Periodicity of certain piecewise affine planar maps

Shigeki Akiyama, Horst Brunotte, Attila Petho, Wolfgang Steiner

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SHIGEKI AKIYAMA, HORST BRUNOTTE, ATTILA PETHŐ, AND WOLFGANG STEINER

Abstract. We determine periodic and aperiodic points of certain piecewise affine maps in the Euclidean plane. Using these maps, we prove for $\lambda \in \{\frac{\pm 1 \pm \sqrt{5}}{2}, \pm\sqrt{2}, \pm\sqrt{3}\}$ that all integer sequences $(a_k)_{k\in\mathbb{Z}}$ satisfying $0 \le a_{k-1} + \lambda a_k + a_{k+1} < 1$ are periodic.

1. Introduction

In the past few decades, discontinuous piecewise affine maps have found considerable interest in the theory of dynamical systems. For an overview, we refer the reader to [1, 7, 12, 13, 17, 18], for particular instances to [29, 16, 25] (polygonal dual billiards), [15] (polygonal exchange transformations), [10, 31, 11, 8] (digital filters) and [19, 21, 22] (propagation of round-off errors in linear systems). The present paper deals with a conjecture on the periodicity of a certain kind of these maps:

Conjecture 1.1. [4, 27] *For every real λ with $|\lambda| < 2$, all integer sequences $(a_k)_{k\in\mathbb{Z}}$ satisfying*

$$0 \le a_{k-1} + \lambda a_k + a_{k+1} < 1 \tag{1.1}$$

for all $k \in \mathbb{Z}$ are periodic.

This conjecture originated on the one hand from a discretization process in a rounding-off scheme occurring in computer simulation of dynamical systems (we refer the reader to [19, 27] and the literature quoted there), and on the other hand in the study of shift radix systems (see [4, 2] for details). Extensive numerical evidence on the periodicity of integer sequences satisfying (1.1) was first observed in [26].

We summarize the situation of the Conjecture 1.1. Since we have approximately

$$\begin{pmatrix} a_k \\ a_{k+1} \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} a_{k-1} \\ a_k \end{pmatrix}$$

and the eigenvalues of the matrix are $\exp(\pm\theta\pi i)$ with $\theta \in [0, 1]$, the sequence may be viewed as a discretized rotation on $\mathbb{Z}^2$, and it is natural to parametrize $-\lambda = 2\cos(\theta\pi)$. There are five different classes of λ of apparently increasing difficulty:

(1) θ is rational and λ is rational.
(2) θ is rational and λ is quadratic.
(3) θ is rational and λ is cubic or of higher degree.
(4) θ is irrational and λ is rational.
(5) None of the above.

The first case consists of the three values $\lambda = -1, 0, 1$, where the conjecture is trivially true. Already in case (2) the problem is far from trivial. A computer assisted proof for $-\lambda = \frac{\sqrt{5}-1}{2}$ was given by Lowenstein, Hatjispyros and Vivaldi [19].[1] A short proof (without use of computers) of the golden mean case $\lambda = \frac{1+\sqrt{5}}{2}$ was given by the authors [3]. The main goal of this paper is to settle the conjecture for all the cases of (2), i.e., the quadratic parameters

$$\lambda = \frac{\pm 1 \pm \sqrt{5}}{2},\ \pm\sqrt{2},\ \pm\sqrt{3}.$$

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[1]Indeed, they showed that all trajectories of the map $(x, y) \mapsto (\lfloor(-\lambda)x\rfloor - y, x)$ on $\mathbb{Z}^2$ are periodic.

The proofs are sensitive to the choice of λ, and we have to work tirelessly in computation and drawings, especially in the last case $\pm\sqrt{3}$. However, an important feature of our proof is that it can basically be checked by hand. The (easiest) case $\frac{1+\sqrt{5}}{2}$ in Section 2 gives a prototype of our discussion and should help the reader to understand the idea for the remaining values.

For case (3), it is possible that Conjecture 1.1 can be proved using the same method, which involves a map on $[0,1)^{2d-2}$, where d denotes the degree of λ. However, it seems to be difficult in case $d \geq 3$ to find self inducing structures, which are essential for this method. In [22], a similar embedding into a higher dimensional torus is used for efficient orbit computations. Goetz [12, 13, 14] found a piecewise $\pi/7$ rotation on an isosceles triangle in a cubic case having a self inducing structure, but we do not see a direct connection to our problem.

The problem currently seems hopeless for cases (4) and (5). However, a nice observation on rational values of λ with prime-power denominator p^n is exhibited in [9]. The authors show that the dynamical system given by (1.1) can be embedded into a p-adic rotation dynamics, by multiplying a p-adic unit. These investigations were extended in [30]. Furthermore, in [27] the case $\lambda = q/p$ with p prime was related to the concept of minimal modules, the lattices of minimal complexity which support periodic orbits.

Now we come back to the content of the present paper. The proof in [19] is based on a discontinuous non-ergodic piecewise affine map on the unit square, which dates back to Adler, Kitchens and Tresser [1]. Let $\lambda^2 = b\lambda + c$ with $b, c \in \mathbb{Z}$. Set $x = \{\lambda a_{k-1}\}$ and $y = \{\lambda a_k\}$, where $\{z\} = z - \lfloor z \rfloor$ denotes the fractional part of z. Then we have $a_{k+1} = -a_{k-1} - \lambda a_k + y$ and

$$\{\lambda a_{k+1}\} = \{-\lambda a_{k-1} - \lambda^2 a_k + \lambda y\} = \{-x + (\lambda - b)y\} = \{-x + cy/\lambda\} = \{-x - \lambda' y\},$$

where λ' is the algebraic conjugate of λ. Therefore we are interested in the map $T : [0,1)^2 \to [0,1)^2$ given by $T(x,y) = (y, \{-x - \lambda' y\})$. Obviously, it suffices to study the periodicity of $(T^k(z))_{k\in\mathbb{Z}}$ for points $z = (x,y) \in (\mathbb{Z}[\lambda] \cap [0,1))^2$ in order to prove the conjecture. Using this map, Kouptsov, Lowenstein and Vivaldi [18] showed for all quadratic λ corresponding to rational rotations $\lambda = \frac{\pm1\pm\sqrt{5}}{2}, \pm\sqrt{2}, \pm\sqrt{3}$ that the trajectories of almost all points are periodic, by heavy use of computers. Of course, such metric results do not settle Conjecture 1.1, which deals with countably many points in $[0,1)^2$, which may be exceptional. The main goal of this article is to show that no point with aperiodic trajectory has coordinates in $\mathbb{Z}[\lambda]$, which proves Conjecture 1.1 for these eight values of λ.

This number theoretical problem is solved by introducing a map S, which is the composition of the first hitting map to the image of a suitably chosen self inducing domain under a (contracting) scaling map and the inverse of the scaling map. A crucial fact is that the inverse of the scaling constant is a Pisot unit in the quadratic number field $\mathbb{Q}(\lambda)$. This number theoretical argument greatly reduces the classification problem of periodic orbits, see e.g. Theorem 2.1. All possible period lengths can be determined explicitly and one can even construct concrete aperiodic points in $(\mathbb{Q}(\lambda) \cap [0,1))^2$. We can associate to each aperiodic orbit a kind of β-expansion with respect to the scaling constant. Note that the set of aperiodic points can be constructed similarly to a Cantor set, and that it is an open question of Mahler [23] whether there exist algebraic points in the triadic Cantor set.

The paper is organized as follows. In Section 2, we reprove the conjecture for the simplest non-trivial case, i.e., where λ equals the golden mean. An exposition of our domain exchange method is given in Section 3, where the ideas of Section 2 are extended to a general setting. In the subsequent seven sections we prove the conjecture for the cases $\lambda = -\gamma, \pm 1/\gamma, \pm\sqrt{2}, \pm\sqrt{3}$. Some parts of the proofs for $\lambda = \pm\sqrt{3}$ are put into the Appendix. We conclude this paper by an observation relating the famous Thue-Morse sequence to the trajectory of points for $\lambda = \pm\gamma, \pm 1/\gamma, \sqrt{3}$.

2. The case $\lambda = \gamma = \frac{1+\sqrt{5}}{2} = -2\cos\frac{4\pi}{5}$

We consider first the golden mean $\lambda = \gamma = \frac{1+\sqrt{5}}{2}$, $\lambda^2 = \lambda + 1$. Note that T is given by

$$T(x,y) = (x,y)A + (0, \lceil x - y/\gamma \rceil) \ \text{ with } \ A = \begin{pmatrix} 0 & -1 \\ 1 & 1/\gamma \end{pmatrix}. \tag{2.1}$$

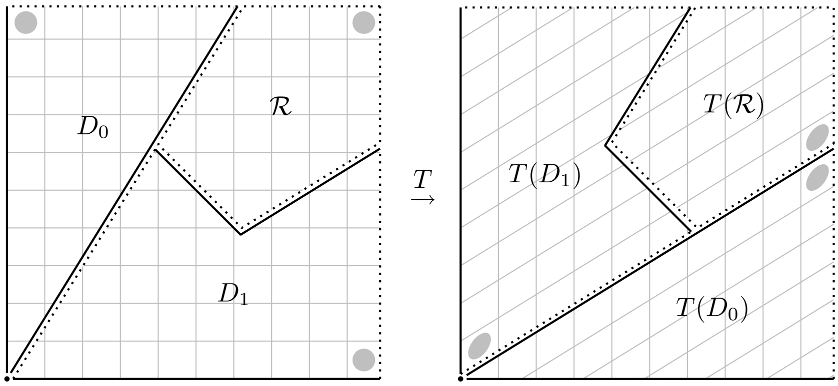

FIGURE 2.1. The piecewise affine map T and the set $\mathcal{R}$, $\lambda = \gamma = \frac{1+\sqrt{5}}{2}$.

Therefore, we have $T(x,y) = (x,y)A$ if $y \geq \gamma x$ and $T(z) = zA + (0,1)$ for the other points $z \in [0,1)^2$, see Figure 2.1. A particular role is played by the set

$$\mathcal{R} = \{(x,y) \in [0,1)^2 : y < \gamma x,\, x+y > 1,\, x < y\gamma\} \cup \{(0,0)\}.$$

If $z \in \mathcal{R}$, $z \neq (0,0)$, then we have $T^{k+1}(z) = T^k(z)A + (0,1)$ for all $k \in \{0,1,2,3,4\}$, hence

$$T^5(z) = zA^5 + (0,1)(A^4 + A^3 + A^2 + A^1 + A^0) = z + (0,1)(A^5 - A^0)(A - A^0)^{-1} = z$$

since $A^5 = A^0$. It can be easily verified that the minimal period length is 5 for all $z \in \mathcal{R}$ except $(\frac{\gamma^2}{\gamma^2+1}, \frac{\gamma^2}{\gamma^2+1})$ and $(0,0)$, which are fixed points of T. Therefore, it is sufficient to consider the domain $\mathcal{D} = [0,1)^2 \setminus \mathcal{R}$ in the following. According to the action of T, we partition $\mathcal{D}$ into two sets D_0 and D_1, with $D_0 = \{(x,y) \in [0,1)^2 : y \geq \gamma x\} \setminus \{(0,0)\}$,

In Figure 2.2, we scale D_0 and D_1 by the factor $1/\gamma^2$ and follow their T-trajectory until the return to $\mathcal{D}/\gamma^2$. Let $\mathcal{P}$ be the set of points in $\mathcal{D}$ which are not eventually mapped to $\mathcal{D}/\gamma^2$, i.e.,

$$\mathcal{P} = D_\alpha \cup T(D_\alpha) \cup D_\beta \cup T(D_\beta) \cup T^2(D_\beta),$$

where D_α is the closed pentagon $\{(x,y) \in D_0 : y \geq 1/\gamma^2, x+y \leq 1, y \leq (1+x)/\gamma\}$ and D_β is the open pentagon $\mathcal{R}/\gamma^2 \setminus \{(0,0)\}$. (In Figure 2.2, D_α is split up into $\{T^k(D_{\tilde{\alpha}}) : k \in \{0,2,4,6,8\}\}$, and D_β is split up into $\{T^k(D_{\tilde{\beta}}) : k \in \{0,3,6,9,12\}\}$.) All points in $\mathcal{P}$ are periodic (with minimal period lengths $2,3,10$ or 15). Figures 2.1 and 2.2 show that the action of the first return map on $\mathcal{D}/\gamma^2$ is similar to the action of T on $\mathcal{D}$, more precisely,

$$\frac{T(z)}{\gamma^2} = \begin{cases} T(z/\gamma^2) & \text{if } z \in D_0, \\ T^6(z/\gamma^2) & \text{if } z \in D_1. \end{cases} \tag{2.2}$$

For $z \in \mathcal{D} \setminus \mathcal{P}$, let $s(z) = \min\{m \geq 0 : T^m(z) \in \mathcal{D}/\gamma^2\}$. (Figure 2.2 shows $s(z) \leq 5$.) By the map

$$S :\ \mathcal{D} \setminus \mathcal{P} \to \mathcal{D}, \quad z \mapsto \gamma^2 T^{s(z)}(z),$$

we can completely characterize the periodic points. For $z \in [0,1)^2$, denote by $\pi(z)$ the minimal period length if $(T^k(z))_{k\in\mathbb{Z}}$ is periodic and set $\pi(z) = \infty$ else.

Theorem 2.1. *$(T^k(z))_{k\in\mathbb{Z}}$ is periodic if and only if $z \in \mathcal{R}$ or $S^n(z) \in \mathcal{P}$ for some $n \geq 0$.*

We postpone the proof to Section 3, where the more general Proposition 3.3 and Theorem 3.4 are proved (with $U(z) = z/\gamma^2$, $R(z) = z$, $\hat{T}(z) = T(z)$, $\hat{\pi}(z) = \pi(z)$, and $z \in D_1$ or $T(z) \in D_1$ for all $z \in \mathcal{D}$, $|\sigma^n(1)| \to \infty$, see below).

(2.2) and Figure 2.2 suggest to define a substitution (or morphism) σ on the alphabet $\mathcal{A} = \{0,1\}$, i.e., a map $\sigma : \mathcal{A} \to \mathcal{A}^*$ (where $\mathcal{A}^*$ denotes the set of words with letters in $\mathcal{A}$), by

$$\sigma :\quad 0 \mapsto 0 \qquad 1 \mapsto 101101$$

in order to code the trajectory of the scaled domains until their return to $\mathcal{D}/\gamma^2$: We have $T^{k-1}(D_\ell/\gamma^2) \subseteq D_{\sigma(\ell)[k]}$ and $T^{|\sigma(\ell)|}(z/\gamma^2) = T(z)/\gamma^2$ for all $z \in D_\ell$, where $w[k]$ denotes the k-th

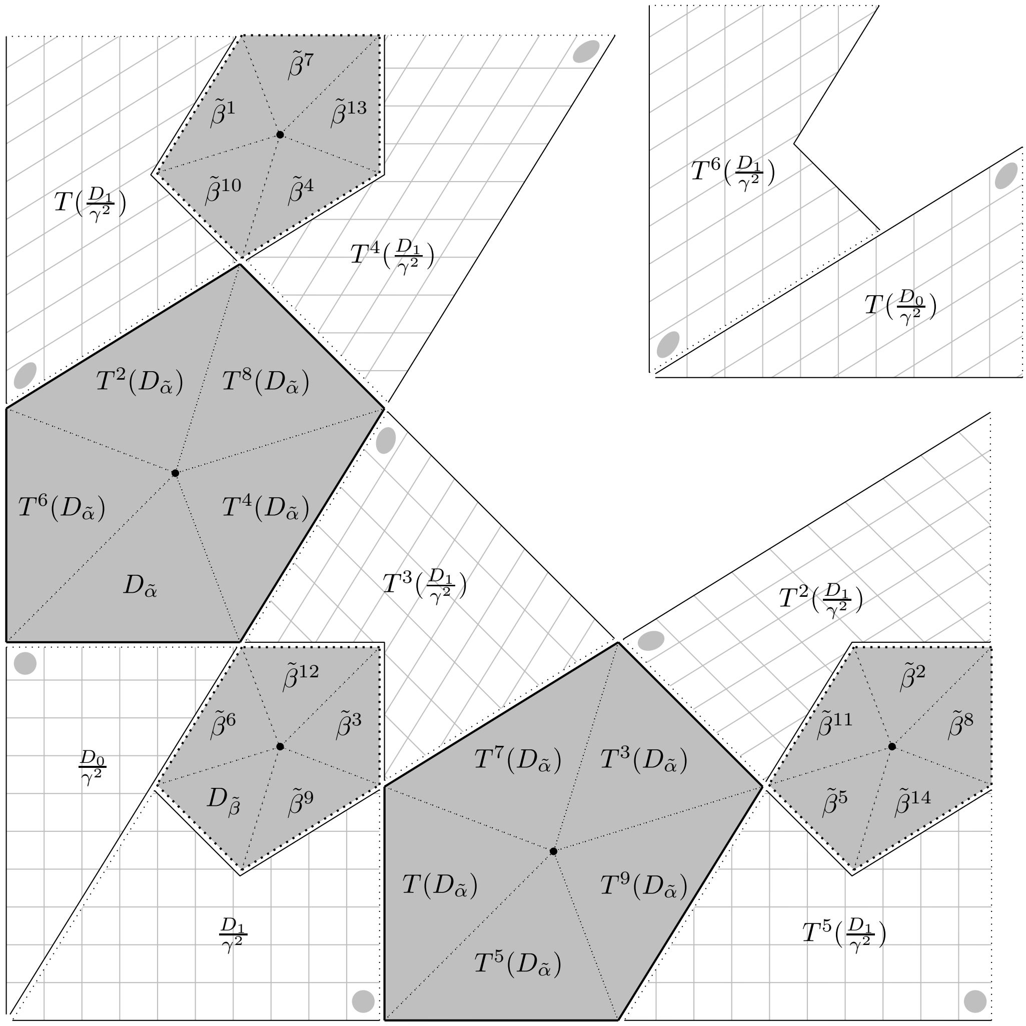

FIGURE 2.2. The trajectory of the scaled domains and the (gray) set $\mathcal{P}$, $\lambda = \gamma$. ($\tilde\beta^k$ stands for $T^k(D_{\tilde\beta})$.)

letter of the word w and $|w|$ denotes its length. Furthermore, we have $T^k(D_\ell/\gamma^2) \cap \mathcal{D}/\gamma^2 = \emptyset$ for $1 \le k < |\sigma(\ell)|$. Extend the definition of σ naturally to words in $\mathcal{A}^*$ by setting $\sigma(vw) = \sigma(v)\sigma(w)$, where vw denotes the concatenation of v and w. Then we get the following lemma, which resembles Proposition 1 by Poggiaspalla [24].

Lemma 2.2. *For every integer $n \ge 0$ and every $\ell \in \{0,1\}$, we have*

- $T^{|\sigma^n(\ell)|}(z/\gamma^{2n}) = T(z)/\gamma^{2n}$ *for all* $z \in D_\ell$,
- $T^{k-1}(D_\ell/\gamma^{2n}) \subseteq D_{\sigma^n(\ell)[k]}$ *for all* k, $1 \le k \le |\sigma^n(\ell)|$
- $T^k(D_\ell/\gamma^{2n}) \cap \mathcal{D}/\gamma^{2n} = \emptyset$ *for all* k, $1 \le k < |\sigma^n(\ell)|$.

The proof is again postponed to Section 3, Lemma 3.1. This lemma allows to determine the minimal period lengths: If $z \in D_\alpha$, then

$$T^{|\sigma^n(0101010101)|}(z/\gamma^{2n}) = T^{|\sigma^n(101010101)|}(T(z)/\gamma^{2n}) = \cdots = T^{10}(z)/\gamma^{2n} = z/\gamma^{2n}$$

for all $n \ge 0$. The only points of the form $T^k(z/\gamma^{2n})$, $1 \le k \le 5|\sigma^n(01)|$, which lie in $\mathcal{D}/\gamma^{2n}$ are the points $T^m(z)/\gamma^{2n}$, $1 \le m \le 9$, which are all different from z/γ^{2n} if $\pi(z) = 10$. Therefore, we obtain $\pi(z/\gamma^{2n}) = 5|\sigma^n(01)|$ in this case. A point $\tilde z$ lies in the trajectory of z/γ^{2n} if and only if $S^n(\tilde z) = T^m(z)$ for some $m \in \mathbb{Z}$, see Lemma 3.2. This implies $\pi(\tilde z) = 5|\sigma^n(01)|$ for these $\tilde z$ as well. The period lengths of all points are given by the following theorem.

Theorem 2.3. *If $\lambda = \gamma$, then the minimal period lengths $\pi(z)$ of $(T^k(z))_{k\in\mathbb{Z}}$ are*

$$\begin{array}{cl} 1 & \text{if } z=(0,0) \text{ or } z=(\frac{\gamma^2}{\gamma^2+1},\frac{\gamma^2}{\gamma^2+1}) \\ 5 & \text{if } z\in\mathcal{R}\setminus\{(0,0),(\frac{\gamma^2}{\gamma^2+1},\frac{\gamma^2}{\gamma^2+1})\} \\ (5\cdot 4^n+1)/3 & \text{if } S^n(z)=T^m(\frac{1/\gamma}{\gamma^2+1},\frac{2}{\gamma^2+1}) \text{ for some } n\ge 0,\ m\in\{0,1\} \\ 5(5\cdot 4^n+1)/3 & \text{if } S^n(z)\in T^m\big(D_\alpha\setminus\{(\frac{1/\gamma}{\gamma^2+1},\frac{2}{\gamma^2+1})\}\big) \text{ for some } n\ge 0,\ m\in\{0,1\} \\ (10\cdot 4^n-1)/3 & \text{if } S^n(z)=T^m(\frac{1}{\gamma^2+1},\frac{1}{\gamma^2+1}) \text{ for some } n\ge 0,\ m\in\{0,1,2\} \\ 5(10\cdot 4^n-1)/3 & \text{if } S^n(z)\in T^m\big(D_\beta\setminus\{(\frac{1}{\gamma^2+1},\frac{1}{\gamma^2+1})\}\big) \text{ for some } n\ge 0,\ m\in\{0,1,2\} \\ \infty & \text{if } S^n(z)\in\mathcal{D}\setminus\mathcal{P} \text{ for all } n\ge 0. \end{array}$$

The minimal period length of $(a_k)_{k\in\mathbb{Z}}$ is $\pi(\{\gamma a_{k-1}\},\{\gamma a_k\})$ (which does not depend on k).

Proof. By Theorem 2.1, Proposition 3.3 and the remarks preceding the theorem, it suffices to calculate $|\sigma^n(0)|$ and $|\sigma^n(1)|$. Clearly, we have $|\sigma^n(0)| = 1$ for all $n \ge 0$ and thus

$$|\sigma^n(1)| = |\sigma^{n-1}(101101)| = 4|\sigma^{n-1}(1)| + 2 = 4(5\cdot 4^{n-1}-2)/3 + 2 = (5\cdot 4^n - 2)/3.$$

If $S^n(z) \in T^m(D_\alpha)$, then $\pi(z) = |\sigma^n(01)|$ and $\pi(z) = 5|\sigma^n(01)|$ respectively. If $S^n(z) \in T^m(D_\beta)$, then $\pi(z) = |\sigma^n(101)|$ and $\pi(z) = 5|\sigma^n(101)|$ respectively. □

Now consider aperiodic points $z \in [0,1)^2$, i.e., $S^n(z) \in \mathcal{D}\setminus\mathcal{P}$ for all $n \ge 0$. We can write

$$S(z) = \gamma^2 T^{s(z)}(z) = \gamma^2\big(zA^{s(z)} + t(z)\big)$$

for some $t(z)$ by using (2.1). Note that $T(z) = zA$ for $z \in D_0$ and $T(z) = zA + (0,1)$ for $z \in D_1$. For $z \in \mathcal{D}/\gamma^2$, we have $s(z) = 0$ and $t(z) = 0$. For $z \in T^k(D_1/\gamma^2)$, $1 \le k \le 5$, we have $s(z) = 6-k$,

$$t(z) = \begin{cases} (0,1) & \text{if } s(z)\in\{1,2\}, \\ (0,1)A^2 + (0,1) = (1/\gamma, 1/\gamma^2) & \text{if } s(z) = 3, \\ (0,1)A^3 + (0,1)A^2 + (0,1) = (0,-1/\gamma) & \text{if } s(z)\in\{4,5\}. \end{cases}$$

We obtain inductively

$$S^n(z) = \gamma^{2n} z A^{s(z)+s(S(z))+\ldots+s(S^{n-1}(z))} + \sum_{k=0}^{n-1} \gamma^{2(n-k)} t(S^k(z)) A^{s(S^{k+1}(z))+\cdots+s(S^{n-1}(z))}.$$

If $z \in \mathbb{Q}(\gamma)^2$, then we have

$$(S^n(z))' = \frac{\Big(zA^{s(z)+s(S(z))+\cdots+s(S^{n-1}(z))}\Big)'}{\gamma^{2n}} + \sum_{k=0}^{n-1} \frac{\Big(t(S^k(z))A^{s(S^{k+1}(z))+\cdots+s(S^{n-1}(z))}\Big)'}{\gamma^{2(n-k)}}$$

$$\|(S^n(z))'\|_\infty \le \frac{\max_{h\in\mathbb{Z}} \|(zA^h)'\|_\infty}{\gamma^{2n}} + \sum_{k=0}^{n-1} \frac{\max_{h\in\mathbb{Z}, w\in\mathcal{D}\setminus\mathcal{P}} \|(t(w)A^h)'\|_\infty}{\gamma^{2n-k}},$$

where $z' = (x', y')$ if $z = (x, y)$ and x', y' are the algebraic conjugates of x, y. Since

$$\begin{aligned} t(z)A^h \in \big\{&(0,0),\ (0,1), (1,1/\gamma), (1/\gamma,-1/\gamma), (-1/\gamma,-1), (-1,0), \\ &(1/\gamma,1/\gamma^2), (1/\gamma^2,-1/\gamma^2), (-1/\gamma^2,-1/\gamma), (-1/\gamma,0), (0,1/\gamma), \\ &(0,-1/\gamma), (-1/\gamma,-1/\gamma^2), (-1/\gamma^2,1/\gamma^2), (1/\gamma^2,1/\gamma), (1/\gamma,0)\big\} \end{aligned}$$

and zA^h takes only the values z, zA, zA^2, zA^3 and zA^4, we obtain

$$\|(S^n(z))'\|_\infty \le \frac{\max_{h\in\mathbb{Z}} \|(zA^h)'\|_\infty}{\gamma^{2n}} + \sum_{k=0}^{n-1} \frac{\gamma^2}{\gamma^{2(n-k)}} < \frac{C(z)}{\gamma^{2n}} + \gamma$$

for some constant $C(z)$. If $z \in (\frac{1}{Q}\mathbb{Z}[\gamma])^2$ for some integer $Q \ge 1$, then $S^n(z) \in (\frac{1}{Q}\mathbb{Z}[\gamma])^2$. Since there exist only finitely many points $w \in (\frac{1}{Q}\mathbb{Z}[\gamma] \cap [0,1))^2$ with $\|w'\|_\infty < C(z) + \gamma$, we must have $\|(S^n(z))'\|_\infty \le \gamma$ for some $n \ge 0$, which proves the following proposition.

