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► **To cite this version:**

Lars Schaefer. Para- tt^* -bundles on the tangent bundle of an almost para-complex manifold. 2007.
hal-00143297

HAL Id: hal-00143297

<https://hal.science/hal-00143297>

Preprint submitted on 25 Apr 2007

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Para- tt^* -bundles on the tangent bundle of an almost para-complex manifold

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May 24, 2006

Abstract

In this paper we study para- tt^* -bundles (TM, D, S) on the tangent bundle of an almost para-complex manifold (M, τ) . We characterise those para- tt^* -bundles with $\nabla = D + S$ induced by the one-parameter family of connections given by $\nabla^\theta = \exp(\theta\tau) \circ \nabla \circ \exp(-\theta\tau)$ and prove a uniqueness result for solutions with a para-complex connection D . Flat nearly para-Kähler manifolds and special para-complex manifolds are shown to be such solutions. We analyse which of these solutions admit metric or symplectic para- tt^* -bundles. Moreover, we give a generalisation of the notion of a para-pluriharmonic map to maps from almost para-complex manifolds (M, τ) into pseudo-Riemannian manifolds and associate to the above metric and symplectic para- tt^* -bundles generalised para-pluriharmonic maps into $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$, respectively into $SO_0(n, n)/U^\pi(C^n)$, where $U^\pi(C^n)$ is the para-complex analogue of the unitary group.

MSC: 53C26, 53C43, 58E20.

Keywords: Para- tt^* -geometry and para- tt^* -bundles; special para-complex and special para-Kähler manifolds; Nearly para-Kähler manifolds; para-pluriharmonic maps; pseudo-Riemannian symmetric spaces.

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1 Introduction

The subject of this work are para- tt^* -bundles (TM, D, S) on the tangent bundle an almost para-complex manifold (M, τ) as base. We generalise the results of the complex geometric setting which can be found in [S3]. In previous work we always supposed a para-complex base manifold. Motivated by the study of the complex case we admit non-integrable para-complex structures, since Levi-Civita flat *nearly para-Kähler manifolds* arise in this way as solutions of para- tt^* -bundles. Nearly para-Kähler manifolds as analogue of nearly Kähler manifolds were introduced in a recent paper of Ivanov and Zamkovoy [IZ]. A further class of solutions is given by **special para-Kähler manifolds** which were studied in [S2]. These geometries arise as one of the special geometries of Euclidean supersymmetry in [CMMS]. This means that para- tt^* -bundles can be seen as a common generalisation of these two geometries and give a kind of duality between them.

Now we discuss the structure of the paper. First we introduce the notion of (metric, symplectic) para- tt^* -bundles and describe these in terms of explicit geometric data. Part of this data is a family of flat connections D^θ on a vector bundle E with $\theta \in \mathbb{R}$ which is defined with help of a second connection D and a field $S \in \Gamma(T^*M \otimes \text{End}(E))$. Given an almost para-complex manifold endowed with a flat connection ∇ we can consider a natural family of flat connections defined by

$$\nabla^\theta = \exp(\theta\tau) \circ \nabla \circ \exp(-\theta\tau) \text{ for } \theta \in \mathbb{R}.$$

We analyse para- tt^* -bundles, such that $D^\theta = \nabla^{\alpha\theta}$ for some $\alpha \in \mathbb{R}$. This choice is motivated by the study of solutions coming from special para-Kähler geometry.

If we restrict to para- tt^* -bundles carrying connections D which are para-complex, i.e. which satisfy $D\tau = 0$, then these para- tt^* -bundles can be recovered uniquely from the pair (∇, τ) . Moreover, we give compatibility conditions on the pair (∇, τ) . These conditions hold for *special para-complex* and nearly para-Kähler manifolds. We discuss two classes of solutions: The first corresponds to special para-complex manifolds with torsion and the second to almost para-complex manifolds endowed with a flat connection ∇ such that (∇, τ) satisfies the nearly para-Kähler condition (with torsion). Further we study whether these para- tt^* -bundles provide *metric* or *symplectic* para- tt^* -bundles. Examples of solutions of the first type are given by special para-Kähler manifolds and solutions of the second type arise on Levi-Civita flat nearly para-Kähler manifolds. In addition we show that neither the special para-Kähler condition is compatible with symplectic para- tt^* -bundles nor the nearly para-Kähler condition is compatible with solutions of metric para- tt^* -bundles.

There exists a relation between para-pluriharmonic maps and para- tt^* -geometry which was studied in [S2]. In this work we generalise the notion of a para-pluriharmonic map to maps from almost para-complex manifolds into pseudo-Riemannian manifolds. Then we introduce \tilde{S}^1 -pluriharmonic maps which generalise S^1 -pluriharmonic maps (cf. [S3]) to para-complex geometry. This is a generalisation of the notion of associated families (see for example [ET]). We relate these \tilde{S}^1 -pluriharmonic maps to the generalisation of para-pluriharmonic maps and prove a result, which relates generalised para-pluriharmonic maps to harmonic maps. With these preparations we are able to associate para-pluriharmonic maps into $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$, respectively into $SO_0(n, n)/U^\pi(C^n)$, to the above metric and symplectic para- tt^* -bundles. Here $U^\pi(C^n)$ is the para-complex analogue of the unitary group.

The author is grateful to his advisor V. Cortés for the support of his work and to M.-A. Lawn-Paillusseau for related discussions.

2 Para-complex differential geometry

In this section we recall facts of para-complex differential geometry, which are needed in the later work. For more information we refer to [CMMS, IZ].

Definition 1 *A para-complex structure on a real finite dimensional vector space V is a non-trivial involution $\tau \in \mathrm{End}(V)$ such that the two eigenspaces $V^\pm := \ker(\mathrm{Id} \mp \tau)$ to the eigenvalues ± 1 of τ have the same dimension. We call the pair (V, τ) a **para-complex vector-space**.*

*An almost para-complex structure on a smooth manifold M is an endomorphism field $\tau \in \Gamma(\mathrm{End}(TM))$ such that, for all $p \in M$, τ_p is a para-complex structure on T_pM . It is called **integrable** if the distributions $T^\pm M = \ker(\mathrm{Id} \mp \tau)$ are integrable. An integrable almost para-complex structure on M is called a **para-complex structure on M** and a manifold M endowed with a para-complex structure is called a **para-complex manifold**.*

The integrability of a para-complex structure is obstructed (cf. [CMMS]) by the vanishing of a tensor, also called **Nijenhuis tensor**, which is defined as

$$N_\tau(X, Y) := [X, Y] + [\tau X, \tau Y] - \tau[X, \tau Y] - \tau[\tau X, Y].$$

The decomposition of the tangent bundle of an (almost) para-complex manifold M into the eigenspaces $T^\pm M$ extends to a bi-grading on the exterior algebra:

$$\Lambda^k T^* M = \bigoplus_{k=p+q} \Lambda^{p+,q-} T^* M \quad (2.1)$$

and induces an obvious bi-grading on exterior forms with values in a vector bundle E .

The **para-complex numbers** are the real algebra, which is generated by 1 and by the **para-complex unit** e with $e^2 = 1$. We denote the para-complex numbers by the symbol C .

For all $z = x + ey \in C$ with $x, y \in \mathbb{R}$ we define the **para-complex conjugation** $\bar{\cdot} : C \rightarrow C$, $x + ey \mapsto x - ey$ and the **real and imaginary parts** of z by

$$\Re(z) := \frac{z + \bar{z}}{2} = x, \quad \Im(z) := \frac{e(z - \bar{z})}{2} = y.$$

The free C -module C^n is a para-complex vector space where its para-complex structure is the multiplication with e and the para-complex conjugation of C extends to $\bar{\cdot} : C^n \rightarrow C^n$, $v \mapsto \bar{v}$.

Note the identity $z\bar{z} = x^2 - y^2$. Therefore the algebra C is sometimes called the **hyper-complex numbers**. The circle $\mathbb{S}^1 = \{z = x + iy \in \mathbb{C} \mid x^2 + y^2 = 1\}$ is replaced by the four hyperbolas $\{z = x + ey \in C \mid x^2 - y^2 = \pm 1\}$. $\tilde{\mathbb{S}}^1$ is defined to be the hyperbola given by the one parameter group $\{z(\theta) = \cosh(\theta) + e \sinh(\theta) \mid \theta \in \mathbb{R}\}$.

The **para-complex dimension** of a para-complex manifold M is the integer $n = \dim_C M := \frac{1}{2} \dim_{\mathbb{R}} M$.

