the essential features of the process and can be used to study the stability and the stabilizability of the non linear process. Such an approach to modeling was proposed by Takagi-Sugeno [32]. Having the property of universal approximation [5, 7], this approach includes the multiple model approach [23] and can be seen also as Polytopic Linear Differential Inclusions (PLDI) [4].

Using a quadratic Lyapunov approach, the stability and stabilization of T-S control systems have been considered extensively. Sufficient stability conditions depending on the existence of a common positive definite matrix are then derived [9, 11, 12, 20, 26, 28, 29]. Convex optimization techniques based on LMIs [4]

have been utilized to derive stability conditions and to solve the multiobjective control problem [15, 22, 31]. In [1] stability conditions are derived in terms of M-matrices and vector Lyapunov functions by regarding the T-S models as an interconnection of LTI subsystems while in [21] they are treated as a linear system having modelling uncertainty. However, if the number of submodels is large, it might be difficult to find a common matrix. Moreover, these constraints are often very conservative and it's well known that, in a lot of cases, such a common positive definite matrix does not exist, whereas the system is stable.

Recently some works have been developed in order to establish new stability conditions by relaxing some of the previous constraints. So one way for obtaining relaxed stability conditions consists of using a piecewise quadratic Lyapunov function formulated as a set of LMIs [6, 10, 17, 25, 27]. While in [17, 27], the continuity of Lyapunov function is carried out by requiring additional constraints, in [25] the function of Lyapunov can be discontinuous. A fuzzy Lyapunov function which is state dependent is also used [3, 8, 16, 18, 30]. This method requires fixing a priori bound on the variation of the state, to prove the stability of the T-S fuzzy systems [8, 16].

Using a nonquadratic Lyapunov function candidate depending on the state of the system, we will derive sufficient condition for stability and stabilization of the continuous T-S system. The first part proposes a method to pick the bound of state variables of T-S continuous models. The second part concerns the stabilization and establishes relaxed conditions using the well-known Parallel Distributed Compensation (PDC) technique [34].

This paper is organized as follows: section 2 presents an overview of T-S systems and gives previous results. In section 3, before giving the main result, propositions to take into account a finite number of rules activated at each time for less of conservatism are developed. At last, using the LMI technique for efficient way of resolution, we develop a method to estimate an upper bound of the state variation to prove the stability of T-S continuous systems. In section 4, using the concept of PDC to perform control laws and LMIs formulation, a similar approach is used for the stabilization of T-S closed loop system. In both cases numerical examples are given to

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illustrate the effectiveness of the proposed stability conditions.

In all that follows, the minimum and maximum eigenvalues of the matrix  $X$  will be respectively denoted by  $\lambda_m(X)$  and  $\lambda_M(X)$ , the norm of  $x$  by  $\|x\|^2 = x^T x$  and the symmetric positive definite matrix (respectively semipositive definite)  $X$  by  $X > 0$  (resp.  $X \geq 0$ ), i.e.  $X \geq Y$  means that the matrix  $X - Y$  is semipositive definite. The following notation is also used:

$$\sum_{i < j} x_i x_j = \sum_{i=1}^n \sum_{j>i}^n x_i x_j, \quad \sum_{i \neq j} x_i x_j = \sum_{i=1}^n \sum_{j=1, j \neq i}^n x_i x_j, \quad \langle x, y \rangle = x^T y$$

where  $(x, y) \in \mathbb{R}^{n,2}$  and  $I_n = \{1, \dots, n\}, \forall n \in \mathbb{N}$

## 2. Continuous Takagi-Sugeno systems and previous results

A T-S fuzzy model [32] is of the following form:

Model rule  $i$ : If  $z_1(t)$  is  $M_{i1}$  and...and  $z_q(t)$  is  $M_{iq}$

$$\text{Then } \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{cases} \quad (1)$$

$z(t) = (z_1(t), \dots, z_q(t))$  is the vector of the premise

variables (called also the decision variable vector),  $M_{ij}$  is a fuzzy set,  $n$  is the number of the If-Then rules (i.e.

the number of submodels),  $x(t) \in \mathbb{R}^p$  is the state vector,

$u(t) \in \mathbb{R}^m$  is the input vector,  $y(t) \in \mathbb{R}^l$  is the output vector,  $A_i \in \mathbb{R}^{p,p}$ ,  $B_i \in \mathbb{R}^{p,m}$  and  $C_i \in \mathbb{R}^{l,p}$ .