Proposition 2.4. *Let $z \in (\frac{1}{Q}\mathbb{Z}[\gamma] \cap [0,1))^2$ be an aperiodic point. Then there exists an aperiodic point $\tilde{z} \in (\frac{1}{Q}\mathbb{Z}[\gamma])^2 \cap \mathcal{D}$ with $\|\tilde{z}'\|_\infty \le \gamma$.*

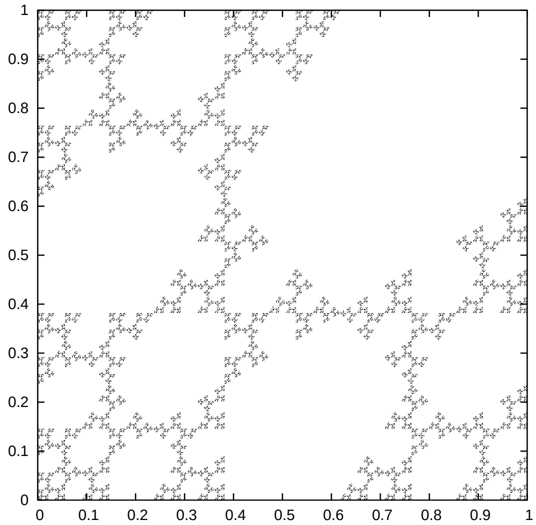

FIGURE 2.3. Aperiodic points, $\lambda = \gamma$.

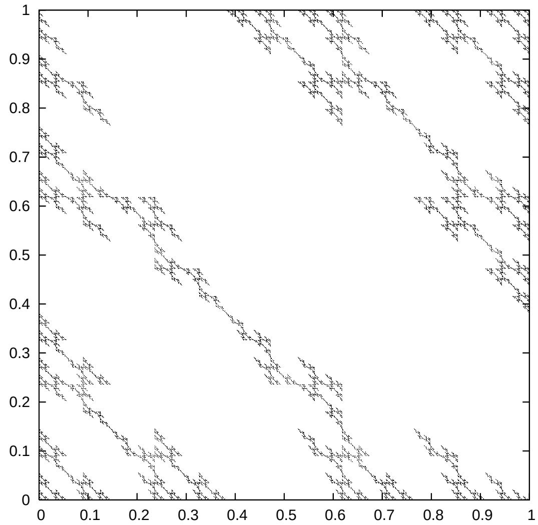

FIGURE 2.4. Aperiodic points, $\lambda = -1/\gamma$.

For every denominator $Q \geq 1$, it is therefore sufficient to check the periodicity of the (finite set of) points $z \in (\frac{1}{Q}\mathbb{Z}[\gamma])^2 \cap \mathcal{D}$ with $\|z'\|_\infty \leq \gamma$ in order to determine if all points in $(\frac{1}{Q}\mathbb{Z}[\gamma] \cap [0,1))^2$ are periodic.

For $Q = 1$, we have to consider $z = (x, y) \in \mathcal{D}$ with $x, y \in \mathbb{Z}[\gamma] \cap [0, 1)$ and $|x'|, |y'| \leq \gamma$, hence $(x, y) \in \{0, 1/\gamma\}^2$. Since $(0, 0)$ and $(1/\gamma, 1/\gamma)$ are in $\mathcal{R}$, it only remains to check the periodicity of $(0, 1/\gamma)$ and $(1/\gamma, 0)$. These two points lie in $\mathcal{P}$, thus Conjecture 1.1 is proved for $\lambda = \gamma$.

For $Q = 2$, a short inspection shows that all points $z \in (\frac{1}{2}\mathbb{Z}[\gamma] \cap [0, 1))^2$ are periodic as well. The situation is completely different for $Q = 3$, and we have

$$S(0, 1/3) = (0, \gamma^2/3), \qquad S(0, \gamma^2/3) = \gamma^2\big((0, \gamma^2/3)A^5 + (0, -1/\gamma)\big) = (0, 2/3),$$
$$S(0, 2/3) = \gamma^2\big((0, 2/3)A^5 + (0, -1/\gamma)\big) = \big(0, 1/(3\gamma^2)\big), \quad S^4(0, 1/3) = S\big(0, 1/(3\gamma^2)\big) = (0, 1/3).$$

This implies $S^n(0, 1/3) \in \mathcal{D} \setminus \mathcal{P}$ for all $n \geq 0$ and $\pi(0, 1/3) = \infty$ by Theorem 2.3.

Theorem 2.5. *$\pi(z)$ is finite for all points $z \in (\mathbb{Z}[\gamma] \cap [0, 1))^2$, but $(T^k(0, 1/3))_{k\in\mathbb{Z}}$ is aperiodic.*

3. General description of the method

In this section, we generalize the method presented in Section 2 in order to make it applicable for $\lambda = -\gamma, \pm 1/\gamma, \pm\sqrt{2}, \pm\sqrt{3}$.

For the moment, we only need that $T : X \to X$ is a bijective map on a set X. Fix $\mathcal{D} \subseteq X$, let

$$\mathcal{R} = \{z \in X : \ T^m(z) \notin \mathcal{D} \text{ for all } m \geq 0\}$$

set $r(z) = \min\{m \geq 0 : T^m(z) \in \mathcal{D}\}$ for $z \in X \setminus \mathcal{R}$, and

$$R : \ X \setminus \mathcal{R} \to \mathcal{D}, \qquad R(z) = T^{r(z)}(z).$$

Let $\hat{T}$ be the first return map (of the iterates by T) on $\mathcal{D}$, i.e.,

$$\hat{T} : \ \mathcal{D} \to \mathcal{D}, \qquad \hat{T}(z) = RT(z) = T^{r(T(z))+1}(z),$$

in particular $\hat{T}(z) = T(z)$ if $T(z) \in \mathcal{D}$. Let $\mathcal{A}$ be a finite set, $\{D_\ell : \ell \in \mathcal{A}\}$ a partition of $\mathcal{D}$ and define a coding map $\iota : \mathcal{D} \to \mathcal{A}^{\mathbb{Z}}$ by $\iota(z) = (\iota_k(z))_{k\in\mathbb{Z}}$ such that $\hat{T}^k(z) \in D_{\iota_k(z)}$ for all $k \in \mathbb{Z}$. Let $U : \mathcal{D} \to \mathcal{D}$, $\varepsilon \in \{-1, 1\}$ and σ a substitution on $\mathcal{A}$ such that, for every $\ell \in \mathcal{A}$ and $z \in D_\ell$,

$$U\hat{T}(z) = \hat{T}^{\varepsilon|\sigma(\ell)|}U(z),$$

$\hat{T}^{\varepsilon k} U(z) \notin U(\mathcal{D})$ for all k, $1 \le k < |\sigma(\ell)|$, and

$$\sigma(\ell) = \begin{cases} \iota_0(U(z))\, \iota_1(U(z)) \cdots \iota_{|\sigma(\ell)|-1}(U(z)) & \text{if } \varepsilon = 1, \\ \iota_{-|\sigma(\ell)|}(U(z)) \cdots \iota_{-2}(U(z))\, \iota_{-1}(U(z)) & \text{if } \varepsilon = -1. \end{cases}$$

Then the following lemma holds.

Lemma 3.1. *For every integer $n \ge 0$, every $\ell \in \mathcal{A}$ and $z \in D_\ell$, we have*

$$U^n \hat{T}(z) = \hat{T}^{\varepsilon^n |\sigma^n(\ell)|} U^n(z),$$

$\hat{T}^{\varepsilon^n k} U^n(z) \notin U^n(\mathcal{D})$ for all k, $1 \le k < |\sigma^n(\ell)|$, and

$$\begin{aligned} \iota_0(U^n(z))\, \iota_1(U^n(z)) \cdots \iota_{|\sigma^n(\ell)|-1}(U^n(z)) &= \sigma^n(\ell) && \text{if } \varepsilon = 1, \\ \iota_0(U^n(z))\, \iota_1(U^n(z)) \cdots \iota_{|\sigma^n(\ell)|-1}(U^n(z)) &= (\sigma\bar{\sigma})^{n/2}(\ell) && \text{if } \varepsilon = -1, \varepsilon^n = 1, \\ \iota_{-|\sigma^n(\ell)|}(U^n(z)) \cdots \iota_{-2}(U^n(z))\, \iota_{-1}(U^n(z)) &= (\sigma\bar{\sigma})^{(n-1)/2}\sigma(\ell) && \text{if } \varepsilon = -1, \varepsilon^n = -1, \end{aligned}$$

where $\bar{\sigma}(\ell) = \ell_m \cdots \ell_2 \ell_1$ if $\sigma(\ell) = \ell_1 \ell_2 \cdots \ell_m$.

Proof. The lemma is trivially true for $n = 0$, and for $n = 1$ by the assumptions on σ. If we suppose inductively that it is true for $n-1$, then let $\sigma(\ell) = \ell_1 \ell_2 \cdots \ell_m$ if $\varepsilon = 1$, $\sigma(\ell) = \ell_m \cdots \ell_2 \ell_1$ if $\varepsilon = -1$, and we obtain (by another induction) for all j, $1 \le j \le m$,

$$\hat{T}^{\varepsilon^n |\sigma^{n-1}(\ell_1 \cdots \ell_{j-1} \ell_j)|} U^n(z) = \hat{T}^{\varepsilon^n |\sigma^{n-1}(\ell_j)|} U^{n-1} \hat{T}^{\varepsilon(j-1)} U(z) = U^{n-1} \hat{T}^{\varepsilon j} U(z). \tag{3.1}$$

If $\varepsilon = 1$, then this follows immediately from the induction hypothesis; if $\varepsilon = -1$, then this follows by setting $k = |\sigma^{n-1}(\ell_j)|$ in

$$\hat{T}^{(-1)^n k} U^{n-1} \hat{T}\big(\hat{T}^{-j} U(z)\big) = \hat{T}^{(-1)^n (k - |\sigma^{n-1}(\ell_j)|)} U^{n-1} \hat{T}^{-j} U(z). \tag{3.2}$$

Therefore, we have

$$\hat{T}^{\varepsilon^n |\sigma^n(\ell)|} U^n(z) = \hat{T}^{\varepsilon^n |\sigma^{n-1}(\ell_1 \cdots \ell_{m-1} \ell_m)|} U^n(z) = U^{n-1} \hat{T}^{\varepsilon m} U(z) = U^{n-1} \hat{T}^{\varepsilon |\sigma(\ell)|} U(z) = U^n \hat{T}(z).$$

If $\varepsilon = 1$, then (3.1) implies that

$$\begin{aligned} \iota_0(U^n(z)) \cdots \iota_{|\sigma^n(\ell)|-1}(U^n(z)) = \big(\iota_0(U^{n-1}U(z)) \cdots \iota_{|\sigma^{n-1}(\ell_1)|-1}(U^{n-1}U(z))\big) \cdots \\ \big(\iota_0(U^{n-1}\hat{T}^{m-1}U(z)) \cdots \iota_{|\sigma^{n-1}(\ell_m)|-1}(U^{n-1}\hat{T}^{m-1}U(z))\big) = \sigma^{n-1}(\ell_1) \cdots \sigma^{n-1}(\ell_m) = \sigma^n(\ell); \end{aligned}$$

if $\varepsilon = -1$ and $\varepsilon^n = 1$, then (3.1) and (3.2) provide

$$\begin{aligned} \iota_0(U^n(z)) \cdots \iota_{|\sigma^n(\ell)|-1}(U^n(z)) &= \big(\iota_{-|\sigma^{n-1}(\ell_1)|}(U^{n-1}T^{-1}U(z)) \cdots \iota_{-1}(U^{n-1}T^{-1}U(z))\big) \\ &\quad \cdots \big(\iota_{-|\sigma^{n-1}(\ell_m)|}(U^{n-1}\hat{T}^{-m}U(z)) \cdots \iota_{-1}(U^{n-1}\hat{T}^{-m}U(z))\big) \\ &= (\sigma\bar{\sigma})^{(n-2)/2}\sigma(\ell_1) \cdots (\sigma\bar{\sigma})^{(n-2)/2}\sigma(\ell_m) = (\sigma\bar{\sigma})^{n/2}(\ell); \end{aligned}$$

if $\varepsilon = -1$ and $\varepsilon^n = -1$, then

$$\begin{aligned} \iota_{-|\sigma^n(\ell)|}(U^n(z)) \cdots \iota_{-1}(U^n(z)) &= \big(\iota_0(U^{n-1}T^{-m}U(z)) \cdots \iota_{|\sigma^{n-1}(\ell_m)|-1}(U^{n-1}T^{-m}U(z))\big) \\ &\quad \cdots \big(\iota_0(U^{n-1}\hat{T}^{-1}U(z)) \cdots \iota_{|\sigma^{n-1}(\ell_1)|}(U^{n-1}\hat{T}^{-1}U(z))\big) \\ &= (\sigma\bar{\sigma})^{(n-1)/2}(\ell_m) \cdots (\sigma\bar{\sigma})^{(n-1)/2}(\ell_1) = (\sigma\bar{\sigma})^{(n-1)/2}\sigma(\ell). \end{aligned}$$

By (3.1), (3.2) and the induction hypothesis, the only points in $(\hat{T}^{\varepsilon^n k} U^n(z))_{1 \le k < |\sigma^n(\ell)|}$ lying in $U^{n-1}(\mathcal{D})$ are $U^n \hat{T}^{\varepsilon j}(z)$, $1 \le j < |\sigma(\ell)|$. Since $\hat{T}^{\varepsilon j}(z) \notin U(\mathcal{D})$ for these j, the lemma is proved. ☐

Remark. If $\tilde{z} = \hat{T}^{-1}(z) \in D_\ell$, then $U^n \hat{T}(\tilde{z}) = \hat{T}^{\varepsilon^n |\sigma^n(\ell)|} U^n(\tilde{z})$, thus $U^n \hat{T}^{-1}(z) = T^{-\varepsilon^n |\sigma^n(\ell)|} U^n(z)$.

As in Section 2, a key role will be played by the map S. Assume that U is injective, let

$$\mathcal{P} = \{z \in \mathcal{D} :\ \hat{T}^m(z) \notin U(\mathcal{D}) \text{ for all } m \in \mathbb{Z}\},$$

fix $\hat{s}(z) = \min\{m \ge 0 : \hat{T}^m(z) \in U(\mathcal{D})\}$ or $\hat{s}(z) = \max\{m \le 0 : \hat{T}^m(z) \in U(\mathcal{D})\}$ for every $z \in \mathcal{D} \setminus \mathcal{P}$, let $s(z) \in \mathbb{Z}$ be such that $\hat{T}^{\hat{s}(z)}(z) = T^{s(z)}(z)$, and define

$$S :\ \mathcal{D} \setminus \mathcal{P} \to \mathcal{D}, \qquad z \mapsto U^{-1} \hat{T}^{\hat{s}(z)}(z) = U^{-1} T^{s(z)}(z).$$

Remark. Allowing $s(z)$ and $\hat{s}(z)$ to be negative decreases the δ in Proposition 3.5 in some cases.

Lemma 3.2. *If $S^n R(z)$ exists, then we have some $m \geq 0$ such that $U^n S^n R(z) = T^m(z)$, and*

$$\tilde{z} = T^m(z) \text{ for some } m \in \mathbb{Z} \text{ if and only if } S^n R(\tilde{z}) = \hat{T}^k S^n R(z) \text{ for some } k \in \mathbb{Z}.$$

Proof. Suppose that $S^n R(z)$ exists. Then we have

$$U^n S^n R(z) = U^{n-1} \hat{T}^{\hat{s}(S^{n-1}R(z))} S^{n-1} R(z) = \hat{T}^{m_1} U^{n-1} S^{n-1} R(z) = \cdots = \hat{T}^{m_1+\cdots+m_n} R(z) = T^m(z)$$

for some $m_1, \ldots, m_n, m \geq 0$.

If $S^n R(\tilde{z}) = \hat{T}^k S^n R(z)$ for some $k \in \mathbb{Z}$, then let $m_1, m_2 \geq 0$ be such that $U^n S^n R(z) = T^{m_1}(z)$, $U^n S^n R(\tilde{z}) = T^{m_2}(\tilde{z})$, and we have

$$T^{m_2}(\tilde{z}) = U^n S^n R(\tilde{z}) = U^n \hat{T}^k S^n R(z) = \hat{T}^{k_1} U^n S^n R(z) = T^{k_2+m_1}(z)$$

for some $k_1, k_2 \in \mathbb{Z}$, hence $\tilde{z} = T^m(z)$ with $m = k_2 + m_1 - m_2$.

If $\tilde{z} = T^m(z)$ for some $m \in \mathbb{Z}$ and $n = 0$, then we have $S^n R(\tilde{z}) = \hat{T}^{k_n} S^n R(z)$ for some $k_n \in \mathbb{Z}$. If we suppose inductively that this is true for $n-1$, then

$$S^n R(\tilde{z}) = S \hat{T}^{k_{n-1}} S^{n-1} R(z) = S \hat{T}^{k_{n-1} - \hat{s}(S^{n-1}R(z))} U S^n R(z) = S U \hat{T}^{k_n} S^n R(z) = \hat{T}^{k_n} S^n R(z)$$

for some $k_{n-1}, k_n \in \mathbb{Z}$, and the statement is proved. □

If rT is constant on every D_ℓ, $\ell \in \mathcal{A}$, then we can define $\tau : \mathcal{A} \to \mathbb{N}$ by $\tau(\ell) = r(T(z)) + 1$ for $z \in D_\ell$ (cf. the definition of $\hat{T}$) and extend τ naturally to words $w \in \mathcal{A}^*$ by $\tau(w) = \sum_{\ell \in \mathcal{A}} |w|_\ell \tau(\ell)$.

Let $\pi(z)$, $\hat{\pi}(z)$ be the minimal period lengths of $(T^k(z))_{k\in\mathbb{Z}}$ and $(\hat{T}^k(z))_{k\in\mathbb{Z}}$ respectively, with $\pi(z) = \infty$, $\hat{\pi}(z) = \infty$ if the sequences are aperiodic. Then the following proposition holds.

Proposition 3.3. *If $\hat{\pi}(S^n R(z)) = p$ and $\ell_1 \cdots \ell_p = \iota_0(S^n R(z)) \cdots \iota_{p-1}(S^n R(z))$, then we have*

$$\hat{\pi}(R(z)) = |\sigma^n(\ell_1 \ell_2 \cdots \ell_p)| \quad \text{and} \quad \pi(z) = \tau(\sigma^n(\ell_1 \ell_2 \cdots \ell_p)) \text{ (if } \tau \text{ is well defined).}$$

Proof. Since $U^n S^n R(z) = T^m(z) = \hat{T}^{\hat{m}} R(z)$ for some $m, \hat{m} \in \mathbb{Z}$, and

$$T^{\tau(\sigma^n(\ell_1\ell_2\cdots\ell_p))} U^n S^n R(z) = \hat{T}^{|\sigma^n(\ell_1\ell_2\cdots\ell_p)|} U^n S^n R(z) = U^n \hat{T}^p S^n R(z) = U^n S^n R(z),$$

we have $\hat{\pi}(R(z)) \leq |\sigma^n(\ell_1 \cdots \ell_p)|$ and $\pi(z) \leq \tau(\sigma^n(\ell_1 \cdots \ell_p))$ (if τ exists). Since p is minimal, we can show similarly to the proof of Lemma 3.1 that these period lengths are minimal. □

We obtain the following characterization of periodic points $z \notin \mathcal{R}$. Note that all points in $\mathcal{P} \cup \mathcal{R}$ are periodic in our cases, hence the characterization is complete.

Theorem 3.4. *Let $R, S, T, \mathcal{D}, \mathcal{P}, \mathcal{R}, \sigma$ be as in the preceding paragraphs of this section. Assume that $\hat{\pi}(z)$ is finite for all $z \in \mathcal{P}$, and that for every $z \in \mathcal{D} \setminus \mathcal{P}$ there exist $m \in \mathbb{Z}$, $\ell \in \mathcal{A}$, such that $\hat{T}^m(z) \in D_\ell$ and $|\sigma^n(\ell)| \to \infty$ for $n \to \infty$. Then we have for $z \notin \mathcal{R}$:*

$$(T^k(z))_{k\in\mathbb{Z}} \text{ is periodic if and only if } S^n R(z) \in \mathcal{P} \text{ for some } n \geq 0.$$

Proof. If $S^n R(z) \in \mathcal{P}$, then we have $\hat{\pi}(R(z)) = \hat{\pi}(S^n R(z)) < \infty$, which implies $\pi(z) < \infty$.

Suppose now that $S^n R(z) \in \mathcal{D} \setminus \mathcal{P}$ for all $n \geq 0$. Then we have $m_n \in \mathbb{Z}$ and $\ell_n \in \mathcal{A}$ such that $\hat{T}^{m_n} S^n R(z) \in D_{\ell_n}$ and $|\sigma^n(\ell_n)| \to \infty$ for $n \to \infty$ (because $\mathcal{A}$ is finite). We have $U^n \hat{T}^{m_n} S^n R(z) = \hat{T}^{\tilde{m}_n} U^n S^n R(z) \in U^n(D_{\ell_n})$ for some $\tilde{m}_n \in \mathbb{Z}$, hence $\hat{T}^{\tilde{m}_n+k} U^n S^n R(z) \notin U^n(\mathcal{D})$ for all k, $1 \leq k < |\sigma^n(\ell_n)|$, which implies $\pi(z) \geq \hat{\pi}(R(z)) = \hat{\pi}(U^n S^n R(z)) \geq |\sigma^n(\ell_n)|$ for all $n \geq 0$, thus $\pi(z) = \infty$. □

Assume now $\lambda \in \{\pm\sqrt{2}, \frac{\pm 1 \pm \sqrt{5}}{2}, \pm\sqrt{3}\}$, let λ' be its algebraic conjugate, $T : [0,1)^2 \to [0,1)^2$,

$$T(x,y) = (x,y)A + (0, \lceil x + \lambda' y \rceil) \text{ with } A = \begin{pmatrix} 0 & -1 \\ 1 & -\lambda' \end{pmatrix}, \tag{3.3}$$

$$U(z) = V^{-1}(\kappa V(z))$$

with $0 < \kappa < 1$, $\kappa \in \mathbb{Z}[\lambda]$, $|\kappa\kappa'| = 1$, and $V(z) = \pm\kappa^n(z - v)$ some $v \in \mathbb{Z}[\lambda]^2$, $n \in \mathbb{Z}$. Let

$$t(z) = V\big(T^{s(z)}(z)\big) - V(z) A^{s(z)}$$

for $z \in \mathcal{D} \setminus \mathcal{P}$. Since $U^{-1}(z) = V^{-1}(V(z)/\kappa)$, we have

$$S(z) = U^{-1} T^{s(z)}(z) = V^{-1} \left(\frac{V(z) A^{s(z)} + t(z)}{\kappa} \right).$$

Note that $A^h = A^0$ for some $h \in \{5, 8, 10, 12\}$,

$$T^{-1}(x, y) = (x, y) A^{-1} + (\lceil \lambda' x + y \rceil, 0) \quad \text{with} \quad A^{-1} = \begin{pmatrix} -\lambda' & 1 \\ -1 & 0 \end{pmatrix},$$

and $T^{-1}(x, y) = (\tilde{x}, \tilde{y})$ with $(\tilde{y}, \tilde{x}) = T(y, x)$. Since $|\hat{s}(z)| < \max_{\ell \in \mathcal{A}} |\sigma(\ell)|$, there exists only a finite number of values for $t(z)$, and we obtain the following proposition.

Proposition 3.5. *Let T, V, κ be as above and the assumptions of Theorem 3.4 be satisfied. Suppose that $\pi(z) = \infty$ for some $z \in (\frac{1}{Q}\mathbb{Z}[\lambda] \cap [0, 1))^2 \setminus \mathcal{R}$, where Q is a positive integer. Then there exists an aperiodic point $\tilde{z} \in (\frac{1}{Q}\mathbb{Z}[\lambda])^2 \cap \mathcal{D}$ with*

$$\|V(\tilde{z})'\|_\infty \le \delta, \quad \text{where } \delta = \frac{\max\{\|(t(z) A^h)'\|_\infty : \, z \in \mathcal{D} \setminus \mathcal{P}, \, \pi(z) = \infty, \, h \in \mathbb{Z}\}}{|\kappa'| - 1}.$$

Proof. First note that δ exists since $t(z)$ and A^h take only finitely many values. If $\pi(z) = \infty$ for some $z \in (\frac{1}{Q}\mathbb{Z}[\lambda] \cap [0, 1))^2 \setminus \mathcal{R}$, then $S^n R(z) \in \mathcal{D} \setminus \mathcal{P}$ for all $n \ge 0$ by Theorem 3.4. In particular, $S^n R(z)$ is aperiodic as well. We use the abbreviations $s_n = s(S^n R(z))$ and $t_n = t(S^n R(z))$. Then we obtain inductively, for $n \ge 1$,

$$V S^n R(z) = \frac{V S^{n-1} R(z) A^{s_{n-1}} + t_{n-1}}{\kappa} = \frac{V R(z) A^{s_0 + s_1 + \cdots + s_{n-1}}}{\kappa^n} + \sum_{k=0}^{n-1} \frac{t_k A^{s_{k+1} + \cdots + s_{n-1}}}{\kappa^{n-k}}.$$

If we look at the algebraic conjugates, then note that $|\kappa'| > 1$, and we obtain

$$\|(V S^n R(z))'\|_\infty < \frac{\left\| \left(V R(z) A^{s_0 + s_1 + \cdots + s_{n-1}} \right)' \right\|_\infty}{|\kappa'|^n} + \delta,$$

thus $\|(V S^n R(z))'\|_\infty \le \delta$ for some $n \ge 0$ (as in Section 2), and we can choose $\tilde{z} = S^n R(z)$. □

Remarks.

- The last proof shows that, for every $z \in (\mathbb{Q}(\lambda) \cap [0, 1))^2 \setminus \mathcal{R}$ with $\pi(z) = \infty$, there are only finitely many possibilities for $V S^n R(z)$, hence $(S^n R(z))_{n \ge 0}$ is eventually periodic.
- For every $z \in \mathcal{D}$ with $\pi(z) = \infty$, we have
$$V(z) = \Big(V S^n(z) \kappa^n - \sum_{k=0}^{n-1} t_k A^{s_{k+1} + \cdots + s_{n-1}} \kappa^k \Big) A^{-s_0 - \cdots - s_{n-1}} = - \sum_{k=0}^{\infty} t_k A^{-\sum_{j=0}^{k} s(S^j(z))} \kappa^k,$$
which is a κ-expansion ($\kappa < 1$) of $V(z)$ with (two-dimensional) "digits" $-t_k A^{-s_0 - s_1 - \cdots - s_k}$.
- As a consequence of Lemma 3.2 and the definition of U, for every aperiodic point $z \in [0, 1)^2 \setminus \mathcal{R}$ and every $c > 0$, there exists some $m \in \mathbb{Z}$ such that $\|T^m(z) - v\|_\infty < c$.
- In all our cases, we have $\varepsilon = \kappa \kappa'$.