Now we consider the **para-complexification** $TM^C = TM \otimes_{\mathbb{R}} C$ of the tangent bundle TM of an almost para-complex manifold (M, τ) and extend $\tau : TM \rightarrow TM$ C -linearly to $\tau : TM^C \rightarrow TM^C$. Then for all $p \in M$ the free C -module $T_p M^C$ decomposes as C -module into the direct sum of two free C -modules

$$T_p M^C = T_p^{1,0} M \oplus T_p^{0,1} M, \quad (2.2)$$

where $T_p^{1,0} M := \{X + e\tau X \mid X \in T_p M\}$ and $T_p^{0,1} M := \{X - e\tau X \mid X \in T_p M\}$. The subbundles $T_p^{1,0} M$ and $T_p^{0,1} M$ can be characterised as the $\pm e$ -eigenbundles of the map $\tau : TM^C \rightarrow TM^C$, i.e. $\tau = e$ on $T^{1,0} M$ and $\tau = -e$ on $T^{0,1} M$.

In the same manner we decompose $T^* M^C = \Lambda^{1,0} T^* M \oplus \Lambda^{0,1} T^* M$ into the $\pm e$ -eigenbundles of the dual para-complex structure $\tau^* : T^* M^C \rightarrow T^* M^C$. This decomposition induces a bi-grading on the C -valued exterior forms

$$\Lambda^k T^* M^C = \bigoplus_{k=p+q} \Lambda^{p,q} T^* M$$

and finally on the C -valued differential forms on M

$$\Omega_C^k(M) = \bigoplus_{k=p+q} \Omega^{p,q}(M).$$

We remark, that for the case $(1, 1)$ and $(1+, 1-)$ the two gradings induced by τ coincide in following sense: $\Lambda^{1,1} T^* M = (\Lambda^{1+,1-} T^* M) \otimes C$.

We need to introduce some additional notions.

Definition 2 Let (V, τ) be a para-complex vector space. A para-hermitian scalar product g on V is a pseudo-Euclidean scalar product for which τ is an anti-isometry, i.e.

$$\tau^*g = g(\tau\cdot, \tau\cdot) = -g(\cdot, \cdot).$$

A para-hermitian vector space (V, τ, g) is a para-complex vector space (V, τ) endowed with a para-hermitian scalar product g . The pair (τ, g) is called para-hermitian structure on the vector space V .

Definition 3 Let (V, τ, g) be a para-hermitian vector space. The para-unitary group of V is the automorphism group

$$U^\pi(V) := \text{Aut}(V, \tau, g) = \{L \in GL(V) \mid [L, \tau] = 0 \text{ and } L^*g = g\}.$$

Definition 4 An almost para-hermitian manifold (M, τ, g) is an almost para-complex manifold (M, τ) endowed with a pseudo-Riemannian metric g such that $\tau^*g = -g$. If τ is integrable, we call (M, τ, g) a para-hermitian manifold. The two-form $\omega := g(\tau\cdot, \cdot)$ is called the fundamental two-form of the almost para-hermitian manifold (M, τ, g) .

3 Para- tt^* -bundles

We generalise the notion of a para- tt^* -bundle which was introduced in [S2], by admitting a base manifold (M, τ) with a non-integrable para-complex structure τ . In this way nearly para-Kähler manifolds appear as solutions of para- tt^* -geometry on the tangent bundle TM . Further we introduce symplectic para- tt^* -bundles.

Definition 5 A para- tt^* -bundle or ptt^* -bundle (E, D, S) over an almost para-complex manifold (M, τ) is a real vector bundle $E \rightarrow M$ endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \text{End } E)$ which satisfy the ptt^* -equation

$$R^\theta = 0 \quad \text{for all } \theta \in \mathbb{R}, \quad (3.1)$$

where R^θ is the curvature tensor of the connection D^θ defined by

$$D_X^\theta := D_X + \cosh(\theta)S_X + \sinh(\theta)S_{\tau X} \quad \text{for all } X \in TM. \quad (3.2)$$

A metric ptt^* -bundle (E, D, S, g) is a ptt^* -bundle (E, D, S) endowed with a possibly indefinite D -parallel fiber metric g such that for all $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \text{for all } X, Y, Z \in T_p M. \quad (3.3)$$

A symplectic ptt^* -bundle (E, D, S, ω) is a ptt^* -bundle (E, D, S) endowed with the structure of a symplectic vector bundle¹ (E, ω) , such that ω is D -parallel and S is ω -symmetric, i.e. for all $p \in M$

$$\omega(S_X Y, Z) = \omega(Y, S_X Z) \quad \text{for all } X, Y, Z \in T_p M. \quad (3.4)$$

¹see D. Mc Duff and D. Salamon [McDS]

Remark 1

If (E, D, S) is a ptt^* -bundle then (E, D, S^θ) is a ptt^* -bundle for all $\theta \in \mathbb{R}$, where

$$S^\theta := D^\theta - D = \cosh(\theta)S + \sinh(\theta)S_\tau. \quad (3.5)$$

The same remark applies to metric ptt^* -bundles and symplectic ptt^* -bundles.

Like for ptt^* -bundles (E, D, S) over a para-complex manifold we find explicit equations for D and S .

Proposition 1 *Let E be a real vector bundle over an almost para-complex manifold (M, τ) endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \text{End } E)$.*

Then (E, D, S) is a ptt^ -bundle if and only if D and S satisfy the following equations:*

$$R^D + S \wedge S = 0, \quad (3.6)$$

$$S \wedge S \text{ is of type } (1,1), \quad (3.7)$$

$$[D_X, S_Y] - [D_Y, S_X] - S_{[X,Y]} = 0, \quad \forall X, Y \in \Gamma(TM), \quad (3.8)$$

$$[D_X, S_{\tau Y}] - [D_Y, S_{\tau X}] - S_{\tau[X,Y]} = 0, \quad \forall X, Y \in \Gamma(TM). \quad (3.9)$$

Fixing a torsion-free connection on (M, τ) the last two equations are equivalent to

$$d^D S = 0 \quad \text{and} \quad d^D S_\tau = 0. \quad (3.10)$$

Proof: (compare [S2]) As for ptt^* -bundles over para-complex manifolds (M, τ) one calculates using the theorems of addition $2 \cosh(\theta) \sinh(\theta) = \sinh(2\theta)$, $2 \cosh^2(\theta) = 1 + \cosh(2\theta)$ and $2 \sinh^2(\theta) = \cosh(2\theta) - 1$, the (finite) decomposition of R^θ in 1 , $\cosh(n\theta)$ and $\sinh(n\theta)$, for $n = 1, 2$. The ptt^* -equation $R^\theta = 0$ is equivalent to the vanishing of all components. This yields the claimed equations. \square

4 Solutions on the tangent bundle of an almost para-complex manifold

In this section we consider para-complex manifolds (M, τ) endowed with a flat connection ∇ . It is natural to regard the one-parameter family of connections ∇^θ , which is defined by

$$\nabla_X^\theta Y = \exp(\theta\tau)\nabla_X(\exp(-\theta\tau)Y) \text{ for } X, Y \in \Gamma(TM), \quad (4.1)$$

where $\exp(\theta\tau) = \cosh(\theta)Id + \sinh(\theta)\tau$. The connection ∇ is flat if and only if ∇^θ is flat.

Definition 6 *Two one-parameter families of connections ∇^θ and D^θ , with $\theta \in \mathbb{R}$, on some vector bundle E are called (linear) equivalent with factor $\alpha \in \mathbb{R}$ if they satisfy the equation $\nabla^\theta = D^{\alpha\theta}$.*

In the sequel we study ptt^* -bundles (TM, D, S) such that the connection D^θ defined in equation (3.2) is linear equivalent to the connection ∇^θ defined in equation (4.1). This ansatz is motivated by our previous study of ptt^* -bundles coming from special para-complex and special para-Kähler manifolds (see [S2]).

Proposition 2 *Given an almost para-complex manifold (M, τ) with a flat connection ∇ . Then a decomposition of $\nabla = D + S$ in a connection D and a section S in $T^*M \otimes \text{End}(TM)$ defines a ptt*-bundle (TM, D, S) , such that the family of connections D^θ is linear equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$ if and only if S and D satisfy*

$$S_{\tau X} = \pm \tau S_X Y$$

and

$$-(D_X \tau)Y = \tau S_X Y + S_X \tau Y =: \{S_X, \tau\}Y$$

for all $X, Y \in \Gamma(TM)$.

