



# Quadratic forms and singularities of genus one or two

Georges Dloussky

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# Quadratic forms and singularities of genus one or two

Georges Dloussky\*

## Abstract

We study singularities obtained by the contraction of the maximal divisor in compact (non-kählerian) surfaces which contain global spherical shells. These singularities are of genus 1 or 2, may be  $\mathbb{Q}$ -Gorenstein, numerically Gorenstein or Gorenstein. A family of polynomials depending on the configuration of the curves computes the discriminants of the quadratic forms of these singularities. We introduce a multiplicative branch topological invariant.

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## 0 Introduction

We are interested in a large class of singularities which generalize cusps, obtained by the contraction of all the rational curves in compact surfaces  $S$  which contain global spherical shells. Particular cases are Inoue-Hirzebruch surfaces with two

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“dual” cycles of rational curves. The duality can be explained by the construction of these surfaces by sequences of blowing-ups [3]. Several authors have studied cusps [8, 10, 18, 19, 14]. In general, the maximal divisor is composed of a cycle with branches. These (non-kählerian) surfaces contain exactly  $n = b_2(S)$  rational curves. The intersection matrices  $M(S)$  have been completely classified [17, 1]; they are negative definite but there is a cycle  $D$  of  $n$  rational curves such that  $D^2 = 0$ . In this article we study normal singularities obtained by contraction of the exceptional divisor and the link between the intersection matrix and global topological or analytical properties of the surface  $S$ . They are elliptic or of genus two in which case they are Gorenstein. Using the existence of global section on  $S$  of  $-mK_S \otimes L$  for a suitable integer  $m \geq 1$  and a flat line bundle  $L \in H^1(S, \mathbb{C}^*)$ , we show that these singularities are  $\mathbb{Q}$ -Gorenstein (resp. numerically Gorenstein) if and only if the global property  $H^0(S, -mK_S) \neq 0$  (resp.  $H^0(S, -K_S \otimes L) \neq 0$ ) holds. The main part of this article is devoted to the study of the discriminant of the quadratic form associated to the singularity. In [1] the quadratic form has been decomposed into a sum of squares. The intersection matrix is completely determined by the sequence of (opposite) self-intersections of the rational curves  $\sigma$  when taken in the canonical order, i.e. the order in which the curves are obtained in a repeated sequence of blowing-ups. Let  $X_\sigma$  be the associated singularity obtained by the contraction of the rational curves. We introduce a family of polynomials  $P_\sigma$  which have integer values on integers, depending on the configuration of the dual graph of the singularity, such that the discriminant is the square of this polynomial. We obtain a multiplicative topological invariant  $\Delta(X_\sigma)$ :  $\Delta(X_{\sigma\sigma'}) = \Delta(X_\sigma)\Delta(X_{\sigma'})$ . We show that it may be computed as the product of the determinants of the branches. We develop here rather the algebraic point of view, however these singularities have deep relations with properties of compact complex surfaces  $S$  containing global spherical shells, the classification of singular contracting germs of mappings and dynamical systems: for instance, the integer  $\Delta(X_\sigma)$  is equal to the integer  $k = k(S)$  which appears in the normal form of contracting germs  $F(z_1, z_2) = (\lambda z_1 z_2^s + P(z_2), z_2^k)$  which define  $S$  [2, 4, 5, 7]. I thank Karl Oeljeklaus for fruitful discussions on that subject.

## 1 Preliminaries

### 1.1 Basic results on singularities

Let  $D_0, \dots, D_{n-1}$  be compact curves on a (not necessarily compact) complex surface  $X$ , such that  $|D| = \cup D_i$  is connected and the intersection matrix  $M$  is negative definite. We denote by  $\mathcal{O}_X$  the structural sheaf of  $X$ ,  $K_X = \det T^*X$  the canonical bundle and by  $\Omega_X^2$  its sheaf of sections. It is well known by Grauert’s theorem that there exists a proper mapping  $\Pi : X \rightarrow \bar{X}$  such that the curves are contracted onto a point  $x$  which is a normal singularity of  $\bar{X}$ . Denote by

$$r : H^0(X, \Omega_X^2) \rightarrow H^0(\bar{X} \setminus \{x\}, \Omega_{\bar{X} \setminus \{x\}}^2)$$

the canonical morphism induced by  $\Pi$ , then we define the **geometric genus** of the singularity  $(\bar{X}, x)$  by

$$p_g = p_g(\bar{X}, x) = h^0(\bar{X}, R^1 \Pi_* \mathcal{O}_X).$$

We have  $p_g = \dim H^0(\bar{X} \setminus \{x\}, \Omega_{\bar{X} \setminus \{x\}}^2) / r H^0(X, \Omega_X^2)$ .

A singularity  $(\bar{X}, x)$  is called **rational** (resp. **elliptic**) if  $p_g(\bar{X}, x) = 0$  (resp.  $p_g(\bar{X}, x) = 1$ ). Therefore a singularity is rational if every 2-form on  $\bar{X} \setminus \{x\}$ , extends to a 2-form on  $X$ .

**Proposition 1. 1** 1)  $p_g$  is independant of the choice of the desingularization.  
 2) We have  $p_g = \chi(\mathcal{O}_{\bar{X}}) - \chi(\mathcal{O}_X)$   
 3) The following sequence

$$0 \rightarrow H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^0(\bar{X}, R^1\Pi_*\mathcal{O}_X) \rightarrow H^2(\bar{X}, \mathcal{O}_{\bar{X}}) \rightarrow H^2(X, \mathcal{O}_X)$$

is exact. In particular, if  $\bar{X}$  is a Stein neighbourhood of  $x$ ,  $p_g = h^1(X, \mathcal{O}_X)$ .

We give now a criterion of rationality [20], p. 152:

**Proposition 1. 2** Let  $\Pi : X \rightarrow \bar{X}$  be the minimal resolution of the singularity  $(\bar{X}, x)$  and denote by  $D_i$  the irreducible components of the exceptional divisor  $D$ . If  $(\bar{X}, x)$  is rational, then:

- i) the curves  $D_i$  are regular and rational
- ii) for  $i \neq j$ ,  $D_i$  meets  $D_j$  transversally and if  $D_i, D_j, D_k$  are distinct irreducible components,  $D_i \cap D_j \cap D_k$  is empty
- iii)  $D$  contains no cycle.

**Definition 1. 3** A singularity  $(\bar{X}, x)$  is called **Gorenstein** if the dualizing sheaf  $\omega_{\bar{X}}$  is trivial, i.e. there exists a neighbourhood  $\bar{U}$  of  $x$  and a non-vanishing holomorphic 2-form on  $\bar{U} \setminus \{x\}$ .

Since there is only a finite number of linearly independant 2-forms in the complement of the exceptional divisor  $D$  modulo  $H^0(S, \Omega^2)$ , a 2-form extends meromorphically across  $D$ . Therefore we have (see [22])

**Lemma 1. 4** Let  $\bar{X}$  be a Gorenstein normal surface and  $\Pi : X \rightarrow \bar{X}$  be the minimal desingularization. Then there is a unique effective divisor  $\Delta$  on  $X$  supported by  $D = \Pi^{-1}(\text{Sing}(X))$  such that

$$\omega_X \simeq \Pi^*\omega_{\bar{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\Delta)$$

## 1.2 Lattices

Here are recalled some well known facts about lattices (see [23]). We call lattice, denoted by  $(L, \langle \cdot, \cdot \rangle)$ , a free  $\mathbf{Z}$ -module  $L$ , endowed with an integral non degenerate symmetric bilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle : L \times L &\longrightarrow \mathbf{Z} \\ (x, y) &\longmapsto \langle x, y \rangle. \end{aligned}$$

If  $B = \{e_1, \dots, e_n\}$  is a basis of  $L$ , the determinant of the matrix

$$(\langle e_i, e_j \rangle)_{1 \leq i, j \leq n},$$

is independent of the choice of the base; this integer, denoted by  $d(L)$  is called the discriminant of the lattice. A lattice is unimodular if  $d(L) = \pm 1$ . Let  $L^\vee := \text{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$  be the dual of  $L$ . The mapping

$$\begin{aligned} \phi : L &\longrightarrow L^\vee \\ x &\longmapsto \langle \cdot, x \rangle \end{aligned}$$

identifies  $L$  with a sublattice of  $L^\vee$  of same rank, since  $d(L) \neq 0$ . Moreover, if  $L_\mathbb{Q} := L \otimes_\mathbb{Z} \mathbb{Q}$ , it is possible to identify  $L^\vee$  with the sub- $\mathbb{Z}$ -module

$$\{x \in L_\mathbb{Q} \mid \forall y \in L, \langle x, y \rangle \in \mathbb{Z}\}$$

of  $L_\mathbb{Q}$ . So, we may write  $L \subset L^\vee \subset L_\mathbb{Q}$ , where  $L$  and  $L^\vee$  have same rank.

**Lemma 1. 5** 1) *The index of  $L$  in  $L^\vee$  is  $|d(L)|$ .*

2) *If  $M$  is a submodule of  $L$  of the same rank, then the index of  $M$  in  $L$  satisfies*

$$[L : M]^2 = d(M) d(L)^{-1}.$$

*In particular  $d(M)$  and  $d(L)$  have same sign.*

### 1.3 Surfaces with global spherical shells

We recall some properties of these surfaces which have been first introduced by Ma. Kato [11] and we refer to [1] for details.

**Definition 1. 6** *Let  $S$  be a compact complex surface. We say that  $S$  contains a global spherical shell, if there is a biholomorphic map  $\varphi : U \rightarrow S$  from a neighbourhood  $U \subset \mathbb{C}^2 \setminus \{0\}$  of the sphere  $S^3$  into  $S$  such that  $S \setminus \varphi(S^3)$  is connected.*

Hopf surfaces are the simplest examples of surfaces with GSS (see [1]), however they contain no rational curves and elliptic curves have self-intersection equal to 0, hence no singularity can be obtained.