4. The case $\lambda = -1/\gamma = \frac{1 - \sqrt{5}}{2} = -2 \cos \frac{2\pi}{5}$

Now we apply the method of Section 3 for $\lambda = -1/\gamma$, i.e., $\lambda' = \gamma$. To this end, set

$$\mathcal{D} = \{(x, y) \in [0, 1)^2 : \, x + y \ge 3 - \gamma\} = D_0 \cup D_1$$

with $D_0 = \{(x, y) \in \mathcal{D} : x + \gamma y > 2\}$, $D_1 = \{(x, y) \in \mathcal{D} : x + \gamma y \le 2\}$. Figure 4.1 shows that $\hat{T}$ is given by $\hat{T}(z) = T^{\tau(\ell)}(z)$ if $z \in D_\ell$, $\ell \in \mathcal{A} = \{0, 1\}$, with $\tau(0) = 1$ and $\tau(1) = 4$. The set which is left out by the iterates of D_0 and D_1 is $\mathcal{R} = \{(0, 0)\} \cup D_A \cup D_B$, with

$$D_A = \{z \in [0, 1)^2 : T^{k+1}(z) = T^k(z) A + (0, 1) \text{ for all } k \ge 0\},$$
$$D_B = \{z \in [0, 1)^2 : T^{k+1}(z) = T^k(z) A + (0, 2) \text{ for all } k \ge 0\}.$$

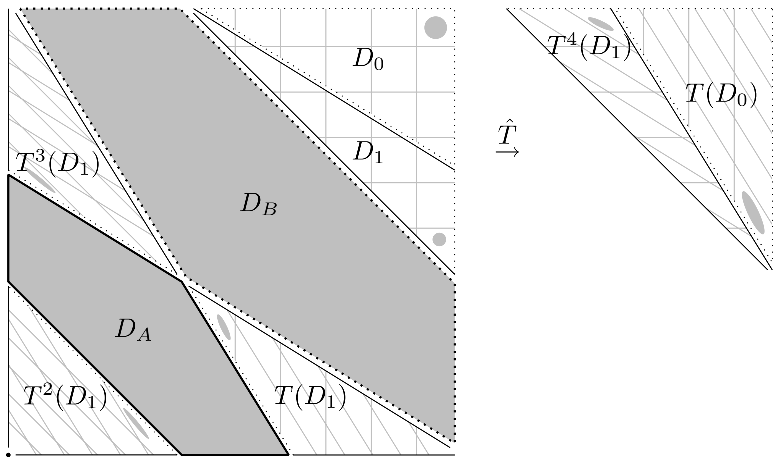

FIGURE 4.1. The map $\hat{T}$, $\hat{T}(D_0) = T(D_0)$, $\hat{T}(D_1) = T^4(D_1)$, and the (gray) set $\mathcal{R}$, $\lambda = -1/\gamma$.

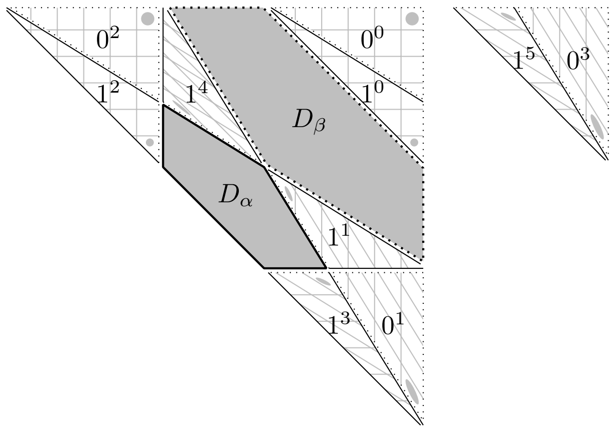

FIGURE 4.2. The trajectory of the scaled domains and $\mathcal{P}$, $\lambda = -1/\gamma$. (ℓ^k stands for $\hat{T}^k U(D_\ell)$.)

As in Section 2, we have $T^5(z) = z$ for all $z \in \mathcal{R}$. If we set

$$U(z) = \frac{z}{\gamma^2} + \Big(\frac{1}{\gamma}, \frac{1}{\gamma}\Big) = (1,1) - \frac{(1,1) - z}{\gamma^2},$$

$V(z) = (1,1) - z$, $\kappa = 1/\gamma^2$, $\varepsilon = 1$, and

$$\sigma: \ 0 \mapsto 010 \qquad 1 \mapsto 01110$$

then Figure 4.2 shows that σ satisfies the conditions in Section 3, and $\mathcal{P} = D_\alpha \cup D_\beta$ with $D_\alpha = U(D_A)$, $D_\beta = U(D_B)$. All points in $\mathcal{P}$ are periodic and $|\sigma^n(\ell)| \to \infty$ as $n \to \infty$ for all $\ell \in \mathcal{A}$. Therefore, all conditions of Proposition 3.3 and Theorem 3.4 are satisfied, and we obtain the following theorem.

Theorem 4.1. *If $\lambda = -1/\gamma$, then the period lengths $\pi(z)$ are*

1	*if* $z \in \{(0,0), (\frac{1}{\gamma^2+1}, \frac{1}{\gamma^2+1}), (\frac{2}{\gamma^2+1}, \frac{2}{\gamma^2+1})\}$
5	*for the other points of the pentagons* D_A *and* D_B
$2(5 \cdot 4^n + 1)/3$	*if* $S^n R(z) = (\frac{\gamma^2}{\gamma^2+1}, \frac{\gamma^2}{\gamma^2+1})$ *for some* $n \ge 0$
$10(5 \cdot 4^n + 1)/3$	*for the other points with* $S^n R(z) \in D_\alpha$ *for some* $n \ge 0$
$(5 \cdot 4^n - 2)/3$	*if* $S^n R(z) = (\frac{3}{\gamma^2+1}, \frac{3}{\gamma^2+1})$ *for some* $n \ge 0$
$5(5 \cdot 4^n - 2)/3$	*for the other points with* $S^n R(z) \in D_\beta$ *for some* $n \ge 0$
∞	*if* $S^n R(z) \in \mathcal{D} \setminus \mathcal{P}$ *for all* $n \ge 0$.

Proof. We easily calculate

$$\begin{pmatrix}|\sigma^n(0)|_0\\|\sigma^n(0)|_1\end{pmatrix} = 4^n\begin{pmatrix}1/3\\1/3\end{pmatrix}+\begin{pmatrix}2/3\\-1/3\end{pmatrix}, \qquad \begin{pmatrix}|\sigma^n(1)|_0\\|\sigma^n(1)|_1\end{pmatrix} = 4^n\begin{pmatrix}2/3\\2/3\end{pmatrix}+\begin{pmatrix}-2/3\\1/3\end{pmatrix},$$

hence $\tau(\sigma^n(0)) = \frac{5}{3}4^n - \frac{2}{3}$, $\tau(\sigma^n(1)) = \frac{10}{3}4^n + \frac{2}{3}$. If $S^nR(z) \in D_\alpha$, then $\pi(z) = \tau(\sigma^n(1))$ and $\pi(z) = \tau(\sigma^n(11111))$ respectively; if $S^nR(z) \in D_\beta$, then $\pi(z) = \tau(\sigma^n(0))$ and $\pi(z) = 5\tau(\sigma^n(0))$ respectively. □

For $z \in U(\mathcal{D})$, we have $\hat{s}(z) = s(z) = 0$ and $t(z) = (0,0)$. For the other $z \in \mathcal{D}\setminus\mathcal{P}$, we choose $\hat{s}(z)$ as follows and obtain the following $s(z), t(z)$:

$$z \in \hat{T}^2U(D_0)\cup\hat{T}^2U(D_1) : \hat{s}(z) = -2,\ s(z) = -5,\ t(z) = V(\hat{T}^{-2}(z)) - V(z) = (-1/\gamma^2, 0)$$
$$z \in \hat{T}U(D_1) : \hat{s}(z) = -1,\ s(z) = -1,\ t(z) = V(\hat{T}^{-1}(z)) - V(z)A^{-1} = (1/\gamma, 0)$$
$$z \in \hat{T}^4U(D_1) : \hat{s}(z) = 1,\ s(z) = 1,\ t(z) = V(\hat{T}(z)) - V(z)A = (0, 1/\gamma)$$
$$z \in \hat{T}U(D_0)\cup\hat{T}^3U(D_1) : \hat{s}(z) = 2,\ s(z) = 5,\ t(z) = V(\hat{T}^2(z)) - V(z) = (0, -1/\gamma^2)$$

Observe the symmetry between positive and negative $\hat{s}(z)$ which is due to the symmetry of $T(x,y)$ and $T^{-1}(y,x)$ and the symmetry of $\mathcal{D}$. With

$$\{(1/\gamma,0)A^h : h \in \mathbb{Z}\} = \{(1/\gamma,0), (0,-1/\gamma), (-1/\gamma,1), (1,-1), (-1,1/\gamma)\},$$

we obtain $\delta \le \max\{\|(t(z)A^h)'\|_\infty : z \in \mathcal{D}\setminus\mathcal{P},\ h \in \mathbb{Z}\}/\gamma = (1/\gamma^2)'/\gamma = \gamma$, as in Section 2. The following theorem shows that aperiodic points with $t(z) = (-1/\gamma^2, 0)$ exist, hence $\delta = \gamma$.

Theorem 4.2. *$\pi(z)$ is finite for all $z \in (\mathbb{Z}[\gamma]\cap[0,1))^2$, but $\pi\big(1-1/(3\gamma), 1-2/(3\gamma)\big) = \infty$.*

Proof. By Proposition 3.5, we have to show that all $z \in \mathbb{Z}[\gamma]^2\cap\mathcal{D}$ with $\|V(z)'\|_\infty \le \gamma$ are periodic. Since $V(\mathcal{D}) = \{(x,y) : x > 0, y > 0, x+y \le 1/\gamma\}$, we have to consider $x, y \in \mathbb{Z}[\gamma]\cap(0,1/\gamma)$ with $|x'|, |y'| \le \gamma$. No such x, y exist, hence the conjecture is proved for $\lambda = -1/\gamma$. Note that $\pi(z)$ is finite for all $z \in (\frac{1}{2}\mathbb{Z}[\gamma]\cap[0,1))^2$ as well. If $V(z) = \big(1/(3\gamma), 2/(3\gamma)\big)$, then we have

$$\begin{aligned}
VS(z) &= \gamma^2\big(V(z)A^5 + (0,-1/\gamma^2)\big) = \big(\gamma/3, 1/(3\gamma^3)\big)\\
VS^2(z) &= \gamma^2\big(VS(z)A^{-5} + (-1/\gamma^2,0)\big) = \big(2/(3\gamma), 1/(3\gamma)\big)\\
VS^3(z) &= \gamma^2\big(VS^2(z)A^{-5} + (0,-1/\gamma^2)\big) = \big(1/(3\gamma^3), \gamma/3)\big)\\
VS^4(z) &= \gamma^2\big(VS^3(z)A^5 + (0,-1/\gamma^2)\big) = \big(1/(3\gamma), 2/(3\gamma)\big) = V(z),
\end{aligned}$$

hence $S^n(z) \in \mathcal{D}\setminus\mathcal{P}$ for all $n \ge 0$ and $\pi(z) = \infty$ by Theorem 4.1. □

5. The case $\lambda = \sqrt{2} = -2\cos\frac{3\pi}{4}$

Let $\lambda = \sqrt{2}$ ($\lambda' = -\sqrt{2}$) and set

$$\mathcal{D} = \{(x,y) \in [0,1)^2 : \sqrt{2}-2 < x - \sqrt{2}y < 0,\ 0 < \sqrt{2}x - y < \sqrt{2}-2\} = \bigcup\nolimits_{\ell\in\mathcal{A}=\{0,1,2,3\}} D_\ell,$$
$$D_0 = \{(x,y)\in\mathcal{D} : x < \sqrt{2}-1\}, \quad D_1 = \{(x,y)\in\mathcal{D} : x > \sqrt{2}-1, y \le \sqrt{2}-1\},$$
$$D_2 = \{(x,y)\in\mathcal{D} : x > \sqrt{2}-1, y > \sqrt{2}-1\}, \quad D_3 = \{(x,y)\in\mathcal{D} : x = \sqrt{2}-1\}.$$

Figure 5.1 shows that $\hat{T}(z) = T^{\tau(\ell)}(z)$ if $z \in D_\ell$, with $\tau(0) = 5$, $\tau(1) = 9$, $\tau(2) = 3$, $\tau(3) = 11$, and $\mathcal{R} = \{(0,0)\}\cup\bigcup_{k=0}^3 T^k(D_A)\cup\bigcup_{k=0}^5 T^k(D_B)$ with $D_A = \{(0,y) : 1-1/\sqrt{2} < y < 1/\sqrt{2}\}$, $D_B = \{(0,1/\sqrt{2})\}$. If we set $U(z) = (\sqrt{2}-1)z$, $V(z) = z$, $\kappa = \sqrt{2}-1$, $\varepsilon = -1$, and

$$\sigma :\ 0 \mapsto 010 \qquad 1 \mapsto 000 \qquad 2 \mapsto 0 \qquad 3 \mapsto 030,$$

then Figure 5.2 shows that σ satisfies the conditions in Section 3, and

$$\mathcal{P} = \{(x,y)\in\mathcal{D} : x,y \ge \sqrt{2}-1\} = D_\alpha \cup D_\beta \cup \hat{T}(D_\beta) \cup D_\zeta$$

with $D_\alpha = D_2$, $D_\beta = \{(x,\sqrt{2}-1) : \sqrt{2}-1 < x < 2-\sqrt{2}\}$ and $D_\zeta = \{(\sqrt{2}-1, \sqrt{2}-1)\}$. All points in $\mathcal{P}$ are periodic and $|\sigma^n(\ell)| \to \infty$ as $n \to \infty$ for all $\ell \in \mathcal{A}$. Therefore, all conditions of Proposition 3.3 and Theorem 3.4 are satisfied, and we obtain the following theorem.

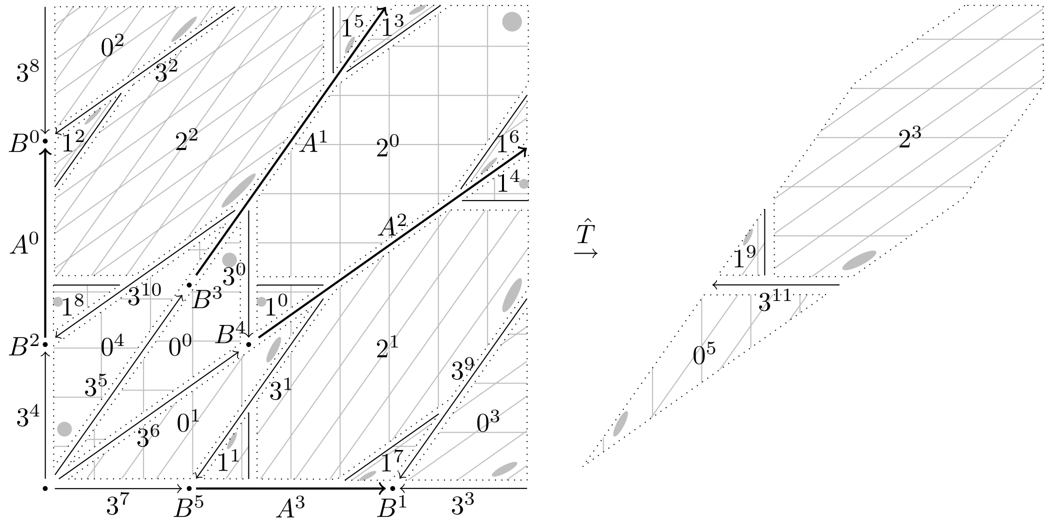

FIGURE 5.1. The map $\hat{T}$ and the set $\mathcal{R}$, $\lambda = \sqrt{2}$. (ℓ^k stands for $T^k(D_\ell)$.)

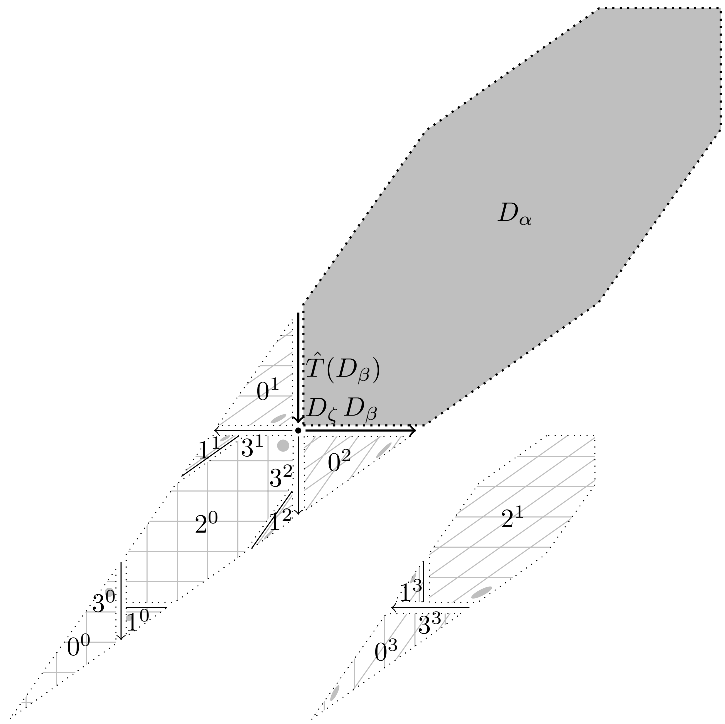

FIGURE 5.2. The trajectory of the scaled domains and $\mathcal{P}$, $\lambda = \sqrt{2}$. (ℓ^k stands for $\hat{T}^{-k}U(D_\ell)$.)

Theorem 5.1. *If $\lambda = \sqrt{2}$, then the minimal period length $\pi(z)$ is*

$$\begin{array}{cl}
1 & \text{if } z = (0,0) \\
4 & \text{if } z = T^m(0,1/2),\ 0 \le m \le 3 \\
8 & \text{for the other points of } T^m(D_A),\ 0 \le m \le 3 \\
6 & \text{if } z = T^m(0,1/\sqrt{2}),\ 0 \le m \le 5 \\
2\cdot 3^n + (-1)^n & \text{if } S^nR(z) = (1/\sqrt{2},1/\sqrt{2}),\ n \ge 0 \\
8(2\cdot 3^n + (-1)^n) & \text{for the other points with } S^nR(z) \in D_\alpha \\
4(3^{n+1}+1+(-1)^n) & \text{if } S^nR(z) \in \{(1/2,\sqrt{2}-1),(\sqrt{2}-1,1/2)\},\ n \ge 0 \\
8(3^{n+1}+1+(-1)^n) & \text{for the other points with } S^nR(z) \in D_\beta \cup \hat{T}(D_\beta) \\
2\cdot 3^{n+1}+4+(-1)^n & \text{if } S^nR(z) = (\sqrt{2}-1,\sqrt{2}-1),\ n \ge 0 \\
\infty & \text{if } S^nR(z) \in \mathcal{D}\setminus\mathcal{P} \text{ for all } n \ge 0.
\end{array}$$

Proof. We easily calculate

$$\begin{pmatrix}|\sigma^n(0)|_0\\|\sigma^n(0)|_1\end{pmatrix} = 3^n\begin{pmatrix}3/4\\1/4\end{pmatrix} + (-1)^n\begin{pmatrix}1/4\\-1/4\end{pmatrix}, \qquad \begin{pmatrix}|\sigma^n(1)|_0\\|\sigma^n(1)|_1\end{pmatrix} = 3^n\begin{pmatrix}3/4\\1/4\end{pmatrix} + (-1)^n\begin{pmatrix}-3/4\\3/4\end{pmatrix}$$

and obtain $\tau(\sigma^n(0)) = 2\cdot 3^{n+1} - (-1)^n$, $\tau(\sigma^n(3)) = \tau(\sigma^{n-1}(030)) = 2\cdot 3^{n+1} + 4 + (-1)^n$. If $S^nR(z) \in D_\alpha$ and $n \ge 1$, then $\pi(z) = \tau(\sigma^n(2)) = \tau(\sigma^{n-1}(0))$ and $\pi(z) = 8\tau(\sigma^{n-1}(0))$ respectively; if $S^nR(z) \in D_\beta$, then $\pi(z) = \tau(\sigma^n(13)) = \tau(\sigma^{n-1}(000030))$ and $\pi(z) = 2\tau(\sigma^{n-1}(000030))$ respectively; if $S^nR(z) = (\sqrt{2}-1,\sqrt{2}-1)$, then $\pi(z) = \tau(\sigma^n(3))$. The given $\pi(z)$ hold for $n = 0$ as well. □

For $z \in \mathcal{D}\setminus(U(\mathcal{D})\cup\mathcal{P})$, we choose $\hat{s}(z)$ as follows and obtain the following $s(z), t(z)$:

$$z \in \hat{T}^{-2}U(D_0\cup D_1\cup D_3) : \hat{s}(z) = -1,\ s(z) = -5,\ t(z) = \hat{T}^{-1}(z) - zA^{-5} = (\sqrt{2}-1, 2-\sqrt{2})$$

$$z \in \hat{T}^{-1}U(D_0\cup D_1\cup D_3) : \hat{s}(z) = 1,\ s(z) = 5,\ t(z) = \hat{T}(z) - zA^{5} = (2-\sqrt{2}, \sqrt{2}-1)$$

This gives $\delta = (2+\sqrt{2})/\sqrt{2} = \sqrt{2}+1$ since

$$\{t(z)A^h : z \in \mathcal{D}\setminus\mathcal{P}, h \in \mathbb{Z}\} = \pm\{(0,0),\ (2-\sqrt{2},\sqrt{2}-1), (\sqrt{2}-1,0), (0,1-\sqrt{2}), (1-\sqrt{2},\sqrt{2}-2)\}.$$

Theorem 5.2. *$\pi(z)$ is finite for all $z \in (\mathbb{Z}[\sqrt{2}]\cap[0,1))^2$, but $(T^k(\frac{3-\sqrt{2}}{4},\frac{2\sqrt{2}-1}{4}))_{k\in\mathbb{Z}}$ is aperiodic.*

Proof. We have to consider $z \in \mathbb{Z}[\sqrt{2}]^2\cap\mathcal{D}$ with $\|z'\|_\infty \le \delta = \sqrt{2}+1$. The only such point is $(\sqrt{2}-1,\sqrt{2}-1) = D_\zeta$, hence Conjecture 1.1 holds for $\lambda = \sqrt{2}$. It can be shown that all points in $(\frac{1}{2}\mathbb{Z}[\sqrt{2}]\cap[0,1))^2$ and $(\frac{1}{3}\mathbb{Z}[\sqrt{2}]\cap[0,1))^2$ are periodic as well. For $z = (\frac{3-\sqrt{2}}{4},\frac{2\sqrt{2}-1}{4})$, we have

$$S(z) = \big(zA^5 + (2-\sqrt{2},\sqrt{2}-1)\big)/\kappa = (\sqrt{2}+1)\Big(\frac{9-6\sqrt{2}}{4},\sqrt{2}-\frac{5}{4}\Big) = \Big(\frac{3\sqrt{2}-3}{4},\frac{3-\sqrt{2}}{4}\Big),$$

$$S^2(z) = \big(S(z)A^5 + (2-\sqrt{2},\sqrt{2}-1)\big)/\kappa = (\sqrt{2}+1)\Big(\frac{5-3\sqrt{2}}{4},\sqrt{2}-\frac{5}{4}\Big) = \Big(\frac{2\sqrt{2}-1}{4},\frac{3-\sqrt{2}}{4}\Big),$$

$S^3(z) = \big(S^2(z)A^{-5} + (\sqrt{2}-1,2-\sqrt{2})\big)/\kappa = (\frac{3-\sqrt{2}}{4},\frac{3\sqrt{2}-3}{4})$ and $S^4(z) = (\frac{3-\sqrt{2}}{4},\frac{2\sqrt{2}-1}{4}) = z$. □

6. The case $\lambda = -\sqrt{2} = -2\cos\frac{\pi}{4}$

Let $\lambda = -\sqrt{2}$ ($\lambda' = \sqrt{2}$) and set

$$\mathcal{D} = \{(x,y) \in [0,1)^2 : \sqrt{2}x + y > 2 \text{ or } x + \sqrt{2}y > 2\} = \bigcup\nolimits_{\ell\in\mathcal{A}=\{0,1,2\}} D_\ell,$$

with $D_0 = \{(x,y) \in \mathcal{D} : x+\sqrt{2}y > 2\}$ and $D_1 = \{(x,y) \in \mathcal{D} : x+\sqrt{2}y < 2\}$. Figure 6.1 shows that $\hat{T}(z) = T^{\tau(\ell)}(z)$ if $z \in D_\ell$, with $\tau(0) = 1$, $\tau(1) = 21$, $\tau(2) = 31$, and

$$\mathcal{R} = \{(0,0)\}\cup D_A\cup D_B\cup\bigcup\nolimits_{k=0}^{3} T^k(D_\Gamma)\cup\bigcup\nolimits_{k=0}^{9} T^k(D_\Delta),$$

$$D_A = \{(x,y) : 0 \le x,y \le 3-2\sqrt{2}\}\setminus\{(0,0),(3-2\sqrt{2},3-2\sqrt{2})\},$$

$$D_B = \{z \in [0,1)^2 : T^{k+1}(z) = T^k(z)A + (0,1) \text{ for all } k \in \mathbb{Z}\},$$

$$D_\Gamma = \{z \in [0,1)^2 : T^{k+1}(z) = T^k(z)A + (0,2) \text{ for all } k \in \mathbb{Z}\},$$

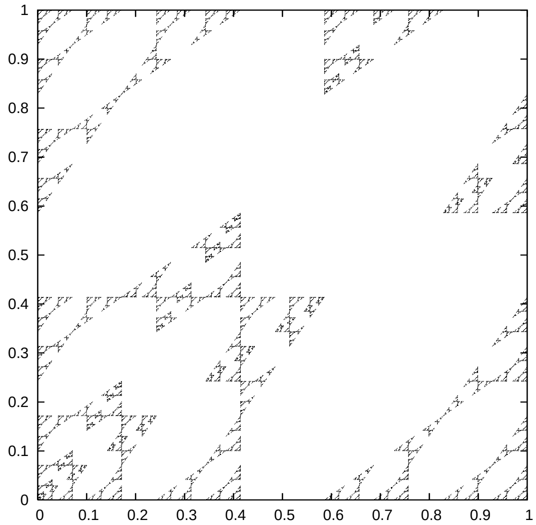

FIGURE 5.3. Aperiodic points, $\lambda = \sqrt{2}$.

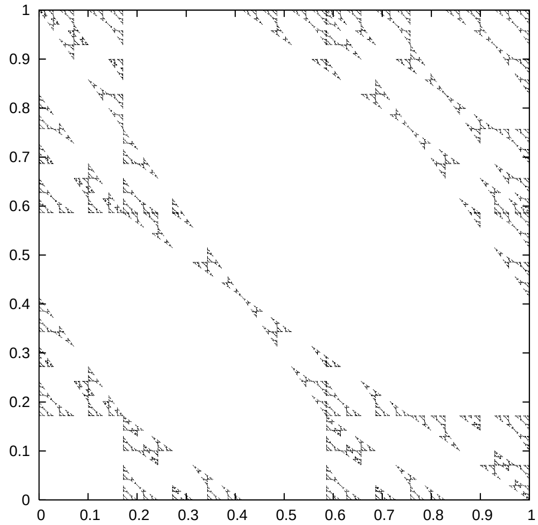

FIGURE 5.4. Aperiodic points, $\lambda = -\sqrt{2}$.