Proof: One has to compute ∇^θ for $X, Y \in \Gamma(TM)$:

$$\begin{aligned} \nabla_X^\theta Y &= \exp(\theta\tau)(D_X + S_X)(\cosh(\theta)Id - \sinh(\theta)\tau)Y \\ &= D_X Y - \exp(\theta\tau) \sinh(\theta)(D_X \tau)Y \\ &\quad + (\cosh(\theta)Id + \sinh(\theta)\tau)S_X(\cosh(\theta)Id - \sinh(\theta)\tau)Y \\ &= D_X Y - (\cosh(\theta) \sinh(\theta) + \sinh^2(\theta)\tau)(D_X \tau)Y + \cosh^2(\theta)S_X Y \\ &\quad - \sinh^2(\theta)\tau S_X \tau Y - \cosh(\theta) \sinh(\theta)[S_X, \tau]Y. \end{aligned}$$

This yields with the theorems of addition, i.e.

$$2 \sinh(\theta) \cosh(\theta) = \sinh(2\theta), \quad 2 \cosh^2(\theta) = 1 + \cosh(2\theta) \quad \text{and} \quad 2 \sinh^2(\theta) = \cosh(2\theta) - 1,$$

the identity

$$\begin{aligned} \nabla_X^\theta Y &= D_X Y - \frac{1}{2} \sinh(2\theta)(D_X \tau)Y - \frac{1}{2}(\cosh(2\theta) - 1)\tau(D_X \tau)Y \\ &\quad + \frac{1}{2}(1 + \cosh(2\theta))S_X Y - \frac{1}{2}(\cosh(2\theta) - 1)\tau S_X \tau Y - \frac{1}{2} \sinh(2\theta)[S_X, \tau]Y \\ &= D_X Y + \frac{1}{2}[S_X + \tau S_X \tau + \tau D_X \tau]Y \\ &\quad + \frac{1}{2} \sinh(2\theta)[[\tau, S_X] - D_X \tau]Y \\ &\quad + \frac{1}{2} \cosh(2\theta)[S_X - \tau S_X \tau - \tau D_X \tau]Y \\ &\stackrel{!}{=} D_X Y + \cosh(\vartheta)T_X Y + \sinh(\vartheta)T_{\tau X} Y \quad \text{with } \vartheta = \pm 2\theta, \end{aligned}$$

where we have to determine $T \in \Gamma(T^*M \otimes \text{End}(TM))$.

Comparing coefficients of 1, $\cosh(2\theta)$, $\sinh(2\theta)$ gives

$$-\tau(D_X \tau)Y = S_X Y + \tau S_X \tau Y, \quad \text{or equivalently} \quad (4.2)$$

$$-(D_X \tau)Y = \tau S_X Y + S_X \tau Y = \{S_X, \tau\}Y,$$

$$T_X Y = \frac{1}{2}(S_X Y - \tau S_X \tau Y - \tau(D_X \tau)Y) \stackrel{(4.2)}{=} S_X Y, \quad (4.3)$$

$$\begin{aligned} T_{\tau X} Y &= \pm \frac{1}{2}([\tau, S_X]Y - (D_X \tau)Y) \\ &\stackrel{(4.2)}{=} \pm \frac{1}{2}(\tau S_X Y - S_X \tau Y + \tau S_X Y + S_X \tau Y) = \pm \tau S_X Y. \quad (4.4) \end{aligned}$$

The constraint on S follows from the last two equations, i.e.

$$S_{\tau X} = \pm \tau S_X Y$$

and the one on D and S from the first equation. \square

We further specialise to para-complex connections D , i.e. connections D satisfying the identity $D\tau = 0$. The existence of such connections was provided by Theorem 1 of [S2].

Corollary 1 *Let (M, τ) be an almost para-complex manifold endowed with a flat connection ∇ and let $\nabla = D + S$ be a given decomposition in a connection D and a section S in $T^*M \otimes \text{End}(TM)$, such that τ is D -parallel. Then (TM, D, S) defines a ptt*-bundle, such that the family of connections D^θ is linear equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$ if and only if S satisfies*

$$S_{\tau X} = \pm \tau S_X \text{ and } \{S_X, \tau\} = 0.$$

Proof: The second condition in proposition 2 yields using $D\tau = 0$ the equation $\{S_X, \tau\} = 0$. The first constraint of proposition 2 remains $S_{\tau X} = \pm \tau S_X \stackrel{\{S_X, \tau\}=0}{=} \mp S_X \tau$. \square

To establish a uniqueness result we need the next lemma.

Lemma 1 *Let (M, τ) be an almost para-complex manifold. Let a connection ∇ on M be given. Then the connection D and the section S of $T^*M \otimes \text{End}(TM)$ defined by*

$$S_X Y = -\frac{1}{2} \tau (\nabla_X \tau) Y \text{ and } D_X Y = \nabla_X Y - S_X Y \text{ for } X, Y \in \Gamma(TM), \quad (4.5)$$

satisfy $D\tau = 0$ and $\{S_X, \tau\} = 0$.

*Otherwise, suppose that ∇ decomposes as $\nabla = D + S$, where D is a connection on M and S is a section in $T^*M \otimes \text{End}(TM)$, such that τ is D -parallel, i.e. $D\tau = 0$ and S anti-commutes with τ , i.e. $\{S_X, \tau\} = 0$ for all $X \in \Gamma(TM)$. Then D and S are uniquely given by equation (4.5).*

Proof: We check $\nabla = D + S$ and $S_X \tau Y = -\frac{1}{2} \tau (\nabla_X \tau) \tau Y = \frac{1}{2} \tau^2 (\nabla_X \tau) Y = -\tau S_X Y$, where the second equality follows from deriving $\tau^2 = Id$. In addition it is

$$(D_X \tau) Y = (\nabla_X \tau) Y - [S_X, \tau] \stackrel{\{S_X, \tau\}=0}{=} (\nabla_X \tau) Y + 2\tau S_X = 0.$$

It rests to show the uniqueness: One starts with D' and S' having same properties and computes

$$0 = (D'_X \tau) Y = (\nabla_X \tau) Y - [S'_X, \tau] Y = (\nabla_X \tau) Y + 2\tau S'_X Y.$$

This further implies

$$S'_X Y = -\frac{1}{2} \tau (\nabla_X \tau) Y = S_X Y \text{ and } D'_X Y = \nabla_X Y - S'_X Y = \nabla_X Y - S_X Y = D_X Y.$$

\square

Corollary 1 and lemma 1 imply the following uniqueness result:

Theorem 1 *Let (M, τ) be an almost para-complex manifold endowed with a flat connection ∇ . If a decomposition of $\nabla = D + S$ in a connection D and a section S in $T^*M \otimes \text{End}(TM)$, such that τ is D -parallel, i.e. $D\tau = 0$, defines a ptt*-bundle (TM, D, S) , such that the family of connections D^θ is linear equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$, then D and S are uniquely determined by the equations $S = -\frac{1}{2}\tau(\nabla\tau)$ and $D = \nabla - S$.*

Moreover, (TM, D, S) as above defines a ptt-bundle, such that the family of connections D^θ is linear equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$, if and only if τ satisfies $(\nabla_{\tau X}\tau) = \pm\tau(\nabla_X\tau)$ and D and S are given by equation (4.5).*

Some classes of examples which satisfy the condition $S_{\tau X} = \pm\tau S_X$ are given in the following propositions.

Proposition 3 *Let an almost para-complex manifold (M, τ) endowed with a connection ∇ be given and let S be the section in $T^*M \otimes \text{End}(TM)$ defined by*

$$S := -\frac{1}{2}\tau(\nabla\tau). \quad (4.6)$$

If the pair (∇, τ) satisfies one of the following conditions

- (i) (∇, τ) is special, i.e. $(\nabla_X\tau)Y = (\nabla_Y\tau)X$ for all $X, Y \in \Gamma(TM)$,*
- (ii) (∇, τ) satisfies the nearly para-Kähler condition, i.e. $(\nabla_X\tau)Y = -(\nabla_Y\tau)X$ for all $X, Y \in \Gamma(TM)$,*

then it holds $S_{\tau X}Y = -\tau S_XY$.

Proof: We prove (i) and (ii) in the same calculation:

$$(\nabla_{\tau X}\tau)Y = \pm(\nabla_Y\tau)\tau X = \mp\tau(\nabla_Y\tau)X = -\tau(\nabla_X\tau)Y.$$

This implies $S_{\tau X}Y = -\frac{1}{2}\tau(\nabla_{\tau X}\tau)Y = \frac{1}{2}\tau^2(\nabla_X\tau)Y = -\tau S_XY$.

□

Proposition 4 *Let a para-complex manifold (M, τ) endowed with a connection ∇ be given and let S be the section in $T^*M \otimes \text{End}(TM)$ defined by*

$$S := -\frac{1}{2}\tau(\nabla\tau). \quad (4.7)$$

If ∇ is (anti-)adapted, i.e. $\nabla_{\tau X} = \pm\tau\nabla_X$ for all para-holomorphic vector-fields X, Y , then it holds $S_{\tau X}Y = \pm\tau S_XY$.