Let  $S$  be a minimal surface containing a GSS with  $n = b_2(S)$ . It is known that  $S$  contains  $n$  rational curves and to each curve it is possible to associate a contracting germ of mapping  $F = \Pi\sigma = \Pi_0 \cdots \Pi_{n-1}\sigma : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  where  $\Pi = \Pi_0 \cdots \Pi_{n-1} : B^\Pi \rightarrow B$  is a sequence of  $n$  blowing-ups. If we want to obtain a minimal surface, the sequence of blowing-ups has to be done in the following way:

- $\Pi_0$  blows up the origin of the two dimensional unit ball  $B$ ,
- $\Pi_1$  blows up a point  $O_0 \in C_0 = \Pi_0^{-1}(0), \dots$
- $\Pi_{i+1}$  blows up a point  $O_i \in C_i = \Pi_i^{-1}(O_{i-1})$ , for  $i = 0, \dots, n-2$ , and
- $\sigma : \bar{B} \rightarrow B^\Pi$  sends isomorphically a neighbourhood of  $\bar{B}$  onto a small ball in  $B^\Pi$  in such a way that  $\sigma(0) \in C_{n-1}$ .

It is easy to see that the homological groups satisfy

$$H_1(S, \mathbb{Z}) \simeq \mathbb{Z}, \quad H_2(S, \mathbb{Z}) \simeq \mathbb{Z}^n$$

In particular,  $b_2(S) = n$ .

The universal covering space  $(\tilde{S}, \omega, S)$  of  $S$  contains only rational curves  $(C_i)_{i \in \mathbb{Z}}$  with a canonical order relation, “the order of creation” ([1], p 29). Following [1], we can associate to  $S$  the following invariants:

- The family of opposite self-intersection of curves of the universal covering space of  $S$ , denoted by

$$a(S) := (a_i)_{i \in \mathbb{Z}} = (-C_i^2)_{i \in \mathbb{Z}}$$

this family is periodic of period  $n$ ,

•

$$\sigma_n(S) := \sum_{i=j}^{j+n-1} a_i = - \sum_{i=0}^{n-1} D_i^2 + 2 \# \{ \text{rational curves with nodes} \}$$

where  $j$  is any index, and the  $D_i$  are the rational curves of  $S$ . It can be seen that  $2n \leq \sigma_n(S) \leq 3n$  ([1], p 43).

- The intersection matrix of the  $n$  rational curves of  $S$ ,

$$M(S) := (D_i \cdot D_j).$$

**Important Remark:** The essential fact to understand the dual graph or the intersection matrix is that

- if  $a_i = -D_i^2 = 2$  then  $D_i$  meets  $D_{i+1}$ ,
- if  $a_i = -D_i^2 = 3$  then  $D_i$  meets  $D_{i+2}, \dots$ ,
- if  $a_i = -D_i^2 = k + 2$  then  $D_i$  meets  $D_{i+k+1}$ ,

the indices beeing in  $\mathbb{Z}/n\mathbb{Z}$ , in particular  $D_i$  may meet itself: we obtain a rational curve with double point.

- $n$  classes of contracting holomorphic germs of mappings  $F = \Pi\sigma : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  ([1], p 32),
- The trace of the surface  $S$ ,

$$tr(S) := trace DF(0)$$

where  $F = \Pi\sigma$ . We know [1], p 44 that  $0 \leq |tr(S)| < 1$  and  $tr(S) \neq 0$  if and only if  $\sigma_n(S) = 2n$ .

**Proposition 1. 7** *Let  $S$  be a surface containing a GSS with  $b_2(S) = n$ ,  $D_0, \dots, D_{n-1}$  the  $n$  rational curves and  $M(S)$  the intersection matrix.*

1) *If  $\sigma_n(S) = 2n$ , then  $detM(S) = 0$ .*

2) *If  $\sigma_n(S) > 2n$ , then  $\sum_{0 \leq i \leq n-1} \mathbb{Z}D_i$  is a complete sublattice of  $H_2(S, \mathbb{Z})$  and its index satisfies*

$$[H_2(S, \mathbb{Z}) : \sum_{0 \leq i \leq n-1} \mathbb{Z}D_i]^2 = detM(S).$$

*In particular,  $detM(S)$  is the square of an integer  $\geq 1$ .*

**Proof:** If  $\sigma_n(S) = 2n$ ,  $S$  is an Inoue surface; if  $\sigma_n(S) > 2n$ ,  $detM(S) \neq 0$  so the sublattice is complete and the result is a mere consequence of lemma 5.  $\square$

In order to give a precise description of the intersection matrix we need the following definitions:

**Definition 1. 8** *Let  $1 \leq p \leq n$ . A  $p$ -uple  $\sigma = (a_i, \dots, a_{i+p-1})$  of  $a(S)$  is called*

- *a singular  $p$ -sequence of  $a(S)$  if*

$$\sigma = (\underbrace{p+2, 2, \dots, 2}_p).$$

*It will be denoted by  $s_p$ .*

- a regular  $p$ -sequence of  $a(S)$  if

$$\sigma = \underbrace{(2, 2, \dots, 2)}_p$$

and  $\sigma$  has no common element with a singular sequence. Such a  $p$ -uple will be denoted by  $r_p$ .

For example  $s_1 = (3), s_2 = (4, 2), s_3 = (5, 2, 2), \dots$  are singular sequences,  $r_3 = (2, 2, 2)$  is a regular sequence. It is easy to see that if we want to have, for example, a curve with self-intersection -4, necessarily, the curve which follows in the sequence of repeated blowing-ups must have self-intersection -2, so it is easy to see ([1], p39), that  $a(S)$  admits a unique partition by  $N$  singular sequences and  $\rho \leq N$  regular sequences of maximal length. More precisely, since  $a(S)$  is periodic it is possible to find a  $n$ -uple  $\sigma$  such that

$$\sigma = \sigma_{p_0} \cdots \sigma_{p_{\rho+N-1}},$$

where  $\sigma_{p_i}$  is a regular or a singular  $p_i$ -sequence with

$$\sum_{i=0}^{N+\rho-1} p_i = n$$

and if  $\sigma_{p_i}$  is regular it is between (mod.  $N + \rho$ ) two singular sequences.

**Notation:** We shall write

$$a(S) = (\bar{\sigma}) = (\overline{\sigma_{p_0} \cdots \sigma_{p_{N+\rho-1}}}).$$

The sequence  $\sigma$  is overlined to indicate that the sequence  $\sigma$  is infinitely repeated to obtain the sequence  $a(S) = (a_i)_{i \in \mathbb{Z}}$ . The sequence  $a(S)$  may be defined by another period. For example

$$a(S) = (\overline{\sigma_{p_1} \cdots \sigma_{p_{N+\rho-1}} \sigma_{p_0}}).$$

If  $\sigma_n(S) = 2n$ ,  $a(S) = (\bar{r}_n)$ ; if  $\sigma_n(S) = 3n$ ,  $a(S)$  is only composed of singular sequences and  $S$  is called a Inoue-Hirzebruch surface. Moreover if  $a(S)$  is composed by the repetition of an even (resp. odd) number of sequences  $\sigma_{p_i}$ , we shall say that  $S$  is an even (resp. odd) Inoue-Hirzebruch surface. An even (resp. odd) Inoue-Hirzebruch surface has exactly 2 cycles (resp. 1 cycle) of rational curves. Another used terminology is respectively hyperbolic Inoue surface and half Inoue surface.

We recall that for any  $VII_0$ -class surface without non constant meromorphic functions, the numerical characters of  $S$  are [10, I p755, II p683]

$$h^{0,1} = 1, h^{1,0} = h^{2,0} = h^{0,2} = 0, -c_1^2 = c_2 = b_2(S), b_2^+ = 0, b_2^- = b_2(S)$$

We shall need in the sequel the explicit description of the dual graph which is composed of a cycle and branches. A branch  $A_s$  determines and is determined by a piece  $\Gamma_s$  of the cycle  $\Gamma$ .

**Theorem 1. 9 ([1] thm 2.39)** *Let  $S$  be a minimal surface containing a GSS,  $n = b_2(S)$ ,  $D_0, \dots, D_{n-1}$  its  $n$  rational curves and  $D = D_0 + \dots + D_{n-1}$ .*

- 1) *If  $\sigma_n(S) = 2n$ , then  $D$  is a cycle and  $D_i^2 = -2$  for  $i = 0, \dots, n-1$ .*
- 2) *If  $2n < \sigma_n(S) < 3n$ , then there are  $\rho(S) \geq 1$  branches and*

$$D = \sum_{s=1}^{\rho(S)} (A_s + \Gamma_s)$$

where

i)  $A_s$  is a branch for  $s = 1, \dots, \rho(S)$ ,

ii)  $\Gamma = \sum_{s=1}^{\rho(S)} \Gamma_s$  is a cycle,

iii)  $A_s$  and  $\Gamma_s$  are defined in the following way: For each sequence

$$(a_{t+1}, \dots, a_{t+l+k_1+\dots+k_p+2}) = (r_l s_{k_1} \dots s_{k_p} 2a_{t+l+k_1+\dots+k_p+2})$$

contained in  $a(S) = (\overline{\sigma_0 \dots \sigma_{N+\rho-1}})$ , where  $l \geq 1$ ,  $p \geq 1$ ,  $k_1 \geq 1, \dots, k_p \geq 1$ ,

$$\left\{ \begin{array}{l} \text{Selfint}(A_s) = (\underbrace{2, \dots, 2}_{k_1-1}, k_2+2, \underbrace{2, \dots, 2}_{k_3-1}, \dots, k_{p-1}+2, \underbrace{2, \dots, 2}_{k_p-1}, 2) \\ \text{Selfint}(\Gamma_s) = (\underbrace{2, \dots, 2}_{l-1}, k_1+2, \underbrace{2, \dots, 2}_{k_2-1}, \dots, k_{p-2}+2, \underbrace{2, \dots, 2}_{k_{p-1}-1}, k_p+2) \end{array} \right. \quad \text{If } p \equiv 1 \pmod{2}$$

$$\left\{ \begin{array}{l} \text{Selfint}(A_s) = (\underbrace{2, \dots, 2}_{k_1-1}, k_2+2, \underbrace{2, \dots, 2}_{k_3-1}, \dots, k_{p-2}+2, \underbrace{2, \dots, 2}_{k_{p-1}-1}, k_p+2) \\ \text{Selfint}(\Gamma_s) = (\underbrace{2, \dots, 2}_{l-1}, k_1+2, \underbrace{2, \dots, 2}_{k_2-1}, \dots, k_{p-1}+2, \underbrace{2, \dots, 2}_{k_p-1}, 2) \end{array} \right. \quad \text{If } p \equiv 0 \pmod{2}$$

iv) The top of the branch  $A_s$  is its first vertex (or curve); the root of  $A_s$  is the first vertex (or curve) of  $\Gamma_t$  where  $t = s+1 \pmod{\rho(S)}$ .