$D_\Delta = \{(1/\sqrt{2}, 0)\}$. Set $\kappa = \sqrt{2} - 1$, $V(z) = ((1,1) - z)/\kappa = (\sqrt{2}+1)((1,1)-z)$, i.e.,

$$U(z) = (1,1) - (\sqrt{2}-1)\big((1,1) - z\big) = (\sqrt{2}-1)z + (2-\sqrt{2}, 2-\sqrt{2}).$$

Then Figure 6.2 shows that the conditions in Section 3 are satisfied by

$$\sigma: \ 0 \mapsto 010 \qquad 1 \mapsto 000 \qquad 2 \mapsto 020$$

with $\varepsilon = -1$ and $\mathcal{P} = D_\alpha \cup \bigcup_{k=0}^{5} \hat{T}^k(D_\beta) \cup \bigcup_{k=0}^{2} \hat{T}^k(D_\zeta)$ with

$$D_\alpha = \{z \in [0,1)^2 : T^{k+1}(z) = T^k(z)A + (0,3) \text{ for all } k \in \mathbb{Z}\},$$

$D_\beta = \{(x, 2-\sqrt{2}x) : 5 - 3\sqrt{2} < x < 2\sqrt{2} - 2\}$ and $D_\zeta = \{(8-5\sqrt{2}, 8-5\sqrt{2})\}$. All points in $\mathcal{P}$ are periodic and $|\sigma^n(\ell)| \to \infty$ as $n \to \infty$ for all $\ell \in \mathcal{A}$. Therefore, all conditions of Proposition 3.3 and Theorem 3.4 are satisfied, and we obtain the following theorem.

Theorem 6.1. *If $\lambda = -\sqrt{2}$, then the minimal period length $\pi(z)$ is*

1	*if* $z \in \{(0,0), (1/\sqrt{2}, 1/\sqrt{2}), (2-\sqrt{2}, 2-\sqrt{2})\}$
4	*if* $z = T^m(3/2-\sqrt{2}, 3/2-\sqrt{2})$ *for some* $m \in \{0,1,2,3\}$
10	*if* $z = T^m(1/\sqrt{2}, 0)$ *for some* $m \in \{0,1,\ldots,9\}$
8	*for the other points in* $\mathcal{R}$
$2\cdot 3^{n+1} - 5(-1)^n$	*if* $S^nR(z) = (3-3/\sqrt{2}, 3-3/\sqrt{2})$ *for some* $n \ge 0$
$8(2\cdot 3^{n+1} - 5(-1)^n)$	*for the other points with* $S^nR(z) \in D_\alpha$
$4(3^{n+2} + 5 - 5(-1)^n)$	*if* $S^nR(z) = \hat{T}^m\big((9-5\sqrt{2})/2, 5-3\sqrt{2}\big)$ *for some* $m \in \{0,\ldots,5\}$, $n \ge 0$
$8(3^{n+2} + 5 - 5(-1)^n)$	*for the other points with* $S^nR(z) \in \hat{T}^m(D_\beta)$
$2\cdot 3^{n+2} + 20 - 5(-1)^n$	*if* $S^nR(z) = \hat{T}^m(8-5\sqrt{2}, 8-5\sqrt{2})$ *for some* $m \in \{0,1,2\}$, $n \ge 0$
∞	*if* $S^nR(z) \in \mathcal{D} \setminus \mathcal{P}$ *for all* $n \ge 0$.

Proof. As for $\lambda = \sqrt{2}$, we have

$$\begin{pmatrix} |\sigma^n(0)|_0 \\ |\sigma^n(0)|_1 \end{pmatrix} = 3^n \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} + (-1)^n \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix}, \qquad \begin{pmatrix} |\sigma^n(1)|_0 \\ |\sigma^n(1)|_1 \end{pmatrix} = 3^n \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} + (-1)^n \begin{pmatrix} -3/4 \\ 3/4 \end{pmatrix},$$

hence $\tau(\sigma^n(0)) = 2\cdot 3^{n+1} - 5(-1)^n$ and $\tau(\sigma^n(2)) = \tau(\sigma^{n-1}(020)) = 2\cdot 3^{n+1} + 20 + 5(-1)^n$. For $S^nR(z) \in D_\alpha$, we have $\pi(z) = \tau(\sigma^n(0))$ and $\pi(z) = 8\tau(\sigma^n(0))$ respectively; if $S^nR(z) \in T^m(D_\beta)$, then $\pi(z) = \tau(\sigma^n(002000))$ and $\pi(z) = 2\tau(\sigma^n(002000))$ respectively; if $S^nR(z) = \hat{T}^m(D_\zeta)$, then $\pi(z) = \tau(\sigma^n(020))$. $\square$

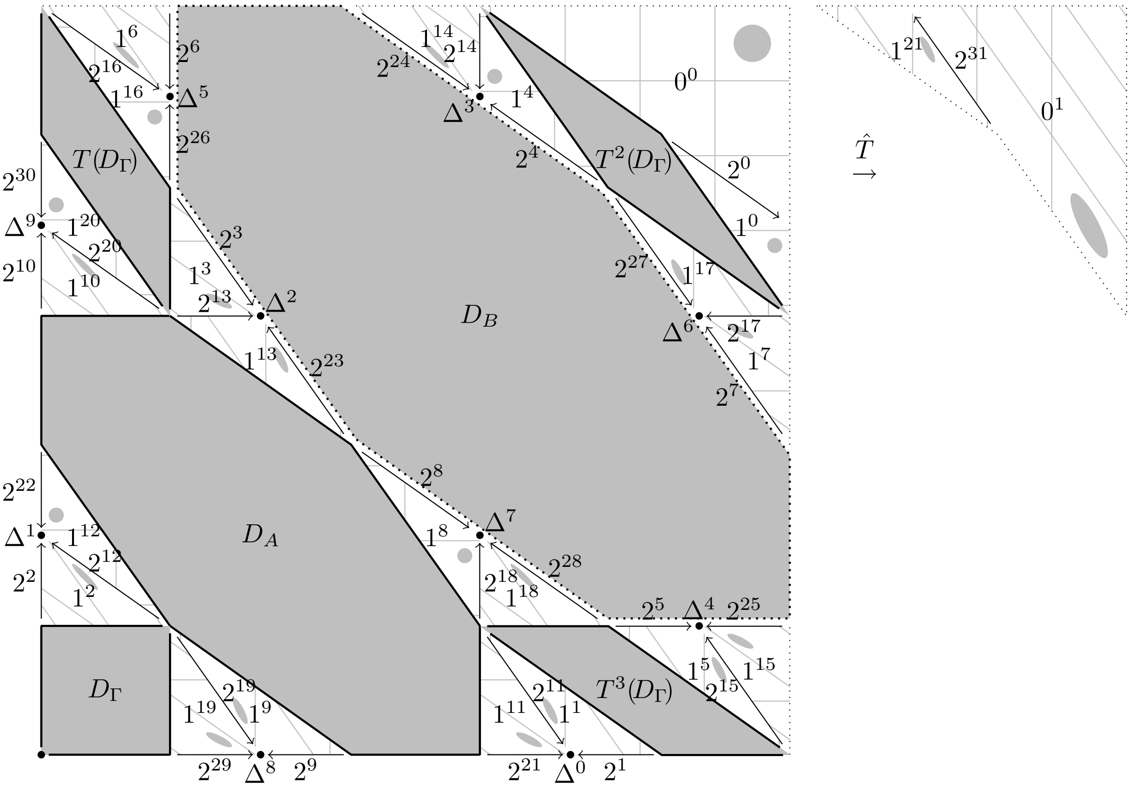

FIGURE 6.1. The map $\hat{T}$ and the set $\mathcal{R}$, $\lambda = -\sqrt{2}$. (ℓ^k stands for $T^k(D_\ell)$.)

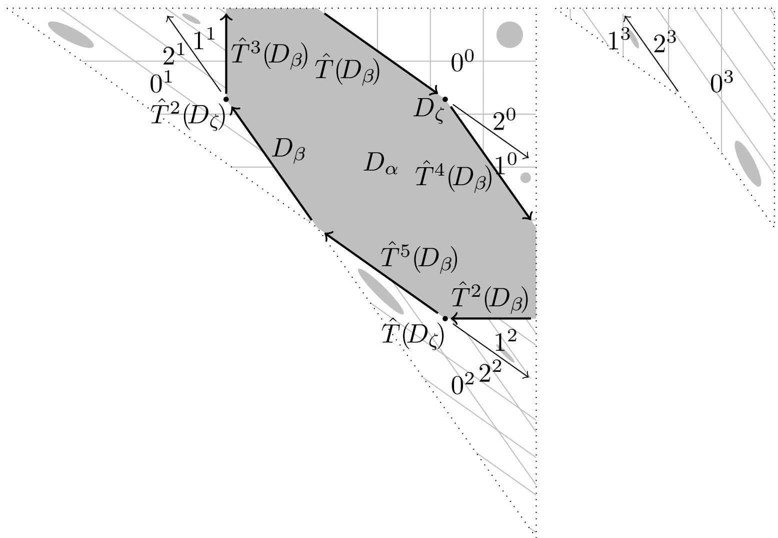

FIGURE 6.2. The trajectory of the scaled domains and $\mathcal{P}$, $\lambda = -\sqrt{2}$. (ℓ^k stands for $\hat{T}^{-k}U(D_\ell)$.)

For $z \in \mathcal{D} \setminus (U(\mathcal{D}) \cup \mathcal{P})$, we choose $\hat{s}(z)$ as follows and obtain the following $s(z), t(z)$:

$$z \in \hat{T}^{-2}U(D_0 \cup D_1 \cup D_2) : \hat{s}(z) = -1,\ s(z) = -1,\ t(z) = V(\hat{T}^{-1}(z)) - V(z)A^{-1} = (1, 0)$$
$$z \in \hat{T}^{-1}U(D_0 \cup D_1 \cup D_2) : \hat{s}(z) = 1,\ s(z) = 1,\ t(z) = V(\hat{T}(z)) - V(z)A = (0, 1)$$

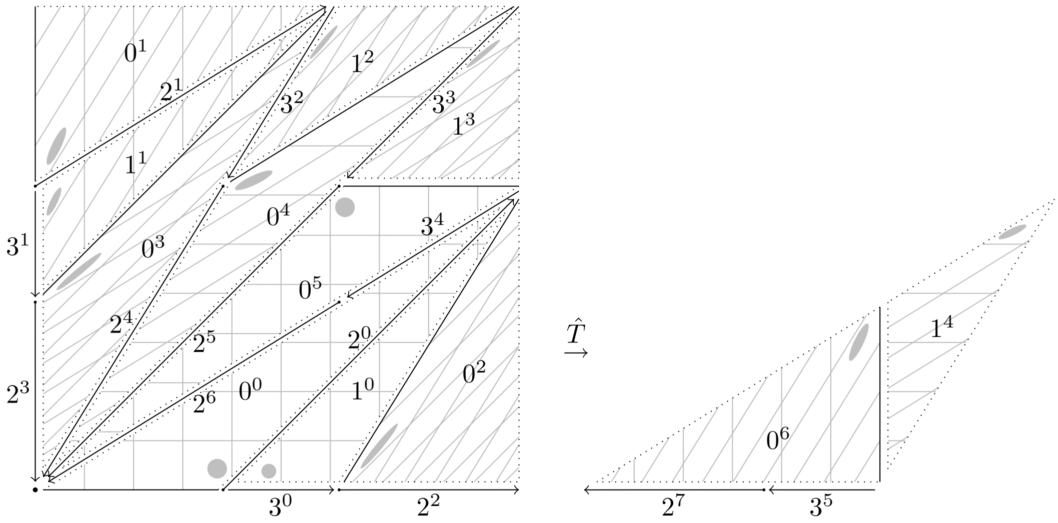

FIGURE 7.1. The map $\hat{T}$, $\lambda = 1/\gamma$. (ℓ^k stands for $T^k(D_\ell)$.)

This gives $\delta = \sqrt{2}/\sqrt{2} = 1$ since

$$\{t(z)A^h : z \in \mathcal{D} \setminus \mathcal{P}, h \in \mathbb{Z}\} = \pm\{(0,0),\ (1,0),(0,1),(1,-\sqrt{2}),(-\sqrt{2},1)\}.$$

Theorem 6.2. *$\pi(z)$ is finite for all $z \in (\mathbb{Z}[\sqrt{2}] \cap [0,1))^2$, but $(T^k(\frac{3}{4}, \frac{5-\sqrt{2}}{4}))_{k\in\mathbb{Z}}$ is aperiodic.*

Proof. Since $V(\mathcal{D}) = \{(x,y) : x > 0, y > 0, x + \sqrt{2}y < 1 \text{ or } \sqrt{2}x + y < 1\}$, there exists no $z \in \mathbb{Z}[\sqrt{2}]^2 \cap \mathcal{D}$ with $\|(V(z))'\|_\infty \le 1$. Therefore Conjecture 1.1 holds for $\lambda = -\sqrt{2}$. It can be shown that all points in $(\frac{1}{2}\mathbb{Z}[\sqrt{2}] \cap [0,1))^2$ and $(\frac{1}{3}\mathbb{Z}[\sqrt{2}] \cap [0,1))^2$ are periodic as well. For $z = (\frac{3}{4}, \frac{5-\sqrt{2}}{4})$, we have $V(z) = (\frac{\sqrt{2}+1}{4}, \frac{1}{4})$,

$$VS(z) = (\sqrt{2}+1)(V(z)A + (0,1)) = (\sqrt{2}+1)\Big(\frac{1}{4}, \frac{3-2\sqrt{2}}{4}\Big) = \Big(\frac{\sqrt{2}+1}{4}, \frac{\sqrt{2}-1}{4}\Big),$$

$VS^2(z) = (\frac{1}{4}, \frac{\sqrt{2}+1}{4})$, $VS^3(z) = (\frac{\sqrt{2}-1}{4}, \frac{\sqrt{2}+1}{4})$ and $VS^4(z) = (\frac{\sqrt{2}+1}{4}, \frac{1}{4}) = V(z)$. □

7. THE CASE $\lambda = 1/\gamma = -2\cos\frac{3\pi}{5}$

Let $\lambda = 1/\gamma$ ($\lambda' = -\gamma$) and set

$$\mathcal{D} = \{(x,y) \in [0,1)^2 : \gamma x - 1 < y < x/\gamma\} = \bigcup\nolimits_{\ell\in\mathcal{A}=\{0,1,2,3\}} D_\ell,$$

with D_0, D_1, D_2, D_3 satisfying the (in)equalities

D_0	D_1	D_2	D_3
$y > x - 1/\gamma^2$	$0 < y < x - 1/\gamma^2$	$y = x - 1/\gamma^2$	$y = 0,\ 1/\gamma^2 < x < 1/\gamma$

Figure 7.1 shows that $\hat{T}(z) = T^{\tau(\ell)}(z)$ if $z \in D_\ell$, with $\tau(0) = 6$, $\tau(1) = 4$, $\tau(2) = 7$, $\tau(3) = 5$, and $\mathcal{R} = \{(0,0)\}$. If we set $U(z) = z/\gamma^2$, $V(z) = z$, $\kappa = 1/\gamma^2$, $\varepsilon = 1$, and

$$\sigma:\ 0 \mapsto 010 \qquad 1 \mapsto 01110 \qquad 2 \mapsto 012 \qquad 3 \mapsto 01112$$

then Figure 7.2 shows that σ satisfies the conditions in Section 3, and

$$\mathcal{P} = D_\alpha \cup D_\beta \cup \bigcup\nolimits_{k=0}^{3} \hat{T}^k(D_\zeta) \cup D_\vartheta \cup \bigcup\nolimits_{k=0}^{1} \hat{T}^k(D_\eta) \cup D_\mu$$

with $D_\alpha = \{z \in \mathcal{D} : \hat{T}^k(z) \in D_0 \text{ for all } k \in \mathbb{Z}\}$, $D_\beta = \{z \in \mathcal{D} : \hat{T}^k(z) \in D_1 \text{ for all } k \in \mathbb{Z}\}$, $D_\zeta = \{(x,0) : 1/\gamma^3 < x < 1/\gamma^2\}$, $D_\eta = D_3$, $D_\vartheta = \{(1/\gamma^3, 0)\}$ and $D_\mu = \{(1/\gamma^2, 0)\}$. All points in $\mathcal{P}$ are periodic and $|\sigma^n(\ell)| \to \infty$ as $n \to \infty$ for all $\ell \in \mathcal{A}$. Therefore, all conditions of Proposition 3.3 and Theorem 3.4 are satisfied, and we obtain the following theorem.

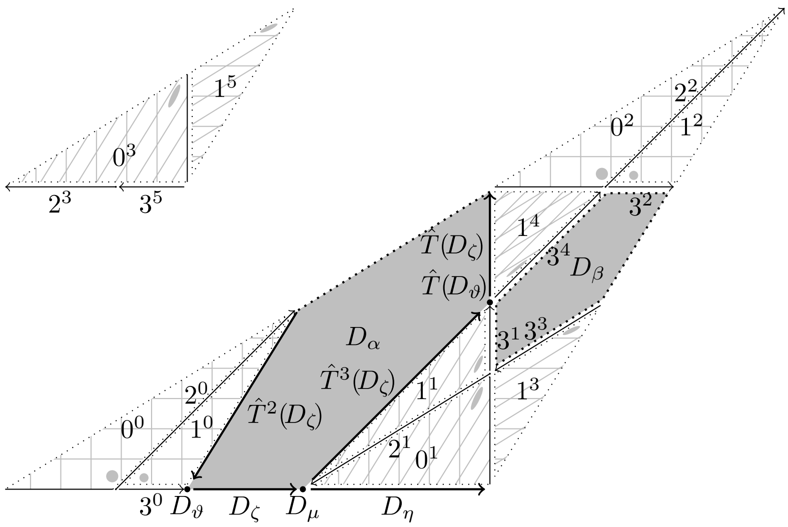

FIGURE 7.2. The trajectory of the scaled domains and $\mathcal{P}$, $\lambda = 1/\gamma$. (ℓ^k stands for $\hat{T}^k U(D_\ell)$.)

Theorem 7.1. *If $\lambda = 1/\gamma$, then the minimal period length $\pi(z)$ is*

$$\begin{array}{cl}
1 & \textit{if } z=(0,0) \\
2(5\cdot 4^n+4)/3 & \textit{if } S^nR(z)=\left(\frac{\gamma}{\gamma^2+1},\frac{1/\gamma}{\gamma^2+1}\right) \textit{ for some } n\ge 0 \\
10(5\cdot 4^n+4)/3 & \textit{for the other points with } S^nR(z)\in D_\alpha \\
4(5\cdot 4^n-2)/3 & \textit{if } S^nR(z)=\left(\frac{\gamma^2}{\gamma^2+1},\frac{1}{\gamma^2+1}\right) \textit{ for some } n\ge 0 \\
20(5\cdot 4^n-2)/3 & \textit{for the other points with } S^nR(z)\in D_\beta \\
5(4^{n+1}-1)/3 & \textit{if } S^nR(z)=(0,1/2) \textit{ for some } n\ge 0 \\
10(4^{n+1}-1)/3 & \textit{for the other points with } S^nR(z)\in D_\vartheta \\
5(2\cdot 4^{n+1}+7)/3 & \textit{if } S^nR(z)=\hat{T}^m(1/(2\gamma),0) \textit{ for some } m\in\{0,1,2,3\} \textit{ and } n\ge 0 \\
10(2\cdot 4^{n+1}+7)/3 & \textit{for the other points with } S^nR(z)\in \hat{T}^m(D_\zeta) \\
(10\cdot 4^n+11)/3 & \textit{if } S^nR(z)=(1/\gamma^2,0) \textit{ for some } n\ge 0 \\
(5\cdot 4^{n+1}+19)/3 & \textit{if } S^nR(z)=\hat{T}^m(1/\gamma^3,0) \textit{ for some } m\in\{0,1\} \textit{ and } n\ge 0 \\
\infty & \textit{if } S^nR(z)\in\mathcal{D}\setminus\mathcal{P} \textit{ for all } n\ge 0.
\end{array}$$

Proof. As for $\lambda = -1/\gamma$, we have

$$\begin{pmatrix}|\sigma^n(0)|_0\\|\sigma^n(0)|_1\end{pmatrix} = 4^n\begin{pmatrix}1/3\\1/3\end{pmatrix}+\begin{pmatrix}2/3\\-1/3\end{pmatrix}, \qquad \begin{pmatrix}|\sigma^n(1)|_0\\|\sigma^n(1)|_1\end{pmatrix} = 4^n\begin{pmatrix}2/3\\2/3\end{pmatrix}+\begin{pmatrix}-2/3\\1/3\end{pmatrix},$$

hence $\tau(\sigma^n(0)) = \frac{10}{3}4^n+\frac{8}{3}$, $\tau(\sigma^n(1)) = \frac{20}{3}4^n-\frac{8}{3}$, $\tau(\sigma^n(2)) = \frac{10}{3}4^n+\frac{11}{3}$, $\tau(\sigma^n(3)) = \frac{20}{3}4^n-\frac{5}{3}$. For $S^nR(z)\in D_\alpha$, we have $\pi(z)=\tau(\sigma^n(0))$ and $\pi(z)=5\tau(\sigma^n(0))$ respectively; if $S^nR(z)\in D_\beta$, then $\pi(z)=\tau(\sigma^n(1))$ and $5\tau(\sigma^n(1))$ respectively; if $S^nR(z)\in D_\eta$, then $\pi(z)=\tau(\sigma^n(3))$ and $2\tau(\sigma^n(3))$ respectively; if $S^nR(z)\in D_\zeta$, then $\pi(z)=\tau(\sigma^n(0002))$ and $2\tau(\sigma^n(0002))$ respectively; if $S^nR(z)=\hat{T}^m(1/\gamma^3,0)$, then $\pi(z)=\tau(\sigma^n(02))$; if $S^nR(z)=(1/\gamma^2,0)$, then $\pi(z)=\tau(\sigma^n(2))$. $\square$

Note that $\hat{T}^mU(D_3)$ plays no role in the calculation of δ since $U(D_3)\subset U(\mathcal{P})$ and thus $\pi(z)<\infty$ for all $z\in\hat{T}^mU(D_3)$. For the other $z\in\mathcal{D}\setminus(\mathcal{P}\cup U(\mathcal{D}))$, we choose $\hat{s}(z)$ as follows:

$$z\in\hat{T}^2U(D_0\cup D_1\cup D_2): \hat{s}(z)=-2,\ s(z)=-10,\ t(z)=\hat{T}^{-2}(z)-z=(-1/\gamma,-1/\gamma^2)$$
$$z\in\hat{T}U(D_1\cup D_2): \hat{s}(z)=-1,\ s(z)=-6,\ t(z)=\hat{T}^{-1}(z)+zA^{-1}=(1,1/\gamma)$$
$$z\in\hat{T}^4U(D_1): \hat{s}(z)=1,\ s(z)=6,\ t(z)=\hat{T}(z)+zA=(1/\gamma,0)$$
$$z\in\hat{T}U(D_0)\cup\hat{T}^3U(D_1): \hat{s}(z)=2,\ s(z)=10,\ t(z)=\hat{T}^2(z)-z=(-1/\gamma^2,0)$$

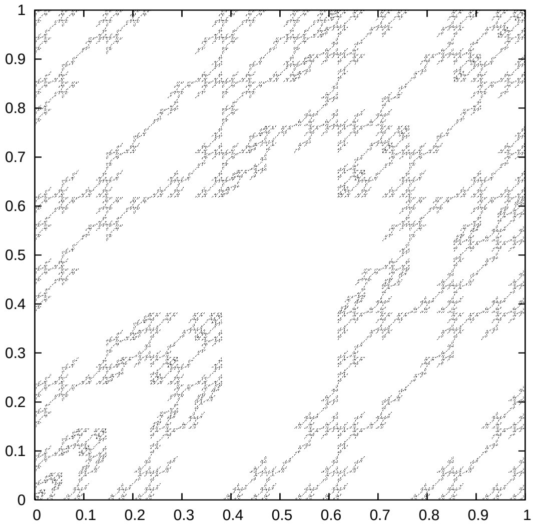

FIGURE 7.3. Aperiodic points, $\lambda = 1/\gamma$.

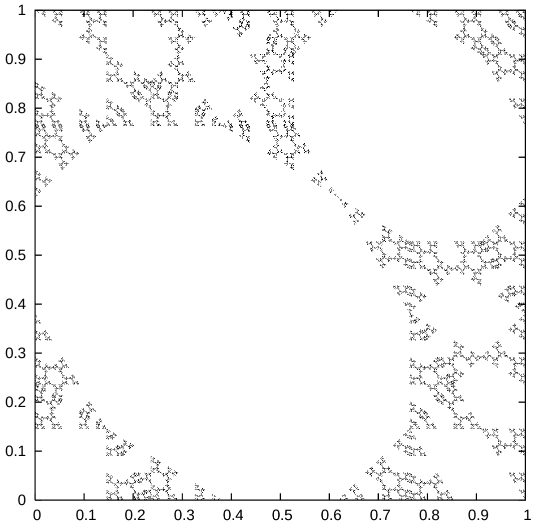

FIGURE 7.4. Aperiodic points, $\lambda = -\gamma$.

This gives again $\delta = \gamma^2/\gamma = \gamma$ since

$$\{(1/\gamma, 0)A^h : h \in \mathbb{Z}\} = \pm\{(1/\gamma, 0), (0, 1/\gamma), (1/\gamma, 1), (1, 1), (1, 1/\gamma)\}.$$

Theorem 7.2. *$\pi(z)$ is finite for all $z \in (\mathbb{Z}[\gamma] \cap [0,1))^2$, but $\big(T^k\big(1/4, 1/(4\gamma^3)\big)\big)_{k\in\mathbb{Z}}$ is aperiodic.*

Proof. Conjecture 1.1 holds for $\lambda = 1/\gamma$ since no $z \in \mathbb{Z}[\gamma]^2 \cap \mathcal{D}$ satisfies $\|z'\|_\infty \le \gamma$. It can be shown that all points in $(\frac{1}{2}\mathbb{Z}[\gamma] \cap [0,1))^2$ and $(\frac{1}{3}\mathbb{Z}[\gamma] \cap [0,1))^2$ are periodic as well. If $z = \big(1/4, 1/(4\gamma^3)\big)$, then we have $S(z) = \big(\gamma^2/4, 1/(4\gamma)\big)$, $S^2(z) = \gamma^2\big(S(z) - (1/\gamma^2, 0)\big) = \big((3\gamma - 2)/4, \gamma/4\big)$ and $S^3(z) = \gamma^2\big(S^2(z) - (1/\gamma, 1/\gamma^2)\big) = \big(1/4, 1/(4\gamma^3)\big) = z$. □

8. THE CASE $\lambda = -\gamma = -2\cos\frac{\pi}{5}$

Let $\lambda = -\gamma$ ($\lambda' = 1/\gamma$) and set

$$\mathcal{D} = \{(x, y) \in [0,1)^2 : x < y,\, \gamma x + y \ge 4 - \gamma\} = D_0 \cup D_1$$

with $D_0 = \{(x, y) \in \mathcal{D} : x > 1 - 1/\gamma^5\}$, $D_1 = \{(x, y) \in \mathcal{D} : x \le 1 - 1/\gamma^5\}$. Figure 8.1 shows that $\hat{T}(z) = T^{\tau(\ell)}(z)$ if $z \in D_\ell$, with $\tau(0) = 42$, $\tau(1) = 28$, and

$$\mathcal{R} = \{(0,0)\} \cup D_A \cup D_B \cup \bigcup\nolimits_{k=0}^{4} T^k(D_\Gamma) \cup \bigcup\nolimits_{k=0}^{1} T^k(D_\Delta) \cup \bigcup\nolimits_{k=0}^{24} T^k(D_E) \cup \bigcup\nolimits_{k=0}^{10} T^k(D_Z)$$

with $D_A = \{z : T^{k+1}(z) = T^k(z)A + (0,1) \text{ for all } k \in \mathbb{Z}\}$, $D_B = \{z : T^{k+1}(z) = T^k(z)A + (0,2)\}$, $D_\Delta = \{z \in [0,1)^2 : T^{2k+1}(z) = T^{2k}(z)A + (0,2),\, T^{2k}(z) = T^{2k-1}(z)A + (0,1) \text{ for all } k \in \mathbb{Z}\}$, $D_\Gamma = \{(x, y) : 0 \le x, y \le 1/\gamma^4\} \setminus \{(0,0), (1/\gamma^4, 1/\gamma^4)\}$, $D_E = \{(x, x) : 1 - 1/\gamma^5 < x < 1\}$, $D_Z = \{(1 - 1/\gamma^5, 1 - 1/\gamma^5)\}$. Set $\kappa = 1/\gamma^2$, $V(z) = \gamma^4\big((1,1) - z\big)$, i.e.

$$U(z) = (1,1) - \big((1,1) - z\big)/\gamma^2 = z/\gamma^2 + (1/\gamma, 1/\gamma).$$

Then Figure 8.2 shows that the conditions in Section 3 are satisfied by $\varepsilon = 1$ and

$$\sigma : \ 0 \mapsto 010 \qquad 1 \mapsto 01110.$$

All points in $\mathcal{P} = D_\alpha \cup D_\beta$ are periodic, with $D_\alpha = \{z \in \mathcal{D} : \hat{T}^k(z) \in D_0 \text{ for all } k \in \mathbb{Z}\}$, $D_\beta = \{z \in \mathcal{D} : \hat{T}^k(z) \in D_1 \text{ for all } k \in \mathbb{Z}\}$. Since $|\sigma^n(\ell)| \to \infty$ as $n \to \infty$ for all $\ell \in \mathcal{A}$, all conditions of Proposition 3.3 and Theorem 3.4 are satisfied, and we obtain the following theorem.