Proof: Since the connection ∇ is (anti-)adapted, we obtain for all para-holomorphic vector-fields X, Y :

$$(\nabla_{\tau X}\tau)Y = \pm\tau(\nabla_X\tau)Y.$$

The following computation finishes the proof

$$S_{\tau X}Y = -\frac{1}{2}\tau(\nabla_{\tau X}\tau)Y = \mp\frac{1}{2}\tau^2(\nabla_X\tau)Y = \pm\tau S_XY.$$

□

Remark 2 *One observes that in proposition 3 the condition (i) is the symmetry of S_XY and condition (ii) is its anti-symmetry. We recall that if the connection ∇ is torsion-free, flat and special then (M, τ, ∇) is a special para-complex manifold, see [CMMS, S2]. Ptt^* -bundles coming from special para-complex manifolds and special para-Kähler manifolds were studied in [S2].*

Moreover we remark, that the second condition arises in nearly para-Kählerian geometry (compare [IZ]) and therefore is quite natural. These geometries as solutions of $p tt^$ -geometry are discussed later in this work.*

Finally, the notion of adapted connections appeared in the study of decompositions on (para-holomorphic) para-complex vector bundles, compare [LS].

5 Solutions on almost para-hermitian manifolds

In this section we study the question under which assumptions almost para-complex manifolds (M, τ) endowed with a flat connection ∇ such that (∇, τ) is special or satisfies the nearly para-Kähler condition define symplectic or metric $p tt^*$ -bundles.

First we recall a lemma from tensor-algebra:

Lemma 2 *Let V be a vector-space and $\alpha \in T^3(V^*)$ an element in the third tensorial power of V^* , the dual space of V . Suppose that $\alpha(X, Y, Z)$ is symmetric (resp. anti-symmetric) in X, Y and Y, Z and $\alpha(X, Y, Z)$ is anti-symmetric (resp. symmetric) in X, Z then $\alpha = 0$.*

Proof: It is $\alpha(X, Y, Z) = \epsilon\alpha(Y, X, Z) = \epsilon\alpha(X, Z, Y)$ with $\epsilon \in \{\pm 1\}$ which implies $\alpha(X, Y, Z) = \epsilon\alpha(Y, X, Z) = \epsilon^2\alpha(Y, Z, X) = \epsilon^3\alpha(Z, Y, X)$. But further it holds $\alpha(X, Y, Z) = -\epsilon\alpha(Z, Y, X)$ and consequently $-\alpha(Z, Y, X) = \epsilon^2\alpha(Z, Y, X) = \alpha(Z, Y, X)$. This shows $\alpha = 0$. □

Proposition 5 *Let (M, τ) be an almost para-complex manifold endowed with a flat connection ∇ , such that (∇, τ) is special. Define S , a section in $T^*M \otimes \text{End}(TM)$, by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \tag{5.1}$$

then $(TM, D = \nabla - S, S)$ defines a $p tt^$ -bundle. If in addition (TM, D, S, ω) is a symplectic $p tt^*$ -bundle, then it is trivial, i.e. $S = 0$.*

Proof: Due to theorem 1 and proposition 3 (TM, D, S) is a ptt^* -bundle.

Suppose, that (TM, D, S, ω) is a symplectic ptt^* -bundle. In order to prove the second part of the proposition, we define the tensor

$$\alpha(X, Y, Z) := \omega(S_X Y, Z) = g(\tau S_X Y, Z).$$

$\alpha(X, Y, Z)$ is symmetric in X, Y , since (∇, τ) is special, i.e. $\nabla\tau$ is symmetric in X, Y . Moreover, it holds

$$\begin{aligned} \alpha(X, Y, Z) &= \omega(S_X Y, Z) = -\omega(Z, S_X Y) \\ &= -\omega(Z, S_Y X) = -\omega(S_Y Z, X) = -\omega(S_Z Y, X) = -\alpha(Z, Y, X). \end{aligned}$$

This is the anti-symmetry of $\alpha(X, Y, Z)$ in X, Z . Finally we compute

$$\begin{aligned} \alpha(X, Y, Z) &= \omega(S_X Y, Z) = \omega(Y, S_X Z) \\ &= \omega(Y, S_Z X) = -\omega(S_Z X, Y) = -\alpha(Z, X, Y) = -\alpha(X, Z, Y), \end{aligned}$$

i.e. the anti-symmetry of $\alpha(X, Y, Z)$ in Y, Z .

Hence α vanishes and consequently S is zero. \square

Proposition 6 *Let an almost para-complex manifold (M, τ) endowed with a flat connection ∇ , such that (∇, τ) satisfies the nearly para-Kähler condition be given. Define S , a section S in $T^*M \otimes \text{End}(TM)$, by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \tag{5.2}$$

then $(TM, D = \nabla - S, S)$ defines a ptt^ -bundle. Suppose, that (TM, D, S, g) is a metric ptt^* -bundle, then it is trivial, i.e. $S = 0$.*

Proof: Due to theorem 1 and proposition 3 (TM, D, S) is a ptt^* -bundle.

Suppose, that it is a metric ptt^* -bundle. The proof is obtained by analysing the symmetries of the tensor

$$\alpha(X, Y, Z) := g(S_X Y, Z).$$

$\alpha(X, Y, Z)$ is anti-symmetric in X, Y , since by the nearly para-Kähler condition $\nabla\tau$ is anti-symmetric in X, Y . In addition it holds

$$\begin{aligned} \alpha(X, Y, Z) &= g(S_X Y, Z) = g(Z, S_X Y) \\ &= -g(Z, S_Y X) = -g(S_Y Z, X) = g(S_Z Y, X) = \alpha(Z, Y, X), \end{aligned}$$

which is the symmetry of $\alpha(X, Y, Z)$ in X, Z . Finally one has

$$\begin{aligned} \alpha(X, Y, Z) &= g(S_X Y, Z) = g(Y, S_X Z) \\ &= -g(Y, S_Z X) = -g(S_Z X, Y) = -\alpha(Z, X, Y) = \alpha(X, Z, Y), \end{aligned}$$

i.e. the symmetry of $\alpha(X, Y, Z)$ in Y, Z .

Hence α vanishes by the above lemma and so does S . \square

The following theorem gives solutions of symplectic para- tt^* -bundles on the tangent bundle, which are more general than the later discussed nearly para-Kähler manifolds in the sense, that we admit the connection ∇ to have torsion, but more special in the sense, that our connection ∇ has to be flat.

Theorem 2 *Let (M, τ, g) be an almost para-hermitian manifold endowed with a flat metric connection ∇ , such that the pair (∇, τ) satisfies the nearly para-Kähler condition. Define S , a section in $T^*M \otimes \text{End}(TM)$, by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \quad (5.3)$$

then $(TM, D = \nabla - S, S, \omega = g(\tau, \cdot))$ defines a symplectic ptt^ -bundle.*

The torsion T^∇ of the connection ∇ and T^D of the connection D are related by

$$T^\nabla - T^D = 2S$$

and it holds $D\tau = 0$.

Proof: Due to theorem 1 and proposition 3 (TM, D, S) is a ptt^* -bundle.

It remains to check $D\omega = 0$ and that S is ω -symmetric.

We observe, that, as g is para-hermitian and $\nabla g = 0$, $\nabla_X \tau$ is skew-symmetric with respect to g . This yields by the following calculation, that S is skew-symmetric with respect to g :

$$\begin{aligned} -2g(S_X Y, Z) &= g(\tau(\nabla_X \tau)Y, Z) = -g((\nabla_X \tau)Y, \tau Z) \\ &= g(Y, (\nabla_X \tau)\tau Z) = -g(Z, \tau(\nabla_X \tau)Y) = 2g(Y, S_X Z). \end{aligned}$$

The definition of $\omega = g(\tau, \cdot)$ and $\{S_X, \tau\} = 0$ yield the ω -symmetry of S_X . Further it holds $D = \nabla + \frac{1}{2}\tau\nabla\tau$, which implies

$$D\tau = \nabla\tau + \frac{1}{2}[\tau\nabla\tau, \tau] = 0.$$

This shows $D\omega = 0$ if and only if $Dg = 0$. But $\nabla g = 0$ and S is skew-symmetric with respect to g , so g is parallel for $D = \nabla - S$.

This shows that $(TM, D = \nabla - S, S, \omega)$ is a symplectic ptt^* -bundle.

From $\nabla = D + S$ we obtain for the torsions

$$T^\nabla(X, Y) = T^D(X, Y) + S_X Y - S_Y X = T^D(X, Y) + 2S_X Y.$$

□

We recall the definition of special para-complex and special para-Kähler manifolds (see [CMMS, S2]):

Definition 7 *A special para-Kähler manifold consists of the data (M, τ, g, ∇) where (M, τ, g) is a para-Kähler manifold with ∇ -parallel para-Kähler form and (M, τ, ∇) is a special para-complex manifold, i.e. (M, τ) is a para-complex manifold endowed with a flat and torsion-free connection ∇ such that (∇, τ) is special.*

The next theorem gives solutions of metric para- tt^* -bundles on the tangent bundle, which are more general than special para-Kähler manifolds in the sense, that we admit connections ∇ with torsion.

