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# Harmonic bundle solutions of Topological-antitopological fusion in para-complex geometry

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## Abstract

In this work we introduce the notion of a para-harmonic bundle, i.e. the generalization of a harmonic bundle [Si] to para-complex differential geometry. We show that para-harmonic bundles are solutions of the para-complex version of metric  $tt^*$ -bundles introduced in [Sch2]. Further we analyze the correspondence between metric para- $tt^*$ -bundles of rank  $2r$  over a para-complex manifold  $M$  and para-pluriharmonic maps from  $M$  into the pseudo-Riemannian symmetric space  $GL(r, \mathbb{R})/O(p, q)$ , which was shown in [Sch2], in the case of a para-harmonic bundle. It is proven, that for para-harmonic bundles the associated para-pluriharmonic maps take values in the totally geodesic subspace  $GL(r, C)/U^\pi(C^r)$  of  $GL(2r, \mathbb{R})/O(r, r)$ . This defines a map  $\Phi$  from para-harmonic bundles over  $M$  to para-pluriharmonic maps from  $M$  to  $GL(r, C)/U^\pi(C^r)$ . The image of  $\Phi$  is also characterized in the paper.

**Keywords:** para-complex geometry, para-harmonic bundles, para- $tt^*$ -geometry, para-pluriharmonic maps and pseudo-Riemannian symmetric spaces

**MSC(2000):** 58E20, 53C43.

# 1 Introduction

Para-complex geometry was first introduced in 1952 by P. Libermann [L]. For a survey on this subject we refer to [CFG]. Examples can be obtained from para-hermitian symmetric spaces, see [B]. These spaces provide examples of para-Kähler manifolds, which are also named bi-Lagrangian manifolds (cf. [EST] for a survey). We remark that para-Kähler manifolds are forced to have a metric of split signature and that consequently the later considered harmonic and para-pluriharmonic maps with a para-Kähler manifold as source-manifold are no longer solutions of an elliptic equation.

The complex notion of  $tt^*$ -geometries originates in the physics of topological field-theories (see [CV]). Motivated by the recent study of the geometries appearing in Euclidean supersymmetry by [CMMS] we generalized in [Sch2] the notion of  $tt^*$ -bundles to para-complex geometry, which we denoted para- $tt^*$ -bundles. We showed that special para-complex and special para-Kähler manifolds are solutions of  $tt^*$ -bundles. Further we roughly gave a correspondence between metric para- $tt^*$ -bundles  $(E, D, S, g)$  and para-pluriharmonic maps into the pseudo-Riemannian symmetric space  $GL(r, \mathbb{R})/O(p, q)$  where  $(p, q)$  is the signature of the bundle metric  $g$  on the vector bundle  $E$ . In the case of a para- $tt^*$ -bundle induced by a special para-Kähler manifold this para-pluriharmonic map was related to the dual Gauß map. Analogous results were shown in [CS, Sch1] for complex geometry.

Another class of  $tt^*$ -bundles in the complex setting are harmonic bundles with positive definite hermitian metric which were introduced by Simpson [Si]. In the same paper Simpson gave a correspondence between harmonic bundles over a compact Kähler manifold  $M$  and harmonic maps from  $M$  into  $GL(r, \mathbb{C})/U(r)$ . In [Sch3] we related this result to the complex version of the above mentioned correspondence between metric  $tt^*$ -bundles and pluriharmonic maps. In fact, we were able to obtain a more general result by applying our correspondence to harmonic bundles with possible indefinite metric over an arbitrary complex manifold. Simpsons result was recovered from this correspondence.

Against this background it arises the question if there exists a notion of harmonic bundle in the para-complex setting, if this generalization of a harmonic bundle supplies a solution of para- $tt^*$ -geometry and finally if there exists a correspondence of these generalized harmonic bundles and para-pluriharmonic maps from  $M$  into the pseudo-Riemannian symmetric space  $GL(r, \mathbb{C})/U^\pi(C^r)$ , which is the para-complex analogue of  $GL(r, \mathbb{C})/U(r)$ . In this paper we answer positively to these questions.

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## 2 Para-complex differential geometry

We shortly recall some notions and facts of para-complex differential geometry. The idea of para-complex geometry is to replace the complex structure  $J$  with  $J^2 = -\mathbb{1}$  (on a finite dimensional vector space  $V$ ) by the **para-complex structure**  $\tau \in End(V)$  satisfying  $\tau^2 = \mathbb{1}$  such that the  $\pm 1$ -eigenspaces have the same dimension. An **almost para-complex structure**

on a smooth manifold  $M$  is an endomorphism-field  $\tau$ , which is a point-wise para-complex structure. If the eigen-distributions  $T^\pm M$  are integrable  $\tau$  is called **para-complex structure on  $M$**  and  $M$  is called a **para-complex manifold**. Analogous to complex geometry there exists a tensor, also called **Nijenhuis tensor**, which is the obstruction to the integrability of the para-complex structure.

The real algebra, which is generated by 1 and by the **para-complex unit**  $e$  with  $e^2 = -1$ , is called the **para-complex numbers** and denoted by  $C$ . For all  $z = x + ey \in C$  with  $x, y \in \mathbb{R}$  we define the **para-complex conjugation** as  $\bar{\cdot} : C \rightarrow C, x + ey \mapsto x - ey$  and the **real and imaginary parts** of  $z$  by  $\Re(z) := x, \Im(z) := y$ . The free  $C$ -module  $C^n$  is a para-complex vector space where its para-complex structure is just the multiplication with  $e$  and the para-complex conjugation of  $C$  extends to  $\bar{\cdot} : C^n \rightarrow C^n, v \mapsto \bar{v}$ .