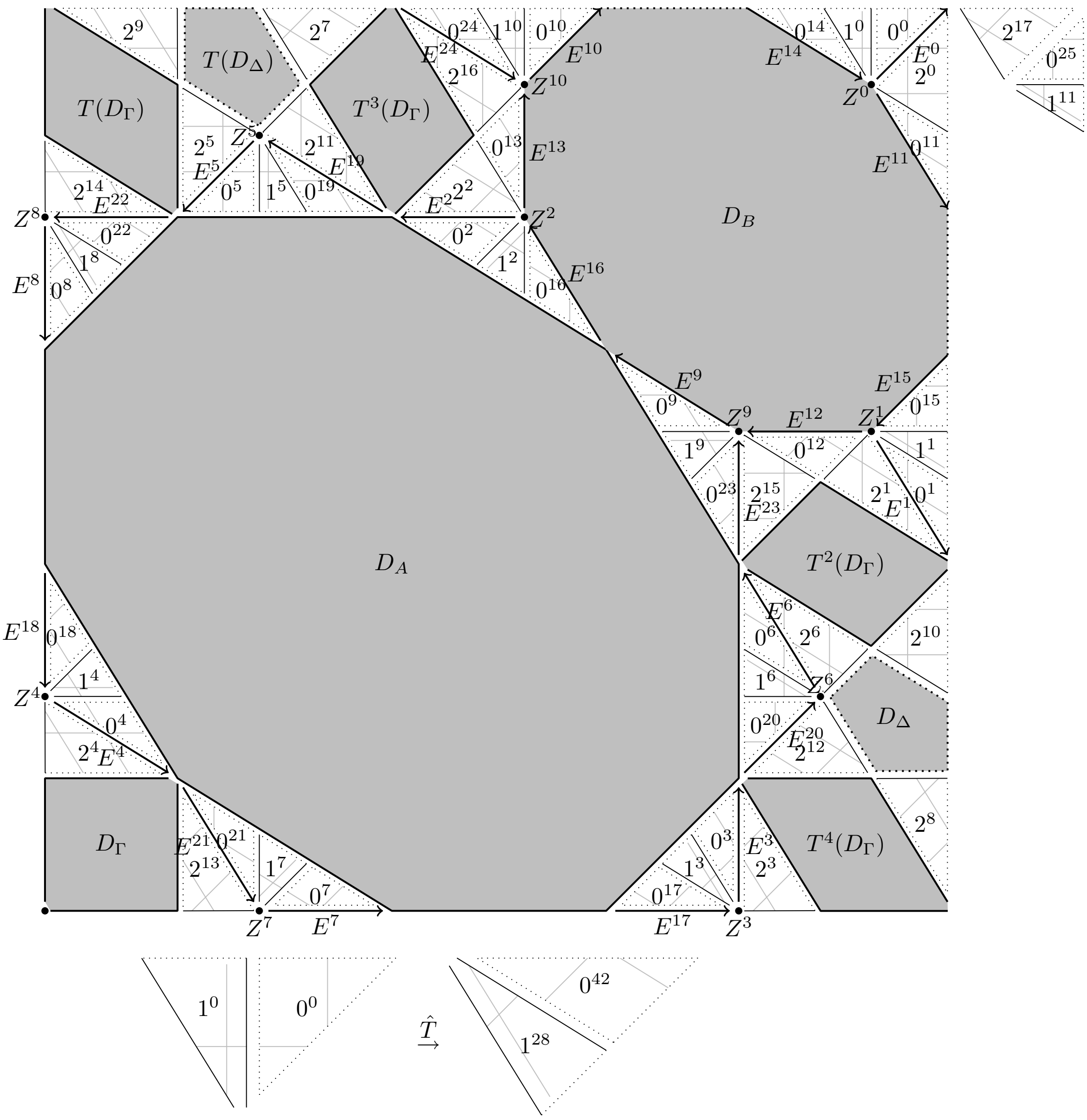

FIGURE 8.1. The map $\hat{T}$ and the set $\mathcal{R}$, $\lambda = -\gamma$. (ℓ^k stands for $T^k(D_\ell)$.)

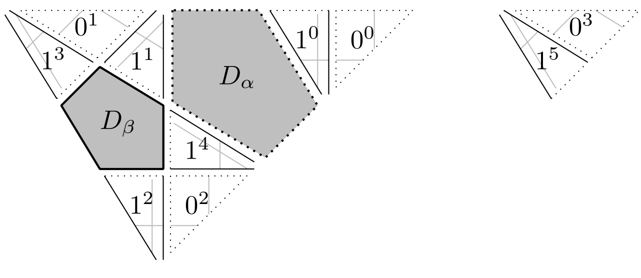

FIGURE 8.2. The trajectory of the scaled domains and $\mathcal{P}$, $\lambda = -\gamma$. (ℓ^k stands for $\hat{T}^k U(D_\ell)$.)

Theorem 8.1. *If $\lambda = -\gamma$, then the minimal period length $\pi(z)$ is*

1	*if* $z \in \{(0,0), (1/\gamma^2, 1/\gamma^2), (2/\gamma^2, 2/\gamma^2)\}$
2	*if* $z \in \{(\frac{5-\gamma}{\gamma^2+1}, \frac{2/\gamma^2}{\gamma^2+1}), (\frac{2/\gamma^2}{\gamma^2+1}, \frac{5-\gamma}{\gamma^2+1})\}$
5	*if* $z = T^m(1/(2\gamma^4), 1/(2\gamma^4))$ *for some* $m \in \{0,1,2,3,4\}$
10	*for the other points of* D_A, D_B, $T^m(D_\Gamma)$, $T^m(D_\Delta)$
11	*if* $z = T^m(1-1/\gamma^5, 1-1/\gamma^5)$ *for some* $m \in \{0,1,\ldots,10\}$
25	*if* $z = T^m(1-1/(2\gamma^5), 1-1/(2\gamma^5))$ *for some* $m \in \{0,1,\ldots,24\}$
50	*for the other points of* $T^m(D_E)$
$2(35\cdot 4^n+28)/3$	*if* $S^nR(z)$ *is the center of* D_α
$10(35\cdot 4^n+28)/3$	*for the other points of* D_α
$4(35\cdot 4^n-14)/3$	*if* $S^nR(z)$ *is the center of* D_β
$20(35\cdot 4^n-14)/3$	*for the other points of* D_β
∞	*if* $S^nR(z) \in \mathcal{D}\setminus\mathcal{P}$ *for all* $n \geq 0$

Proof. As for $\lambda = -1/\gamma$ and $\lambda = 1/\gamma$, we have

$$\begin{pmatrix} |\sigma^n(0)|_0 \\ |\sigma^n(0)|_1 \end{pmatrix} = 4^n \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}, \qquad \begin{pmatrix} |\sigma^n(1)|_0 \\ |\sigma^n(1)|_1 \end{pmatrix} = 4^n \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} + \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix},$$

hence $\tau(\sigma^n(0)) = (70\cdot 4^n + 56)/3$, $\tau(\sigma^n(1)) = (140\cdot 4^n - 56)/3$. For $S^nR(z) \in D_\alpha$, we have $\pi(z) = \tau(\sigma^n(0))$ and $5\tau(\sigma^n(0))$ respectively; if $S^nR(z) \in D_\beta$, then $\pi(z) = \tau(\sigma^n(1))$ and $5\tau(\sigma^n(1))$ respectively. □

We choose $\hat{s}(z)$ as follows and obtain the following $s(z), t(z)$:

$$\begin{aligned} z \in \hat{T}^2U(D_0\cup D_1) &: \hat{s}(z) = -2,\ s(z) = -70,\ t(z) = V(\hat{T}^{-2}(z)) - V(z) = (-1/\gamma^2, -1/\gamma^2) \\ z \in \hat{T}U(D_1) &: \hat{s}(z) = -1,\ s(z) = -42,\ t(z) = V(\hat{T}^{-1}(z)) - V(z)A^{-2} = (1/\gamma, 1/\gamma) \\ z \in \hat{T}^4U(D_1) &: \hat{s}(z) = 1,\ s(z) = 42,\ t(z) = V(\hat{T}(z)) - V(z)A^2 = (1,0) \\ z \in \hat{T}U(D_0)\cup\hat{T}^3U(D_1) &: \hat{s}(z) = 2,\ s(z) = 70,\ t(z) = V(\hat{T}^2(z)) - V(z) = (-1/\gamma, 0) \end{aligned}$$

This gives again $\delta = \gamma^2/\gamma = \gamma$ since

$$\{(1,0)A^h : h \in \mathbb{Z}\} = \pm\{(1,0), (0,1), (1,-1/\gamma), (1/\gamma, 1/\gamma), (1/\gamma, -1)\}.$$

Theorem 8.2. *$\pi(z)$ is finite for all $z \in (\mathbb{Z}[\gamma]\cap[0,1))^2$, but $\pi\big(1-1/(3\gamma^2), 1-1/(3\gamma^5)\big) = \infty$.*

Proof. Since $V(\mathcal{D}) = \{(x,y) : x > y > 0, \gamma x + y \leq \gamma\}$, we have no point $z \in \mathbb{Z}[\gamma]^2 \cap \mathcal{D}$ with $\|V(z)'\|_\infty \leq \gamma$, and Conjecture 1.1 holds for $\lambda = -\gamma$. If $V(z) = \big(\gamma^2/3, 1/(3\gamma)\big)$, then we have

$$VS(z) = \gamma^2\Big(V(z) - \Big(\frac{1}{\gamma}, 0\Big)\Big) = \Big(\frac{2}{3}, \frac{\gamma}{3}\Big),\ VS^2(z) = \gamma^2\Big(VS(z) - \Big(\frac{1}{\gamma^2}, \frac{1}{\gamma^2}\Big)\Big) = \Big(\frac{\gamma^2+1}{3\gamma}, \frac{2}{3\gamma}\Big),$$

$VS^3(z) = \gamma^2\big(VS^2(z) - \big(\frac{1}{\gamma^2}, \frac{1}{\gamma^2}\big)\big) = \big(\frac{3\gamma-2}{3}, \frac{1}{3\gamma^3}\big)$ and $VS^4(z) = \gamma^2\big(VS^3(z) - \big(\frac{1}{\gamma}, 0\big)\big) = V(z)$. □

9. The case $\lambda = \sqrt{3} = -2\cos\frac{5\pi}{6}$

The case $\lambda = \sqrt{3}$ is much more involved than the previous cases. Therefore we show only that all points in $(\mathbb{Z}[\sqrt{3}]\cap[0,1))^2$ are periodic and refrain from calculating the period lengths. Furthermore we postpone the determination of $\hat{T}$ and $\mathcal{R}$ to Appendix A. Let

$$\mathcal{D} = \{(x,y) : 2x-\sqrt{3}y < 2-\sqrt{3}, 2y-\sqrt{3}x < 2-\sqrt{3}, y-\sqrt{3}x < 195-113\sqrt{3}, x-\sqrt{3}y < 195-113\sqrt{3}\}$$

and $\mathcal{D}_1 = \mathcal{D}\setminus\mathcal{D}_2$, where $\mathcal{D}_2$ is defined by the inequalities

$$2x-\sqrt{3}y > 267-154\sqrt{3},\ 2y-\sqrt{3}x > 267-154\sqrt{3},\ y-\sqrt{3}x > 98-57\sqrt{3},\ x-\sqrt{3}y > 98-57\sqrt{3}.$$

The sets $\mathcal{D}_1$ and $\mathcal{D}_2$ have to be treated separately because their trajectories are disjoint, and both sets contain aperiodic points. The trajectories of aperiodic points in $\mathcal{D}_1$ come arbitrarily

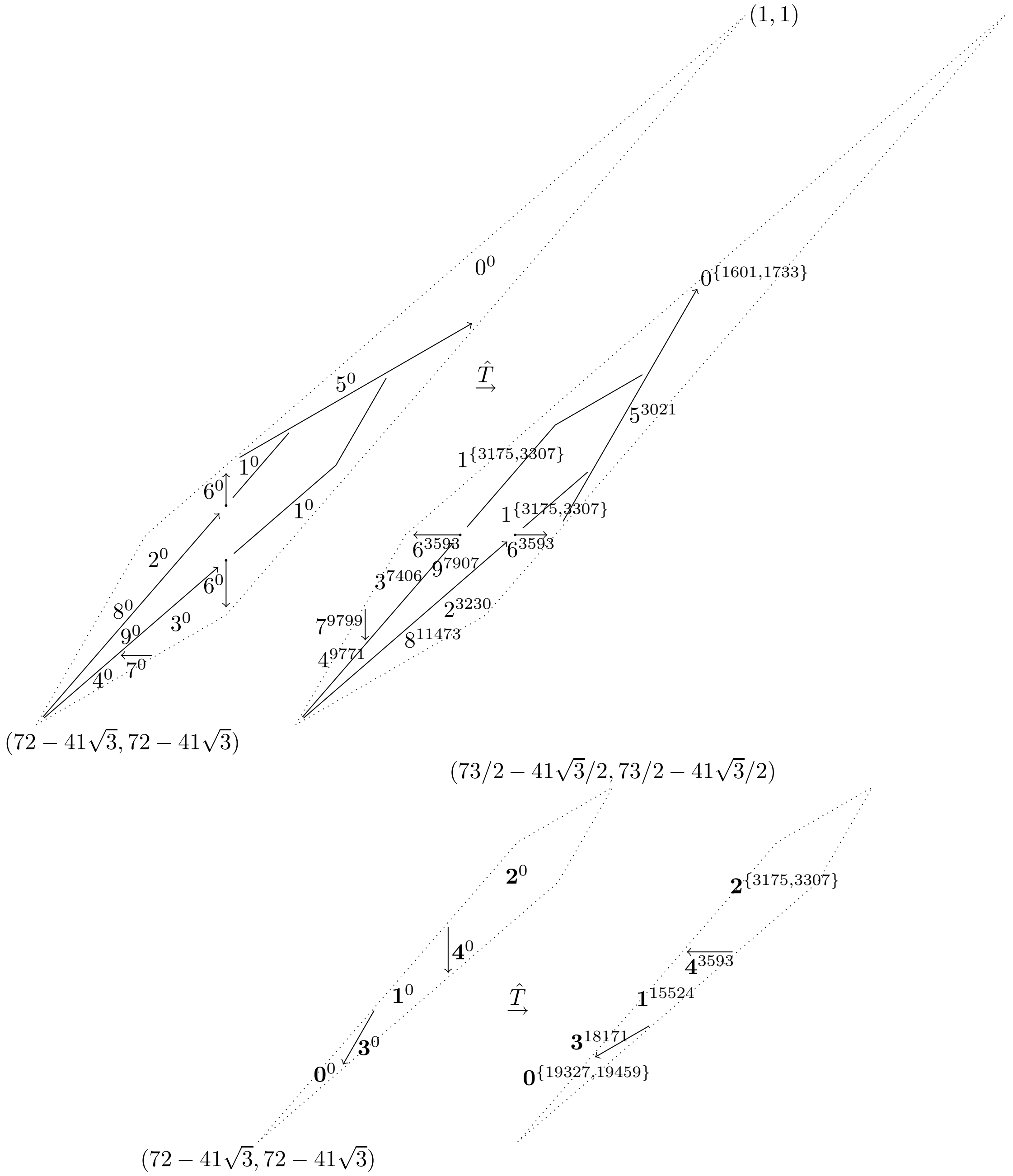

FIGURE 9.1. The first return map on $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, $\lambda = \sqrt{3}$. (ℓ^k stands for $T^k(D_\ell)$.)

close to $(1, 1)$, whereas $(72 - 41\sqrt{3}, 72 - 41\sqrt{3})$ is a limit point in $\mathcal{D}_2$. (Note that $72 - 41\sqrt{3} = 1 - (\sqrt{3} + 1)(2 - \sqrt{3})^4 \approx 0.9859$.) The scaling maps are

$$U_1(z) = (2 - \sqrt{3})z + (\sqrt{3} - 1, \sqrt{3} - 1) = V_1^{-1}(\kappa V_1(z)) \qquad \text{for } z \in \mathcal{D}_1,$$

$$U_2(z) = (2 - \sqrt{3})z + (113\sqrt{3} - 95, 113\sqrt{3} - 195) = V_2^{-1}(\kappa V_2(z)) \qquad \text{for } z \in \mathcal{D}_2,$$

with $\kappa = 2 - \sqrt{3}$, $V_1(z) = \big((1, 1) - z\big)/\kappa^4$, $V_2(z) = \big(z - (72 - 41\sqrt{3}, 72 - 41\sqrt{3})\big)/\kappa^5$. Then we have

$$V_1(\mathcal{D}) = \{(x, y) : 2x > \sqrt{3}y,\ 2y > \sqrt{3}x,\ x > \sqrt{3}y - 2,\ y > \sqrt{3}x - 2\},$$

$$V_2(\mathcal{D}_2) = \{(x, y) : 2x > \sqrt{3}y,\ 2y > \sqrt{3}x,\ x > \sqrt{3}y - 2 - \sqrt{3},\ y > \sqrt{3}x - 2 - \sqrt{3}\}.$$

The first return map $\hat{T}$ induces a partition of $\mathcal{D}_1$ into sets $D_0, \dots, D_9$ and a partition of $\mathcal{D}_2$ into sets $D_{\mathbf{0}}, \dots, D_{\mathbf{4}}$, as in Figure 9.1. These sets are defined by the following (in)equalities:

$V_1(D_0)$	$V_1(D_1)$	$V_1(D_2)$	$V_1(D_3)$	$V_1(D_4)$
$x > \sqrt{3}y - 1$	$x > \sqrt{3}y - 1$, $x < 2$	$2x > \sqrt{3}y + \sqrt{3} - 1$, $x > 2$	$2y > \sqrt{3}x + \sqrt{3} - 1$, $x > 2,\ y < 2\sqrt{3} - 1$	$2y > \sqrt{3}x + \sqrt{3} - 1$, $y > 2\sqrt{3} - 1$

$V_1(D_5)$	$V_1(D_6)$	$V_1(D_7)$	$V_1(D_8)$	$V_1(D_9)$
$x = \sqrt{3}y - 1$	$x = 2$	$y = 2\sqrt{3} - 1$, $x < 3 - 1/\sqrt{3}$	$2x = \sqrt{3}y + \sqrt{3} - 1$, $x > 2$	$2y = \sqrt{3}x + \sqrt{3} - 1$, $x > 2$

$V_2(D_{\mathbf{0}})$	$V_2(D_{\mathbf{1}})$	$V_2(D_{\mathbf{2}})$	$V_2(D_{\mathbf{3}})$	$V_2(D_{\mathbf{4}})$
$y > \sqrt{3}x - 1$	$y < \sqrt{3}x - 1,\ x < \sqrt{3} + 1$	$x > \sqrt{3} + 1$	$y = \sqrt{3}x - 1$	$x = \sqrt{3} + 1$

The return times of $z \in D_\ell$ to $\mathcal{D}$ are given by the following tables.

D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9
1601, 1733	3175, 3307	3230	7406	9771	3021	3593	9799	11473	7907

$D_{\mathbf{0}}$	$D_{\mathbf{1}}$	$D_{\mathbf{2}}$	$D_{\mathbf{3}}$	$D_{\mathbf{4}}$
19459	15524	3175, 3307	18171	3593

Note that the return times are not constant on all D_ℓ. E.g., the return time for $z \in D_0$ is 1601 if $V_1(z) = (1, y)$ and 1733 else, see Appendix A for details. Since we do not calculate the period lengths, it is not necessary to distinguish between the parts of D_ℓ with different period lengths.

9.1. **The scaling domain $\mathcal{D}_1$.** Figure 9.2 shows the trajectory of the open scaled sets in $\mathcal{D}_1$. Here, $V_1(\mathcal{D}_1)$ is split up into the three stripes $x < \sqrt{3} - 1$, $\sqrt{3} - 1 < x < 2$ and $x > 2$, and $D_{\bar{1}}$ denotes the set given by $V_1(D_{\bar{1}}) = \{(x, y) \in V_1(\mathcal{D}) : x > \sqrt{3}y - 1, x < 2\}$. We see that

$$\sigma_1 : \begin{array}{ccl} 0 & \mapsto & 010 \\ 1 & \mapsto & 01110 \\ 2 & \mapsto & 01210 \end{array} \quad \begin{array}{ccl} 3 & \mapsto & 012100001210 \\ 4 & \mapsto & 01210000000001210 \\ & & \end{array} \quad \begin{array}{ccl} 5 & \mapsto & 01510 \\ 6 & \mapsto & 01610 \\ & & \end{array} \quad \begin{array}{ccl} 7 & \mapsto & 01210000500001210 \\ 8 & \mapsto & 01210012621001210 \\ 9 & \mapsto & 0121005001210 \end{array}$$

codes the trajectory of $U_1(D_\ell)$, $\ell \in \{0, 1, 2, 3, 4\}$, with $\hat{T}^{|\sigma_1(\ell)|} U_1(z) = U_1 \hat{T}(z)$ for $z \in D_\ell$. All points in D_α, D_β and D_γ are periodic. Figure 9.4 shows that $D_{\tilde{\varepsilon}}, D_{\tilde{\zeta}}, D_{\tilde{\eta}}$ and the grey part of $U_1(\mathcal{D}_{\bar{1}})$ split up further, but all their points are periodic as well.

The trajectory of the scaled lines is depicted in Figure 9.3, where again $V_1(\mathcal{D}_1)$ is split up into the stripes $x < \sqrt{3} - 1$, $\sqrt{3} - 1 \le x < 2$ and $x \ge 2$. Here, $D_{\bar{1}}$ denotes boundary lines of D_1, and $D_{\bar{6}}$ is given by $V_1(D_{\bar{6}}) = \{(2, y) \in V_1(\mathcal{D})\}$. We see that σ_1 codes the trajectory of $U_1(D_\ell)$, $\ell \in \{5, 6, 7, 8, 9\}$, as well and satisfies the conditions in Section 3 (with respect to $\mathcal{D}_1$). All points in $D_\iota, D_\kappa, D_\lambda, D_\mu, D_\nu, D_\xi, D_o, D_\pi, D_\rho$ (and their orbits) are periodic. The finitely many remaining points in $\mathcal{P}_1 = \{z \in \mathcal{D}_1 : \hat{T}^m(z) \notin U_1(\mathcal{D}_1)$ for all $m \in \mathbb{Z}\}$ are clearly periodic as well. Since $|\sigma_1^n(\ell)| \to \infty$ for all $\ell \in \{0, \dots, 9\}$, we can use Proposition 3.5 to show the following proposition.