The final output of the T-S systems is interpolated as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^n \mu_i(z(t))(A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^n \mu_i(z(t)) C_i x(t) \end{cases} \quad (2)$$

$$\mu_i(z(t)) = \frac{\omega_i(z(t))}{\sum_{i=1}^n \omega_i(z(t))} \quad \text{and} \quad \omega_i(z(t)) = \prod_{j=1}^q M_{ij}(z(t))$$

where  $\mu_i(z(t))$  is the normalized activation function in relation with the  $i^{\text{th}}$  submodel is such that:

$$\begin{cases} \sum_{i=1}^n \mu_i(z(t)) = 1 \\ \mu_i(z(t)) \geq 0 \quad \forall i \in I_n \end{cases} \quad (3)$$

The choice of the variable  $z(t)$  leads to different class

of systems. It can depend on the measurable state variables and possibly on the input; in this case, the system (2) describes a nonlinear system. It can also be an unknown constant value, system (2) then represents a polytopic linear differential inclusion (PLDI). More detail about this type of representation can be found in [2].

It is assumed in this paper that the activation functions depend only on the measurable state of the system and that only  $r$  local models are activated for a given  $x(t)$ , depending on the structure of the activation functions  $\mu_i(x(t))$ .

The unforced T-S system of (2) is defined as follows

$$\dot{x}(t) = \sum_{i=1}^n \mu_i(x(t)) A_i x(t) \quad (4)$$

Let us recall first the basic stability conditions of the system (4) derived using quadratic Lyapunov function [29]. If there exists a common matrix  $P > 0$  such that

$$A_i^T P + P A_i < 0 \quad \forall i \in I_n \quad (5)$$

Then the system described by (4) is globally asymptotically stable. The existence of such a common positive definite matrix  $P$  is a key issue to check the stability of a T-S model. Inequalities (5) give a sufficient condition for ensuring stability of the system described by (4). However, it is well known that, in a lot of cases, a common matrix  $P > 0$  does not exist whereas the T-S model is stable. The following lemma gives sufficient conditions for the non existence of a such matrix  $P$  [4].

**Lemma 1:** If there exists matrices  $X_i, \forall i \in I_n$  not all zero such that

$$X_i \geq 0 \quad \text{and} \quad \sum_{i=1}^n A_i^T X_i + X_i A_i \geq 0$$

then the inequalities (5) do not admit a solution  $P > 0$ .

In some cases, one way to reduce the conservativeness of the quadratic analysis is the use of the so-called S-procedure together with piecewise quadratic Lyapunov functions [10, 17, 25, 27] or the use of multiple Lyapunov functions [6].

## 3. Stability analysis using nonquadratic Lyapunov function

Consider a Lyapunov function candidate of the form:

$$V(x(t)) = x(t)^T P(x(t)) x(t), \quad P(x(t)) = \sum_{i=1}^n \mu_i(x(t)) P_i, \quad P_i > 0 \quad (6)$$

and  $\mu_i(x(t))$  has the properties (3). This is a radially unbounded Lyapunov function [19] since that  $\forall x(t) \in \mathbb{R}^p$

$$c_1 \|x(t)\|^2 \leq V(x(t)) \leq c_2 \|x(t)\|^2 \quad (7a)$$

with

$$c_1 = \max(\lambda_m(P_1), \dots, \lambda_m(P_n)) > 0, c_2 = \min(\lambda_M(P_1), \dots, \lambda_M(P_n)) > 0 \quad (7b)$$

The time-derivative of the Lyapunov function candidate (6) is

$$\dot{V}(x(t)) = \dot{x}(t)^T P(x(t))x(t) + x(t)^T P(x(t))\dot{x}(t) + x(t)^T \dot{P}(x(t))x(t) \quad (8)$$

*Assumption H1:* In the sequel, without loss of generality, the index of the matrices  $P_i, \forall i \in I_n$  are ordered as follows:  $P_1 \geq P_2 \geq \dots \geq P_n$ .

### 3.1. Stability analysis

To study the stability of the unforced T-S system (4), we need the following propositions.

**Proposition 1:** Taking into account the assumption *H1*, the last term of (8) can be bounded as follows

$$x(t)^T \dot{P}(x(t))x(t) \leq \gamma \|\dot{x}(t)\| \|x(t)\|^2 \sum_{i=1}^r \lambda_M(P_i)$$

where  $\gamma = \max_{i \in I_n} \left( \left\| \frac{\partial \mu_i(x(t))}{\partial x(t)} \right\| \right)$  and  $r$  is the number of rules simultaneously activated at each time such that  $2 \leq r \leq n$ .