3) If  $\sigma_n(S) = 3n$ ,  $D$  has no branch and

i) If  $a(S) = (\overline{s_{k_1} \dots s_{k_{2p}}})$  then

$$D = \Gamma + \Gamma'$$

where  $\Gamma$  and  $\Gamma'$  are two cycles

$$\left\{ \begin{array}{l} \text{Selfint}(\Gamma) = (k_1+2, \underbrace{2, \dots, 2}_{k_2-1}, k_3+2, \underbrace{2, \dots, 2}_{k_4-1}, \dots, k_{2p-1}+2, \underbrace{2, \dots, 2}_{k_{2p}-1}) \\ \text{Selfint}(\Gamma') = (\underbrace{2, \dots, 2}_{k_1-1}, k_2+2, \underbrace{2, \dots, 2}_{k_3-1}, k_4+2, \dots, \underbrace{2, \dots, 2}_{k_{2p-1}-1}, k_{2p}+2) \end{array} \right.$$

ii) If  $a(S) = (\overline{s_{k_1} \dots s_{k_{2p+1}}})$  then  $D$  contains only one cycle and

$$\begin{aligned} \text{Selfint}(D) = & (k_1+2, \underbrace{2, \dots, 2}_{k_2-1}, k_3+2, \underbrace{2, \dots, 2}_{k_4-1}, \dots, k_{2p+1}+2, \\ & \underbrace{2, \dots, 2}_{k_1-1}, k_2+2, \underbrace{2, \dots, 2}_{k_3-1}, \dots, k_{2p}+2, \underbrace{2, \dots, 2}_{k_{2p+1}-1}) \end{aligned}$$



## 1.4 Intersection matrix of the exceptional divisor

Let  $\sigma = \sigma_0 \cdots \sigma_{N+\rho-1}$  where  $\sigma_i = r_{p_i} = (2, 2, \dots, 2)$  is a regular sequence of length  $p_i$  or  $\sigma_i = s_{p_i} = (p_i + 2, 2, \dots, 2)$  is a singular sequence of length  $p_i$ ,  $i = 0, \dots, N + \rho - 1$ . We suppose that

- there are  $N$  singular sequences and  $\rho \leq N$  regular sequences
- if  $\sigma_i$  is regular, then  $\sigma_{i-1}$  and  $\sigma_{i+1}$  are singular, indices being in  $\mathbb{Z}/(N+\rho)\mathbb{Z}$ .

Let  $n = \sum_{i=0}^{N+\rho-1} p_i$  be the number of integers in the sequence  $\sigma$ .

**Examples 1. 10** For  $N \geq 0$  we have the following possible sequences:

- If  $N = 0$ ,  $\sigma = r_n$ ,
- If  $N = 1$ ,  $\sigma = s_n$  or  $\sigma = s_p r_m$ ,  $p + m = n$ ,
- If  $N = 2$ ,  $\sigma = s_{p_0} s_{p_1}$ ,  $\sigma = s_{p_0} s_{p_1} r_{m_0}$ ,  $\sigma = s_{p_0} r_{m_0} s_{p_1}$ ,  $\sigma = s_{p_0} r_{m_0} s_{p_1} r_{m_1}$ ,
- If  $N = 3$ ,  $\sigma = s_{p_0} s_{p_1} s_{p_2}$   
 $\sigma = s_{p_0} r_{m_0} s_{p_1} s_{p_2}$ ,  $\sigma = s_{p_0} s_{p_1} r_{m_0} s_{p_2}$ ,  $\sigma = s_{p_0} s_{p_1} s_{p_2} r_{m_0}$ ,  
 $\sigma = s_{p_0} r_{m_0} s_{p_1} r_{m_1} s_{p_2}$ ,  $\sigma = s_{p_0} s_{p_1} r_{m_0} s_{p_2} r_{m_1}$ ,  $\sigma = s_{p_0} r_{m_0} s_{p_1} s_{p_2} r_{m_1}$ ,  
 $\sigma = s_{p_0} r_{m_0} s_{p_1} r_{m_1} s_{p_2} r_{m_2}$ .

To a sequence  $\sigma$  we associate a symmetric matrix of type  $(n, n)$ ,  $M(\sigma) = (m_{ij})$  “written on a torus”, with indices in  $\mathbb{Z}/n\mathbb{Z}$  defined in the following way: if  $\sigma = \sigma_0 \cdots \sigma_{N+\rho-1} = (a_0, \dots, a_{n-1})$

$$\text{i) } m_{ii} = \begin{cases} a_i & \text{if } a_i \neq n+1 \\ n-1 & \text{if } a_i = n+1 \end{cases}$$

ii) For  $0 \leq i < j \leq n-1$ ,

$$m_{ij} = m_{ji} = \begin{cases} -2 & \text{if } j = i + m_{ii} - 1 \text{ and } i = j + m_{jj} - 1 \pmod{n} \\ -1 & \text{if } j = i + m_{ii} - 1 \text{ or else } i = j + m_{jj} - 1 \pmod{n} \\ 0 & \text{in all other cases} \end{cases}$$

**Theorem 1. 11 (D1,N1)** 1) Let  $S$  be a minimal complex compact surface containing a GSS with  $n = b_2(S) > 0$ , then  $S$  contains  $n$  rational curves  $D_0, \dots, D_{n-1}$  and there exists  $\sigma$  such that the intersection matrix  $M(S)$  of the rational curves in  $S$  satisfy

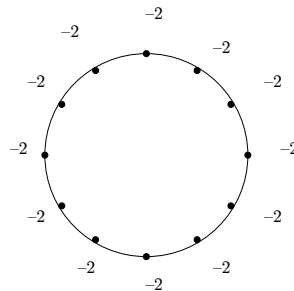
$$M(S) = -M(\sigma).$$

Moreover the curve  $D_i$  is non-singular if and only if  $a_i \neq n+1$ .

Conversely, for any  $\sigma$  there exists a surface  $S$  containing a GSS such that  $M(S) = -M(\sigma)$ .

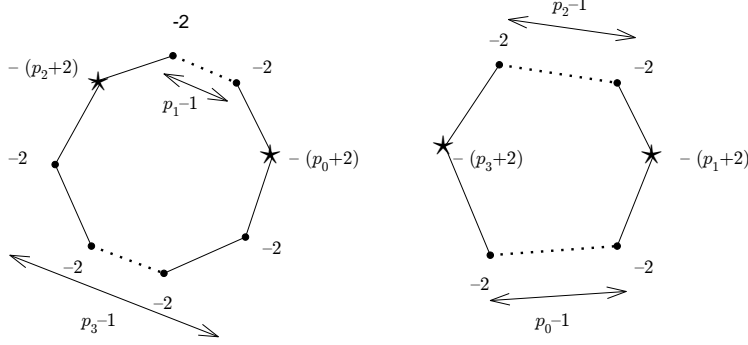
2) For any  $\sigma \neq r_n$ ,  $M(\sigma)$  is positive definite.

**Examples 1. 12** 1) For  $\sigma = r_n$ ,  $M(\sigma)$  is not positive definite. The dual graph of the curves has  $n$  vertices

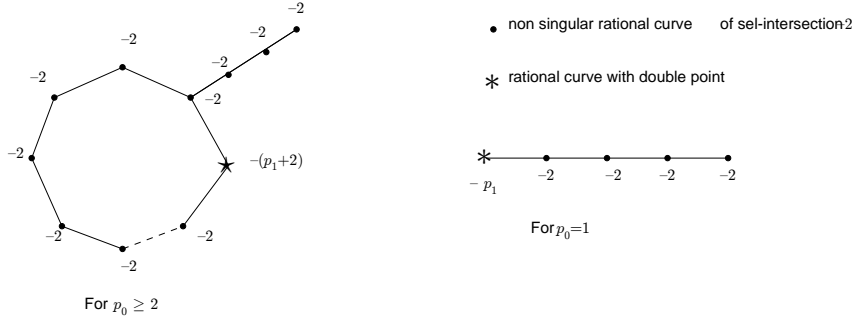


This configuration of curves appear on Enoki surfaces [6, 16, 1].

2) If  $\sigma = s_{p_0} \cdots s_{p_{N-1}}$  we obtain respectively one or two cycles if  $N$  is odd (resp. even). The singularities are cusps and surfaces are odd (resp. even) Inoue-Hirzebruch surfaces [9, 16, 1]. When there are two cycles, one of the two cycles determines the other. For example, if  $\sigma = s_{p_0} s_{p_1} s_{p_2} s_{p_3}$ , we obtain a cycle with  $p_1 + p_3$  curves and another with  $p_0 + p_2$  curves.



3) The intermediate case [16, 1, 4]. There are branches and the number of branches is equal to the number of regular sequences in  $\sigma$ . For example, if  $\sigma = r_{p_0} s_{p_1}$  the dual graph is



## 2 Normal singularities associated to surfaces with GSS

### 2.1 Genus of the singularities

If  $S$  is a Inoue-Hirzebruch surface we obtain by contraction of a cycle, a singularity called a cusp. They appear also in the compactification of Hilbert modular surfaces [8]. We are interested here in the general situation of any surface containing a GSS.