Proposition 9.1. *$\pi(z)$ is finite for all $z \in \mathbb{Z}[\sqrt{3}]^2 \cap \mathcal{D}_1$, but $\pi(V_1^{-1}(\sqrt{3} + 1/4, 7/4)) = \infty$.*

Proof. First we show that only D_0 and D_1 contain aperiodic points: D_3, D_4, D_7, D_8, D_9 lie in $\mathcal{P}_1$. The only part of D_2 which is not in $\mathcal{P}_1$ or $\hat{T}^m U_1(\mathcal{P}_1)$, lies in $\hat{T}^2 U_1(D_2)$. By iterating this argument on $\hat{T}^2 U_1(D_2)$, the possible set of aperiodic points in D_2 becomes smaller and smaller, and converges to $V_1^{-1}(2, \sqrt{3}) \notin D_2$. A similar reasoning shows that all points in D_5 and D_6 are periodic. Therefore it is sufficient to determine $t(z)$ for points in the trajectories of $U_1(D_0 \cup D_1)$.

$$z \in \hat{T} U_1(D_0) \cup \hat{T}^3 U_1(D_1) : \hat{s}(z) = 2,\ s(z) \equiv 0 \bmod 12,\ t(z) = (1 - \sqrt{3})(\sqrt{3}, 2)$$
$$z \in \hat{T}^4 U_1(D_1) : \hat{s}(z) = 1,\ s(z) \equiv 5 \bmod 12,\ t(z) = V_1(\hat{T}(z)) - V_1(z) A^5 = (\sqrt{3}, 2)$$
$$z \in \hat{T} U_1(D_1) : \hat{s}(z) = -1,\ s(z) \equiv -5 \bmod 12,\ t(z) = (2, \sqrt{3})$$
$$z \in \hat{T}^2 U_1(D_0) \cup \hat{T}^2 U_1(D_1) : \hat{s}(z) = -2,\ s(z) \equiv 0 \bmod 12,\ t(z) = (1 - \sqrt{3})(2, \sqrt{3})$$

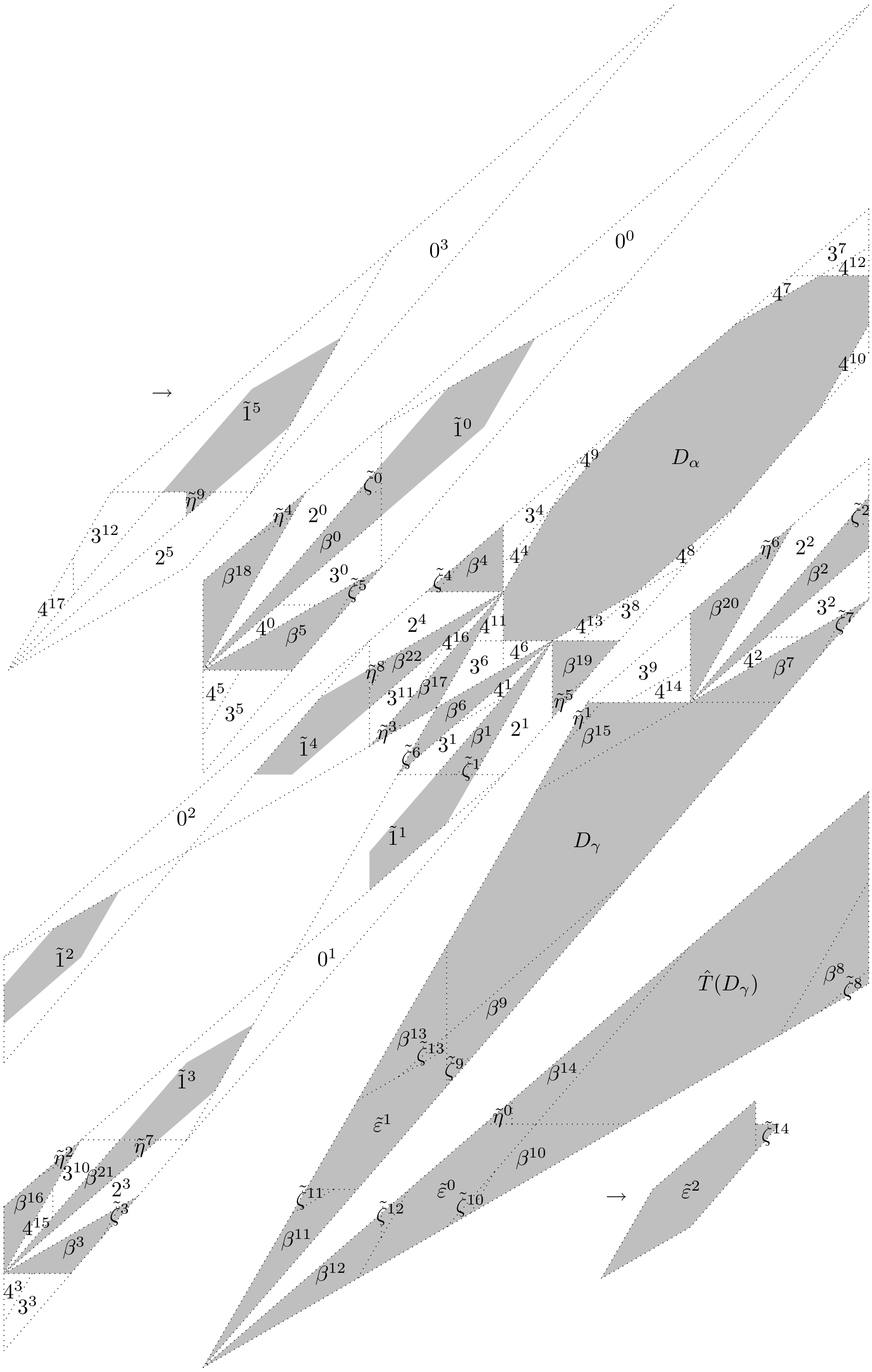

FIGURE 9.2. The trajectory of the open scaled sets in $\mathcal{D}_1$ and the set $\mathcal{P}_1$, $\lambda = \sqrt{3}$. (ℓ^k stands for $\hat{T}^k U_1(D_\ell)$ if $\ell \in \{0, \tilde{1}, 2, 3, 4\}$, for $\hat{T}^k(D_\ell)$ else.)

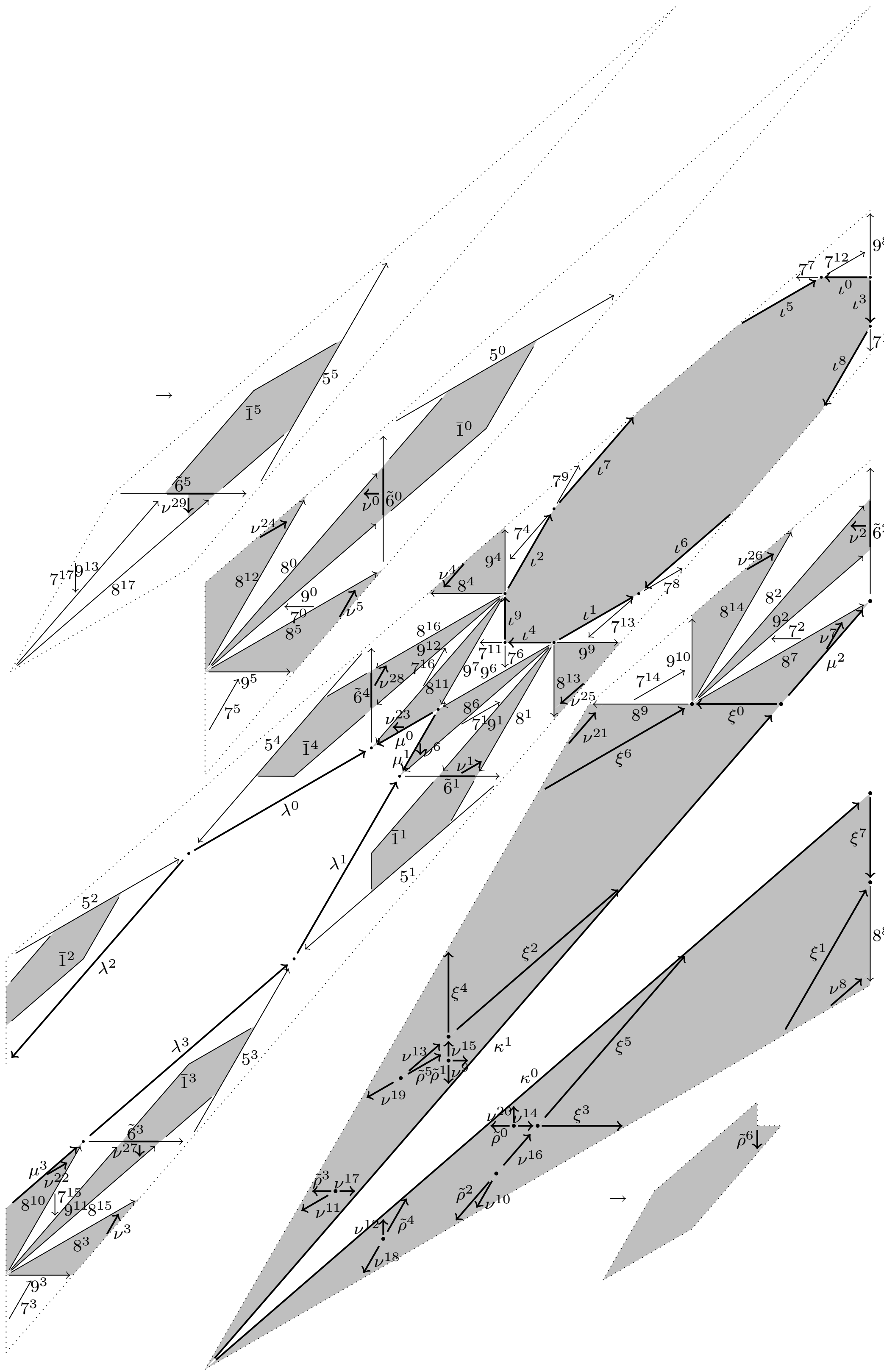

FIGURE 9.3. The trajectory of the scaled lines and the set $\mathcal{P}_1$, $\lambda = \sqrt{3}$. (ℓ^k stands for $\hat{T}^k U_1(D_\ell)$ if $\ell \in \{\bar{1}, 5, \tilde{6}, 7, 8, 9\}$, for $\hat{T}^k(D_\ell)$ else.)

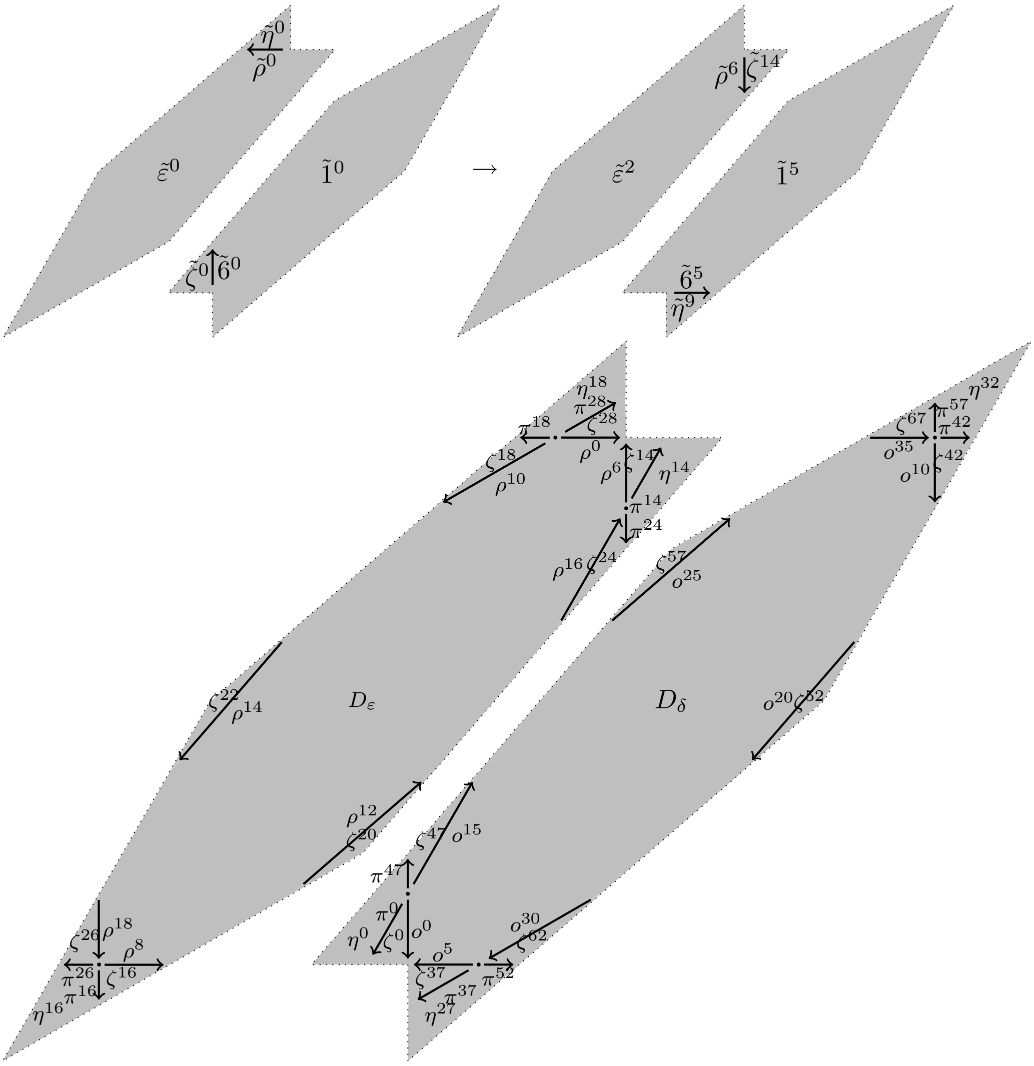

FIGURE 9.4. Small parts of $\mathcal{P}_1$, $\lambda = \sqrt{3}$. (ℓ^k stands for $\hat{T}^k(D_\ell)$ for $\ell \notin \{\tilde{1}, \tilde{6}\}$.)

We have $\delta_1 = (\sqrt{3}+1)2/(\sqrt{3}+1) = 2$ since

$$\{(\sqrt{3},2)A^h : h \in \mathbb{Z}\} = \pm\{(\sqrt{3},2),\, (2,\sqrt{3}),\, (\sqrt{3},1),\, (1,0),\, (0,1),\, (1,\sqrt{3})\}.$$

The only point $z \in V_1(\mathbb{Z}[\sqrt{3}]^2 \cap \mathcal{D}_1)$ with $\|z'\|_\infty \le 2$ is $(1,1) \in V_1(D_\alpha)$.

If $V_1(z) = (\sqrt{3}+1/4, 7/4)$, then we have

$$V_1S(z) = (2+\sqrt{3})\big(V_1(z) + (1-\sqrt{3})(2,\sqrt{3})\big) = (3/2+\sqrt{3}/4, 3\sqrt{3}/4+1/2),$$
$$V_1S^2(z) = (2+\sqrt{3})\big(V_1S(z) + (1-\sqrt{3})(2,\sqrt{3})\big) = (7/4, \sqrt{3}+1/4),$$

$V_1S^3(z) = (2+\sqrt{3})\big(V_1S^2(z)+(1-\sqrt{3})(\sqrt{3},2)\big) = (3\sqrt{3}/4+1/2, 3/2+\sqrt{3}/4)$, $V_1S^4(z) = V_1(z)$. □

Remark. The primitive part of σ_1 is again $0 \mapsto 010$, $1 \mapsto 01110$.

9.2. **The scaling domain $\mathcal{D}_2$.** Figure 9.5 shows the trajectory of the the scaled domains in $\mathcal{D}_2$. Here, $V_2(\mathcal{D}_2)$ is split up into $x \le \sqrt{3}+1$ and $x > \sqrt{3}+1$. With $\varepsilon_2 = 1$ and

$$\sigma_2 : \begin{array}{lll lll} \mathbf{0} & \mapsto & \mathbf{01222222210} & \mathbf{3} & \mapsto & \mathbf{012242210} \\ \mathbf{1} & \mapsto & \mathbf{012210} & \mathbf{4} & \mapsto & \mathbf{030} \\ \mathbf{2} & \mapsto & \mathbf{0} & & & \end{array}$$

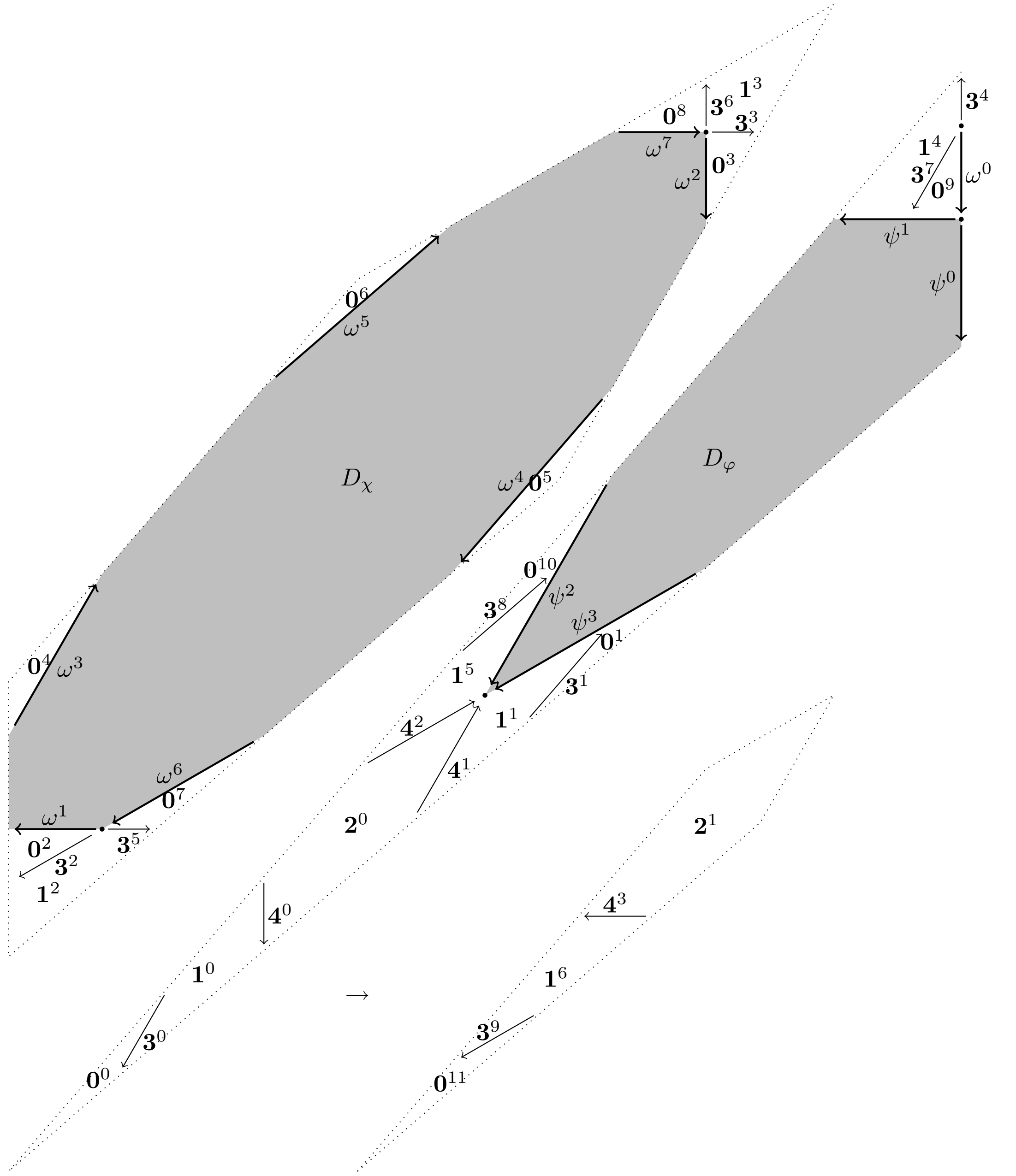

FIGURE 9.5. The trajectory of the scaled domains in $\mathcal{D}_2$ and the set $\mathcal{P}_2$, $\lambda = \sqrt{3}$. (ℓ^k stands for $\hat{T}^k(D_\ell)$ if $\ell \in \{\psi, \omega\}$, for $\hat{T}^k U_2(D_\ell)$ else.)

the conditions in Section 3 are satisfied. The set $\mathcal{P}_2 = \{z \in \mathcal{D}_2 : \hat{T}^m(z) \notin U_2(\mathcal{D}_2) \text{ for all } m \in \mathbb{Z}\}$ consists of the orbits of $D_\varphi, D_\chi, D_\psi, D_\omega$ and several isolated (periodic) points. Since $|\sigma_2^n(\ell)| \to \infty$ for all $\ell \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$, we can use Proposition 3.5 to show the following proposition.

Proposition 9.2. *$\pi(z)$ is finite for all $z \in \mathbb{Z}[\sqrt{3}]^2 \cap \mathcal{D}_2$, but $\pi(V_2^{-1}(5/7, 3\sqrt{3}/7)) = \infty$.*

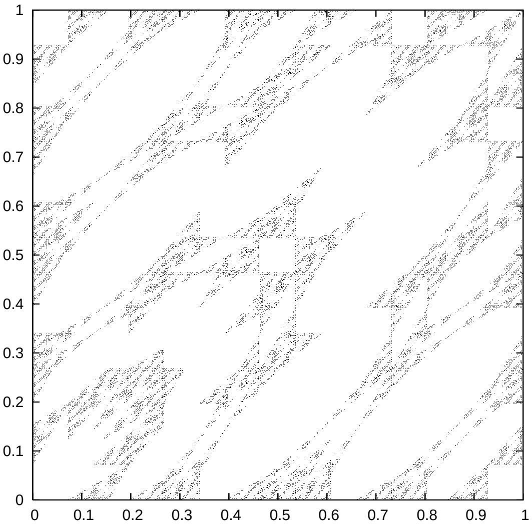

FIGURE 9.6. Aperiodic points, $\lambda = \sqrt{3}$.

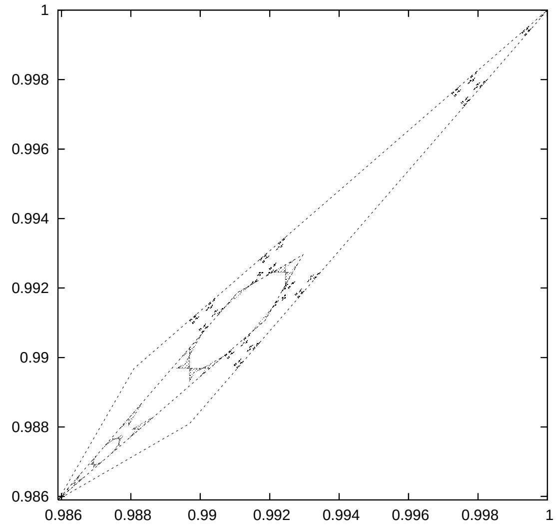

FIGURE 9.7. Aperiodic points in $\mathcal{D}_1 \cup \mathcal{D}_2$, $\lambda = \sqrt{3}$.

Proof. Similarly to $\mathcal{D}_1$, we see that all points in $D_{\mathbf{3}}$ and $D_{\mathbf{4}}$ are periodic. Choose $\hat{s}(z)$ as follows:

$$z \in \hat{T}^{10} U_2(D_{\mathbf{0}}) \cup \hat{T}^5 U_2(D_{\mathbf{1}}) : \hat{s}(z) = 1,\ s(z) \equiv 7 \bmod 12,\ t(z) = (2, \sqrt{3})$$
$$z \in \hat{T}^{9} U_2(D_{\mathbf{0}}) \cup \hat{T}^4 U_2(D_{\mathbf{1}}) : \hat{s}(z) = 2,\ s(z) \equiv 3 \bmod 12,\ t(z) = (1-\sqrt{3}, \sqrt{3}-1)$$
$$z \in \hat{T}^{8} U_2(D_{\mathbf{0}}) \cup \hat{T}^3 U_2(D_{\mathbf{1}}) : \hat{s}(z) = 3,\ s(z) \equiv 10 \bmod 12,\ t(z) = (1-\sqrt{3}, -3)$$
$$z \in \hat{T}^{7} U_2(D_{\mathbf{0}}) : \hat{s}(z) = 4,\ s(z) \equiv 5 \bmod 12,\ t(z) = \sqrt{3}(\sqrt{3}, 2)$$
$$z \in \hat{T}^{6} U_2(D_{\mathbf{0}}) : \hat{s}(z) = 5,\ s(z) \equiv 0 \bmod 12,\ t(z) = -2(\sqrt{3}, 2)$$

For the remaining $z \in \hat{T}^m U_2(D_{\mathbf{0}} \cup D_{\mathbf{1}})$, $\hat{s}(z), s(z)$ and $t(z)$ are obtained by symmetry. The sets $\{(1-\sqrt{3}, \sqrt{3}-1)A^h : h \in \mathbb{Z}\}$ and $\{(\sqrt{3}-1, 3)A^h : h \in \mathbb{Z}\}$ are

$$\pm\{(1-\sqrt{3}, \sqrt{3}-1), (\sqrt{3}-1, 2), (2, \sqrt{3}+1), (\sqrt{3}+1, \sqrt{3}+1), (\sqrt{3}+1, 2), (2, \sqrt{3}-1)\},$$
$$\pm\{(\sqrt{3}-1, 3), (3, 2\sqrt{3}+1), (2\sqrt{3}+1, 3+\sqrt{3}), (3+\sqrt{3}, 2+\sqrt{3}), (2+\sqrt{3}, \sqrt{3}), (\sqrt{3}, 1-\sqrt{3})\},$$

hence $\delta_2 = 4/(\sqrt{3}+1) = 2(\sqrt{3}-1)$. The only $x \in \mathbb{Z}[\sqrt{3}]$ with $0 < x < 5$ and $|x'| \le 2(\sqrt{3}-1)$ are $1, 1+\sqrt{3}, 2+\sqrt{3}$ and $3+\sqrt{3}$. Therefore the only $z \in V_2(\mathbb{Z}[\sqrt{3}]^2 \cap \mathcal{D}_2)$ with $\|z'\|_\infty \le 2(\sqrt{3}-1)$ are $(1,1)$, the center of $V_2U_2(D_\chi)$, $(1+\sqrt{3}, 1+\sqrt{3})$, the center of $D_{\mathbf{4}}$, $(2+\sqrt{3}, 2+\sqrt{3})$, the center of D_χ, and $(3+\sqrt{3}, 3+\sqrt{3})$, a fixed point of $\hat{T}^3$.

If $V_2(z) = (5/7, 3\sqrt{3}/7)$, then we have $V_2S(z) = (2+\sqrt{3})V_2(z)$ and
$V_2S^2(z) = (2+\sqrt{3})\big(V_2S(z)A^3 + (1-\sqrt{3}, \sqrt{3}-1)\big) = (5/7, 3\sqrt{3}/7) = V_2(z)$. □

By combining Propositions 9.1 and 9.2 and the fact that all points in $\mathcal{R}$ are periodic (see Appendix A), we obtain the following theorem.

Theorem 9.3. *Conjecture 1.1 holds for $\lambda = \sqrt{3}$.*

Remark. The eigenvalues corresponding to the primitive part of σ_2 ($\ell \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$) are $5, -2$ and 1.

10. THE CASE $\lambda = -\sqrt{3} = -2\cos\frac{\pi}{6}$

Let $\mathcal{D} = \{(x,y) \in [0,1)^2 : x + \sqrt{3}y > 5\sqrt{3} - 6 \text{ or } y + \sqrt{3}x > 5\sqrt{3} - 6\}$, U_1 as in Section 9 and

$$U(z) = U_1^2(z) = (2-\sqrt{3})^2 z + (4\sqrt{3}-6, 4\sqrt{3}-6) = V^{-1}(\kappa V(z)),$$

$\kappa = (2-\sqrt{3})^2$, $V(z) = \big((1,1) - z\big)/\kappa$. Then we have

$$V(\mathcal{D}) = \{(x,y) :\ x > 0,\ y > 0,\ x + \sqrt{3}y < 1 \text{ or } y + \sqrt{3}x < 1\}.$$

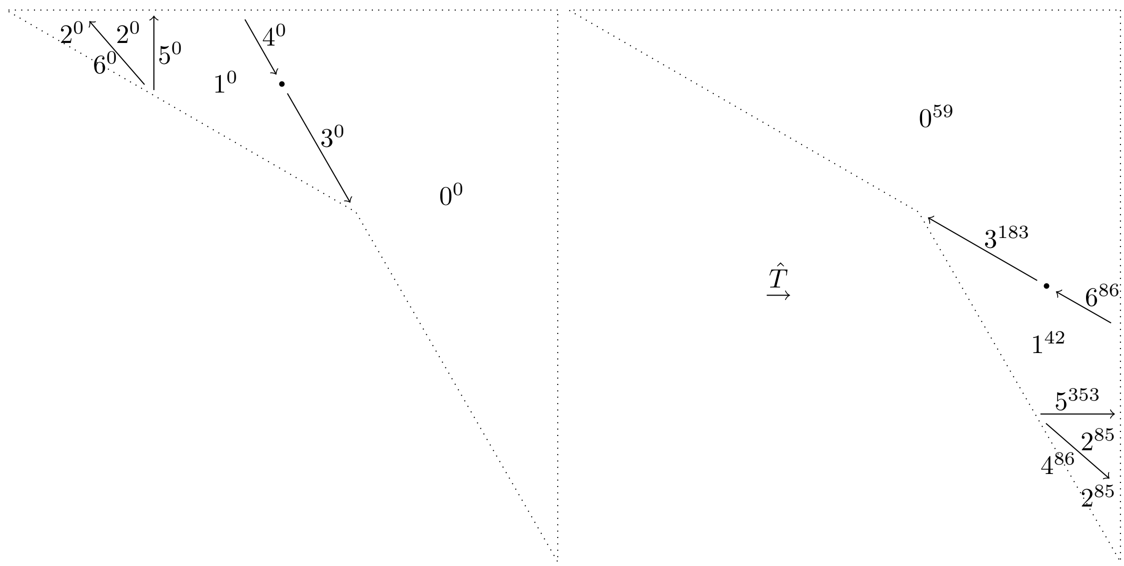

FIGURE 10.1. The map $\hat{T}$ on $\mathcal{D}$, $\lambda = -\sqrt{3}$. (ℓ^k stands for $T^k(D_\ell)$.)