*Proof:* The last term of the derivative of the state dependent matrix (8) can be written as follows

$$\dot{P}(x(t)) = \sum_{i=1}^n \left\langle \frac{\partial \mu_i(x(t))}{\partial x(t)}, \frac{\partial x(t)}{\partial t} \right\rangle P_i \quad (9)$$

Considering the upper bound of the derivative of the normalized activation functions  $\mu_i(x(t))$ , denoted by

$$\gamma = \max_{i \in I_n} \left\| \frac{\partial \mu_i(x(t))}{\partial x(t)} \right\| \quad (10)$$

and using the assumption *H1*, we obtain

$$x(t)^T \dot{P}(x(t))x(t) \leq \gamma \|\dot{x}(t)\| \sum_{i=1}^r x(t)^T P_i x(t)$$

The result of the proposition 1 follows directly from the above inequality using the properties of the largest eigenvalue of a symmetric positive definite matrix.

**Theorem 1:** Suppose that there exists matrix  $Q > 0$  and  $P_i > 0$  such that the following LMIs hold:

$$A_i^T P_i + P_i A_i < -Q \quad (11a)$$

$$A_i^T P_j + P_j A_i + A_j^T P_i + P_i A_j < -2Q \quad (11b)$$

$\forall (i, j) \in I_n^2, i < j$  and  $\mu_i(x(t))\mu_j(x(t)) \neq 0$ . Then the T-S model (4) is exponentially stable if the following constraint on the state variation is satisfied:

$$\|\dot{x}(t)\| < \frac{\lambda_m(Q)}{\gamma \sum_{i=1}^r \lambda_M(P_i)} \quad (12)$$

*Proof:* Consider the Lyapunov function candidate (6). In order to be a Lyapunov function, its time-derivative (8) along the trajectories of the T-S system (6) must be negative. This time-derivative is

$$\begin{aligned} \dot{V}(x(t)) &= x(t)^T \sum_{i=1}^n \sum_{j=1}^n \mu_i(x(t))\mu_j(x(t)) (A_i^T P_j + P_j A_i) x(t) + \\ &\quad x(t)^T \dot{P}(x(t))x(t) \\ &= x(t)^T \dot{P}(x(t))x(t) + x^T \sum_{i=1}^n \mu_i^2(x) (A_i^T P_i + P_i A_i) x + \\ &\quad x(t)^T \sum_{i < j} \mu_i(x(t))\mu_j(x(t)) (A_i^T P_j + P_j A_i + A_j^T P_i + P_i A_j) x(t) \end{aligned} \quad (13)$$

From (11) and the properties of the normalized activation functions (3) we deduce

$$\begin{aligned} \dot{V}(x(t)) &< -x(t)^T Q x(t) \left( \sum_{i=1}^n \mu_i^2(x(t)) + 2 \sum_{i < j} \mu_i(x(t))\mu_j(x(t)) \right) + \\ &\quad x(t)^T \dot{P}(x(t))x(t) \\ &= -x(t)^T Q x(t) + x(t)^T \dot{P}(x(t))x(t) \end{aligned} \quad (14)$$

Proposition 1 leads to

$$\dot{V}(x(t)) < -x(t)^T Q x(t) + \gamma \|\dot{x}(t)\| \|x(t)\|^2 \sum_{i=1}^r \lambda_M(P_i) \quad (15)$$

Thus, it is easy to conclude from (15) that the system (4) is asymptotically stable if the conditions (12) holds. Moreover, some complementary calculus given in appendix shows that the system is exponentially stable.

### 3.2. Design example

Consider the following example of the form (4) composed of two LTI local models

$$\dot{x}(t) = \sum_{i=1}^2 \mu_i(x_1(t)) A_i x(t) \quad (16a)$$

The state evolution matrices and the normalized activation functions are respectively

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix} \quad (16b)$$

$$\mu_1(x_1(t)) = 1 - \frac{x_1(t)^2}{3}, \mu_2(x_1(t)) = \frac{x_1(t)^2}{3}$$

with:

$$|x_1(t)| \leq \sqrt{3} \quad (16c)$$

Whereas the simulation results indicate that the system (16) is stable (the proof of stability can be derived using the Popov criterion as it is given in appendix 2), the resolution of the LMIs obtained from (5) shows that there is no quadratic Lyapunov function ensuring stability of the above T-S model. This

verification can be also made by solving the dual problem stated in lemma 1. Stability conditions stated in theorem 1 give the following LMIs:

$$\begin{aligned} A_1^T P_1 + P_1 A_1 &< -Q \\ A_2^T P_2 + P_2 A_2 &< -Q \\ A_1^T P_2 + P_2 A_1 + A_2^T P_1 + P_1 A_2 &< -2Q \end{aligned} \quad (17)$$