**Proposition 2.13** *Let  $S$  be a compact complex surface of class  $VII_0$  without non constant meromorphic. It is supposed that  $n := b_2(S) > 0$ , the maximal divisor  $D$  is not trivial and the intersection matrix  $M(S)$  is negative definite. Denote by  $\Pi : S \rightarrow \bar{S}$  the contraction of the curves onto isolated singular points. Then the following properties are equivalent:*

- i)  $D$  contains a cycle of rational curves;
- ii)  $H^1(\bar{S}, \mathcal{O}_{\bar{S}}) = 0$ .

Proof: **i)  $\Rightarrow$  ii)** By Proposition 1,

$$(*) \quad 0 \rightarrow H^1(\bar{S}, \mathcal{O}_{\bar{S}}) \rightarrow H^1(S, \mathcal{O}_S) \rightarrow H^0(\bar{S}, R^1\Pi_*\mathcal{O}_S) \rightarrow H^2(\bar{S}, \mathcal{O}_{\bar{S}}) \rightarrow 0.$$

If  $D$  contains a cycle then  $h^0(\bar{S}, R^1\Pi_*\mathcal{O}_S) \geq 1$ . We suppose that  $h^1(\bar{S}, \mathcal{O}_{\bar{S}}) = 1$  and we shall derive a contradiction. With these assumptions,  $h^0(\bar{S}, \omega_{\bar{S}}) = h^2(\bar{S}, \mathcal{O}_{\bar{S}}) = 1$  and  $h^0(\bar{S}, R^1\Pi_*\mathcal{O}_S) = 1$  since  $\bar{S}$  has no non constant meromorphic functions. If  $x_i$ ,  $i = 0, \dots, p$  are the singular points of  $\bar{S}$ ,  $\Gamma_i = \Pi^{-1}(x_i)$  and  $p_g(S, x_i)$  the geometric genus of  $(S, x_i)$ ,  $\sum p_g(S, x_i) = h^0(\bar{S}, R^1\Pi_*\mathcal{O}_S) = 1$ , therefore there are Gorenstein rational singular points and one elliptic Gorenstein singular point, i.e. rational double points with trivial canonical divisor and one minimal elliptic singularity,  $(S, x_0)$  with canonical divisor  $\Gamma_0$ . Since there is a global meromorphic 2-form on  $S$ ,  $n = -K_S^2 = -\Gamma_0^2$ . By [16],  $S$  is a odd Inoue-Hirzebruch surface (i.e. with one cycle); but such a surface has no canonical divisor (see for example [3])... a contradiction.

**ii)  $\Rightarrow$  i)** By the exact sequence  $(*)$ ,  $h^0(\bar{S}, R^1\Pi_*\mathcal{O}_S) \leq 2$  without any assumption and  $1 \leq h^0(\bar{S}, R^1\Pi_*\mathcal{O}_S)$  by ii). Therefore there is a singular point, say  $(S, x_0)$  such that  $p_g(S, x_0) \geq 1$ . If  $\Gamma_0$  where simply connected, then taking a 3-cover space of  $S$  we would obtain a contradiction since  $h^0(\bar{S}, R^1\Pi_*\mathcal{O}_S) \leq 2$ .  $\square$

**Lemma 2. 14** *Let  $S$  be a surface with a GSS and such that  $b_2(S) > 0$ . Let  $D$  be the maximal divisor of  $S$  and  $p : S \rightarrow \bar{S}$  the contraction of  $D$ . Then*

$$0 \rightarrow H^1(S, \mathcal{O}_S) \rightarrow H^0(\bar{S}, R^1\Pi_*\mathcal{O}_S) \rightarrow H^2(\bar{S}, \mathcal{O}_{\bar{S}}) \rightarrow 0$$

and

$$(\dagger). \quad 1 \leq h^0(\bar{S}, R^1\Pi_*\mathcal{O}_S) = h^0(\bar{S}, \omega_{\bar{S}}) + 1 \leq 2$$

Proof: By Proposition 13 we have the desired exact sequence. Since  $S$  has no non constant meromorphic functions, the dimension of  $H^0(\bar{S}, \omega_{\bar{S}})$  is 0 or 1.  $\square$

The proof of the following theorem follows the arguments of [15] Corollaire.

**Theorem 2. 15** *Let  $S$  be a surface with a GSS, and  $C$  a connected component of the maximal divisor  $D$ . Suppose that  $C$  is exceptional and let  $\Pi : S \rightarrow \bar{S}$  the contraction of  $C$ ,  $\{x\} = p(C)$ . Then:*

- 1)  $p_g(\bar{S}, x) = 1$  or  $2$ .
- 2) If  $S$  is a Inoue surface and  $C = E$  is the elliptic curve,  $(\bar{S}, x)$  is elliptic.
- 3) If  $2n < \sigma_n(S) < 3n$   $|D|$  is connected and the following conditions are equivalent:
  - i)  $p_g(\bar{S}, p) = 2$
  - ii) the dualizing sheaf of  $\bar{S}$  is trivial i.e.  $\omega_{\bar{S}} \simeq \mathcal{O}_{\bar{S}}$
  - iii) the anticanonical bundle  $-K$  is defined by a positive divisor  $\Gamma$  i.e.  $\omega_S \simeq \mathcal{O}_S(-\Gamma)$  where  $\Gamma > 0$ .
  - iv)  $(\bar{S}, p)$  is a Gorenstein singularity.
- 4) If  $S$  is an even Inoue-Hirzebruch surface, each cycle gives a minimally elliptic singularity and the dualizing sheaf of  $\bar{S}$  is trivial. In particular singularities are Gorenstein.
- 5) If  $S$  is an odd Inoue-Hirzebruch surface the cycle gives a minimally elliptic singularity but the dualizing sheaf of  $\bar{S}$  is not trivial. The singularity is still Gorenstein.

Proof: 1) and 2) By Lemma 2.14 and Proposition 2, it remains to consider Inoue surfaces. If  $p_g(\bar{S}, x) = 2$ , there were by  $(\dagger)$ , a non vanishing holomorphic 2-form  $\omega$  on  $S \setminus E$ . But  $K = [-E - \Gamma]$ , therefore there is a meromorphic 2-form  $\omega'$  on  $S \setminus E$  with a pole of order 1 along  $\Gamma$ . The meromorphic function  $\bar{f} = \frac{p_*\omega}{p_*\omega'}$  extends across the normal singularity. Set  $f = \bar{f} \circ p$ , then  $f$  is a meromorphic function on  $S$ , therefore is constant, so we have a contradiction.

3)  $i) \iff ii)$ : Notice that a global section of  $\omega_{\bar{S}}$  cannot vanish since there is no curve. Therefore by  $(\dagger)$   $p_g(\bar{S}, p) = 2$  if and only if  $\omega_{\bar{S}}$  is trivial.

$ii) \Rightarrow iii)$  By Lemma 4.

$iii) \Rightarrow ii)$  Let  $\bar{U} = \bar{S} \setminus \{p\}$ ,  $U = \Pi^{-1}(\bar{U})$  and  $i : \bar{U} \hookrightarrow U$  the inclusion. We have since  $\bar{S}$  is normal

$$\omega_{\bar{S}} = i_*\omega_{\bar{U}} \simeq i_*\Pi_*\omega_U \simeq i_*\Pi_*\mathcal{O}_U \simeq i_*\mathcal{O}_{\bar{U}} \simeq \mathcal{O}_{\bar{S}}$$

Trivially  $ii) \Rightarrow iv)$ , we shall prove  $iv) \Rightarrow i)$ . In fact, suppose that  $p_g(\bar{S}, p) = 1$ , then by [13] theorem 3.10, the singularity would be minimally elliptic, but it is impossible since in the case  $2n < \sigma_n(S) < 3n$  the maximal divisor contains a cycle with at least one branch [1] p113.

4) Suppose that  $S$  is an even Inoue-Hirzebruch surface then the sheaf  $R^1\Pi_*\mathcal{O}_S$  is supported by two points. By  $(\dagger)$  and Proposition 2,  $h^0(\bar{S}, R^1\Pi_*\mathcal{O}_S) = 2$ , and both singularities are minimally elliptic (see [13]).

5) It is well known ([9] or [3] Prop.2.14) that the canonical line bundle  $K$  of an odd Inoue-Hirzebruch surface is not given by a divisor, therefore the unique singularity of  $\bar{S}$  is minimally elliptic thanks to the assertion 3). The surface  $S$  admits a double covering by an even Inoue-Hirzebruch surface. By 4) the singularity is Gorenstein.  $\square$

**Remark 2. 16** Conditions i) and iv) are local conditions, though ii) and iii) are global ones.

## 2.2 $\mathbb{Q}$ -Gorenstein and numerically Gorenstein singularities

**Definition 2. 17** Let  $D$  be a connected exceptional divisor in  $X$  and  $\pi : X \rightarrow \bar{X}$  the contraction onto  $x = p(D) \in \bar{X}$ . Then  $(\bar{X}, x)$  is a **numerically Gorenstein** (resp.  **$\mathbb{Q}$ -Gorenstein**) singularity if the positive numerically anticanonical  $\mathbb{Q}$ -divisor  $D_{-K}$  is a divisor (resp. there exists an integer  $m$  such that the  $m$ -anticanonical bundle  $K^{-m}$  has a section).