Theorem 3 *Let an almost para-hermitian manifold (M, τ, g) endowed with a flat connection ∇ be given. Suppose that (∇, τ) is special and the two-form $\omega = g(\tau \cdot, \cdot)$ is ∇ -parallel. Define S , a section in $T^*M \otimes \text{End}(TM)$, by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \quad (5.4)$$

then $(TM, D = \nabla - S, S, g)$ defines a metric ptt^ -bundle. Moreover, the connections D and ∇ have the same torsion and we have $D\tau = 0$.*

If in addition ∇ is torsion-free, then D is the Levi-Civita connection of g , (M, τ, g) is a para-Kähler manifold and (M, τ, g, ∇) is a special para-Kähler manifold.

Proof: Due to theorem 1 and proposition 3 (TM, D, S) defines a ptt^* -bundle.

It remains to prove $Dg = 0$ and that S is g -symmetric.

First we remark that $\omega(\tau X, Y) = -\omega(X, \tau Y)$ as g is para-hermitian. This implies by $\nabla\omega = 0$ the ω -skew-symmetry of $\nabla_X\tau$, which yields that $S_X = -\frac{1}{2}\tau(\nabla_X\tau)$ is ω -skew-symmetric, since $\tau(\nabla_X\tau) = -(\nabla_X\tau)\tau$. Finally $\{S_X, \tau\} = 0$ shows the g -symmetry of S_X . Moreover, we compute

$$D\tau = \nabla\tau + \frac{1}{2}[\tau\nabla\tau, \tau] = 0$$

and consequently $Dg = 0$ is equivalent to $D\omega = 0$.

From $\nabla\omega = 0$ and the ω -skew-symmetry of S it follows $D\omega = (\nabla - S)\omega = 0$.

From the symmetry of $\nabla\tau$, i.e. $(\nabla_X\tau)Y = (\nabla_Y\tau)X$ for all $X, Y \in TM$ we obtain $S_X Y = S_Y X$. This shows, that the torsions of the connections ∇ and D coincide.

Suppose now that ∇ is torsion-free. This implies, that $D = \nabla - S$ is torsion-free and consequently the Levi-Civita-connection of g . Further the equation $\nabla\omega = 0$ implies $d\omega = 0$, since ∇ is torsion-free. As $D\tau = 0$ and D is torsion-free, τ is integrable. Hence (M, τ, g) is para-Kähler. In addition (M, τ, ∇) is special para-complex by the conditions on ∇ and τ . Summarising (M, τ, g, ∇) is a special para-Kähler manifold. \square

In [S2] we treated special para-Kähler solutions of ptt^* -geometry in more details.

Now we are going to apply the above results to nearly para-Kähler manifolds. In order to do this we recall the notion of a nearly para-Kähler manifold which was recently introduced by Ivanov and Zamkovoy [IZ].

Definition 8 *An almost para-hermitian manifold (M, τ, g) is called nearly para-Kähler manifold, if its Levi-Civita connection $\nabla = \nabla^g$ satisfies the equation*

$$(\nabla_X\tau)Y = -(\nabla_Y\tau)X, \quad \forall X, Y \in \Gamma(TM). \quad (5.5)$$

A nearly para-Kähler manifold is called strict, if $\nabla\tau \neq 0$.

We recall that the tensor $\nabla\tau$ defines two tensors A and B

$$A(X, Y, Z) := g((\nabla_X\tau)Y, Z) \text{ and } B(X, Y, Z) := g((\nabla_X\tau)Y, \tau Z) \text{ with } X, Y, Z \in TM,$$

which are both (real) three-forms of type $(3, 0) + (0, 3)$.

A connection of particular importance in nearly para-Kähler geometry is the connection $\bar{\nabla}$ defined by

$$\bar{\nabla}_X Y := \nabla_X Y - \frac{1}{2}(\nabla_X \tau)\tau Y, \text{ for all } X, Y \in \Gamma(TM). \quad (5.6)$$

We remark, that $\bar{\nabla}$ is the unique connection with totally skew-symmetric torsion satisfying $\bar{\nabla}g = 0$ and $\bar{\nabla}\tau = 0$ (compare [IZ]).

The torsion of the connection $\bar{\nabla}$ is given by

$$T^{\bar{\nabla}}(X, Y) = -(\nabla_X \tau)\tau Y, \text{ for all } X, Y \in \Gamma(TM) \quad (5.7)$$

and it vanishes if and only if (M, τ, g) is a para-Kähler manifold.

Corollary 2 *Let (M, τ, g) be a nearly para-Kähler manifold, such that its Levi-Civita connection ∇ is flat and let S be the section in $T^*M \otimes \text{End}(TM)$ defined by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \quad (5.8)$$

then $(TM, \bar{\nabla}, S)$ defines a ptt^ -bundle. Suppose, that $(TM, \bar{\nabla}, S, g)$ is a metric ptt^* -bundle, then it is trivial, i.e. $S = 0$ and consequently (M, τ, g) is para-Kähler.*

Proof: If one puts $D = \bar{\nabla}$ we are in the situation of proposition 6. □

Remark 3 *In common work with V. Cortés [CS2] we gave a constructive classification of flat nearly pseudo-Kähler manifolds. The application of the methods of [CS2] to nearly para-Kähler manifolds is work in progress [CS3].*

Theorem 4 *Let (M, τ, g) be a nearly para-Kähler manifold, such that its Levi-Civita connection ∇ is flat. Let S be the section in $T^*M \otimes \text{End}(TM)$ defined by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \quad (5.9)$$

then $(TM, \bar{\nabla}, S, \omega := g(\tau\cdot, \cdot))$ is a symplectic ptt^ -bundle. Further it holds*

$$B(X, Y, Z) = 2g(S_X Y, Z) \text{ and } \bar{\nabla}\tau = 0. \quad (5.10)$$

Proof: By setting $D = \bar{\nabla}$ we are in the situation of theorem 2. In addition it holds

$$2g(S_X Y, Z) = -g(\tau(\nabla_X \tau)Y, Z) = g((\nabla_X \tau)Y, \tau Z) = B(X, Y, Z).$$

□

6 Para-pluriharmonic maps from almost para-complex manifolds into pseudo-Riemannian manifolds

In this section we generalise the notion of a para-pluriharmonic map to maps from almost para-complex manifolds to pseudo-Riemannian manifolds and we introduce the para-complex analogue of an associated family (compare [ET] for the complex setting). Afterwards we show that maps admitting the para-complex analogue of an associated family give rise to a para-pluriharmonic map and we give conditions under which a para-pluriharmonic map is harmonic.

Let (M, τ) be an almost para-complex manifold of real dimension $2n$. In complex geometry it is well-known (compare [KN]) that on every almost complex manifold (M, J) there exists a connection with torsion $T = \frac{1}{4}N_J$ where N_J is the Nijenhuis tensor of the complex structure J . This result was generalised to para-complex geometry in Theorem 1 of [S2], which states that on an almost para-complex manifold (M, τ) there exists a para-complex connection with torsion $T = -\frac{1}{4}N_\tau$ where N_τ is the Nijenhuis tensor of the para-complex structure τ .

Definition 9 *Let (M, τ) be an almost para-complex manifold. A connection D on the tangent bundle of M is called nice if it is para-complex and its torsion T satisfies $T = \lambda N_\tau$ with a smooth function λ on M .*

We introduce the notion of a para-pluriharmonic map from an almost para-complex manifold:

Definition 10 *Let (M, τ, D) be an almost para-complex manifold endowed with a nice connection D on TM and N be a smooth manifold endowed with a connection ∇^N . Denote by ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^N .*

A smooth map $f : M \rightarrow N$ is called para-pluriharmonic if and only if it satisfies the equation

$$(\nabla df)^{1,1} = 0. \tag{6.1}$$

Remark 4 *First we remark, that for a para-complex manifold (M, τ) and a pseudo-Riemannian target manifold (N, h) with its Levi-Civita connection ∇^h the para-pluriharmonic equation (6.1) does not depend on the connection D if D is chosen in an appropriate class (compare [S2]). In fact nice connections on para-complex manifolds belong to this class. A famous case are para-Kähler manifolds (M, τ, g) , where D is taken to be the Levi-Civita connection.*

To deal with associated families of para-pluriharmonic maps we need to recall an integrability condition satisfied by the differential of a smooth map. Denote by N a smooth manifold with a connection ∇^N on its tangent bundle having torsion tensor T^N . Let a second smooth manifold M and a smooth map $f : M \rightarrow N$ be given. The differential $F := df : TM \rightarrow f^*TN = E$ of the map f induces a vector bundle homomorphism

between the tangent bundle TM of M and the pull-back of TN via f . The bundle homomorphism $T^E : \Lambda^2 E \rightarrow E$ induced by the torsion T^N of N satisfies the identity

$$\nabla_V^E F(W) - \nabla_W^E F(V) - F([V, W]) = T^E(F(V), F(W)), \quad (6.2)$$

where $\nabla^E = f^* \nabla^N$ denotes the pull-back connection, i.e. the connection which is induced on E by ∇^N and where $V, W \in \Gamma(TM)$.