Note, that  $z\bar{z} = x^2 - y^2$ . Therefore the algebra  $C$  is sometimes called the **hypercomplex numbers**. The circle  $\mathbb{S}^1 = \{z = x + iy \in \mathbb{C} \mid x^2 + y^2 = 1\}$  is replaced by the four hyperbolas  $\{z = x + ey \in C \mid x^2 - y^2 = \pm 1\}$ . We define  $\tilde{\mathbb{S}}^1$  to be the hyperbola given by the one parameter group  $\{z(\theta) = \cosh(\theta) + e \sinh(\theta) \mid \theta \in \mathbb{R}\}$ .

A para-complex vector space  $(V, \tau)$  endowed with a pseudo-Euclidean metric  $g$  is called **para-hermitian vector space**, if  $g$  is  $\tau$ -anti-invariant, i.e.  $\tau^*g = -g$ . The **para-unitary group** of  $V$  is defined as the group of automorphisms

$$U^\pi(V) := \text{Aut}(V, \tau, g) := \{L \in GL(V) \mid [L, \tau] = 0 \text{ and } L^*g = g\}$$

and its Lie-algebra is denoted by  $\mathfrak{u}^\pi(V)$ . For  $C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$  the **standard para-hermitian structure** is defined by the above para-complex structure and the metric  $g = \text{diag}(\mathbb{1}, -\mathbb{1})$  (cf. Example 7 of [CMMS]). The corresponding para-unitary group is given by (cf. Proposition 2 of [CMMS]):

$$U^\pi(C^n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \in \text{End}(\mathbb{R}^n), A^T A - B^T B = \mathbb{1}_n, A^T B - B^T A = 0 \right\}.$$

There exist two bi-gradings on the exterior algebra: The one is induced by the splitting in  $T^\pm M$  and denoted by  $\Lambda^k T^* M = \bigoplus_{k=p+q} \Lambda^{p+, q-} T^* M$  and induces an obvious bi-

grading on exterior forms with values in a vector bundle  $E$ . The second is induced by the decomposition of the **para-complexified tangent bundle**  $TM^C = TM \otimes_{\mathbb{R}} C$  into the subbundles  $T_p^{1,0} M$  and  $T_p^{0,1} M$  which are defined as the  $\pm e$ -eigenbundles of the para-complex linear extension of  $\tau$ . This induces a bi-grading on the  $C$ -valued exterior forms noted  $\Lambda^k T^* M^C = \bigoplus_{k=p+q} \Lambda^{p, q} T^* M$  and finally on the  $C$ -valued differential forms on  $M$

$\Omega_C^k(M) = \bigoplus_{k=p+q} \Omega^{p, q}(M)$ . In the case  $(1, 1)$  and  $(1+, 1-)$  the two gradings induced by

$\tau$  coincide, in the sense that  $\Lambda^{1,1} T^* M = (\Lambda^{1+, 1-} T^* M) \otimes C$ . The bundles  $\Lambda^{p, q} T^* M$  are para-complex vector bundles in the following sense: A **para-complex vector bundle** of rank  $r$  over a para-complex manifold  $(M, \tau)$  is a smooth real vector bundle  $\pi : E \rightarrow M$  of rank  $2r$  endowed with a fiberwise para-complex structure  $\tau^E \in \Gamma(\text{End}(E))$ . We denote it by  $(E, \tau^E)$ . In the following text we always identify the fibers of a para-complex vector bundle  $E$  of rank  $r$  with the free  $C$ -module  $C^r$ .

One has a notion of para-holomorphic vector bundles, too. These were extensively studied in a common work with M.-A. Lawn-Paillusseau [LS].

### 3 Para-pluriharmonic maps

In this section we discuss para-pluriharmonic maps, which are the analogue of pluriharmonic maps in para-complex geometry.

Throughout this section we consider a para-complex manifold  $(M, \tau)$ . Given a pseudo-Riemannian manifold  $(N, h)$  a map  $f : M \rightarrow N$  is called **para-pluriharmonic** if  $f|_C$  is harmonic for every para-complex curve  $C \subset M$ , i.e. for every real two-dimensional para-complex submanifold. We remark, that the harmonicity of  $f|_C$  is independent of the choice of a pseudo-Riemannian metric in the conformal class of  $C$ .

Let us further fix an **adapted connection**  $D$  on  $(M, \tau)$ , i.e. a connection which satisfies

$$D_{\tau Y} X = \tau D_Y X \quad (3.1)$$

for all vector fields  $X, Y$  with  $\mathcal{L}_X \tau = 0$  (i.e. for which  $X + e\tau X$  is para-holomorphic). The notion of adapted connections on para-holomorphic vector bundles over para-complex manifolds was introduced in [LS]. The existence of adapted connections on para-complex manifolds was assured in proposition 9 of [Sch2].

A map  $f : M \rightarrow N$  is para-pluriharmonic if and only if it satisfies the following equation

$$(\nabla df)^{(1,1)} = 0, \quad df \in \Gamma(T^*M \otimes f^*TN), \quad (3.2)$$

where  $\nabla$  is induced by the adapted connection  $D$  and the Levi-Civita connection of  $h$ . Later, we use a special class of para-pluriharmonic morphisms:

**Proposition 1** *Let  $(M, \tau)$  be a para-complex manifold,  $X, Y$  be pseudo-Riemannian manifolds and  $\Psi : X \rightarrow Y$  be a totally geodesic immersion. Then a map  $f : M \rightarrow X$  is para-pluriharmonic if and only if  $\Psi \circ f : M \rightarrow Y$  is para-pluriharmonic.*

*Proof:* It is well-known (cf. Eells and Sampson [ES]) that a map  $f : M \rightarrow X$  is harmonic if and only if  $\Psi \circ f : M \rightarrow Y$  is harmonic. The definition of para-pluriharmonic maps finishes the proof.  $\square$

We are going to apply this result to the symmetric spaces  $G/K$  with  $G = GL(2r, \mathbb{R})$  and  $K = O(r, r)$  or  $G = GL(r, \mathbb{C})$  and  $K = U^\pi(C^r)$ . We discuss this for the second example, since the first can be found in [Sch2] and is very similar.