**Proposition 2. 18** Let  $S$  be a compact complex surface containing a GSS of intermediate type, i.e  $2n < \sigma_n(S) < 3n$ ,  $\Pi : S \rightarrow \bar{S}$  the contraction of the maximal divisor and  $x = \Pi(D)$  the singular point of  $\bar{S}$ . Then

i)  $(\bar{S}, x)$  is numerically Gorenstein if and only if there exists a unique  $\kappa \in \mathbb{C}^*$  such that

$$H^0(S, K^{-1} \otimes L^\kappa) \neq 0,$$

ii)  $(\bar{S}, x)$  is  $\mathbb{Q}$ -Gorenstein if and only if there exists an integer  $m \geq 1$  such that

$$H^0(S, K^{-m}) \neq 0.$$

**Proof:** i) The sufficient condition is evident and the necessary condition derives from [4] thm 4.5.

ii) The sufficient condition is evident. Conversely, suppose that there exists an open neighbourhood  $U$  of  $D$  with  $0 \neq \theta \in H^0(U, K_U^{-m})$ , non vanishing outside the exceptional divisor. Since the curves are a base of  $H^2(S, \mathbb{Q})$ ,  $K_S^{-m}$  is numerically equivalent to a positive divisor. The exponential exact sequence for surfaces of class VII<sub>0</sub> [12], yields the exact sequence

$$1 \rightarrow H^1(S, \mathbb{C}^\star) \rightarrow H^1(S, \mathcal{O}_S^\star) \xrightarrow{c_1} H^2(S, \mathbb{Z}) \rightarrow 0$$

where  $\mathbb{C}^\star \simeq H^1(S, \mathbb{C}^\star)$ . Therefore there exists a unique  $\kappa \in \mathbb{C}^\star$  such that

$$H^0(S, K_S^{-m} \otimes L^\kappa) \neq 0.$$

Let  $0 \neq \omega \in H^0(S, K_S^{-m} \otimes L^\kappa)$ . Flat line bundles are defined by a representation of  $\pi_1(S)$  in  $\mathbb{C}^\star$ . Since in the intermediate case the cycle  $\Gamma$  of rational curves fulfils  $H_1(\Gamma, \mathbb{Z}) = H_1(S, \mathbb{Z})$ , the restriction  $H^1(S, \mathbb{C}^\star) \rightarrow H^1(U, \mathbb{C}^\star)$  is an isomorphism. Then  $\theta/\omega \in H^0(U, L^{1/\kappa})$  may vanish or may have a pole only on the exceptional divisor. Since the intersection matrix is negative definite,  $\theta/\omega$  cannot vanish and  $L|_U^{1/\kappa}$  is holomorphically trivial, hence  $\kappa = 1$ .  $\square$

In the exemple [4] 4.9, there is a family of surfaces with two rational curves, one rational curve with double point  $D_0$  and a non-singular rational curve  $D_1$ ,  $D_0^2 = -1$ ,  $D_1^2 = -2$  and  $D_0 D_1 = 1$ . The obtained singularity is Gorenstein elliptic for  $\alpha = \pm i$  and deforms into numerically Gorenstein elliptic for other values of the parameter  $\alpha$ .

### 3 Discriminant of the singularities

#### 3.1 A family $\mathfrak{P}$ of polynomials

For an integer  $N \geq 1$ , we denote  $\mathbb{Z}/N\mathbb{Z} = \{\dot{0}, \dot{1}, \dots, \overline{N-1}\}$ . Let

$$A = \{\dot{a}_1, \dots, \dot{a}_p\} \subset \mathbb{Z}/N\mathbb{Z}$$

a subset with  $p$  elements,  $0 \leq p \leq N$ . We may suppose that we have

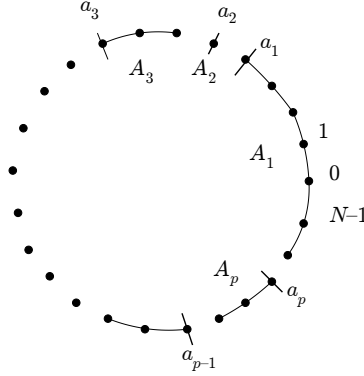
$$0 \leq a_1 < a_2 < \dots < a_p \leq N-1$$

which allows to define a partition  $\mathcal{A} = (A_i)_{1 \leq i \leq p}$  of  $\mathbb{Z}/N\mathbb{Z}$ , where

$$A_1 := \{\dot{k} \in \mathbb{Z}/N\mathbb{Z} \mid 0 \leq k \leq a_1 \text{ or } a_p < k \leq N-1\}$$

$$A_i := \{\dot{k} \in \mathbb{Z}/N\mathbb{Z} \mid a_{i-1} < k \leq a_i\} \text{ for } 2 \leq i \leq p.$$

When  $A = \emptyset$ ,  $\mathcal{A}$  is the trivial partition and  $A_1 = \mathbb{Z}/N\mathbb{Z}$ .



**Definition 3. 19** Let  $N \geq 1$ ,  $A \subset \mathbb{Z}/N\mathbb{Z}$  and  $B \not\subset \mathbb{Z}/N\mathbb{Z}$ .

1) We shall say that  $B$  is a **generating allowed subset** relatively to  $A$  if  $B$  satisfies one of the following conditions:

i)  $B = \{\dot{a}\}$  with  $\dot{a} \in A$ .

ii)  $B = \{\dot{k}, \overline{k+1}\}$  and there exists  $1 \leq i \leq p$  such that  $B \subset A_i$ .

2) We shall say that  $B$  is an **allowed subset** relatively to  $A$  if  $B$  admits a (possibly empty) partition into generating allowed subsets.

This set will be denoted by  $\mathcal{P}_A$ .

**Definition 3. 20** For every  $N \geq 0$ , let  $\mathfrak{P}_N$  be the family of polynomials defined in the following way:  $\mathfrak{P}_0 = \{0\}$ .

If  $N \geq 1$ ,  $\mathfrak{P}_N \subset \mathbb{Z}[X_0, \dots, X_{N-1}]$  is the set of polynomials

$$P_A(X_0, \dots, X_{N-1}) = \sum_{B \in \mathcal{P}_A} \prod_{i \notin B} X_i \quad \text{for } A \subset \mathbb{Z}/N\mathbb{Z}$$

We shall denote

$$\mathfrak{P} = \bigcup_{N \geq 0} \mathfrak{P}_N$$

the union of all these polynomials.

**Examples 3. 21** For  $N = 1$ , there is only one polynomial  $\mathfrak{P}_1 = \{X\}$ .

For  $N = 2$ ,

$$\mathfrak{P}_2 = \{P_\emptyset(X_0, X_1) = X_0X_1, P_{\{0\}}(X_0, X_1) = X_0X_1 + X_1,$$

$$P_{\{1\}}(X_0, X_1) = X_0X_1 + X_0, P_{\{0,1\}}(X_0, X_1) = X_0X_1 + X_0 + X_1\}$$

For  $N = 3$ ,  $\mathfrak{P}_3$  contains the following polynomials

$$\left\{ \begin{array}{l} P_\emptyset(X_0, X_1, X_2) = X_0X_1X_2 + X_0 + X_1 + X_2, \\ P_{\{0\}}(X_0, X_1, X_2) = X_0X_1X_2 + X_1X_2 + X_0 + X_2, \\ P_{\{0,1\}}(X_0, X_1, X_2) = X_0X_1X_2 + X_1X_2 + X_0X_2 + X_1 + X_2, \\ P_{\{0,1,2\}}(X_0, X_1, X_2) = X_0X_1X_2 + X_1X_2 + X_0X_2 + X_0X_1 + X_0 + X_1 + X_2 \end{array} \right.$$

and those obtained by circular permutation of the variables.

Next proposition 22 gives the first properties of polynomials of  $\mathfrak{P}$ , lemma 26 shows that by vanishing of variables corresponding to an allowed subset, we shall still obtain polynomials of  $\mathfrak{P}$ , proposition 27 shows that these polynomials are irreducible, finally proposition 29 gives a characterization of the family  $\mathfrak{P}$ .

**Proposition 3. 22** 1) If  $N \neq N'$ , then  $\mathfrak{P}_N \cap \mathfrak{P}_{N'} = \emptyset$   
 2) For  $N \geq 2$ , the mapping

$$\begin{array}{ccc} \mathfrak{P}(\mathbb{Z}/N\mathbb{Z}) & \longrightarrow & \mathfrak{P}_N \\ A & \mapsto & P_A \end{array}$$

is a bijection from the set  $\mathfrak{P}(\mathbb{Z}/N\mathbb{Z})$  of subsets of  $\mathbb{Z}/N\mathbb{Z}$  on  $\mathfrak{P}_N$ . In particular, if  $N \geq 2$ ,  $\mathfrak{P}_N$  has  $2^N$  elements.

3) If  $A \subset \mathbb{Z}/N\mathbb{Z}$ , then:

i)  $\deg P_A = N$  and  $\prod_{i=0}^{N-1} X_i$  is the only monomial of  $P_A$  of degree  $N$ .

ii) For  $N \geq 2$ , homogeneous part of  $P_A$  of degree  $N - 1$  has  $\text{Card} A$  monomials and these are

$$\prod_{i \neq a} X_i \quad \text{for every } a \in A$$

In particular homogeneous part of  $P_A$  of degree  $N - 1$  determines  $A$  and  $P_A$  uniquely.

iii)  $P_A(0) = 0$ .

4) If  $P(X_0, \dots, X_{N-1}) \in \mathfrak{P}_N$  and  $\alpha$  is a circular permutation of  $\{0, \dots, N - 1\}$  then  $P(X_{\alpha(0)}, \dots, X_{\alpha(N-1)}) \in \mathfrak{P}_N$ .

**Proof :** 1) derives from 3) i); 2) from 3) ii). Besides, the only monomial of degree  $N$  is obtained for  $B = \emptyset \in \mathcal{P}_A$ , monomials of degree  $N - 1$  are obtained for one element subsets  $\{a\} \in \mathcal{P}_A$ . The integer  $N$  being fixed, these monomials determine  $A$  and  $P_A$ . Finally, an allowed subset is by definition different from  $\mathbb{Z}/N\mathbb{Z}$ , so we have the assertion 3) iii). Assertion 4) is evident.  $\square$

**Lemma and Definition 3. 23** Let  $A \subset \mathbb{Z}/N\mathbb{Z}$ ,  $\mathcal{A} = (A_i)_{1 \leq i \leq p}$  the partition of  $\mathbb{Z}/N\mathbb{Z}$  defined by  $A$  and  $B \in \mathcal{P}_A$ .