In the rest of this section we denote by D a nice connection on the almost para-complex manifold (M, τ) . Under this assumption we restate the condition (6.2):

$$\begin{aligned} T^E(F(V), F(W)) &= \nabla_V^E F(W) - \nabla_W^E F(V) - F([V, W]) \\ &= \nabla_V^E F(W) - \nabla_W^E F(V) \\ &\quad - F(D_V W) + F(D_W V) + F(T(V, W)) \\ &= \nabla_V^E F(W) - \nabla_W^E F(V) \\ &\quad - F(D_V W) + F(D_W V) + \lambda F(N_\tau(V, W)) \\ &= (\nabla_V F)W - (\nabla_W F)V + \lambda F(N_\tau(V, W)), \end{aligned} \quad (6.3)$$

where ∇ is the connection induced on $T^*M \otimes E$ by D and ∇^E .

Later in this work we consider the case where N is a pseudo-Riemannian symmetric space with its Levi-Civita connection ∇^N .

Let $\alpha \in \mathbb{R}$ and define $\mathcal{R}_\alpha : TM \rightarrow TM$ as

$$\mathcal{R}_\alpha(X) = \cosh(\alpha)X + \sinh(\alpha)\tau X.$$

This is a parallel endomorphism field on the tangent bundle TM of M . The eigenvalues of which are seen to be $e^{e\alpha}$ on $T^{1,0}M$ and $e^{-e\alpha}$ on $T^{0,1}M$.

An associated family for f is a family of maps $f_\alpha : M \rightarrow N$ with $\alpha \in \mathbb{R}$ and $f_0 = f$, such that

$$\Phi_\alpha \circ df_\alpha = df \circ \mathcal{R}_\alpha, \quad \forall \alpha \in \mathbb{R}, \quad (6.4)$$

for some bundle isomorphism $\Phi_\alpha : f_\alpha^* TN \rightarrow f^* TN$, $\alpha \in \mathbb{R}$, which is parallel with respect to ∇^N , i.e. Φ_α satisfies

$$\Phi_\alpha \circ (f_\alpha^* \nabla^N) = (f^* \nabla^N) \circ \Phi_\alpha.$$

One can check, that each map f_α of an associated family itself admits an associated family.

Theorem 5 *Let (M, τ) be an almost para-complex manifold endowed with a nice connection D , N be a smooth manifold endowed with a torsion-free connection ∇^N and $f : (M, D, \tau) \rightarrow (N, \nabla^N)$ be a smooth map admitting an associated family f_α , then f is para-pluriharmonic. More precisely, each map of the associated family f_α is para-pluriharmonic.*

Proof: Since Φ_α is parallel with respect to ∇^N , ∇^N is torsion-free and D is nice, we are able to apply equation (6.3) to the family $df_\alpha = F_\alpha = \Phi_\alpha^{-1} \circ df \circ \mathcal{R}_\alpha$ and find

$$(\nabla_V F_\alpha)W - (\nabla_W F_\alpha)V + \lambda F_\alpha(N_\tau(V, W)) = 0.$$

As \mathcal{R}_α is D -parallel we obtain

$$(\nabla_X F_\alpha) = \Phi_\alpha^{-1} \circ (\nabla_X F) \circ \mathcal{R}_\alpha.$$

If $Z = X - e\tau X$ and $W = Y + e\tau Y$ have different type it holds $N_\tau(Z, W) = 0$, where we extended the Nijenhuis tensor para-complex linearly. This yields

$$(\nabla_V F_\alpha)W = (\nabla_W F_\alpha)V, \quad \forall \alpha \in \mathbb{R}$$

and further we get

$$\begin{aligned} (\nabla_Z F_\alpha)W &= e^{e\alpha} \Phi_\alpha^{-1}(\nabla_Z F)W, \\ (\nabla_W F_\alpha)Z &= e^{-e\alpha} \Phi_\alpha^{-1}(\nabla_W F)Z = e^{-e\alpha} \Phi_\alpha^{-1}(\nabla_Z F)W, \end{aligned}$$

for all $\alpha \in \mathbb{R}$. Since this should coincide, it follows $(\nabla df)^{(1,1)} = 0$, i.e. $f : (M, D, \tau) \rightarrow (N, \nabla^N)$ is para-pluriharmonic. The rest follows, since each map of the associated family f_α admits an associated family $g_\beta = f_{(\alpha+\beta)}$. \square

This is the motivation of the following definition:

Definition 11 *Let (M, τ) be an almost para-complex manifold endowed with a nice connection D , N be a smooth manifold endowed with a torsion-free connection ∇^N . A smooth map $f : (M, D, \tau) \rightarrow (N, \nabla^N)$ is said to be \tilde{S}^1 -pluriharmonic if and only if it admits an associated family.*

Given a para-hermitian metric g on M then in general a nice connection D is not the Levi-Civita connection ∇^g of g . Therefore the para-pluriharmonic equation (6.1) does not imply the harmonicity of f . If in addition the tensor $D - \nabla^g$ is trace-free the para-pluriharmonic equation implies the harmonic equation. This is true in the case of a special para-Kähler manifold (M, τ, g, ∇) and for a nearly para-Kähler manifold, where $D = \bar{\nabla}$ and $\bar{\nabla} - \nabla^g$ is skew-symmetric.

Proposition 7 *Let (M, τ, g) be an almost para-hermitian manifold endowed with a nice connection D , N be a pseudo-Riemannian manifold with its Levi-Civita connection ∇^N . Suppose, that the tensor $S = \nabla^g - D$ is trace-free. Then a para-pluriharmonic map $f : M \rightarrow N$ is harmonic.*

Proof: We calculate the expression

$$\begin{aligned} \text{tr}_g(\nabla df) &= \sum_i g(e_i, e_i) [\nabla_{e_i}^E df(e_i) - df(D_{e_i} e_i)] \\ &= \sum_i g(e_i, e_i) [\nabla_{e_i}^E df(e_i) - df((\nabla^g - S)_{e_i} e_i)] \\ &= \sum_i g(e_i, e_i) [\nabla_{e_i}^E df(e_i) - df(\nabla_{e_i}^g e_i)] \\ &= \text{tr}_g(\tilde{\nabla}^g df) \end{aligned}$$

where $\tilde{\nabla}^g$ is the connection induced on $T^*M \otimes E$ by ∇^g and ∇^E and e_i is an orthogonal basis for g on TM . Since g is para-hermitian, we obtain from the para-pluriharmonic equation

$$\text{tr}_g(\nabla df) = \text{tr}_g(\nabla df^{(1,1)}) = 0.$$

\square

7 Related para-pluriharmonic and harmonic maps

7.1 The classifying map of a flat nearly para-Kähler manifold

In this section we consider simply connected almost para-hermitian manifolds (M, τ, g) endowed with a flat metric connection ∇ such that (∇, τ) satisfies the nearly para-Kähler condition.

In particular, simply connected flat nearly para-Kähler manifolds (M^{2n}, τ, g) , i.e. nearly para-Kähler manifolds (M, τ, g) with flat Levi-Civita connection ∇^g are of this type.

Since (M, g, ∇) is simply connected and flat, we may identify by fixing a ∇ -parallel frame s_0 its tangent bundle TM with $(M \times V, \langle \cdot, \cdot \rangle)$, where $V = C^n = (\mathbb{R}^{2n}, j_0)$ is endowed with the standard scalar product $\langle \cdot, \cdot \rangle$ of signature (n, n) on $V = \mathbb{R}^{2n}$.

The compatible para-complex structure τ defines via this identification a map

$$\tau : M \rightarrow \mathcal{P}(V, \langle \cdot, \cdot \rangle),$$

where $\mathcal{P}(V, \langle \cdot, \cdot \rangle)$ is the set of para-complex structures on V which are compatible with $\langle \cdot, \cdot \rangle$ and the orientation.