In the following subsection we identify  $C^r$  with  $\mathbb{R}^r \oplus e\mathbb{R}^r = \mathbb{R}^{2r}$ . The multiplication with  $e$  equals the automorphism  $E = \begin{pmatrix} 0 & \mathbb{1}_r \\ \mathbb{1}_r & 0 \end{pmatrix}$  and  $GL(r, \mathbb{C})$  (respectively  $\mathfrak{gl}_r(\mathbb{C})$ ) consists of the elements in  $GL(2r, \mathbb{R})$  (respectively  $\mathfrak{gl}_{2r}(\mathbb{R})$ ) commuting with  $E$ .

First, we have to discuss some linear algebra: We start with the notion of para-hermitian sesquilinear scalar products on para-complex vector spaces:

#### Definition 1

1. A **para-hermitian sesquilinear scalar product** is a non-degenerate sesquilinear form  $h : C^r \times C^r \rightarrow C$ , i.e. (i)  $h$  is non-degenerate: Given  $w \in C^r$  such that for all  $v \in C^r$  it holds  $h(v, w) = 0$ , then it follows  $w = 0$ , (ii)  $h(v, w) = \overline{h(w, v)}$ ,  $\forall v, w \in C^r$ , (iii)  $h(\lambda v, w) = \lambda h(v, w)$ ,  $\forall \lambda \in C; v, w \in C^r$ .

2. Let  $z = (z^1, \dots, z^r)$  and  $w = (w^1, \dots, w^r)$  be two elements of  $C^r$ , then one defines the standard  $C$ -bilinear scalar product on  $C^r$  by  $z \cdot w := \sum_{i=1}^r z^i w^i$  and the standard para-hermitian sesquilinear scalar product by  $(z, w)_{C^r} := z \cdot \bar{w}$ .
3. Given a matrix  $C$  of  $\text{End}(C^r) = \text{End}_C(C^r)$ , we define the para-hermitian conjugation by  $C \mapsto C^h = \bar{C}^t$ . We call  $C$  para-hermitian if and only if  $C^h = C$ . We denote by  $\text{herm}(C^r)$  the set of para-hermitian endomorphisms and by  $\text{Herm}(C^r) = \text{herm}(C^r) \cap \text{GL}(r, C)$ .

We remark, that there is **no** notion of para-hermitian signature, since from  $h(v, v) = -1$  for a  $v \in C^r$  we obtain  $h(ev, ev) = 1$ .

### Proposition 2

- (a) Given an element  $C$  of  $\text{End}(C^r)$  then it holds  $(Cz, w)_{C^r} = (z, C^h w)_{C^r}$ ,  $\forall z, w \in C^r$ .
- (b) The set  $\text{herm}(C^r)$  is a real vector space.
- (c) There is a bijective correspondence between  $\text{Herm}(C^r)$  and para-hermitian sesquilinear scalar products  $h$  on  $C^r$  given by  $H \mapsto h(\cdot, \cdot) := (H\cdot, \cdot)_{C^r}$ .

We now analyze the map which corresponds to taking the real part  $\text{Re } h$  of  $h$ . This is the map  $\mathcal{R}$  satisfying  $\text{Re } h = (\mathcal{R}(H)\cdot, \cdot)_{\mathbb{R}^{2r}}$ , where  $(\cdot, \cdot)_{\mathbb{R}^{2r}}$  is the Euclidean standard scalar product on  $\mathbb{R}^{2r}$ .

First we identify some usual operations in the above identification  $C^r = \mathbb{R}^r \oplus e\mathbb{R}^r = \mathbb{R}^{2r}$ : An endomorphism  $C \in \text{End}(C^r)$  decomposes in its real part  $A$  and its imaginary part  $B$ , i.e.  $C = A + eB$  where  $A, B \in \text{End}(\mathbb{R}^r)$  and  $C$  is identified via a map, which we denote by  $\iota$ , with the matrix

$$\iota(C) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The para-complex conjugation of  $C$ , i.e.  $\bar{C} = A - eB$ , and the transposition  $C^t = A^t + eB^t$  correspond to

$$\iota(\bar{C}) = \begin{pmatrix} A & -B \\ -B & A \end{pmatrix} \text{ and } \iota(C^t) = \begin{pmatrix} A^t & B^t \\ B^t & A^t \end{pmatrix} = \iota(C)^T,$$

where  $\cdot^T$  is the transposition in  $\text{End}(\mathbb{R}^{2r})$  and the adjoint with respect to  $(\cdot, \cdot)_{C^r}$  is  $C^h = \bar{C}^t$  which corresponds to

$$\iota(C^h) = \begin{pmatrix} A^t & -B^t \\ -B^t & A^t \end{pmatrix} \stackrel{(*)}{=} \mathbb{1}_{r,r} \iota(C)^T \mathbb{1}_{r,r}.$$

The equality in  $(*)$  is due to the calculation:

$$\begin{aligned} \iota(C^h) \mathbb{1}_{r,r} &= \begin{pmatrix} A^t & -B^t \\ -B^t & A^t \end{pmatrix} \mathbb{1}_{r,r} = \begin{pmatrix} A^t & B^t \\ -B^t & -A^t \end{pmatrix} \\ &= \mathbb{1}_{r,r} \begin{pmatrix} A^t & B^t \\ B^t & A^t \end{pmatrix} = \mathbb{1}_{r,r} \iota(C)^T = \mathbb{1}_{r,r} \iota(C^t) \end{aligned} \tag{3.3}$$

with

$$\mathbb{1}_{r,r} = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & -\mathbb{1}_r \end{pmatrix}.$$

A para-hermitian sesquilinear scalar product  $h$  corresponds to a para-hermitian matrix  $H \in \text{Herm}(C^r)$  (compare with proposition 2) defined by  $h(\cdot, \cdot) = (H\cdot, \cdot)_{C^r}$ . The condition  $C^h = C$ , i.e.  $C$  is para-hermitian, means in our model that  $C = A + eB$  such that the real matrices  $A$  and  $B$  satisfy  $A = A^t$  and  $B = -B^t$ .