Figure 10.1 shows the first return map $\hat{T}$ on $\mathcal{D}$, which is determined in Appendix B. The sets $D_0, \ldots, D_6$ satisfy the (in)equalities

$V(D_0)$	$V(D_1)$	$V(D_2)$
$\sqrt{3}x + y < 1$	$\sqrt{3}x + y > 1$, $x < \sqrt{3} - 1$	$x > \sqrt{3} - 1$, $2x + \sqrt{3}y \neq \sqrt{3}$

$V(D_3)$	$V(D_4)$	$V(D_5)$	$V(D_6)$
$\sqrt{3}x + y = 1$, $x < 1/2$	$\sqrt{3}x + y = 1$, $x > 1/2$	$x = \sqrt{3} - 1$	$2x + \sqrt{3}y = \sqrt{3}$

The remaining point $z = V^{-1}(1/2, 1 - \sqrt{3}/2)$ has return time 183 and satisfies $\hat{T}^{10}(z) = z$.

Figure 10.2 shows that the first return map on $U_1(\mathcal{D})$ differs from $U_1 \hat{T} U_1^{-1}$ on several lines. Therefore we add the lines D_7, D_8, D_9 satisfying the following (in)equalities

$V(D_7)$	$V(D_8)$	$V(D_9)$
$\sqrt{3}x + y = 1$	$\sqrt{3}x + 2y = 1$, $x > 2 - \sqrt{3}$	$\sqrt{3}x + 2y = 1$, $x < 2 - \sqrt{3}$

and define $D_{\tilde{0}} = D_0 \setminus V^{-1}(\{(x, y) : \sqrt{3}x + 2y = 1\})$, $D_{\tilde{2}} = D_2 \cup D_6$. For $z \in D_{\tilde{\ell}}$, $\ell \in \{0, 2\}$ and $z \in D_\ell$, $\ell = 1$, we have $\hat{T}^{|\sigma_1(\ell)|} U_1(z) = U_1 \hat{T}(z)$ with

$$\sigma_1 : \begin{array}{llllllllllll} 0 & \mapsto & 020 & 1 & \mapsto & 010^4 10 & 2 & \mapsto & 010^9 10 & & & \\ 5 & \mapsto & 010^4 40 & 7 & \mapsto & 050 & 8 & \mapsto & 060^4 10 & 9 & \mapsto & 060^9 30^9 40 \end{array}$$

Figure 10.3 shows that the substitution σ given by $\sigma(\ell) = \sigma_1 \sigma_2(\ell)$ with

$$\sigma_2 : \begin{array}{llllllllllll} 0 & \mapsto & 020 & 1 & \mapsto & 010^4 10 & 2 & \mapsto & 010^9 10 & & & \\ 3 & \mapsto & 050^5 90^5 80 & 4 & \mapsto & 050^4 10 & 5 & \mapsto & 010^4 70^4 10 & 6 & \mapsto & 010^4 80 \end{array}$$

satisfies the conditions in Section 3 (with $\varepsilon = 1$). The coding of the return path of the remaining point is $\sigma_1(050^4 70^4 80)$.

Theorem 10.1. *$\pi(z)$ is finite for all $z \in (\mathbb{Z}[\sqrt{3}] \cap [0, 1))^2$, but $\pi(V^{-1}(2/7, \sqrt{3}/7 + 1/7) = \infty$.*

Proof. First we show that all points on the lines $U_1^n(D_\ell)$, $\ell \in \{3, \ldots, 9\}$, $n \geq 0$, are periodic. The only possibly aperiodic part of D_5 is $\hat{T} U_1(D_7)$, and the only possibly aperiodic part of $U_1(D_7)$ is $\hat{T}^{23} U_1^2(D_5)$. Inductively, the set of aperiodic points in D_5 converges to $V^{-1}(\sqrt{3}-1, 1-1/\sqrt{3}) \notin D_5$ and is therefore empty. Therefore, all points in $U^n(D_5)$ and $U^n U_1(D_7)$ are periodic. Similar arguments show that all points in $U^n(D_3)$ in $U^n U_1(D_9)$ are periodic, then the same holds for $U^n(D_4)$ and $U^n U_1(D_5)$, for $U^n(D_6)$ and $U^n U_1(D_8)$, and finally for $U^n(D_8)$ and $U^n U_1(D_6)$. Then it is clear that all points in $U^n U_1(D_3 \cup D_4)$ and $U^n(D_7 \cup D_9)$ are periodic as well.

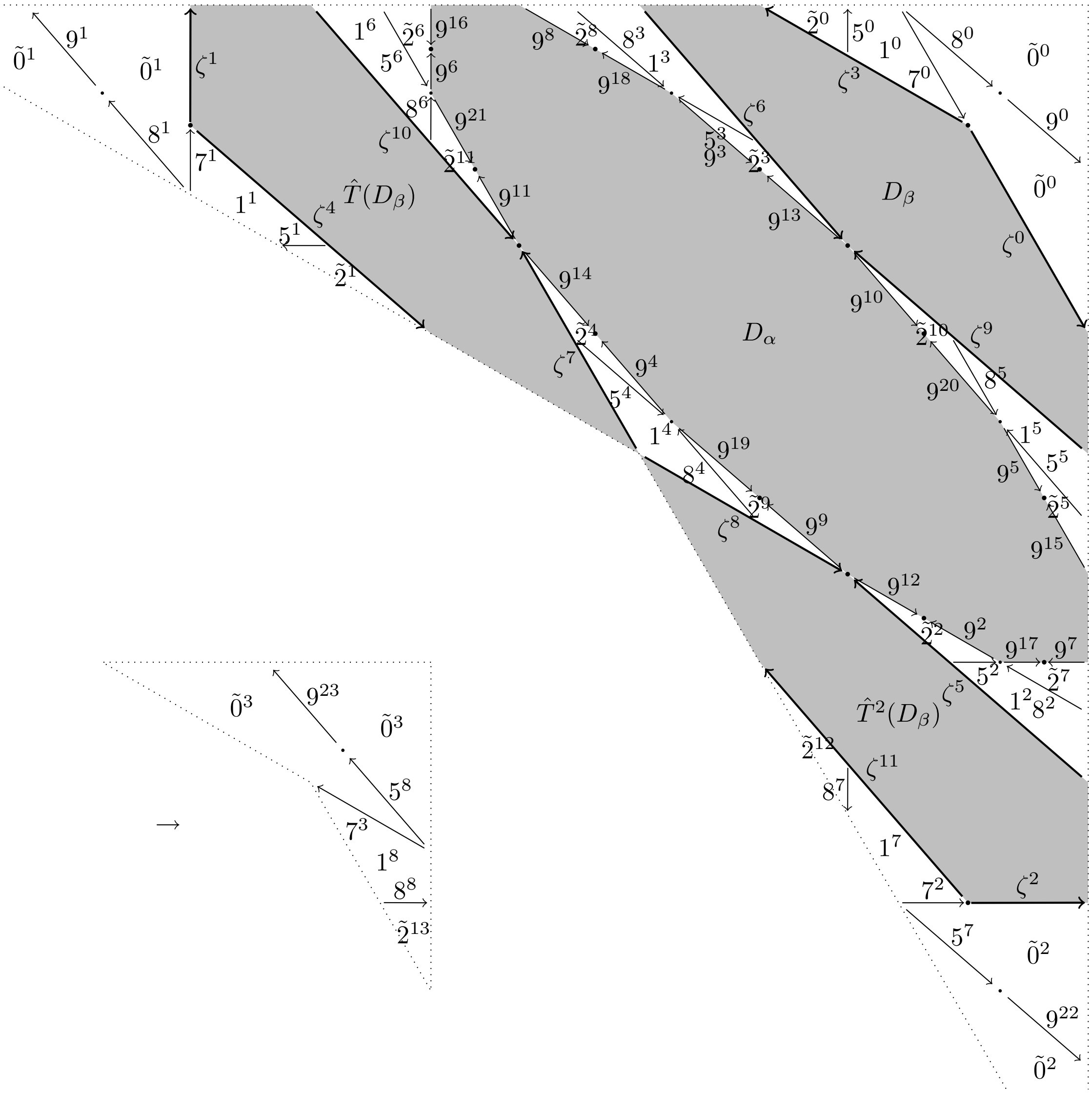

FIGURE 10.2. Trajectory of $U_1(\mathcal{D})$ and large parts of $\mathcal{P}$, $\lambda = -\sqrt{3}$. (ℓ^k stands for $\hat{T}^k U_1(D_\ell)$.)

Therefore we can limit our considerations to $U_1^n(D_{\tilde{0}} \cup D_1 \cup D_2)$, and consider the scaling map U_1 instead of U. If we define $\hat{s}_1(z), s_1(z)$ and $t_1(z)$ accordingly, we obtain:

$$\begin{aligned}
z \in \hat{T}^{-1}U_1(\mathcal{D}) &: \hat{s}_1(z) = 1,\ s_1(z) \equiv 11 \bmod 12,\ t_1(z) = V(\hat{T}(z)) - V(z)A^{-1} = (1,0) \\
z \in \hat{T}^{6}U_1(D_1) \cup \hat{T}^{11}U_1(D_2) &: \hat{s}_1(z) = 2,\ s_1(z) \equiv 5 \bmod 12,\ t_1(z) = (-1, \sqrt{3}-1) \\
z \in \hat{T}^{5}U_1(D_1) \cup \hat{T}^{10}U_1(D_2) &: \hat{s}_1(z) = 3,\ s_1(z) \equiv 4 \bmod 12,\ t_1(z) = (\sqrt{3}-1, \sqrt{3}-2) \\
z \in \hat{T}^{4}U_1(D_1) \cup \hat{T}^{9}U_1(D_2) &: \hat{s}_1(z) = 4,\ s_1(z) \equiv 3 \bmod 12,\ t_1(z) = (\sqrt{3}-1)(-\sqrt{3}, 2) \\
z \in \hat{T}^{8}U_1(D_2) &: \hat{s}_1(z) = 5,\ s_1(z) \equiv 2 \bmod 12,\ t_1(z) = (2-\sqrt{3})(\sqrt{3}, -2) \\
z \in \hat{T}^{7}U_1(D_2) &: \hat{s}_1(z) = 6,\ s_1(z) \equiv 1 \bmod 12,\ t_1(z) = (2\sqrt{3}-4, 3\sqrt{3}-4)
\end{aligned}$$

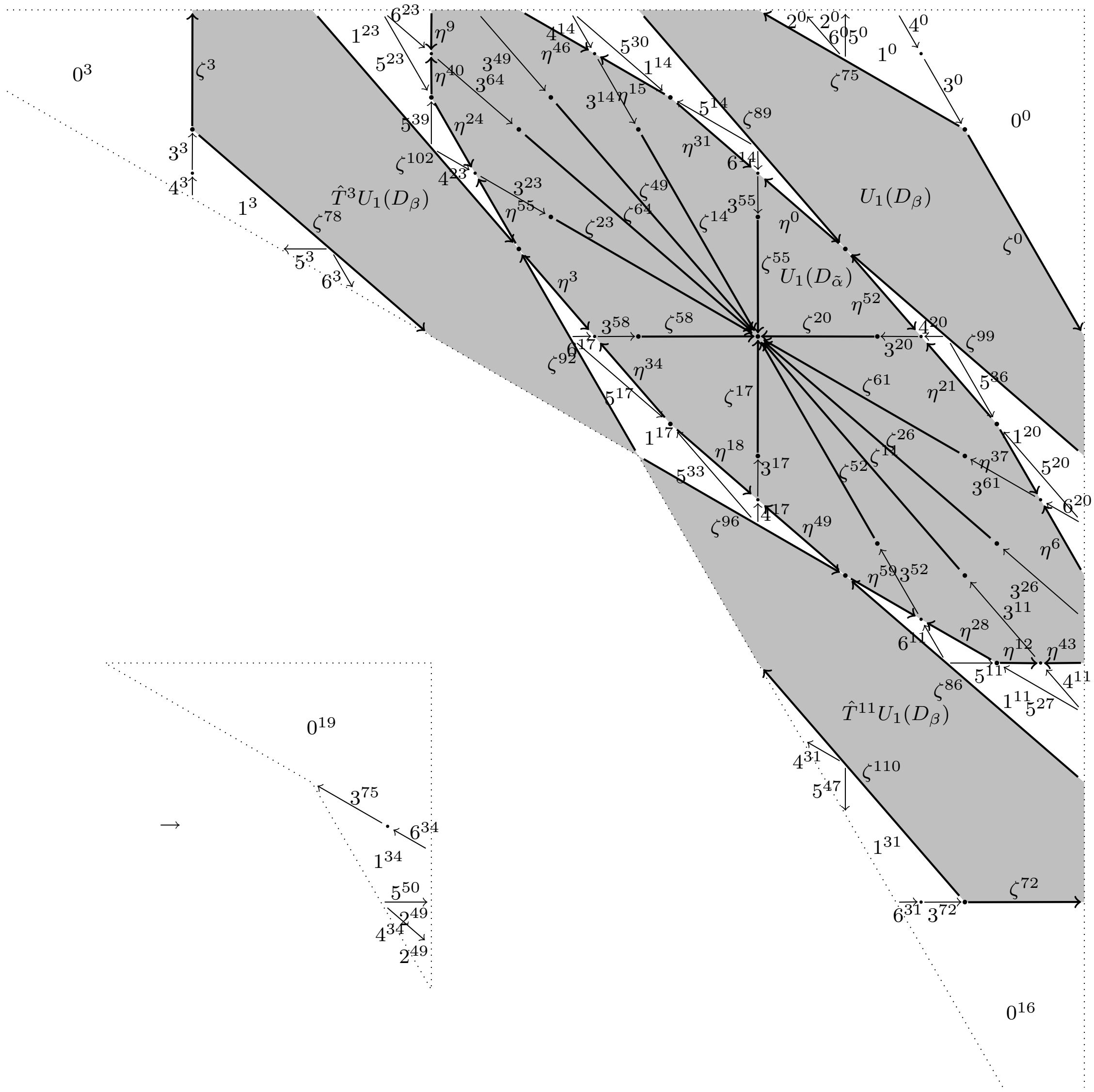

FIGURE 10.3. Trajectory of $U(\mathcal{D})$ and small parts of $\mathcal{P}$, $\lambda = -\sqrt{3}$. (ℓ^k stands for $\hat{T}^k U(D_\ell)$.)

For the remaining z, $\hat{s}_1(z), s_1(z)$ and $t_1(z)$ are given symmetrically. By looking at the following sets $\{t_1(z)A^h : h \in \mathbb{Z}\}$, we obtain $\delta_1 = (3\sqrt{3}+4)/(\sqrt{3}+1) = (5+\sqrt{3})/2$:

$$\pm\{(1,0),\, (0,1),\, (1,-\sqrt{3}),\, (-\sqrt{3},2),\, (2,-\sqrt{3}),\, (-\sqrt{3},1)\},$$
$$\pm\{(1,1-\sqrt{3}),\, (1-\sqrt{3},2-\sqrt{3}),\, (2-\sqrt{3},2-\sqrt{3}),\, (2-\sqrt{3},1-\sqrt{3}),\, (1-\sqrt{3},1),\, (1,-1)\},$$
$$\pm\{(2\sqrt{3}-4,3\sqrt{3}-4),\, (3\sqrt{3}-4,2\sqrt{3}-5),\, (2\sqrt{3}-5,2\sqrt{3}-2),$$
$$(2\sqrt{3}-2,-1),\, (-1,\sqrt{3}-2),\, (2-\sqrt{3},4-2\sqrt{3})\}.$$

The only $x \in \mathbb{Z}[\sqrt{3}]$ with $0 < x < 1$ and $|x'| \le (5+\sqrt{3})/2$ is $\sqrt{3}-1$. Therefore no point $z \in V(\mathbb{Z}[\sqrt{3}]^2 \cap \mathcal{D})$ satisfies $\|z'\|_\infty \le \delta_1$, and Conjecture 1.1 holds for $\lambda = -\sqrt{3}$.

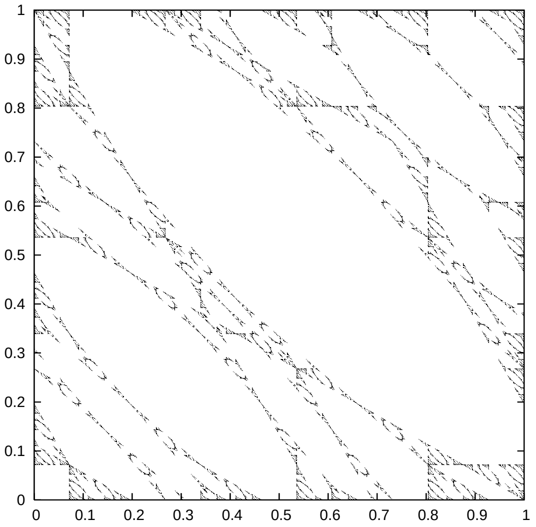

FIGURE 10.4. Aperiodic points, $\lambda = -\sqrt{3}$.

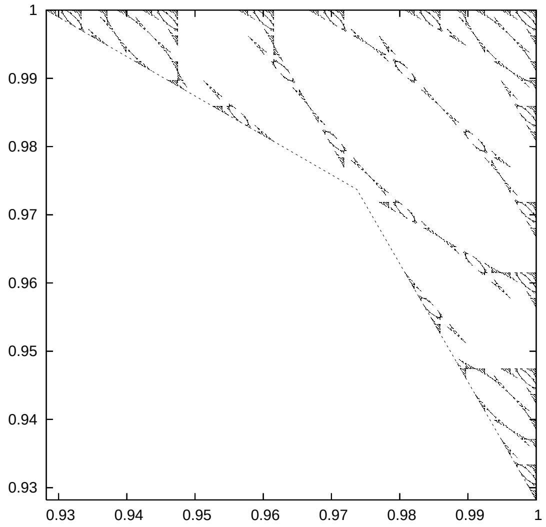

FIGURE 10.5. Aperiodic points in $\mathcal{D}$, $\lambda = -\sqrt{3}$.

If $V(z) = (2/7, \sqrt{3}/7 + 1/7)$, then we have

$$VS_1(z) = (2+\sqrt{3})\big(V(z)A^3 + (\sqrt{3}-1)(-\sqrt{3}, 2) = (3\sqrt{3}/7 - 5/7, 5\sqrt{3}/7 - 3/7),$$
$$VS_1^2(z) = (2+\sqrt{3})\big(VS_1(z)A^{11} + (1,0) = (\sqrt{3}/7 + 2/7, \sqrt{3}/7 - 1/7),$$
$$VS_1^3(z) = (2+\sqrt{3})\big(VS_1^2(z)A^5 + (-1, \sqrt{3}-1) = (\sqrt{3}/7 - 1/7, 3\sqrt{3}/7),$$
$$VS_1^4(z) = (2+\sqrt{3})\big(VS_1^3(z)A^{11} + (1,0) = (2/7, \sqrt{3}/7 + 1/7) = V(z). \qquad \square$$

Remark. The eigenvalues corresponding to the primitive part of σ_1 ($\ell \in \{0,1,2\}$) are 5, -2 and 1.

11. The Thue-Morse sequence, the golden mean and $\sqrt{3}$

We conclude by exhibiting a relation between the Thue-Morse sequence and substitutions we used in golden mean cases (see [6] for a survey on links between fractal objects and automatic sequences). The Thue-Morse sequence is a fixed point of the substitution $0 \mapsto 01$, $1 \mapsto 10$:

$$0\,1\,10\,1001\,10010110\,1001011001101001\,10010110011010010110100110010110\cdots$$

It can be written as

$$0^1 1^2 0^1 1^1 0^2 1^2 0^2 1^1 0^1 1^2 0^1 1^1 0^2 1^1 0^1 1^2 0^2 1^2 0^1 1^1 0^2 1^2 0^2 1^1 0^1 1^2 0^2 1^2 0^1 1^1 0^2 1^1 0^1 1^2 0^1 1^1 0^2 1^2 0^2 1^1 0^1 1^2 0^1 \cdots$$

By subtracting 1 from each term of the sequence of exponents (the run-lengths of 0's and 1's) we obtain the sequence

$$0\,10\,01110010\,0100111001110011100100100111010\cdots$$

which is easily shown to be the fixed point of the substitution $0 \mapsto 010$, $1 \mapsto 01110$ (see [5]), which is equal to σ in the cases $\lambda = -1/\gamma$, $\lambda = 1/\gamma$, $\lambda = -\gamma$, and to σ_1 in the case $\lambda = \sqrt{3}$. In case $\lambda = \gamma$, we have that $\sigma^\infty(1)$ is the image of this word by the morphism $0 \mapsto 10$, $1 \mapsto 110$ since $\sigma(10) = (10)(110)(10)$ and $\sigma(110) = (10)(110)(110)(110)(10)$.

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Dep. of Mathematics, Faculty of Science Niigata University, Ikarashi 2-8050, Niigata 950-2181, Japan
akiyama@math.sc.niigata-u.ac.jp

Haus-Endt-Strasse 88, D-40593 Düsseldorf, Germany
brunoth@web.de

Department of Computer Science, University of Debrecen, P.O. Box 12, H-4010 Debrecen, Hungary
pethoe@inf.unideb.hu

LIAFA, CNRS, Université Paris 7, Case 7014, 2 place Jussieu, 75251 Paris Cedex 05, France
steiner@liafa.jussieu.fr

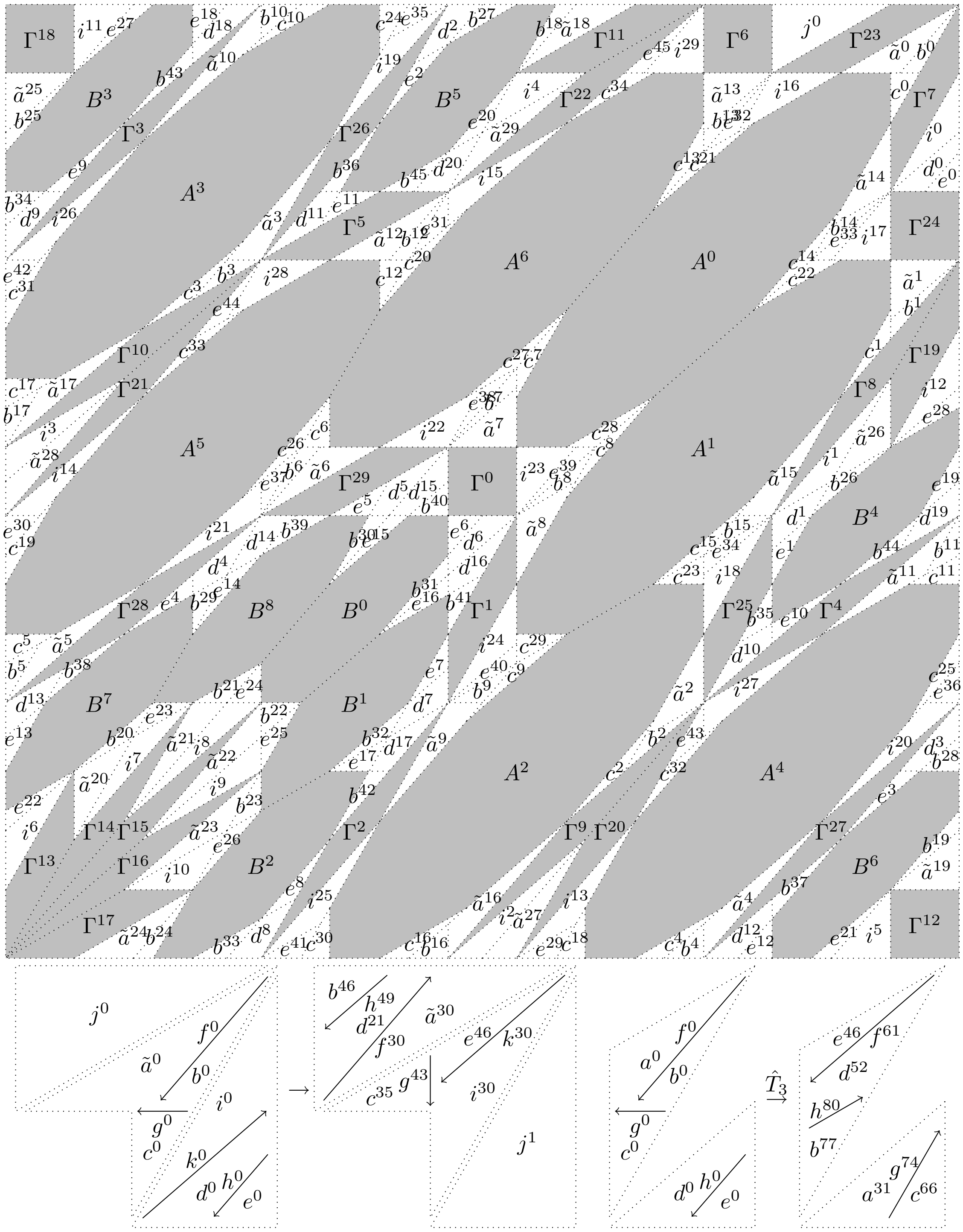

Figure A.1. The first return map $\hat{T}_3$ and large parts of $\mathcal{R}$, $\lambda = \sqrt{3}$. (ℓ^k stands for $T^k(D_\ell)$.)

Appendix A. The map $\hat{T}$ for $\lambda = \sqrt{3}$.