One can consider $\mathcal{P}(V, \langle \cdot, \cdot \rangle)$ as a subset in the vector space $\mathfrak{so}(n, n) = \mathfrak{so}(V) \subset \text{Mat}(\mathbb{R}^{2n})$ characterised by the equation

$$f(j) = \mathbb{1}_{2n}, \tag{7.1}$$

where $f : \text{Mat}(\mathbb{R}^{2n}) \rightarrow \text{Mat}(\mathbb{R}^{2n})$ is given by $f : A \mapsto A^2$. The differential of this map is $df_A(H) = \{A, H\}$ for $A, H \in \text{Mat}(\mathbb{R}^{2n})$. We remark, that elements satisfying the equation (7.1) define automatically para-complex structures, since they are trace-free and hence their eigenspaces to the eigenvalues ± 1 have the same dimension. The differential df has constant rank in points j satisfying the equation (7.1), since one sees

$$\begin{aligned} \ker df_j &= \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\}, \\ \text{im } df_j &\cong \{A \in \mathfrak{so}(V) \mid [j, A] = 0\} \cong \mathfrak{u}^\pi(C^n). \end{aligned}$$

Applying the regular value theorem $\mathcal{P}(V, \langle \cdot, \cdot \rangle)$ is shown to be a submanifold of $\mathfrak{so}(V)$. Its tangent space at $j \in \mathcal{P}(V, \langle \cdot, \cdot \rangle)$ is

$$T_j \mathcal{P}(V, \langle \cdot, \cdot \rangle) = \ker df_j = \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\}. \tag{7.2}$$

Moreover, $\mathcal{P}(V, \langle \cdot, \cdot \rangle)$ can be identified with the pseudo-Riemannian symmetric space $SO_0(n, n)/U^\pi(C^n)$, where $SO_0(n, n)$ is the identity component of the special pseudo-orthogonal group $SO(n, n)$ and $U^\pi(C^n)$ is the para-unitary group, by the map

$$\begin{aligned} \Phi &: SO_0(n, n)/U^\pi(C^n) \rightarrow \mathcal{P}(V, \langle \cdot, \cdot \rangle), \\ gK &\mapsto g j_0 g^{-1}, \end{aligned}$$

which maps the canonical base point $o = eK$ to j_0 .

An element $j \in \mathcal{P}(V, \langle \cdot, \cdot \rangle)$ defines a symmetric decomposition of $\mathfrak{so}(V)$ by

$$\begin{aligned} \mathfrak{p}(j) &= \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\}, \\ \mathfrak{k}(j) &= \{A \in \mathfrak{so}(V) \mid [j, A] = 0\} \cong \mathfrak{u}^\pi(C^n). \end{aligned}$$

In particular we note $\mathfrak{k}(j_0) = \mathfrak{u}^\pi(C^n)$. Moreover, one observes $T_j \mathcal{P}(V, \langle \cdot, \cdot \rangle) = \mathfrak{p}(j)$. Let $\tilde{j} \in SO_0(n, n)/U^\pi(C^n)$ and $j = \Phi(\tilde{j})$, then $T_{\tilde{j}}SO_0(n, n)/U^\pi(C^n)$ is canonically identified with $\mathfrak{p}(j)$. We determine now the differential of the above identification:

Proposition 8 *Let $\Psi = \Phi^{-1} : \mathcal{P}(V, \langle \cdot, \cdot \rangle) \rightarrow SO_0(n, n)/U^\pi(C^n)$. Then it holds at $j \in \mathcal{P}(V, \langle \cdot, \cdot \rangle)$*

$$d\Psi : T_j \mathcal{P}(V, \langle \cdot, \cdot \rangle) \ni X \mapsto -\frac{1}{2}j^{-1}X \in \mathfrak{p}(j). \quad (7.3)$$

The next theorem constructs the desired para-pluriharmonic map.

Theorem 6 *Let (M, τ, g) be a simply connected almost para-hermitian manifold endowed with a flat metric connection ∇ such that (∇, τ) satisfies the nearly para-Kähler condition, then $(TM, D = \nabla - S, S = -\frac{1}{2}\tau(\nabla\tau), \omega = g(\tau\cdot, \cdot))$ defines a symplectic ptt*-bundle and the matrix of τ in a D^θ -flat frame $s^\theta = (s_i^\theta)$ defines an \tilde{S}^1 -pluriharmonic map $\tilde{\tau}^\theta : M \rightarrow \mathcal{P}(V, \langle \cdot, \cdot \rangle) \rightarrow SO_0(n, n)/U^\pi(C^n)$.*

In particular, given a nice connection D on M the map

$$\tilde{\tau}^\theta : (M, \tau, D) \rightarrow SO_0(n, n)/U^\pi(C^n)$$

is para-pluriharmonic.

Proof: One observes $D^\theta g = 0$, as one has $\nabla g = 0$ and $S_X^\theta := \cosh(\theta)S_X + \sinh(\theta)S_{\tau X}$ takes values in $\mathfrak{so}(V)$. This means we can choose for each θ the D^θ -flat frame s^θ orthonormal, such that $s^{\theta=0} = s_0$. Using $D\tau = 0$ (compare theorem 1 and lemma 1) we obtain

$$Xg(\tau s_i^\theta, s_j^\theta) = g(D_X^\theta(\tau s_i^\theta), s_j^\theta) = g((D_X^\theta\tau)s_i^\theta, s_j^\theta) = g([S_X^\theta, \tau]s_i^\theta, s_j^\theta) = -2g(\tau S_X^\theta s_i^\theta, s_j^\theta).$$

Let S^{s^θ} and τ^{s^θ} be the representation of S and τ in the frame s^θ , then we have

$$(\tau^{s^\theta})^{-1}X(\tau^{s^\theta}) = -2S^{s^\theta}$$

or

$$d\tilde{\tau}^\theta = (s^\theta)^{-1} \circ S^\theta \circ s^\theta.$$

Here the frame s^θ is seen as a map $s^\theta : M \times V \rightarrow TM$. For $X \in \Gamma(TM)$ we obtain

$$\begin{aligned} d\tilde{\tau}^\theta(X) &= (s^\theta)^{-1} \circ S_X^\theta \circ (s^\theta) = (s^\theta)^{-1} \circ S_{\mathcal{R}_\theta X} \circ (s^\theta) \\ &= ((s^\theta)^{-1}s^\theta) \circ d\tilde{\tau}(\mathcal{R}_\theta X) \circ ((s^\theta)^{-1}s^\theta) \\ &= Ad_{\alpha_\theta}^{-1} \circ d\tilde{\tau}(\mathcal{R}_\theta X) = \Phi_\theta^{-1} \circ d\tilde{\tau}(\mathcal{R}_\theta X), \end{aligned}$$

where $\alpha_\theta = (s^\theta)^{-1}s^0$ is the frame change from s_0 to s_θ and $\Phi_\theta = Ad_{\alpha_\theta}$. Φ_θ is parallel with respect to the Levi-Civita connection on $SO_0(n, n)/U^\pi(C^n)$ and hence $\tilde{\tau}^\theta$ is \tilde{S}^1 -pluriharmonic. Given a nice connection D on M theorem 5 shows that $\tilde{\tau}^\theta$ is para-pluriharmonic. \square

The next corollary emphasises the nearly para-Kähler setting:

Corollary 3 *Let (M, τ, g) be a simply connected flat nearly para-Kähler manifold and $(TM, \bar{\nabla} = \nabla^g - S, S = -\frac{1}{2}\tau(\nabla\tau), \omega(\cdot, \cdot) = g(\tau\cdot, \cdot))$ the associated symplectic ptt*-bundle, then the matrix of τ in a D^θ -flat frame $s^\theta = (s_i^\theta)$ defines an $\tilde{\mathcal{S}}^1$ -pluriharmonic map $\tilde{\tau}^\theta : M \rightarrow \mathcal{P}(V, \langle \cdot, \cdot \rangle) \rightarrow SO_0(n, n)/U^\pi(C^n)$.*

For nearly para-Kähler manifolds one can get more precise information about the map $\tilde{\tau}^\theta$:

Theorem 7 *Let (M, τ, g) be a simply connected flat nearly para-Kähler manifold and $(TM, \bar{\nabla} = \nabla^g - S, S = -\frac{1}{2}\tau(\nabla\tau), \omega(\cdot, \cdot) = g(\tau\cdot, \cdot))$ the associated symplectic ptt*-bundle. Then the connection $\bar{\nabla}$ is nice and the matrix of τ in a D^θ -flat frame $s^\theta = (s_i^\theta)$ defines a para-pluriharmonic map $\tilde{\tau}^\theta : (M, \tau, \bar{\nabla}) \rightarrow \mathcal{P}(V, \langle \cdot, \cdot \rangle) \rightarrow SO_0(n, n)/U^\pi(C^n)$. Moreover, the map $\tilde{\tau}^\theta$ is harmonic.*

Proof: We have to prove, that $\bar{\nabla}$ is nice. Therefore we rewrite the Nijenhuis tensor

$$\begin{aligned} N_\tau(X, Y) &= (\nabla_{\tau X}\tau)Y - (\nabla_{\tau Y}\tau)X - \tau(\nabla_X\tau)Y + \tau(\nabla_Y\tau)X \\ &= -4\tau(\nabla_X\tau)Y, \end{aligned}$$

where the second equality follows from the nearly para-Kähler condition and by

$$(\nabla_{\tau X}\tau)Y = -(\nabla_Y\tau)\tau X = \tau(\nabla_Y\tau)X = -\tau(\nabla_X\tau)Y.$$

On the other hand the torsion of $\bar{\nabla}$ was given in equation (5.7) by

$$T^{\bar{\nabla}}(X, Y) = \tau(\nabla_X\tau)Y$$

and consequently $\bar{\nabla}$ is nice.