The group  $GL(r, C)$  operates on  $\text{Herm}(C^r)$  via

$$GL(r, C) \times \text{Herm}(C^r) \rightarrow \text{Herm}(C^r), \quad (g, B) \mapsto g \cdot B := (g^{-1})^h B g^{-1}.$$

Let us denote by  $\text{Sym}_{p,q}(\mathbb{R}^r)$  the symmetric  $r \times r$  matrices of symmetric signature  $(p, q)$  with  $r = p + q$ . We consider the  $GL(r, \mathbb{R})$ -action on  $\text{Sym}_{p,q}(\mathbb{R}^r)$  given by  $(g^{-1}, S) \mapsto g^{-1} \cdot S = g^T S g$  with  $g \in GL(r, \mathbb{R})$  and  $S \in \text{Sym}_{p,q}(\mathbb{R}^r)$ .

**Lemma 1** *The map  $\mathcal{R}$  is given by  $\mathcal{R} = \mathbb{1}_{r,r} \circ \iota$  and it defines an equivariant immersion  $\mathcal{R} : \text{Herm}(C^r) \rightarrow \text{Sym}_{r,r}(\mathbb{R}^{2r})$ .*

*Proof:* First we determine  $\mathcal{R}$ : With  $z, w \in C^r$  we have  $\beta(z, w) := \text{Re}(z, w)_{C^r} = \frac{1}{2}(z \cdot \bar{w} + \bar{z} \cdot w)$  and  $\text{Re } h(z, w) = \text{Re}(Hz, w)_{C^r} = \frac{1}{2}[(Hz) \cdot \bar{w} + (\overline{Hz}) \cdot w] = \beta(Hz, w)$ . Further we remark that  $\beta(\cdot, \cdot) = \text{Re}(\cdot, \cdot)_{C^r} = (\cdot, \cdot)_{\mathbb{R}^{r,r}}$ , where  $(\cdot, \cdot)_{\mathbb{R}^{r,r}} = (\mathbb{1}_{r,r} \cdot, \cdot)_{\mathbb{R}^{2r}}$  is the (pseudo-)Euclidean standard scalar product of signature  $(r, r)$  on  $\mathbb{R}^{2r}$ . This yields  $\text{Re } h(z, w) = (Hz, w)_{\mathbb{R}^{r,r}} = (\mathbb{1}_{r,r} Hz, w)_{\mathbb{R}^{2r}}$  and for  $H = A + eB$  with  $A, B \in \text{End}(\mathbb{R}^r)$

$$\mathcal{R}(H) = \mathbb{1}_{r,r} \iota(H) = \mathbb{1}_{r,r} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}.$$

Since  $H$  is para-hermitian, we obtain  $\mathcal{R}(H)^T = \mathcal{R}(H)$ . The symmetric signature of the symmetric matrix  $\mathcal{R}(H)$  is  $(r, r)$ , as it is the real part of a para-hermitian sesquilinear scalar product. Summarizing we have  $\mathcal{R} : \text{Herm}(C^r) \rightarrow \text{Sym}_{r,r}(\mathbb{R}^{2r})$ ,  $H \mapsto \mathcal{R}(H) = \mathbb{1}_{r,r} \iota(H)$ . Moreover one sees, that the map  $\mathcal{R}$  has maximal rank.

We show, that  $\mathcal{R}$  is equivariant with respect to the above mentioned actions. In fact,

$$\begin{aligned} \mathcal{R}(g^{-1} \cdot H) &= \mathcal{R}(g^h H g) = \mathbb{1}_{r,r} \iota(g^h H g) \\ &= \mathbb{1}_{r,r} \iota(g^h) \iota(H) \iota(g) \stackrel{(3.3)}{=} \iota(g)^T \mathbb{1}_{r,r} \iota(H) \iota(g) \\ &= \iota(g)^T \mathcal{R}(H) \iota(g) = \iota(g)^{-1} \cdot \mathcal{R}(H). \end{aligned}$$

□

**Lemma 2** *The inclusion  $GL(r, C)/U^\pi(C^r) \hookrightarrow GL(2r, \mathbb{R})/O(r, r)$  is totally geodesic.*

*Proof:* The decomposition  $\mathfrak{gl}_{2r}(\mathbb{R}) = \text{sym}_{r,r}(\mathbb{R}^{2r}) \oplus \mathfrak{o}(r, r)$ , where  $\text{sym}_{r,r}(\mathbb{R}^{2r})$  are the symmetric matrices with respect to  $(\cdot, \cdot)_{\mathbb{R}^{r,r}}$ , is a symmetric decomposition associated to the symmetric space  $GL(2r, \mathbb{R})/O(r, r)$  and hence

$$[[\text{sym}_{r,r}(\mathbb{R}^{2r}), \text{sym}_{r,r}(\mathbb{R}^{2r})], \text{sym}_{r,r}(\mathbb{R}^{2r})] \subset \text{sym}_{r,r}(\mathbb{R}^{2r}).$$

Let  $A, B, C \in \text{herm}(C^r)$ . From  $[A, E] = [B, E] = [C, E] = 0$ , we conclude with the Jacobi identity  $[[A, B], E] = 0$  and  $[[[A, B], C], E] = 0$ . Hence  $T_{\mathbb{1}_r}GL(r, C)/U^\pi(C^r) = \text{herm}(C^r)$  is a Lie-triple-system in  $T_{\mathbb{1}_r, r}\text{Sym}_{r, r}(\mathbb{R}^{2r}) = \text{sym}_{r, r}(\mathbb{R}^{2r})$ , i.e.