1) Consider subsets of  $B$  of the type  $I = \{\overline{j+1}, \dots, \overline{j+k}\}$  such that:

i)  $\overline{j+k} \in A$ ,

ii)  $I \subset B$  is maximal for inclusion,

Then  $I$  is an allowed subset relatively to  $A$  which will be called an **allowed subset fixed to  $A$** . The element  $\overline{j}$  will be called the **spring** of  $I$ .

2) Let  $S_B$  be the set of springs of allowed subsets fixed to  $A$ , then we have  $S_B \cap B = \emptyset$ .

3) Consider subsets of  $B$  of the type  $J = \{\overline{j+1}, \dots, \overline{j+2k}\}$  such that:

i) there exists  $i$ ,  $1 \leq i \leq p$  such that  $J \subset A_i$ ,

ii)  $J \subset B$  is maximal for inclusion,

iii) For every allowed subset  $I$ , fixed to  $A$ , we have  $J \cap I = \emptyset$

then  $J$  is an allowed subset relatively to  $A$ , which be called a **wandering allowed subset**.

4)  $B$  admits a unique partition by fixed allowed subsets and wandering allowed subsets. This partition will be called the **canonical partition** of  $B$ .

**Proof:** clear.  $\square$

**Remark 3. 24** If  $X \subset \mathbb{Z}/N\mathbb{Z}$  is not empty and  $N' = \text{Card } X$ , canonical action of  $\mathbb{Z}/N\mathbb{Z}$  on itself induces an action of  $\mathbb{Z}/N'\mathbb{Z}$  on  $X$ , denoted by  $\dot{+}$ , defined in the following way: If  $\dot{x} \in X$ , let  $j \geq 1$  be the least integer such that  $\overline{x + j} \in X$ ; we set  $\dot{x} \dot{+} \dot{1} = \overline{x + j}$ .

**Lemma 3. 25** Let  $A \subset \mathbb{Z}/N\mathbb{Z}$ , and  $B \in \mathcal{P}_A$ . Denote by  $B'$  the complement of  $B$  in  $\mathbb{Z}/N\mathbb{Z}$ ,  $N' = \text{Card } B'$  and let  $\varphi : B' \rightarrow \mathbb{Z}/N'\mathbb{Z}$  a bijection compatible with action of  $\mathbb{Z}/N'\mathbb{Z}$  on  $B'$  and on  $\mathbb{Z}/N'\mathbb{Z}$ . If

$$A' = \varphi(A \cap B') \bigcup \varphi(S_B)$$

where  $S_B$  is the set of springs of  $B$ , then:  
the mapping

$$\begin{aligned} \bar{\varphi} : \{C \in \mathcal{P}_A \mid B \subset C\} &\longrightarrow \mathfrak{P}(\mathbb{Z}/N'\mathbb{Z}) \\ C &\longmapsto \varphi(C \cap B') \end{aligned}$$

is a bijection from  $\{C \in \mathcal{P}_A \mid B \subset C\}$  on  $\mathcal{P}_{A'}$ .

**Proof:** 1)  $\bar{\varphi}$  is clearly injective.

2) Let  $C \in \mathcal{P}_A$  such that  $B \subset C$ ; to show that  $\bar{\varphi}(C) \in \mathcal{P}_{A'}$ , it is sufficient to show that if  $I$  (resp.  $J$ ) is an allowed subset fixed to  $A$  (resp. a wandering allowed subset) belonging to the canonical partition of  $C$ , then  $\varphi(I \cap B') \in \mathcal{P}_{A'}$  (resp.  $\varphi(J \cap B') \in \mathcal{P}_{A'}$ ). On this purpose, we notice that if the last element of  $I$  belongs to  $A \cap B$ , then  $\varphi(I \cap B')$  is an allowed subset with last element in  $\varphi(S_B)$ ; if the last element of  $I$  is in  $A \cap B'$ ,  $\varphi(I \cap B')$  is an allowed subset with the same last element in  $\varphi(A \cap B')$ . Therefore in both cases  $\varphi(I \cap B')$  is an allowed subset fixed to  $A'$ . Besides,  $J \cap A = \emptyset$  and  $J \cap S_B = \emptyset$ , hence  $\varphi(J \cap B')$  is contained in an interval of the partition of  $\mathbb{Z}/N'\mathbb{Z}$  associated to  $A'$ ;  $J$  has an even number of elements and  $J \cap B'$  also. Finally,  $J \cap B'$  is a wandering allowed subset.

3) Let  $C' \in \mathcal{P}_{A'}$  and  $C = \varphi^{-1}(C') \cup B$ . Then  $C \in \mathcal{P}_A$ , therefore  $\bar{\varphi}$  is surjective.  $\square$

**Lemma 3. 26** Let  $P_A \in \mathfrak{P}_N$ ,  $B \subset \mathbb{Z}/N\mathbb{Z}$  an allowed subset relatively to  $A$ ,  $B'$  the complement of  $B$  in  $\mathbb{Z}/N\mathbb{Z}$  and  $N' = \text{Card } B'$ . Then, identifying  $\mathbb{Z}[X_i, i \in B']$  with  $\mathbb{Z}[X_0, \dots, X_{N'-1}]$ , there exists  $A' \subset \mathbb{Z}/N'\mathbb{Z}$  such that

$$P_A(X_i = 0, i \in B) = P_{A'}.$$

**Proof :** In  $P_A(X_i = 0, i \in B)$  remain only monomials  $\prod_{i \notin C} X_i$  of  $P_A$  such that  $B \subset C$ ; we then conclude by lemma 25.  $\square$

**Proposition 3. 27** 1) If  $A = \emptyset$  and  $N$  is even (resp. odd),  $P_A$  has only monomials of even (resp. odd) degrees.

2) If  $N \geq 3$  and  $P_A \in \mathfrak{P}_N$ ,  $P_A$  is irreducible in  $\mathbb{Z}[X_0, \dots, X_{N-1}]$ .



**Proof :** 1) If  $B \in \mathcal{P}_A$  then  $\text{Card } B = 0 \pmod 2$ .

2)**First case:**  $A = \emptyset$ . The polynomial  $P = P_A$  is invariant by circular permutation of the variables. Suppose that  $P = P_1 P_2$ , with  $P_j \in \mathbb{Q}[X_0, \dots, X_{N-1}]$ ,  $j = 1, 2$  and  $P_1$  irreducible,  $P_2 \notin \mathbb{Q}$ . Fix a variable, say  $X_i$ , then  $\deg_{X_i} P = 1$ ; therefore the degree of one polynomial is zero and the degree of the other is one. Hence  $P_1$  and  $P_2$  have different variables. Denote by  $I_j$ ,  $j = 1, 2$  the subsets of indices  $i$  such that  $P_j$  depends on  $X_i$ . By 22 1) and 3),

$$P_j(X_i, i \in I_j) = \lambda_j \prod_{i \in I_j} X_i \pmod{(X_i, i \in I_j)^{\text{Card } I_j - 2}}, \quad \lambda_1 \lambda_2 = 1,$$

and  $P_j$  contains only monomials the degree of which is of the same parity as  $\text{Card } I_j$ , because  $P_1$  and  $P_2$  depend on different variables.

We show now that  $P_1$  cannot depend on two consecutive variables: in fact, we could choose  $X_i$  and  $X_{i+1}$  in such a way that  $P_1$  should not depend on  $X_{i+2}$ . However  $P$  is stable by circular permutation, then

$$P(X) = P_1(X_i, i \in I_1) P_2(X_i, i \in I_2) = P_1(X_{i+1}, i \in I_1) P_2(X_{i+1}, i \in I_2)$$

where  $P_1(X_{i+1}, i \in I_1)$  is irreducible but cannot divide neither  $P_1(X_i, i \in I_1)$  neither  $P_2(X_i, i \in I_2)$ , which is impossible since  $\mathbb{Q}[X_0, \dots, X_{N-1}]$  is factorial. Finally we fix two consecutive indices  $i \in I_1$  and  $i+1 \in I_2$ . Then  $\{i, i+1\}$  is an allowed subset, then by lemma 26,

$$P(X_i = X_{i+1} = 0) \in \mathfrak{P}_{N-2},$$

and by 22 1),  $\deg P(X_i = X_{i+1} = 0) = N - 2$ . Then

$$\deg P_1(X_i = X_{i+1} = 0) = \deg P_1(X_i = 0) \leq \text{Card } I_1 - 2$$

$$\deg P_2(X_i = X_{i+1} = 0) = \deg P_2(X_{i+1} = 0) \leq \text{Card } I_2 - 2$$

which yields

$$N - 2 = \deg P_1(X_i = X_{i+1} = 0) + \deg P_2(X_i = X_{i+1} = 0) \leq N - 4$$

a contradiction.

**Second case:**  $A \neq \emptyset$ . We prove the result by induction on  $N \geq 3$ . The result for  $N = 3$  is true by example 21. Let  $N \geq 4$  and suppose, in order to simplify the notations, that  $N - 1 \in A$ . We have

$$P_A(X_0, \dots, X_{N-1}) = X_{N-1} (P_A(X_{N-1} = 1) - P_A(X_{N-1} = 0)) + P_A(X_{N-1} = 0),$$

with

$$\begin{aligned} Q(X_0, \dots, X_{N-2}) &:= P_A(X_{N-1} = 1) - P_A(X_{N-1} = 0) \\ &= \prod_{i \neq N-1} X_i \pmod{(X_0, \dots, X_{N-2})^{N-2}} \end{aligned}$$

$$R(X_0, \dots, X_{N-2}) := P_A(X_{N-1} = 0) = \prod_{i \neq N-1} X_i \pmod{(X_0, \dots, X_{N-2})^{N-2}}.$$

Since  $\{N - 1\}$  is an allowed subset for  $A$ ,  $R \in \mathfrak{P}_{N-1}$  by 26. Now, by induction hypothesis,  $R \in \mathbb{Z}[X_0, \dots, X_{N-2}]$  is irreducible. By Eisenstein criterion, it is sufficient to prove that  $R$  does not divide  $Q$ . But  $\deg R = \deg Q = N - 1$  and both polynomials have the same dominant monomial. Therefore we have to check that  $R \neq Q$ .