By solving LMIs (17) the following values are obtained

$$P_1 = \begin{bmatrix} 0.77 & 0.20 \\ 0.20 & 0.51 \end{bmatrix}, P_2 = \begin{bmatrix} 2.86 & 0.10 \\ 0.10 & 0.44 \end{bmatrix}, Q = \begin{bmatrix} 5.56 & 3.08 \\ 3.08 & 1.77 \end{bmatrix} \quad (18a)$$

Then the inequality (12) allows the condition of stability, namely the upper bound of the state variation of the system, to be deduced:

$$\|\dot{x}(t)\| < 7.64 \quad (18b)$$

Figure 2 shows that, with respect to the constraint (18), all trajectories of the above T-S systems converge to zero.

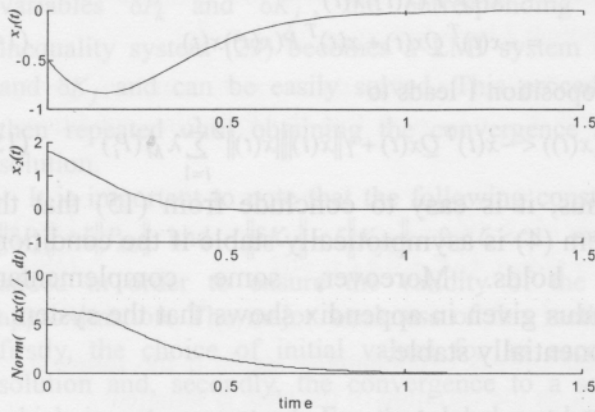


Figure 2. States trajectories of (16) with initial values

## 4. Nonquadratic stabilizability

### 4.1. Stabilizability analysis using a nonquadratic Lyapunov function

In this section, the well-known PDC technique [29] is utilized to design controllers in order to stabilize the T-S system (2). The PDC controller is represented by

$$u(t) = - \sum_{i=1}^n \mu_i(x(t)) K_i x(t) \quad (19)$$

In the following, the pairs  $(A_i, B_i), \forall i \in I_n$  are assumed to be stabilizable. By substituting (19) into (2), the closed-loop T-S system is expressed as:

$$\dot{x}(t) = \sum_{i=1}^n \sum_{j=1}^n \mu_i(x(t)) \mu_j(x(t)) \tilde{A}_{ij} x(t) \quad (20a)$$

$$\tilde{A}_{ij} = A_i - B_i K_j \quad (20b)$$

To study the stability of the closed-loop T-S system (20) and find appropriate  $K_i$  gains, the

following propositions are developed.

**Proposition 2:** Given normalized activation functions as described in (3), the following inequality holds:

$$\sum_{i \neq j}^n \mu_i(x(t)) \mu_j(x(t)) \leq 1 - \frac{1}{r}, \quad \forall 2 \leq r < n \quad (21)$$

where  $r$  is the number of rules simultaneously activated at each time.

**Proposition 3:** Given normalized activation functions as described in (3), the following inequality holds:

$$\sum_{i=1}^n \mu_i^3(x(t)) \geq \frac{1}{r} - \sum_{i \neq j}^n \mu_i(x(t)) \mu_j^2(x(t)), \quad \forall 2 \leq r < n \quad (22)$$

*Proof:* In the following, the proof of the proposition 2 and proposition 3 are successively established. From the properties of the normalized activation function (3), we have

$$\left( \sum_{i=1}^n \mu_i(x(t)) \right)^2 = \sum_{i=1}^n \mu_i^2(x(t)) + \sum_{i \neq j}^n \mu_i(x(t)) \mu_j(x(t))$$

and then

$$\sum_{i=1}^n \mu_i^2(x(t)) = 1 - \sum_{i \neq j}^n \mu_i(x(t)) \mu_j(x(t)) \quad (23)$$