$$[[\text{herm}(C^r), \text{herm}(C^r)], \text{herm}(C^r)] \subset \text{herm}(C^r)$$

and consequently  $GL(r, C)/U^\pi(C^r)$  is a totally geodesic submanifold of  $GL(2r, \mathbb{R})/O(r, r)$ .  
□

**Lemma 3** *The  $GL(r, C)$ -action on  $\text{Herm}(C^r)$  induces a diffeomorphism*

$$\begin{aligned} \Psi &: GL(r, C)/U^\pi(C^r) \xrightarrow{\sim} \text{Herm}(C^r) \subset GL(r, C), \\ gU^\pi(C^r) &\mapsto g \cdot \mathbb{1}_r = (g^{-1})^h \mathbb{1}_r g^{-1} = (g^{-1})^h g^{-1}. \end{aligned} \quad (3.4)$$

*Proof:* The stabilizer of  $\mathbb{1}_r$  under the  $GL(r, C)$ -action on  $\text{Herm}(C^r)$  is shown by a short calculation to be  $GL(r, C)_{\mathbb{1}_r} = \{g \in GL(r, C) \mid g \cdot \mathbb{1}_r = (g^{-1})^h \mathbb{1}_r g^{-1} = \mathbb{1}_r\} = U^\pi(C^r)$ . If the operation  $\cdot$  is transitive we obtain, by the orbit stabilizer theorem, a diffeomorphism

$$\begin{aligned} \Psi &: GL(r, C)/U^\pi(C^r) \xrightarrow{\sim} \text{Herm}(C^r) \subset GL(r, C), \\ gU^\pi(C^r) &\mapsto g \cdot \mathbb{1}_r = (g^{-1})^h \mathbb{1}_r g^{-1} = (g^{-1})^h g^{-1}. \end{aligned} \quad (3.5)$$

The transitivity is due to the following argument: Any para-hermitian sesquilinear scalar product is uniquely determined by its real part, which lies in  $\text{Sym}_{r, r}(\mathbb{R}^{2r})$ . On this space  $GL(2r, \mathbb{R})$  acts transitively.

**We claim:**  $h' = g \cdot h$  with some para-hermitian sesquilinear scalar product  $h$  and an element  $g \in GL(2r, \mathbb{R})$  is a para-hermitian sesquilinear scalar product if and only if  $g \in GL(r, C)$ .

*Proof:* This claim follows from a short calculation: Let  $v, w \in C^r$  and  $\lambda \in C$ : On the one hand it holds  $h'(\lambda v, w) = \lambda h'(v, w) = \lambda(g \cdot h)(v, w) = h(\lambda g^{-1}v, g^{-1}w)$  and on the other hand  $h'(\lambda v, w) = (g \cdot h)(\lambda v, w) = h(g^{-1}\lambda v, g^{-1}w)$ . Subtracting these two equations yields  $h((g^{-1}\lambda - \lambda g^{-1})v, g^{-1}w) = 0$ . Setting  $w = gw'$  with arbitrary  $w' \in C^r$  we obtain  $h((g^{-1}\lambda - \lambda g^{-1})v, w') = 0$ . Since  $g$  is invertible and  $h$  is non-degenerate, we conclude  $g^{-1}\lambda v = \lambda g^{-1}v$ , which implies the  $C$ -linearity of  $g$ . □

We are now going to analyze para-pluriharmonic maps into these spaces:

**Proposition 3** *Let  $(M, \tau)$  be a para-complex manifold and endow  $GL(r, C)/U^\pi(C^r)$  with the (pseudo-)metric induced by the trace-form on  $GL(r, C)$ . Then the map  $\Psi : GL(r, C)/U^\pi(C^r) \xrightarrow{\sim} \text{Herm}(C^r)$  defined in equation (3.4) is totally geodesic and a map  $\phi : M \rightarrow GL(r, C)/U^\pi(C^r)$  is para-pluriharmonic if and only if*

$$\psi = \Psi \circ \phi : M \rightarrow GL(r, C)/U^\pi(C^r) \xrightarrow{\sim} \text{Herm}(C^r) \subset GL(r, C)$$

*is para-pluriharmonic.*

*Proof:* To prove this we define  $\sigma : GL(r, C) \rightarrow GL(r, C)$ ,  $g \mapsto (g^{-1})^h$ . The map  $\sigma$  is a homomorphism and an involution satisfying  $GL(r, C)^\sigma = U^\pi(C^r)$ . Hence the Cartan immersion can be written as

$$i : GL(r, C)/U^\pi(C^r) \rightarrow GL(r, C), \quad g \mapsto g\sigma(g^{-1}) = gg^h = gg^h = \Psi \circ \Lambda(g),$$



where  $\Lambda$  is the map induced on  $GL(r, C)/U^\pi(C^r)$  by  $\tilde{\Lambda} : GL(r, C) \rightarrow GL(r, C)$ ,  $g \mapsto (g^{-1})^h$  which is an isometry of the invariant metric, since  $g \mapsto g^h = \mathbb{1}_{r,r} g^T \mathbb{1}_{r,r}$  and  $g \mapsto g^{-1}$  are isometries of the invariant metric. Therefore  $\Psi$  is totally geodesic, since  $i$  is totally geodesic. Proposition 1 finishes the proof.  $\square$

To be complete we mention the related symmetric decomposition:

$$\mathfrak{h} = \{A \in \mathfrak{gl}_r(C) \mid A^h = -A\} = \mathfrak{u}^\pi(C^r) \text{ and } \mathfrak{p} = \{A \in \mathfrak{gl}_r(C) \mid A^h = A\} = \text{herm}(C^r).$$