As the scaling domain $\mathcal{D}$ is very small in case $\lambda = \sqrt{3}$, the determination of $\hat{T}$ is done in several steps. Figure A.1 shows the action of $\hat{T}_3$, which is the first return map on the domain

$$\{x > 4\sqrt{3}-6, 2y < \sqrt{3}x+2-\sqrt{3}, y > \sqrt{3}x+1-\sqrt{3}\}\cup\{x < 1, y > 12\sqrt{3}x-20, y > \sqrt{3}x+6\sqrt{3}-11\},$$

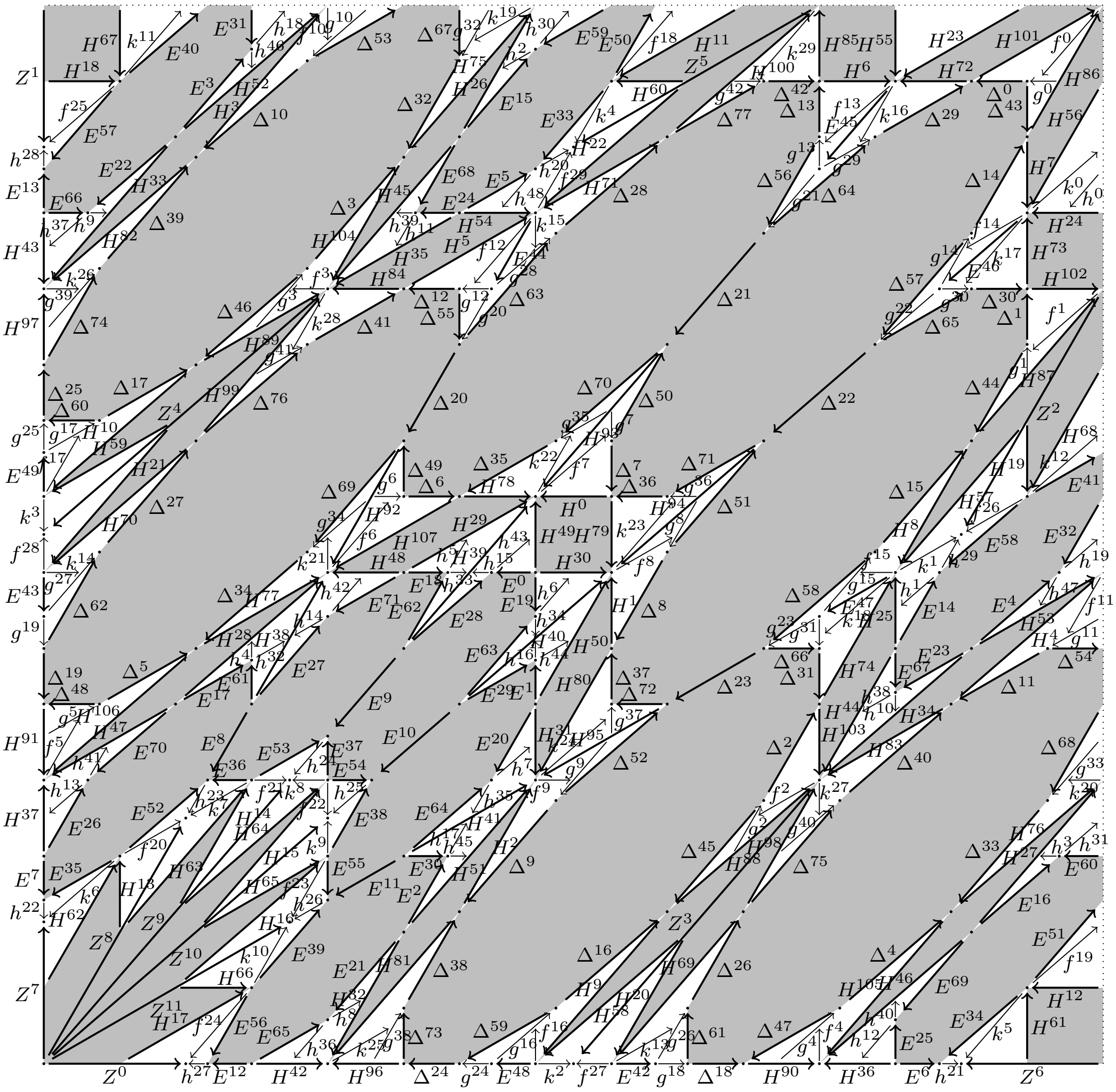

FIGURE A.2. Trajectories of long lines in $\mathcal{R}$ and D_f, D_g, D_h, D_k, $\lambda = \sqrt{3}$. (ℓ^k stands for $T^k(D_\ell)$.)

on sets $D_a, \ldots, D_h$. To this end, we first determine the trajectory of sets $D_{\tilde{a}}, D_b, \ldots, D_k$, which partition a symmetric version of this domain. Figure A.1 shows the trajectory of the open sets $D_{\tilde{a}}, D_b, D_c, D_d, D_e, D_i, D_j$, Figure A.2 completes the picture with the trajectories of the lines D_f, D_g, D_h, D_k. All points which are not on these trajectories are periodic. ¿From the symmetric first return map, it is easy to determine $\hat{T}_3$.

Next, we consider the first return map on

$$\{(x,y) : 2y < \sqrt{3}x + 2 - \sqrt{3},\, 2x < \sqrt{3}y + 2 - \sqrt{3},\, x \geq 30\sqrt{3} - 51 \text{ or } y \geq 30\sqrt{3} - 51\}$$

in Figures A.3 and A.4, partitioned into open sets $D_k, D_{l_1}, D_{l_2}, D_{\tilde{m}}, D_{\tilde{n}}$ and lines $D_{l_3}, D_o, D_{\tilde{p}}, D_{\tilde{q}}$. ¿From this map, we easily obtain the first return map $\hat{T}_4$ on $\{(x,y) : 2y < \sqrt{3}x + 2 - \sqrt{3},\, 2x < \sqrt{3}y + 2 - \sqrt{3},\, x \geq 30\sqrt{3} - 51 \text{ and } y \geq 30\sqrt{3} - 51\}$, which is partitioned into the sets $D_l, \ldots, D_q$. Observe that the return time on D_l is not constant since the trajectories of the three parts $D_{l_1}, D_{l_2}, D_{l_3}$ are different. This implies that the return times on D_0, D_1 and $D_{\mathbf{0}}$ are not constant.

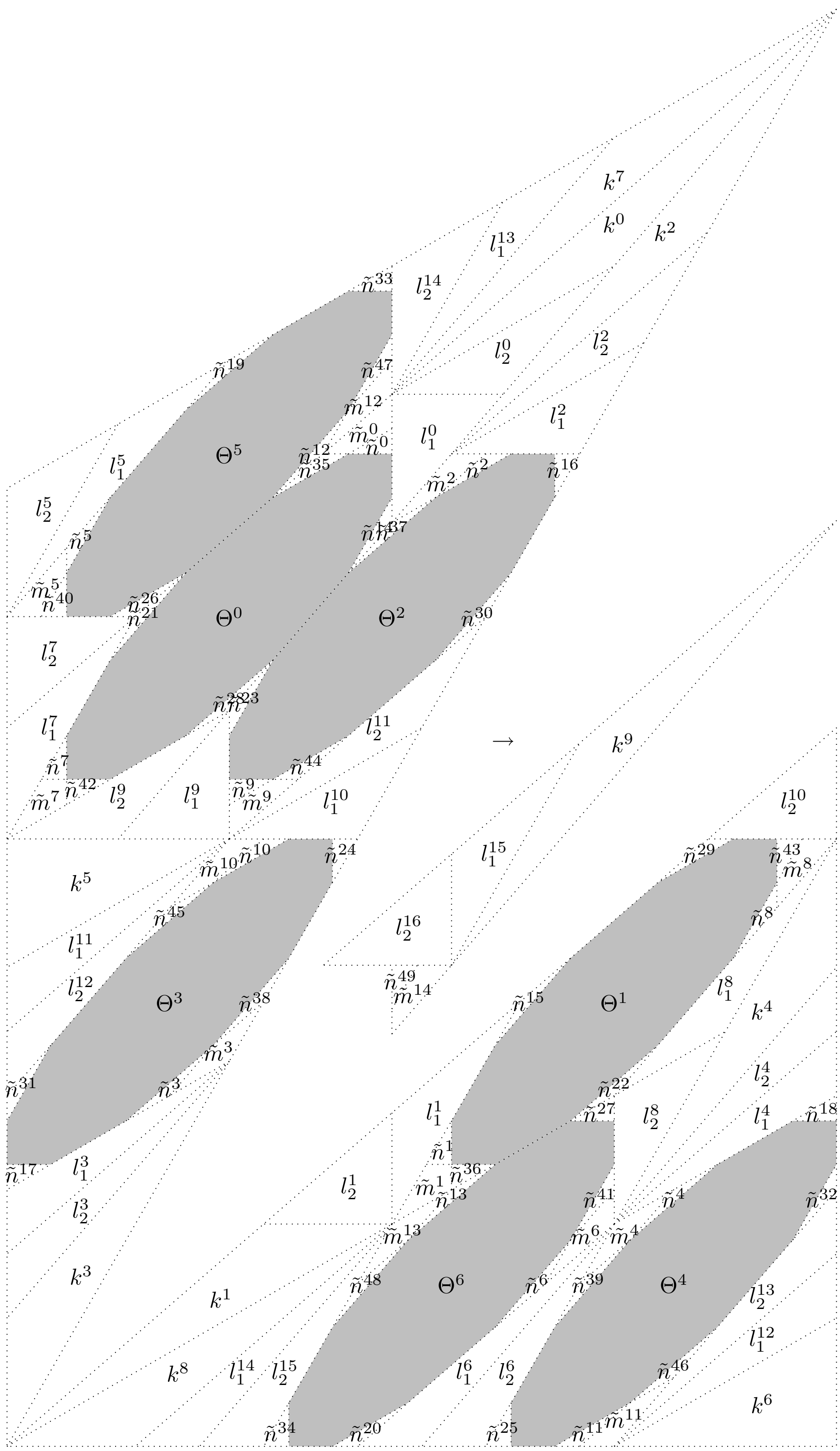

FIGURE A.3. An intermediate first return map, $\lambda = \sqrt{3}$. (ℓ^k stands for $\hat{T}_3^k(D_\ell)$.)

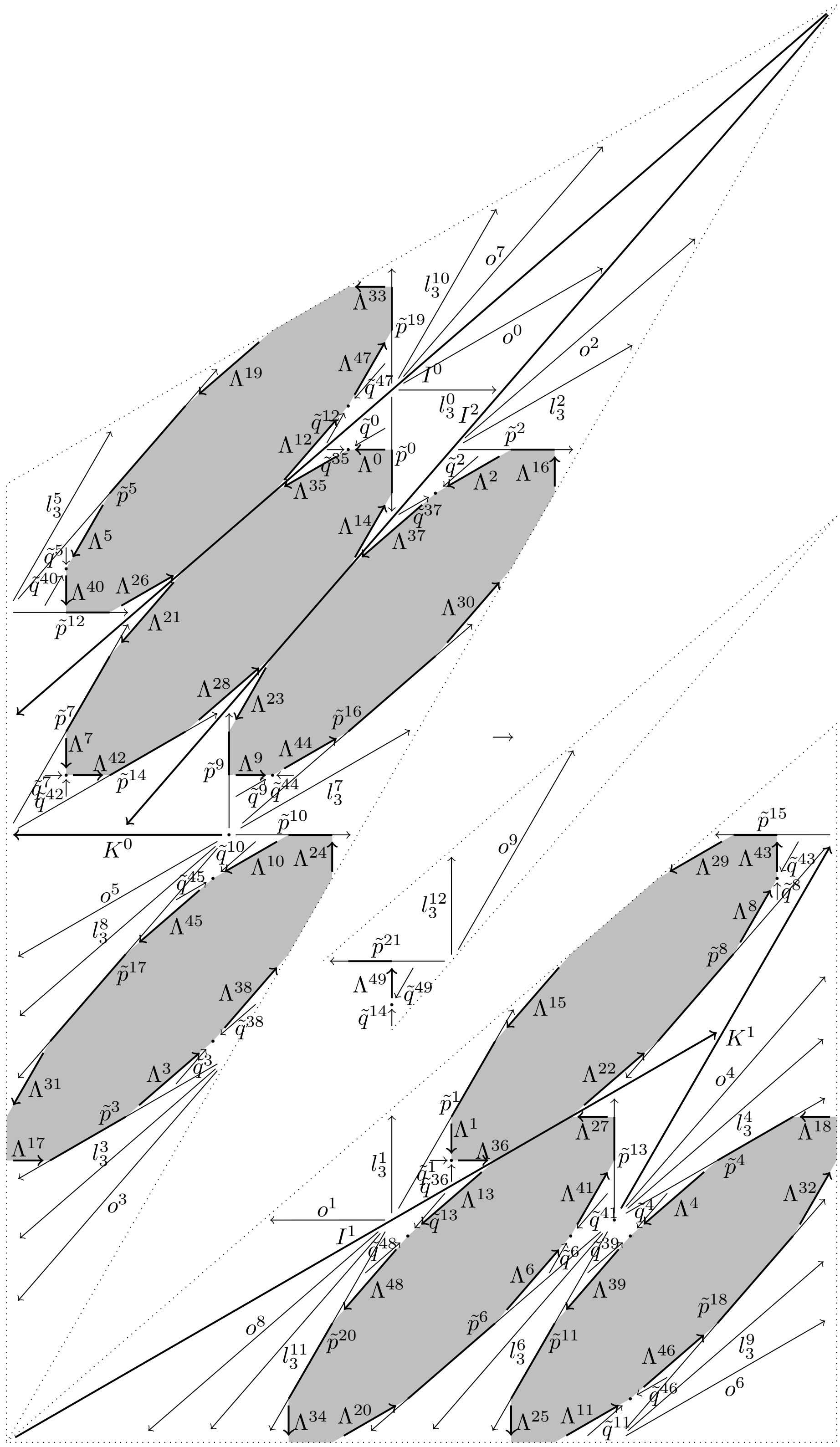

FIGURE A.4. The trajectory of the lines, $\lambda = \sqrt{3}$. (ℓ^k stands for $\hat{T}_3^k(D_\ell)$.)

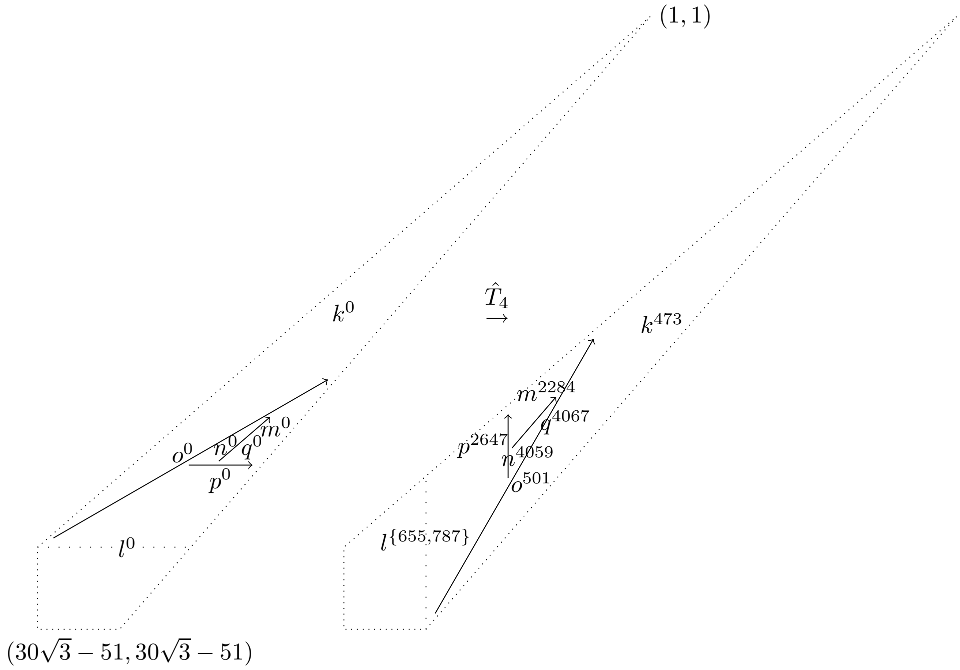

FIGURE A.5. The map $\hat{T}_4$, $\lambda = \sqrt{3}$. (ℓ^k stands for $T^k(D_\ell)$.)

Finally, we consider in Figure A.6 the first return map on

$$\{(x,y) : 2y < \sqrt{3}x + 2 - \sqrt{3},\ 2x < \sqrt{3}y + 2 - \sqrt{3},\ x \geq 72 - 41\sqrt{3} \text{ or } y \geq 72 - 41\sqrt{3}\},$$

partitioned into sets $D_0, D_{\tilde{1}}, \ldots, D_{\tilde{9}}, D_{\tilde{\mathbf{0}}}, D_{\tilde{\mathbf{1}}}, D_{\tilde{\mathbf{3}}}$, from which it is easy to deduce $\hat{T}$ on $\mathcal{D}$.

APPENDIX B. THE MAP $\hat{T}$ FOR $\lambda = -\sqrt{3}$.

For $\lambda = -\sqrt{3}$, we consider in Figure B.1 the first return map on

$$\{(x,y) \in [0,1)^2 : 2x + \sqrt{3}y > 3\sqrt{3} - 2 \text{ or } \sqrt{3}x + 2y > 3\sqrt{3} - 2\},$$

partitioned into sets $D_a, \ldots, D_i$. Figure B.2 provides the first return map $\hat{T}$ on $\mathcal{D}$. Again, all points in $\mathcal{R}$ are periodic.

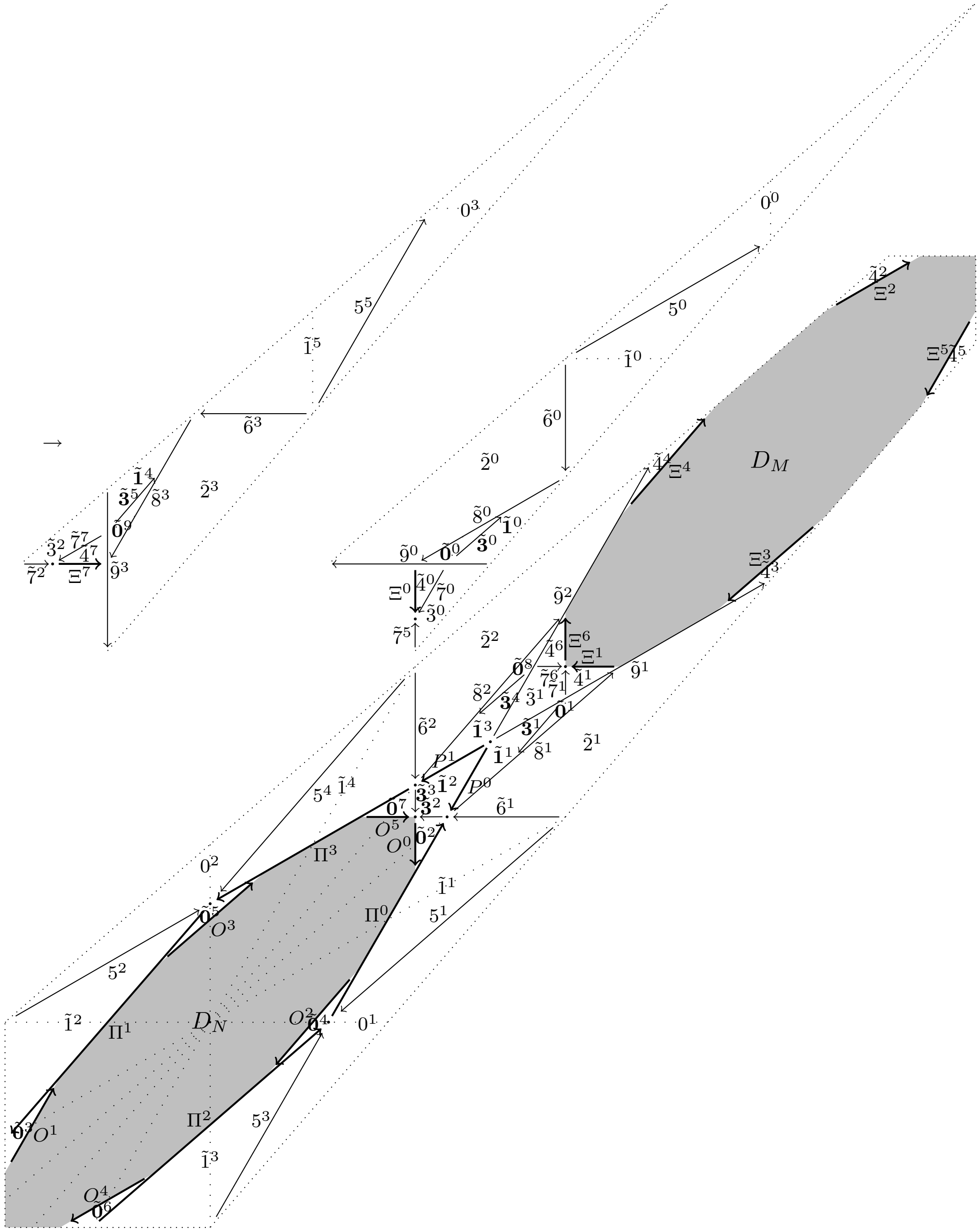

FIGURE A.6. Almost the map $\hat{T}$, $\lambda = \sqrt{3}$. (ℓ^k stands for $\hat{T}_4^k(D_\ell)$.)

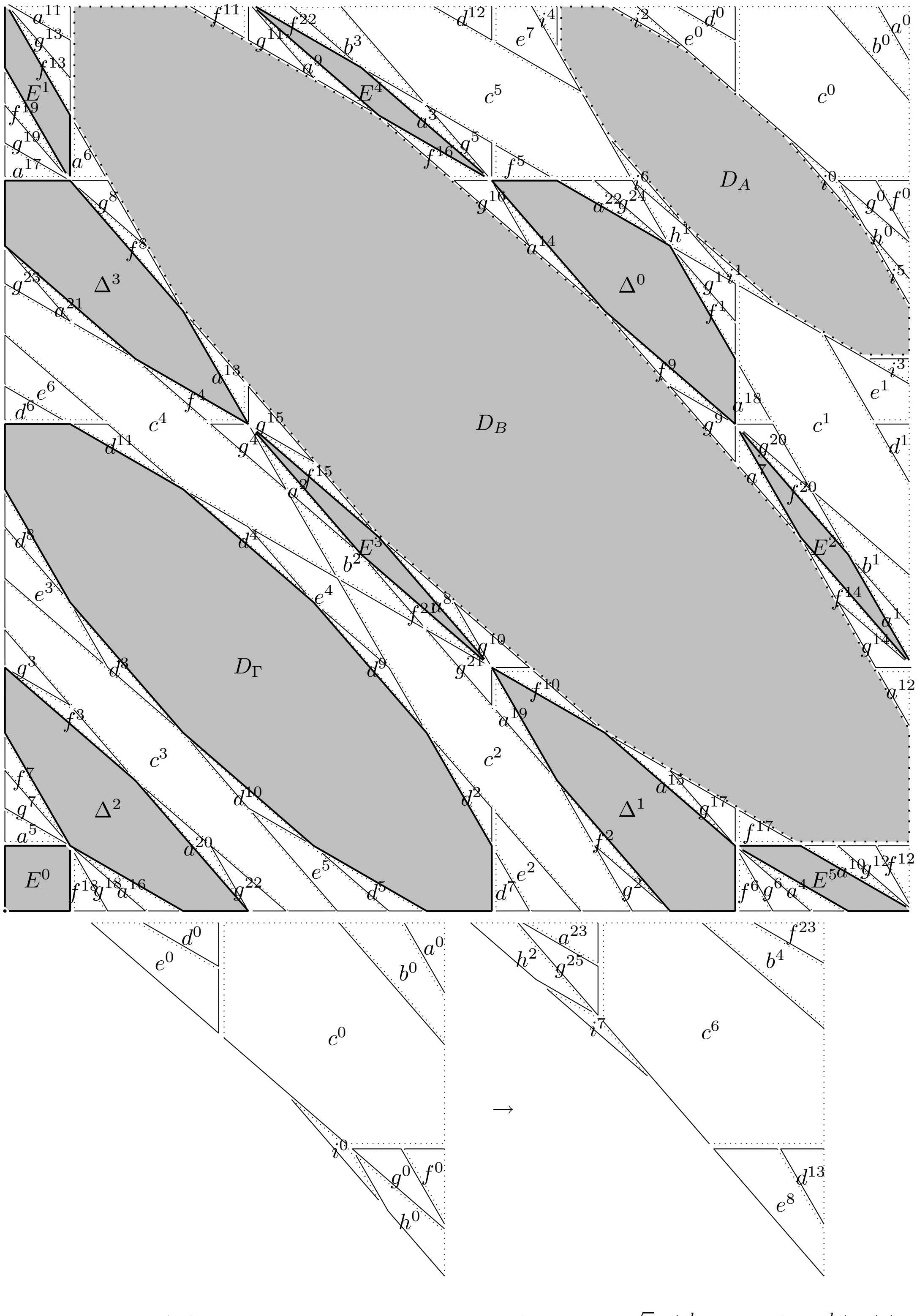

FIGURE B.1. A first return map and large parts of $\mathcal{R}$, $\lambda = -\sqrt{3}$. (ℓ^k stands for $T^k(D_\ell)$.)

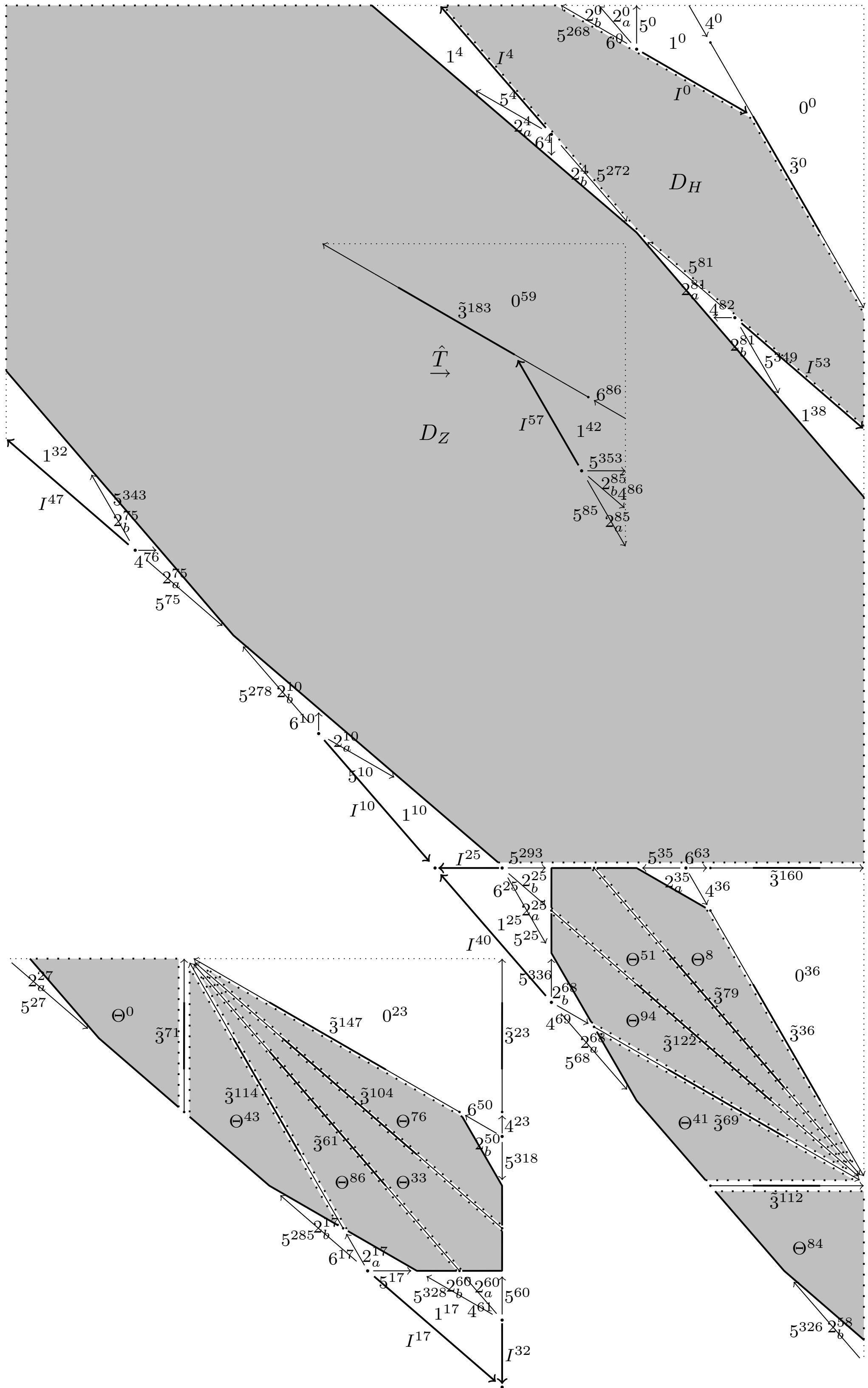

FIGURE B.2. The map $\hat{T}$ and small parts of $\mathcal{R}$, $\lambda = -\sqrt{3}$. (ℓ^k stands for $T^k(D_\ell)$.)