Due to corollary 3 the map $\tilde{\tau}^\theta$ is $\tilde{\mathcal{S}}^1$ -pluriharmonic. Since $\bar{\nabla}$ is nice, theorem 5 implies that $\tilde{\tau}^\theta$ is para-pluriharmonic. From the skew-symmetry of S and proposition 7 we obtain that $\tilde{\tau}^\theta$ is harmonic. \square

7.2 The dual Gauß map of a special para-Kähler manifold with torsion

In this subsection we consider a simply connected almost para-hermitian manifold (M, τ, g) with a flat connection ∇ , such that (∇, τ) is special and the two-form $\omega = g(\tau\cdot, \cdot)$ is ∇ -parallel.

Using the flat connection ∇ we identify by fixing a ∇ -parallel symplectic frame s_0 the tangent bundle (TM, ω) with $(M \times V, \omega_0)$ where $V = \mathbb{R}^{2n}$ and ω_0 is its standard symplectic form.

The compatible para-complex structure τ is seen as a map

$$\tau : M \rightarrow \mathcal{P}(V, \omega_0),$$

where $\mathcal{P}(V, \omega_0)$ is the set of para-complex structures on V which are compatible with ω_0 . Now we discuss the differential geometry of $\mathcal{P}(V, \omega_0)$, where ω_0 is the standard symplectic

form on $V = C^n = (\mathbb{R}^{2n}, j_0)$.

First, we consider $\mathcal{P}(V, \omega_0)$ as a subset of the vector space $\mathfrak{sp}(\mathbb{R}^{2n}) \subset \text{Mat}(\mathbb{R}^{2n})$ characterised by the set of equations

$$f(j) = \mathbb{1}_{2n}, \quad (7.4)$$

where $f : \text{Mat}(\mathbb{R}^{2n}) \rightarrow \text{Mat}(\mathbb{R}^{2n})$ is given as in the last section. Again, df has constant rank in points j satisfying the equation (7.4), since one sees

$$\begin{aligned} \ker df_j &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\}, \\ \text{im } df_j &\cong \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid [j, A] = 0\} \cong \mathfrak{u}^\pi(C^r). \end{aligned}$$

Applying the regular value theorem we obtain that $\mathcal{P}(V, \omega_0)$ is a submanifold of $\mathfrak{sp}(\mathbb{R}^{2n})$. Its tangent space at $j \in \mathcal{P}(V, \omega_0)$ is

$$T_j \mathcal{P}(V, \omega_0) = \ker df_j = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\}. \quad (7.5)$$

In addition the manifold $\mathcal{P}(V, \omega_0)$ can be identified with the pseudo-Riemannian symmetric space $\text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ by the map

$$\begin{aligned} \Phi : \text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n) &\rightarrow \mathcal{P}(V, \omega_0), \\ gK &\mapsto g j_0 g^{-1}, \end{aligned}$$

which maps the canonical base point $o = eK$ to j_0 .

Any $j \in \mathcal{P}(V, \omega_0)$ defines a symmetric decomposition of $\mathfrak{sp}(\mathbb{R}^{2n})$ by

$$\begin{aligned} \mathfrak{p}(j) &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\}, \\ \mathfrak{k}(j) &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid [j, A] = 0\} \cong \mathfrak{u}^\pi(C^r). \end{aligned}$$

In particular it is $\mathfrak{k}(j_0) = \mathfrak{u}^\pi(C^r)$. Moreover, one observes $T_j \mathcal{P}(V, \omega_0) = \mathfrak{p}(j)$.

Let $\tilde{j} \in \text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ and $j = \Phi(\tilde{j})$, then $T_{\tilde{j}} \text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ is canonically identified with $\mathfrak{p}(j)$ and for the differential of the identification we have:

Proposition 9 *Let $\Psi = \Phi^{-1} : \mathcal{P}(V, \omega_0) \rightarrow \text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$. Then it holds at $j \in \mathcal{P}(V, \omega_0)$*

$$d\Psi : T_j \mathcal{P}(V, \omega_0) \ni X \mapsto -\frac{1}{2}j^{-1}X \in \mathfrak{p}(j). \quad (7.6)$$

Recall, that under the above assumptions $(TM, D = \nabla - S, S = -\frac{1}{2}\tau(\nabla\tau), g)$ defines a metric $p\tilde{t}t^*$ -bundle. Analogue to the last section we obtain:

Theorem 8 *Let (M, τ, g) be a simply connected almost para-hermitian manifold endowed with a flat connection ∇ , such that (∇, τ) is special and the two-form $\omega = g(\tau, \cdot)$ is ∇ -parallel and let $(TM, D = \nabla - S, S = -\frac{1}{2}\tau(\nabla\tau), g)$ be the associated metric $p\tilde{t}t^*$ -bundle. Then the matrix of τ in a D^θ -flat frame $s^\theta = (s_i^\theta)$ defines an \tilde{S}^1 -pluriharmonic map $\tilde{\tau}^\theta : M \rightarrow \mathcal{P}(V, \omega_0) \rightarrow \text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$.*

In particular, given a nice connection D on (M, τ) then the map $\tilde{\tau}^\theta : (M, \tau, D) \rightarrow \text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ is para-pluriharmonic.

Proof: Since we have $D^0\omega = \nabla\omega = (D + S)\omega = 0$ and $S_X^\theta := \cosh(\theta)S_X + \sinh(\theta)S_{\tau X}$ is skew-symmetric with respect to ω , we obtain $D\omega = 0$ and $D^\theta\omega = 0$. Hence one can choose for each θ the D^θ -parallel frame s^θ as a symplectic frame, such that $s^{\theta=0} = s_0$. This yields using $D\tau = 0$ (cf. theorem 1 and lemma 1)

$$X.\omega(\tau s_i^\theta, s_j^\theta) = \omega(D_X^\theta(\tau s_i^\theta), s_j^\theta) = \omega((D_X^\theta\tau)s_i^\theta, s_j^\theta) = \omega([S_X^\theta, \tau]s_i^\theta, s_j^\theta) = -2\omega(\tau S_X^\theta s_i^\theta, s_j^\theta).$$

Denote by $S^{s^\theta}, \tau^{s^\theta}$ the representation of S and τ in the frame s^θ , then we get

$$(\tau^{s^\theta})^{-1}X(\tau^{s^\theta}) = -2S^{s^\theta}$$

or

$$d\tilde{\tau}^\theta = (s^\theta)^{-1} \circ S^\theta \circ s^\theta,$$

where the frame s^θ is seen as a map $s^\theta : M \times V \rightarrow TM$. This shows for $X \in \Gamma(TM)$

$$\begin{aligned} d\tilde{\tau}^\theta(X) &= (s^\theta)^{-1} \circ S_X^\theta \circ (s^\theta) = (s^\theta)^{-1} \circ S_{\mathcal{R}_\theta X} \circ (s^\theta) \\ &= ((s^\theta)^{-1}s^0) \circ d\tilde{\tau}(\mathcal{R}_\theta X) \circ ((s^0)^{-1}s^\theta) \\ &= Ad_{\alpha_\theta}^{-1} \circ d\tilde{\tau}(\mathcal{R}_\theta X) = \Phi_\theta^{-1} \circ d\tilde{\tau}(\mathcal{R}_\theta X), \end{aligned}$$

where $\alpha_\theta = (s^\theta)^{-1}s^0$ denotes the frame change from s_0 to s_θ and $\Phi_\theta = Ad_{\alpha_\theta}$ which is parallel with respect to the Levi-Civita connection on $\text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$. In other words we have found an associated family. Given a nice connection D on (M, τ) theorem 5 shows that $\tilde{\tau}^\theta$ is para-pluriharmonic. \square

In the case where the above ptt^* -bundle comes from a special para-Kähler manifold we have the theorem:

Theorem 9 *Let (M, τ, g, ∇) be a simply connected special para-Kähler manifold and $(TM, D = \nabla - S, S = -\frac{1}{2}\tau\nabla\tau, g)$ be the associated metric ptt^* -bundle, then the matrix of τ in a D^θ -flat frame $s^\theta = (s_i^\theta)$ defines a para-pluriharmonic map $\tilde{\tau}^\theta : (M, \tau, D) \rightarrow \text{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$. Further $\tilde{\tau}^\theta$ is harmonic.*

Proof: Using theorem 8 the map $\tilde{\tau}^\theta$ is \tilde{S}^1 -pluriharmonic. In the special para-Kähler case we know that D is the Levi-Civita connection and hence it is torsion-free. The para-complex structure τ is integrable and hence it holds $N_\tau = 0$. This means, that D is nice and theorem 5 shows that $\tilde{\tau}^\theta$ is para-pluriharmonic. Since S is trace-free we get from proposition 7 that $\tilde{\tau}^\theta$ is harmonic. \square

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