Let  $\tilde{\Psi} : GL(r, \mathbb{R})/O(p, q) \xrightarrow{\sim} \text{Sym}_{p,q}(\mathbb{R}^r) \subset GL(r, \mathbb{R})$  be the identification obtained from the above mentioned action of  $GL(r, \mathbb{R})$  on  $\text{Sym}_{p,q}(\mathbb{R}^r)$ . With a similar argumentation one can prove (compare [Sch2]) that  $\tilde{\Psi}$  is a totally geodesic immersion. Summarizing the above information and the results of [Sch2], we have the commutative diagram:

$$\begin{array}{ccc} & \frac{GL(r, C)}{U^\pi(C^r)} & \xrightarrow{[i]} & \frac{GL(2r, \mathbb{R})}{O(r, r)} & \\ & \nearrow \tilde{h} & & \downarrow \tilde{\Psi} & \\ M & & & & \\ & \searrow h & & \downarrow \Psi & \\ & \text{Herm}(C^r) & \xrightarrow{\mathcal{R}} & \text{Sym}_{r,r}(\mathbb{R}^{2r}), & \end{array} \quad (3.6)$$

where  $[i]$  is induced by the inclusion  $i : GL(r, C) \hookrightarrow GL(2r, \mathbb{R})$ . Since all other maps in the square of this diagram are totally geodesic, the map

$$\mathcal{R} : \text{Herm}(C^r) \rightarrow \text{Sym}_{r,r}(\mathbb{R}^{2r}), H \mapsto \mathbb{1}_{r,r} \iota(H)$$

is a totally geodesic map.

**Notation:** In the following work we use the notations  $S(p, q) = GL(r, \mathbb{R})/O(p, q)$  and  $H(r) = GL(r, C)/U^\pi(C^r)$ .

Using the commutative diagram gives the proposition:

**Proposition 4** *A map  $h : M \rightarrow \text{Herm}(C^r)$  is para-pluriharmonic, if and only if  $g = \text{Re } h : M \rightarrow \text{Sym}_{r,r}(\mathbb{R}^{2r})$  is para-pluriharmonic.*

*A map  $\tilde{h} : M \rightarrow H(r) = GL(r, C)/U^\pi(C^r)$  is para-pluriharmonic, if and only if  $\tilde{g} = [i] \circ h : M \rightarrow S(r, r)$  is para-pluriharmonic.*

*Proof:* As discussed in Lemma 1, 2 and 3 of this section the map  $\mathcal{R} : \text{Herm}(C^r) \rightarrow \text{Sym}_{r,r}(\mathbb{R}^{2r})$  is totally geodesic and an immersion. This means that we are in the situation of proposition 1.

The second claim follows from the square of the commutative diagram (3.6) and from the information, that the composition of a map  $f$  from  $M$  to  $\text{Herm}(C^r)$  (respectively  $\text{Sym}_{r,r}(\mathbb{R}^{2r})$ ) with  $\Psi^{-1}$  (respectively  $\tilde{\Psi}^{-1}$ ) is para-pluriharmonic, if and only if  $f$  is para-pluriharmonic.  $\square$

## 4 Para- $tt^*$ -bundles and associated para-pluriharmonic maps

In this section we recall the notion of (metric) para- $tt^*$ -bundles and explain the correspondence between metric para- $tt^*$ -bundles and para-pluriharmonic maps, which was given in [Sch2].

**Definition 2** A para- $tt^*$ -bundle or  $ptt^*$ -bundle  $(E, D, S)$  over a para-complex manifold  $(M, \tau)$  is a real vector bundle  $E \rightarrow M$  endowed with a connection  $D$  and a section  $S \in \Gamma(T^*M \otimes \text{End } E)$  which satisfy the  $ptt^*$ -equation

$$R^\theta = 0 \quad \text{for all } \theta \in \mathbb{R}, \quad (4.1)$$

where  $R^\theta$  is the curvature tensor of the connection  $D^\theta$  defined by

$$D_X^\theta := D_X + \cosh(\theta)S_X + \sinh(\theta)S_{\tau X} \quad \text{for all } X \in TM. \quad (4.2)$$

A metric  $ptt^*$ -bundle  $(E, D, S, g)$  is a  $ptt^*$ -bundle  $(E, D, S)$  endowed with a possibly indefinite  $D$ -parallel fiber metric  $g$  such that for all  $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \text{for all } X, Y, Z \in T_p M. \quad (4.3)$$

**Remark 1** 1) If  $(E, D, S)$  is a  $ptt^*$ -bundle then  $(E, D, S^\theta)$  is a  $ptt^*$ -bundle for all  $\theta \in \mathbb{R}$ , where  $S^\theta := D^\theta - D = \cosh(\theta)S + \sinh(\theta)S_\tau$ . The same remark applies to metric  $ptt^*$ -bundles.

2) The flatness of the connection  $D^\theta$  can be expressed in a set of equations on  $D$  and  $S$  which can be found in proposition 7 of [Sch2].

Given a metric  $ptt^*$ -bundle  $(E, D, S, g)$ , we consider the flat connection  $D^\theta$  for a fixed  $\theta \in \mathbb{R}$ . Any  $D^\theta$ -parallel frame  $s = (s_1, \dots, s_r)$  of  $E$  defines a map

$$G = G^{(s)} : M \rightarrow \text{Sym}_{p,q}(\mathbb{R}^r); \quad x \mapsto G(x) := (g_x(s_i(x), s_j(x))), \quad (4.4)$$

where  $(p, q)$  is the signature of the metric  $g$ .

Let  $G/K$  be a pseudo-Riemannian symmetric space with associated symmetric decomposition  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ . We recall that a map  $f : (M, \tau) \rightarrow G/K$  is said to be **admissible**, if the para-complex linear extension of its differential maps  $T_x^{1,0}M$  (respectively  $T_x^{0,1}M$ ) to an Abelian subspace of  $\mathfrak{p}^C = \mathfrak{p} \otimes C$  for all  $x \in M$ .

If  $M$  is simply-connected then it was shown in [Sch2] Theorem 3, that  $G : M \rightarrow \text{Sym}_{p,q}(\mathbb{R}^r)$  is para-pluriharmonic and that it induces an admissible para-pluriharmonic map  $\tilde{G} : M \xrightarrow{G} \text{Sym}_{p,q}(\mathbb{R}^r) \xrightarrow{\sim} S(p, q)$ .