Using the corollary 4 in [29]:

$$\sum_{i=1}^n \mu_i^2(x(t)) \geq \frac{1}{r-1} \sum_{i \neq j}^n \mu_i(x(t)) \mu_j(x(t)) \quad (24)$$

the proposition 2 is deduced. After summation of (23) multiplied by  $\mu_i(x(t))$ , we have

$$\sum_{i=1}^n \mu_i(x(t)) \sum_{i=1}^n \mu_i^2(x(t)) = 1 - \sum_{i \neq j}^n \mu_i(x(t)) \mu_j(x(t)) \quad (26)$$

The left-hand side of (26) can be developed as

$$\sum_{i=1}^n \mu_i(x(t)) \sum_{i=1}^n \mu_i^2(x(t)) = \sum_{i=1}^n \mu_i^3(x(t)) + \sum_{i \neq j}^n \mu_i^2(x(t)) \mu_j(x(t))$$

which leads to

$$\sum_{i=1}^n \mu_i^3(x(t)) = 1 - \sum_{i \neq j}^n \mu_i^2(x(t)) \mu_j(x(t)) - \sum_{i \neq j}^n \mu_i(x(t)) \mu_j(x(t)) \quad (27)$$

Then, taking into account the result (21), the proposition 3 is deduced.

In the following, for less of conservatism, we make the following assumption:

**Assumption H2:** It is assumed that only two rules (sub-models) are activated simultaneously at each time ( $r = 2$ ).

This hypothesis is a very common one especially when using triangular or trapezoidal type activation functions. Let us now define the following matrices where  $\tilde{A}_{ij}$  are defined by (20b):

$$H_{ijk} = \tilde{A}_{ij}^T P_k + P_k \tilde{A}_{ij} \quad (28)$$

**Theorem 2:** Suppose that there exists matrices  $Q > 0$ ,  $P_i > 0$  and  $K_i, i \in I_n$  such that the following conditions hold:

$$H_{iii} < -Q \quad (29a)$$

$$H_{iji} + H_{jii} + H_{ijj} < -Q \quad (29b)$$

$\forall (i, j) \in I_n^2, i \neq j$  and  $\mu_i(x(t))\mu_j(x(t)) \neq 0$ . Then the closed-loop model (20) is exponentially stable if the following constraint on the state variation is satisfied:

$$\|\dot{x}(t)\| < \frac{\lambda_m(Q)}{r \sum_{i=1}^n \lambda_M(P_i)} \quad (30)$$

*Proof:* The time-derivative of the Lyapunov function candidate (8) along the trajectories of the T-S system (20) is

$$\dot{V}(x(t)) = x(t)^T \dot{P}(x(t))x(t) + x(t)^T R(x(t))x(t) \quad (31)$$

with:

$$R(x(t)) = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \mu_i(x(t))\mu_j(x(t))\mu_k(x(t)) (\tilde{A}_{ij}^T P_k + P_k \tilde{A}_{ij}) \quad (32)$$

Let us define the sets of index:

$$U = \{(i, j, k) \in I_n^3 : i \neq j, i \neq k, j \neq k\}$$

The equality (32) can be also written as

$$R(x(t)) = \sum_{i \neq j} \mu_i^2(x(t))\mu_j(x(t))(H_{ijj} + H_{ijj} + H_{iji}) + \quad (33)$$

$$\sum_{(i,j,k) \in U} \mu_i(x(t))\mu_j(x(t))\mu_k(x(t))H_{ijk} + \sum_{i=1}^n \mu_i^3(x(t))H_{iii}$$

Taking into account the assumption H2, the second term of the RHS of (33) can be ignored leading to the following simplified expression

$$R(x(t)) = \sum_{i \neq j} \mu_i^2(x(t))\mu_j(x(t))(H_{ijj} + H_{ijj} + H_{iji}) + \sum_{i=1}^n \mu_i^3(x(t))H_{iii} \quad (34)$$

The inequality (29b) allows to write

$$R(x(t)) < \sum_{i=1}^n \mu_i^3(x(t))H_{iii} - Q \sum_{i \neq j} \mu_i^2(x(t))\mu_j(x(t)) \quad (35)$$

From the proposition 3, the expression (35) can be rewritten as

$$R(x(t)) < \sum_{i=1}^n \mu_i^3(x(t))H_{iii} - Q \left( \frac{1}{r} - \sum_{i=1}^n \mu_i^3(x(t)) \right) \\ = \sum_{i=1}^n \mu_i^3(x(t))(H_{iii} + Q) - \frac{1}{r} Q$$

and from (29a), we obtain  $R(x(t)) < -\frac{1}{r}Q$ . Thus considering (31), and according to proposition 1, we obtain

$$\dot{V}(x(t)) < -x(t)^T \frac{1}{r} Q x(t) + \gamma \|\dot{x}(t)\| \|x(t)\|^2 \sum_{i=1}^r \lambda_M(P_i) \quad (36)$$

Consequently, the condition on the upper bound of state variation (30) guarantees the exponential stability of the closed loop T-S system (20).