- If  $A \neq \mathbb{Z}/N\mathbb{Z}$ , we may suppose that  $N - 2 \notin A$  and  $N - 1 \in A$ , then  $\{N - 2, N - 1\} \in \mathcal{P}_A$  and the monomial  $M_{N-3} = \prod_{0 \leq i \leq N-3} X_i$  is in  $P_A$ , hence in  $R$ , however  $M_{N-3}X_{N-1}$  is not in  $P_A$  hence not in  $Q$ .
- If  $A = \mathbb{Z}/N\mathbb{Z}$ ,  $P_A$  contains  $X_{N-1}$ , therefore  $Q(0, \dots, 0) = 1$  though  $R(0, \dots, 0) = 0$ .  $\square$

**Remark 3. 28** If  $N = 2$  second assertion of the preceeding proposition is wrong as it can be seen in example 21.

**Proposition 3. 29** Let  $\mathfrak{P}' = \bigcup_{N \geq 0} \mathfrak{P}'_N$  be a family of polynomials where

$$\mathfrak{P}'_N \subset \mathbb{Q}[X_0, \dots, X_{N-1}]$$

satisfy the following conditions:

- i) For every  $0 \leq N \leq 2$ ,  $\mathfrak{P}'_N = \mathfrak{P}_N$ ,
- ii) For every  $N \geq 0$ ,  $\text{Card } \mathfrak{P}'_N = \text{Card } \mathfrak{P}_N$ ,
- iii) If  $P \in \mathfrak{P}'_N$ , then  $\deg P = N$ , and its homogeneous part of degree  $N$  is  $\prod_{0 \leq i \leq N-1} X_i$ ,
- iv) If  $N \geq 3$  and  $P \in \mathfrak{P}'_N$ , there exists  $A = A_P \subset \mathbb{Z}/N\mathbb{Z}$  such that for every generating allowed subset  $B \in \mathcal{P}_A$  we have

$$P(X_i = 0, i \in B) \in \mathfrak{P}'_{N - \text{Card } B}.$$

Moreover, for every monomial  $\lambda \prod_{i \notin C} X_i$  of  $P$ , where  $C \neq \emptyset$  and  $\lambda \in \mathbb{Q}$ , there exists a generating allowed subset  $B$  such that  $B \subset C$ .

Then, for every  $N \geq 0$ ,  $\mathfrak{P}'_N = \mathfrak{P}_N$ .

**Proof:** We show by induction on  $N \geq 2$  that  $\mathfrak{P}'_N = \mathfrak{P}_N$ . By i) let  $N \geq 3$ . Let  $P \in \mathfrak{P}'_N$ . By condition iv), there exists  $A = A_P \subset \mathbb{Z}/N\mathbb{Z}$  such that for every  $B \in \mathcal{P}_A$

$$P(X_i = 0, i \in B) \in \mathfrak{P}_{N - \text{Card } B}.$$

We are going to show that  $P = P_A$ . Both polynomials have the same dominant monomials  $\prod_{0 \leq i \leq N-1} X_i$ . Let  $B \in \mathcal{P}_A$  and  $\prod_{i \notin B} X_i$  one of the monomials of  $P_A$ . By iv), induction hypothesis and Proposition 22, 3),

$$P(X_i = 0, i \in B) = \prod_{i \notin B} X_i \mod (X_i, i \notin B)^{N - \text{Card } B - 1}$$

hence this monomials belongs to  $P$  and by iii) each monomial of  $P_A$  belongs to  $P$ . Conversely let  $\lambda \prod_{i \notin C} X_i$  be a monomial of  $P$ , let  $B \in \mathcal{P}_A$  such that  $B \subset C$ . Denoting by  $B'$  the complement of  $B$  in  $\mathbb{Z}/N\mathbb{Z}$  and  $N' = \text{Card } B'$ , there exists  $A' \subset \mathbb{Z}/N'\mathbb{Z}$  for which

$$P(X_i = 0, i \in B) = P_{A'}$$

and by lemma 25,  $C \in \mathcal{P}_A$  and  $\lambda = 1$ .

We have now,  $\mathfrak{P}'_N \subset \mathfrak{P}_N$ . We conclude by ii).  $\square$

To end this section we give a property of these polynomials which will allow to compute the discriminant of singularities whose exceptional set is associated to concatenation of sequences  $\sigma = \sigma' \sigma''$ .

**Proposition 3. 30** *Let  $A' = \{a'_1, \dots, a'_{p'}\} \subset \mathbb{Z}/N'\mathbb{Z}$  and  $A'' = \{a''_1, \dots, a''_{p''}\} \subset \mathbb{Z}/N''\mathbb{Z}$ . We identify  $A'$  (resp.  $A''$ ) with the subset of  $\mathbb{Z}/N\mathbb{Z}$  (denoted in the same way)*

$$A' = \{a'_1, \dots, a'_{p'}\} \subset \mathbb{Z}/N\mathbb{Z} \quad (\text{resp. } A'' = \{a''_1 + N', \dots, a''_{p''} + N'\} \subset \mathbb{Z}/N\mathbb{Z})$$

where

$$N = N' + N'', \quad \text{and} \quad 0 \leq a'_1 < \dots < a'_{p'} < N' \leq a''_1 + N' < \dots < a''_{p''} + N' < N.$$

Setting  $A = A' \cup A'' \subset \mathbb{Z}/N\mathbb{Z}$  we have

$$\begin{aligned} P_A(X_0, \dots, X_{N-1}) &= P_{A'}(X_0, \dots, X_{N'-1}) P_{A''}(X_{N'}, \dots, X_{N'+N''-1}) \\ &\quad + P_{A'}(X_0, \dots, X_{N'-1}) + P_{A''}(X_{N'}, \dots, X_{N'+N''-1}). \end{aligned}$$

**Proof:** With the same identification as in the statement we have

$$\begin{aligned} \mathcal{P}_A &= \{B' \cup B'' \mid B' \in \mathcal{P}_{A'}, B'' \in \mathcal{P}_{A''}\} \\ &\quad \cup \{B' \cup \{N', \dots, N-1\} \mid B' \in \mathcal{P}_{A'}\} \cup \{\{0, \dots, N'-1\} \cup B'' \mid B'' \in \mathcal{P}_{A''}\} \end{aligned}$$

and this gives the three terms of the decomposition.  $\square$

### 3.2 Main results

**Theorem 3. 31 (Main theorem)** *Let  $\sigma = \sigma_0 \dots \sigma_i \dots \sigma_{N+\rho}$  be a sequence of integers such that there are  $N \geq 1$  singular sequences  $\sigma_{i_j} = s_{k_j}$ ,  $0 \leq j \leq N-1$  and  $0 \leq \rho \leq N$  regular sequences  $r_{m_l}$ ,  $0 \leq l \leq \rho-1$ . Let  $A \subset \mathbb{Z}/N\mathbb{Z}$  defined by*

$$A = A(\sigma) := \{0 \leq j \leq N-1 \mid \sigma_i \text{ is a regular sequence if } i = i_j + 1 \pmod{N+\rho}\}.$$

Then we have

$$\det M(\sigma) = P_A(k_0, \dots, k_{N-1})^2.$$

**Corollary 3. 32** *Let  $S$  be a minimal surface containing a GSS with  $n = b_2(S)$ , rational curves  $D_0, \dots, D_{n-1}$  and intersection matrix  $M(S) = (D_i D_j) = -M(\sigma)$ . Then*

i) *The index of the sublattice  $\sum_{i=0}^{n-1} \mathbb{Z} D_i$  in  $H_2(S, \mathbb{Z})$  is*

$$\left[ H_2(S, \mathbb{Z}) : \sum_{i=0}^{n-1} \mathbb{Z} D_i \right] = P_{A(S)}(k_0, \dots, k_{N-1});$$

ii) *The curves  $D_0, \dots, D_{n-1}$  form a basis of  $H_2(S, \mathbb{Q})$  if and only if  $\sigma \neq r_n$ ;*

iii) *The curves  $D_0, \dots, D_{n-1}$  form a basis of  $H_2(S, \mathbb{Z})$  if and only if  $\sigma = s_1 r_{n-1} = (3, 2, \dots, 2)$  for  $n \geq 1$  or  $\sigma = s_1 s_1 = (3, 3)$  if  $n = 2$ .*

*In this case we have the following matrices:*

- $n = 1$ ,  $M(S) = -1$ ,
- $n = 2$ ,  $M(S) = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$

- $n \geq 3$ ,

$$\begin{pmatrix} -3 & 0 & 1 & 0 & \dots & 0 & 1 \\ 0 & -2 & 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & -2 & 1 & \ddots & & \vdots \\ 0 & 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 & 1 & -2 \end{pmatrix}$$

**Corollary 3. 33** *Let  $S$  be an even Inoue-Hirzebruch surface with intersection matrices  $M(S) = -M(\sigma)$  and  $\sigma = s_{k_0} \cdots s_{k_{2q-1}}$ . Let  $\Gamma$  and  $\Gamma'$  be the two cycles with intersection matrices  $M(\Gamma)$  and  $M(\Gamma')$ , then*

$$\begin{aligned} [H^2(\Gamma, \mathbb{Z}) : H_2(\Gamma, \mathbb{Z})] &= |\det M(\Gamma)| = P_{\mathbb{Z}/2q\mathbb{Z}}(k_0, \dots, k_{2q-1}) = |\det M(\Gamma')| \\ &= [H^2(\Gamma', \mathbb{Z}) : H_2(\Gamma', \mathbb{Z})]. \end{aligned}$$