Conversely, we constructed in Theorem 4 of [Sch2] a metric  $tt^*$ -bundle  $(E = M \times \mathbb{R}^{2r}, D = \partial - S, S = d\tilde{G}, g = \langle G \cdot, \cdot \rangle_{\mathbb{R}^{2r}})$  over a simply-connected manifold from an admissible para-pluriharmonic map  $\tilde{G} = \tilde{\Psi}^{-1} \circ G : M \rightarrow S(p, q)$ . If  $M$  is not simply-connected, then we have to replace the maps  $G$  and  $\tilde{G}$  by twisted para-pluriharmonic maps (cf. [Sch2] Theorems 5 and 6).

## 5 Para-harmonic bundles as solutions of $ptt^*$ -geometries

In this section we introduce the notion of a para-harmonic bundle and show that every such bundle gives a solution of  $ptt^*$ -geometry.

**Definition 3** A para-harmonic bundle  $(E \rightarrow M, D, C, \bar{C}, h)$  consists of the following data: A para-complex vector bundle  $E$  over a para-complex manifold  $(M, \tau)$ , a para-hermitian metric  $h$ , i.e. a smooth fiberwise para-hermitian sesquilinear scalar product, a metric connection  $D$  with respect to  $h$  and two  $C^\infty$ -linear maps  $C : \Gamma(E) \rightarrow \Gamma(\Lambda^{1,0}T^*M \otimes E)$  and  $\bar{C} : \Gamma(E) \rightarrow \Gamma(\Lambda^{0,1}T^*M \otimes E)$ , such that the connection

$$D^{(\lambda)} = D + \lambda C + \bar{\lambda} \bar{C}$$

is flat for all  $\lambda \in \tilde{\mathbb{S}}^1$  and  $h(C_Z a, b) = h(a, \bar{C}_{\bar{Z}} b)$  with  $a, b \in \Gamma(E)$  and  $Z \in \Gamma(T^{1,0}M)$ .

**Theorem 1** Let  $(E \rightarrow M, D, C, \bar{C}, h)$  be a para-harmonic bundle over the para-complex manifold  $(M, \tau)$ , then  $(E, D, S, g = \text{Re}h)$ , with  $S_X := C_Z + \bar{C}_{\bar{Z}}$  for  $X = Z + \bar{Z} \in TM$  and  $Z \in T^{1,0}M$ , is a metric  $ptt^*$ -bundle.

*Proof:* For  $\lambda = \cosh(\alpha) + e \sinh(\alpha) \in \tilde{\mathbb{S}}^1$  we compute  $D^{(\lambda)}$  :

$$\begin{aligned} D_X^{(\lambda)} &= D_X + \lambda C_Z + \bar{\lambda} \bar{C}_{\bar{Z}} = D_X + \cosh(\alpha)(C_Z + \bar{C}_{\bar{Z}}) + \sinh(\alpha)(eC_Z - e\bar{C}_{\bar{Z}}) \\ &= D_X + \cosh(\alpha)S_X + \sinh(\alpha)(C_{\tau Z} + \bar{C}_{\tau \bar{Z}}) \\ &= D_X + \cosh(\alpha)S_X + \sinh(\alpha)S_{\tau X} = D_X^\alpha. \end{aligned}$$

Hence we have

$$D^\alpha = D^{(\lambda)} \tag{5.1}$$

and  $D^\alpha$  is flat if and only if  $D^{(\lambda)}$  is flat.

Further we show, that  $S$  is  $g$ -symmetric. With  $X = Z + \bar{Z}$  for  $Z \in T^{1,0}M$  one finds

$$h(S_X \cdot, \cdot) = h(C_Z + \bar{C}_{\bar{Z}} \cdot, \cdot) = h(\cdot, C_Z + \bar{C}_{\bar{Z}} \cdot) = h(\cdot, S_X \cdot)$$

and consequently the symmetry of  $S$  with respect to  $g = \text{Re}h$ . Finally a direct calculation shows  $Dg = 0$ . This proves, that  $(E, D, S, g = \text{Re}h)$  is a metric  $ptt^*$ -bundle.  $\square$

**Remark:** From the  $ptt^*$ -equations we know that  $R^D + S \wedge S = 0$  and that  $S \wedge S$  is of type  $(1, 1)$ . Hence the  $(0, 2)$ -part of  $R^D$  vanishes. By the para-complex generalization of a well-known theorem of complex geometry, the connection  $D$  induces a para-holomorphic structure such that  $D$  is adapted to this para-holomorphic structure (compare [LS] Proposition 2 and [E] for the case where  $M$  is a surface.). From  $Dh = 0$  we obtain that  $D$  is the para-complex analogue of the Chern connection, which was introduced in [E].

## 6 The para-pluriharmonic maps associated to a para-harmonic bundle

In the last section we have shown, that every para-harmonic bundle induces a metric  $ptt^*$ -bundle and hence a para-pluriharmonic map to  $S(r, r) = GL(2r, \mathbb{R})/O(r, r)$  where  $r = \text{rk}_C(E)$  is the para-complex rank of  $E$ . Afterwards, we use the additional information of the para-harmonic bundle structure to restrict the target of the para-pluriharmonic map to  $H(r) = GL(r, C)/U^\pi(C^r)$ . Finally we get a correspondence between para-harmonic bundles and admissible para-pluriharmonic maps into  $H(r)$ . Summarizing our current knowledge yields the corollary:

**Corollary 1** *Let  $(E \rightarrow M, D, C, \bar{C}, h)$  be a para-harmonic bundle over the simply connected para-complex manifold  $(M, \tau)$ , then the representation of  $g = \text{Re } h$  in a  $D^{(\lambda)}$ -flat frame defines an admissible para-pluriharmonic map  $\Phi_g : M \rightarrow S(r, r)$ .*

*Proof:* This follows from the identity (5.1), i.e.  $D_X^{(\lambda)} = D_X^\alpha$  for  $\lambda = \cosh(\alpha) + e \sinh(\alpha) \in \tilde{\mathbb{S}}^1$  and from Theorem 3 of [Sch2].  $\square$

To restrict the image of  $\Phi_g$  to  $H(r)$  we analyze the real-part of  $h$  by using proposition 4.