If the input matrices are "positively linearly dependant" i.e.

*Assumption H3:*  $\exists B \in \mathbb{R}^{p \times m}$  and  $\alpha_i > 0, i \in I_n$  such that  $B_i = \alpha_i B$

An interesting control law of the form [24]

$$u(t) = - \frac{\sum_{i=1}^n \mu_i(x(t)) \alpha_i K_i x(t)}{\sum_{i=1}^n \mu_i(x(t)) \alpha_i} \quad (37)$$

should be considered instead of the PDC controller (19). Consequently, by substituting (37) into (2), the closed-loop T-S system can be represented as

$$\dot{x}(t) = \sum_{i=1}^n \mu_i(x(t)) \tilde{A}_{ii} x(t) \quad (38)$$

where  $\tilde{A}_{ii}$  is defined in (20b). Notice that in this case, the system (38) is written without the coupled terms (i.e.  $i \neq j$ ) and the stability conditions are obtained directly from theorem 2 by substituting  $A_i$  by  $\tilde{A}_{ii}$ :

**Theorem 3:** Suppose that there exists matrices  $Q > 0$ ,  $P_i > 0$  and  $K_i, i \in I_n$  such that the following conditions hold:

$$\tilde{A}_{ii}^T P_i + P_i \tilde{A}_{ii} < -Q \quad (39a)$$

$$\tilde{A}_{ii}^T P_j + P_j \tilde{A}_{ii} + \tilde{A}_{jj}^T P_i + P_i \tilde{A}_{jj} < -2Q \quad (39b)$$

$\forall (i, j) \in I_n^2, i < j$  and  $\mu_i(x(t))\mu_j(x(t)) \neq 0$ . Then the T-S model (38) is exponentially stable if the following constraint on the variation of the state is satisfied:

$$\|\dot{x}(t)\| < \frac{\lambda_m(Q)}{r \sum_{i=1}^n \lambda_M(P_i)} \quad (40)$$

*Proof:* see theorem 2.

**Remarks:** If the assumption H3 holds

- (1) The control law (37) leads to  $n \frac{(n+1)}{2}$  constraints instead of  $n^2$  with the PDC control law (19).
- (2) The assumption H2 is not required: More precisely, no constraint on the number of simultaneously activated rules is required and the bound on the state variation in the case of the control law (37) is twice more bigger than these with PDC control law.

It should be noted that constraints (29) and (39) are Bilinear Matrix Inequalities (BMIs) in  $P_i$  and  $K_i, \forall i \in I_n$ . We know also that many control problems that require the solution to BMIs can be formulated as LMIs, which may be solved very efficiently. Unfortunately our problem is not convex, its LMI

formulation is very difficult and may have multiple local solutions. In this paper, we use the path-following method, developed in [13], for solving locally BMI problem ((29) or (39)). This method utilizes a first order perturbation approximation to linearize the BMI problem. Hence, the BMIs are converted into a series of LMIs iteratively solved until a desired performance is achieved. Some examples are given in [13, 14], the employed procedure is the following.

Let  $P_{k0}$  and  $K_{j0}$  be initial values such that

$$P_k = P_{k0} + \delta P_k, \quad K_j = K_{j0} + \delta K_j \quad (41)$$

Thus, using the definitions (20b) and neglecting the second order terms  $\delta P_k B_i \delta K_j$  and  $B_i \delta K_j \delta P_k$ , it is possible to express a first order approximation  $\tilde{H}_{ijk}$  of  $H_{ijk}$ :

$$\begin{aligned} \tilde{H}_{ijk} = & A_i^T P_{k0} + P_{k0} A_i - (B_i K_{j0})^T P_{k0} - P_{k0} B_i K_{j0} + A_i^T \delta P_k + \\ & \delta P_k A_i - (B_i K_{j0})^T \delta P_k - \delta P_k B_i K_{j0} - \delta K_j^T (P_{k0} B_i)^T - P_{k0} B_i \delta K_j \end{aligned} \quad (42)$$

As this last equation is linear with regard to the variables  $\delta P_k$  and  $\delta K_j$ , the corresponding matrix inequality system (29) becomes a LMI system in  $\delta P_k$  and  $\delta K_j$  and can be easily solved. This procedure is then repeated until obtaining the convergence of the solution.