**Theorem 2** *Let  $(E \rightarrow M, D, C, \bar{C}, h)$  be a para-harmonic bundle over the simply connected para-complex manifold  $(M, \tau)$ . Then the representation of  $h$  in a  $D^{(\lambda)}$ -flat frame defines a para-pluriharmonic map  $\phi_h : M \rightarrow \text{Herm}(C^r)$  which itself induces an admissible para-pluriharmonic map  $\tilde{\phi}_h = \Psi^{-1} \circ \phi_h : M \rightarrow H(r)$ .*

*Proof:* The para-pluriharmonicity of  $\phi_h$  and  $\tilde{\phi}_h$  follows from corollary 1 and proposition 4. For the second part we observe, that the differential of  $\mathcal{R} : \mathfrak{gl}_r(C) \rightarrow \mathfrak{gl}_{2r}(\mathbb{R})$  is a homomorphism of Lie-algebras and therefore preserves the vanishing of the Lie-bracket.  $\square$

The following theorem gives the converse statement:

**Theorem 3** *Let  $(M, \tau)$  be a simply connected para-complex manifold and  $E = M \times C^r$ . An admissible para-pluriharmonic map  $\tilde{\phi}_h : M \rightarrow H(r)$  induces an admissible para-pluriharmonic map  $\tilde{\phi}_g = [i] \circ \tilde{\phi}_h : M \rightarrow S(r, r)$ . Moreover, the admissible para-pluriharmonic map  $\tilde{\phi}_h$  defines a para-harmonic bundle  $(E, D = \partial - C - \bar{C}, C = (d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{C^r})$ , where  $\partial$  is the para-complex linear extension on  $TM^C$  of the flat connection on  $E = M \times C^r$ .*

*Proof:* Thanks to proposition 4 the map  $\tilde{\phi}_g$  is para-pluriharmonic. The same argument as in theorem 2 shows that  $\tilde{\phi}_g$  is admissible. Therefore we obtain from Theorem 4 of [Sch2] a  $ptt^*$ -bundle  $(E = M \times \mathbb{R}^{2r}, D = \partial - S, S = d\tilde{\phi}_g, g = \langle \phi_g \cdot, \cdot \rangle_{\mathbb{R}^{2r}})$ . Utilizing the additional information, which we have from the fact, that the map  $\phi_g$  comes from  $\phi_h$ , we

show that  $(E, D = \partial - C - \bar{C}, C = (d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{C^r})$  is a para-harmonic bundle. The para-hermitian metric  $h$  is defined by

$$h = g + e\omega$$

with  $\omega = g(E \cdot, \cdot)$ . This is the standard relation between para-hermitian sesquilinear scalar products on para-complex vector spaces and the para-hermitian scalar products on the underlying real vector spaces.

It is  $D_X E = [\partial_X - S_X, E] = [S_X, E] = 0$  with  $X \in \Gamma(TM)$ , since  $S$  is the differential of a map from  $M$  to  $GL(r, C)$  and consequently commutes with  $E$ . Hence,  $D\omega = 0$  follows from  $Dg = 0$  and  $Dh = 0$  from  $D\omega = 0$  and  $Dg = 0$ .

The definition of  $S$  and  $S_\tau$  in theorem 1, i.e.  $S_X = C_Z + \bar{C}_{\bar{Z}}$  and  $S_{\tau X} = C_{\tau Z} + \bar{C}_{\tau \bar{Z}}$  for  $X = Z + \bar{Z}$  and  $Z \in T^{1,0}M$ , yields the definition of  $C$  by  $2C_Z = S_X + ES_{\tau X}$  and  $2C_{\bar{Z}} = S_X - ES_{\tau X}$ . The identity  $D_X^{(\lambda)} = D_X^\alpha$  for  $\lambda = \cosh(\alpha) + e \sinh(\alpha) \in \tilde{S}^1$  implies further the equivalence between the flatness of  $D^{(\lambda)}$  and  $D^\alpha$ .

We recall the relations  $E^*g = -g$  and  $(*)g(E \cdot, \cdot) = -g(\cdot, E \cdot)$ , which implies the anti-symmetry of  $\omega = g(E \cdot, \cdot)$  and  $(*)\omega(E \cdot, \cdot) = -\omega(\cdot, E \cdot)$ . Further we use the identities  $(**) [S, E] = [S_\tau, E] = 0$  and that  $(***)S, S_\tau$  are  $g$ -symmetric. Due to  $(**)$  and  $(***)$  we obtain:  $(***S, S_\tau)$   $\omega$ -symmetric. These identities imply after a short computation  $h(C_Z \cdot, \cdot) = h(\cdot, \bar{C}_{\bar{Z}} \cdot)$ .

Using  $S = -d\tilde{\phi}_g = -d([i] \circ \tilde{\phi}_h) = -d\tilde{\phi}_h$  we find extending  $S$  on  $TM^C$  to  $S^C$  for  $Z \in T^{1,0}M$  the equations  $C_Z = S_Z^C = -d\tilde{\phi}_h(Z)$  and  $\bar{C}_{\bar{Z}} = -d\tilde{\phi}_h(\bar{Z})$ .  $\square$

One can obtain similar results if the manifold  $(M, \tau)$  is not simply connected. In fact, one has to replace the para-pluriharmonic maps by twisted para-pluriharmonic maps and to apply Theorems 5 and 6 of [Sch2] instead of Theorems 3 and 4 of the same reference.

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