It is important to note that the following constraints:  $\|\delta P_k\| < \zeta \|P_{k0}\|$  and  $\|\delta K_j\| < \zeta \|K_{j0}\|$ ,  $0 < \zeta \ll 1$ , must be added in order to ensure the validity of the linear approximation. The major weakness of this method is, firstly, the choice of initial values for an acceptable solution and, secondly, the convergence to a solution which is not guaranteed. For the global resolution of BMIs one will consult the reference [2].

#### 4.2. Design example

Consider the following T-S model of the form (2)

$$\dot{x}(t) = \sum_{i=1}^2 \mu_i(x(t)) (A_i x(t) + B_i u(t)) \quad (43)$$

with  $r = n = 2$ ,

$$\mu_1(x_1(k)) = \frac{(1 - \tanh(x_1(k)))}{2}, \quad \mu_2(x_1(k)) = 1 - \mu_1(x_1(k)),$$

$$\begin{aligned} A_1 = & \begin{bmatrix} 2.31 & -71.83 & 227.6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & B_1 = & \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \\ A_2 = & \begin{bmatrix} 1.604 & -107 & 180 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & B_2 = & \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \end{aligned} \quad (44)$$

One note that a common quadratic Lyapunov function to design the T-S controller and then prove the stabilization of the above closed loop T-S system does

not exist [20, 29]. From conditions (29) given in theorem 2 and with definitions (28) and (20b), we obtain the following four BMIs problem in  $Q$ ,  $P_1$ ,  $P_2$ ,  $K_1$  and  $K_2$

$$\begin{cases} P_1 > 0, P_2 > 0, Q > 0 \\ H_{iii} < -Q \quad \forall i \in I_2 \\ H_{iji} + H_{jii} + H_{ijj} < -Q \quad \forall i \neq j \in I_2 \end{cases} \quad (45)$$

Using the linearization method (42), we obtain:

$$P_1 = \begin{bmatrix} 2.1 & 7.6 & 33.8 \\ 7.6 & 142.1 & 146.8 \\ 33.8 & 146.8 & 1136.8 \end{bmatrix}, P_2 = \begin{bmatrix} 3.3 & 9.8 & 68.8 \\ 9.8 & 155.5 & 201.7 \\ 68.8 & 201.7 & 2320.6 \end{bmatrix} \quad (46a)$$

$$Q = 10^3 \begin{bmatrix} 0.33 & 1.5 & 11.10 \\ 1.5 & 7.08 & 50.63 \\ 11.10 & 50.63 & 375.82 \end{bmatrix} \quad (46b)$$

$$K_1 = [7.39 \ 36.07 \ 258.38], \quad K_2 = [-8.34 \ -24.36 \ -279.52] \quad (47)$$

The condition (30) allows the upper bound of the state variation guaranteeing the exponential stability of the closed loop T-S systems (43) to be computed:  $\|\dot{x}(t)\| < 109.40$

Figure 3 shows that, with respect to the constraint of the state variation, all trajectories of the above T-S systems converge to zero.

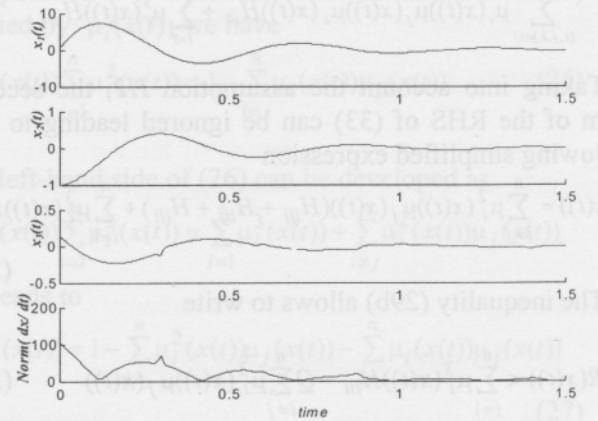


Figure 3: States trajectories with initial values  $x(0) = [1 \ -1 \ 0.1]$

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### Appendix

#### Exponential stability

If constraint (12) holds, the inequality (15) can be written as

$$\dot{V}(x(t)) < -|\beta|x(t)^T x(t) \leq -\sigma\|x(t)\|^2 \quad (48)$$

where  $|\beta| = \lambda_M(Q) - \|\dot{x}\| \sum_{i=1}^r \lambda_M(P_i)$  and  $\sigma$  is a positive real such that  $\sigma \geq |\beta|, \forall x(t)$ . With the property (7), the inequality (48) can be rewritten as:

$$\dot{V}(x(t)) \leq -\frac{\sigma}{c_2} V(x(t)).$$

The solution of the above differential equation is:

$$V(x(t)) \leq V(0)e^{-\frac{\sigma}{c_2}t} \quad (49)$$

which leads to  $\|x(t)\|^2 c_1 \leq \|x(0)\|^2 c_2 e^{-\frac{\sigma}{c_2}t}$ . Hence

$$\|x(t)\| \leq \|x(0)\| \left(\frac{c_2}{c_1}\right)^{1/2} e^{-\frac{\sigma}{2c_2}t} \quad (50)$$

This last expression shows that the constraint (12) ensures the exponential stability of the T-S model (4).

#### Stability domain

The Popov method makes the connection between the Nyquist harmonic analysis and the Lyapunov temporal analysis. The Popov criterion uses a Lyapunov function candidate of the Lure-type:

$$V(x) = x^T P x + 2\mu \int_0^{y=Cx} \varphi(v) dv$$

where  $\mu \geq 0$  is to be chosen. The T-S model (16) can be rewritten as

$$\begin{cases} \dot{x}(t) = Ax(t) - B\varphi(y(t)) \\ y(t) = Cx(t) \end{cases} \quad (52)$$

where  $A = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}; B = \begin{bmatrix} -3 \\ -21 \end{bmatrix}; C = [1 \ 0]; \varphi(y) = \frac{y^3}{3}$

with the following properties:  $A$  is Hurwitz,  $(A, B)$  is controllable, and  $(A, C)$  is observable.

The non linear function  $\varphi(\cdot)$  (figure 4) is such that:  $y \cdot \varphi(y) \geq 0; 0 \cdot y \leq \varphi(y) < K \cdot y$ , where  $K > 0$ . The transfer function of the linear part of the system is

$$G(s) = C(sI - A)^{-1}B, \text{ that is: } G(s) = \frac{-3s + 78}{s^2 + 7s + 6}.$$

The frequency condition for the Popov criterion [19] is given by

$$\Re[(1 + j\omega\mu)G(j\omega)] > \frac{-1}{K}, \forall \omega \in \mathbb{R} \quad (53)$$

If we plot  $\Re[G(j\omega)]$  versus  $\omega \cdot \Im[G(j\omega)]$  (Popov plot)

with  $\omega$  as a parameter, the condition (53) is satisfied if the plot lies to the right of the straight line that intercepts the point  $\frac{-1}{K} + j \cdot 0$  with slope  $\frac{1}{\mu}$  (figure 5).

Indeed one can chose  $K = 1.81$  and  $\mu = 0.14$ , which give the domain of stability:  $x_1 \in [-2.33, +2.33], \forall x_2(t)$

It is instructive to note that, for this value of  $K$ , the circle criterion fails because it is related to the existence of a common Lyapunov function which do not exist for this system.

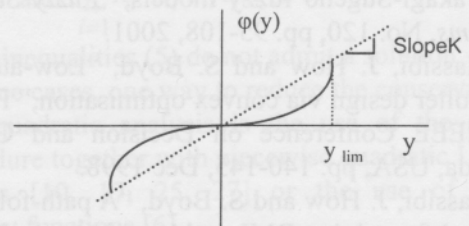


Figure 4. Local sector non linearity

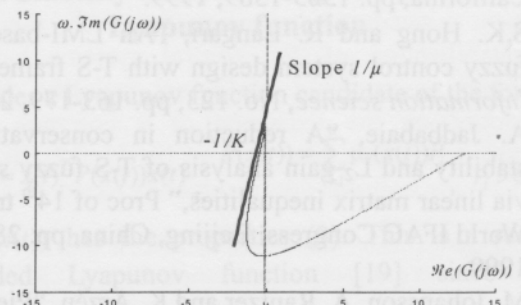
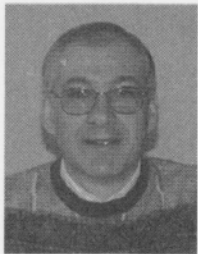


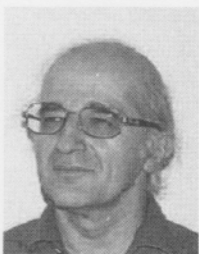
Figure 5. The Popov plot.



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