### 3.3 A multiplicative invariant associated to singularities

**Definition 3. 34** *A simple sequence  $\sigma$  is a sequence of the form  $\sigma = s_{k_0} \cdots s_{k_{N-1}} r_m$  with  $N \geq 1$ . A singularity is called simple if it is obtained by the contraction of a divisor whose dual graph is associated to a simple sequence. Of course any sequence  $\sigma = \sigma_0 \cdots \sigma_{N+\rho-1}$  where  $\sigma_i$  is singular or regular splits into  $\rho$  simple sequences.*

*The polynomial associated to the family of singularities of type  $\sigma$  is defined by*

$$\Delta(X_\sigma) := P_\sigma(X_0, \dots, X_{p-1}) + 1.$$

**Lemma 3. 35** *Let  $\sigma'$  and  $\sigma''$  two sequences of the above form. Then*

$$\Delta(X_{\sigma'\sigma''}) = \Delta(X'_\sigma)\Delta(X_{\sigma''}).$$

**Proof:** By the main theorem,  $P_{\sigma'} = P_{A'}$ ,  $P_{\sigma''} = P_{A''}$  with

$$A' \subset \mathbb{Z}/N'\mathbb{Z}, \quad A'' \subset \mathbb{Z}/N''\mathbb{Z}.$$

For  $N = N' + N''$  and  $A = A' \cup A'' \subset \mathbb{Z}/N\mathbb{Z}$ , we have by 30,

$$\begin{aligned} \Delta(X_{\sigma\sigma'}) &= P_{\sigma\sigma'}(X_0, \dots, X_{N+N'-1}) + 1 = P_A(X_0, \dots, X_{N+N'-1}) + 1 \\ &= P_{A'}(X_0, \dots, X_{N'-1})P_{A''}(X_{N'}, \dots, X_{N-1}) + P_{A'}(X_0, \dots, X_{N'-1}) \\ &\quad + P_{A''}(X_{N'}, \dots, X_{N-1}) + 1 \\ &= (P_{A'}(X_0, \dots, X_{N'-1}) + 1)(P_{A''}(X_{N'}, \dots, X_{N-1}) + 1) \\ &= \Delta(X'_\sigma)\Delta(X_{\sigma''}). \end{aligned}$$

□

Now we shall express the invariant  $\Delta(X_\sigma)$  for  $\sigma$  simple, thanks to the determinant of the branch:

**Lemma 3. 36** *Let  $\sigma$  be a simple sequence with branch  $A$  defined by*

$$Selfint(A) = \begin{cases} (\underbrace{2, \dots, 2}_{k_0-1}, k_1+2, \underbrace{2, \dots, 2}_{k_2-1}, \dots, k_{p-2}+2, \underbrace{2, \dots, 2}_{k_{p-1}-1}, 2) \\ \quad \text{if } p \equiv 1 \pmod{2} \\ (\underbrace{2, \dots, 2}_{k_0-1}, k_1+2, \underbrace{2, \dots, 2}_{k_2-1}, \dots, k_{p-3}+2, \underbrace{2, \dots, 2}_{k_{p-2}-1}, k_{p-1}+2) \\ \quad \text{if } p \equiv 0 \pmod{2} \end{cases}$$

then

$$P_\sigma(k_0, \dots, k_{p-1}) + 1 = \det A,$$

where  $\det A$  is the determinant of the intersection matrix of the curves in  $A$ .

**Proof:** For  $p = 1$ ,  $Selfint(A) = (\underbrace{2, \dots, 2}_{k_0})$  and  $\det A = k_0 + 1 = P_\sigma(k_0) + 1$ .

For  $p = 2$ ,  $Selfint(A) = (\underbrace{2, \dots, 2}_{k_0-1}, k_2+2)$  and  $\det A = k_0 k_1 + k_0 + 1 = P_\sigma(k_0, k_1) + 1$

(see Example 21). By induction: we suppose that  $p$  is odd, i.e.  $p = 2q + 1$ ; the even case is left to the reader. We shall use notations in (20). Since there is only one branch we have  $\sigma = (s_{k_0} \cdots s_{k_{2q}} r_m)$ ,  $N = 2q + 1$ . For  $\{2q\} \subset \mathbb{Z}/(2q+1)\mathbb{Z}$ , we denote the allowed subsets by  $\mathcal{P}_{2q+1}$ . For the sequel we need the following observation: Let  $B \in \mathcal{P}_{2q+1}$ , then:

- if  $2q \notin B$  and  $2q-1 \notin B$ ,  $B \in \mathcal{P}_{2q-1}$  and  $\sharp(B)$  is even;
- if  $2q \in B$  and  $2q-1 \notin B$ ,  $B = \{2q\} \cup B'$ ,  $B' \in \mathcal{P}_{2q-1}$  and  $\sharp(B)$  is even;
- if  $2q \notin B$  and  $2q-1 \in B$ ,  $B = \{2q-1, 2q-2\} \cup B'$ ,  $B' \in \mathcal{P}_{2q-1}$  and  $\sharp(B)$  is even;
- if  $2q \in B$  and  $2q-1 \in B$ ,  $B = \{2q, 2q-1\} \cup B'$ ,  $B' \in \mathcal{P}_{2q-1}$  and  $\sharp(B)$  is odd or even.

Denote by  $\Delta_{2q+1}$  the determinant of the branch when  $\sigma$  contains  $2q+1$  singular sequences. Applying lemma (39) below, we have

$$\begin{aligned} \Delta_{2q+1}(k_0, \dots, k_{2q}) &= (k_{2q} + 1) \left\{ k_{2q-1} \Delta_{2q-1}(k_0, \dots, k_{2q-2} - 1) \right. \\ &\quad \left. + \Delta_{2q-1}(k_0, \dots, k_{2q-2}) \right\} - k_{2q} \Delta_{2q-1}(k_0, \dots, k_{2q-2} - 1) \\ &= k_{2q} k_{2q-1} \Delta_{2q-1}(k_0, \dots, k_{2q-2} - 1) \\ &\quad + k_{2q} \left\{ \Delta_{2q-1}(k_0, \dots, k_{2q-2}) - \Delta_{2q-1}(k_0, \dots, k_{2q-2} - 1) \right\} \\ &\quad + k_{2q-1} \Delta_{2q-1}(k_0, \dots, k_{2q-2} - 1) \\ &\quad + \Delta_{2q-1}(k_0, \dots, k_{2q-2}) \end{aligned}$$

In the sequel  $\sum_{B' \in \mathcal{P}_{2q-1}} \prod_{i \notin B'} k_i$  is shortened into  $\sum_{B' \in \mathcal{P}_{2q-1}}$ . Recall that  $B' \in$

$\mathcal{P}_{2q-1}$ , i.e.  $B' \subset \{0, \dots, 2q-2\} = \mathbb{Z}/(2q-1)\mathbb{Z}$ . By induction hypothesis,

$$\begin{aligned}
\Delta_{2q+1}(k_0, \dots, k_{2q}) &= k_{2q}k_{2q-1} \left\{ \sum_{\substack{B' \in \mathcal{P}_{2q-1} \\ 2q-2 \in B'}} \prod_{i \notin B'} k_i + \sum_{\substack{B' \in \mathcal{P}_{2q-1} \\ 2q-2 \notin B'}} \left( \prod_{\substack{i \notin B' \\ i < 2q-2}} k_i \right) (k_{2q-2} - 1) + 1 \right\} \\
&\quad + k_{2q} \left\{ \sum_{\substack{B' \in \mathcal{P}_{2q-1} \\ 2q-2 \in B'}} + \sum_{\substack{B' \in \mathcal{P}_{2q-1} \\ 2q-2 \notin B'}} - \sum_{\substack{B' \in \mathcal{P}_{2q-1} \\ 2q-2 \in B'}} - \sum_{\substack{B' \in \mathcal{P}_{2q-1} \\ 2q-2 \notin B'}} \left( \prod_{\substack{i \notin B' \\ i < 2q-2}} k_i \right) (k_{2q-2} - 1) \right\} \\
&\quad + k_{2q-1} \left\{ \sum_{\substack{B' \in \mathcal{P}_{2q-1} \\ 2q-2 \in B'}} + \sum_{\substack{B' \in \mathcal{P}_{2q-1} \\ 2q-2 \notin B'}} \left( \prod_{\substack{i \notin B' \\ i < 2q-2}} k_i \right) (k_{2q-2} - 1) + 1 \right\} + \sum_{B' \in \mathcal{P}_{2q-1}} + 1 \\
&= k_{2q}k_{2q-1} \left\{ \sum_{2q-2 \in B'} + \sum_{2q-2 \notin B'} - \sum_{2q-2 \notin B'} \left( \prod_{i \notin \{2q-2\} \cup B'} k_i \right) + 1 \right\} + k_{2q} \left\{ \sum_{\substack{2q-2 \in B' \\ \#(B') \text{ odd}}} \right\} \\
&\quad + k_{2q-1} \left\{ \sum_{2q-2 \in B'} + \sum_{2q-2 \notin B'} - \sum_{2q-2 \notin B'} \left( \prod_{i \notin \{2q-2\} \cup B'} k_i \right) + 1 \right\} + \sum_{B' \in \mathcal{P}_{2q-1}} + 1 \\
&= k_{2q}k_{2q-1} \left\{ \sum_{\substack{2q-2 \in B' \\ \#(B') \text{ even}}} + \sum_{2q-2 \notin B'} + 1 \right\} + k_{2q} \left\{ \sum_{\substack{2q-2 \in B' \\ \#(B') \text{ odd}}} \right\} \\
&\quad + k_{2q-1} \left\{ \sum_{\substack{2q-2 \in B' \\ \#(B') \text{ even}}} + \sum_{2q-2 \notin B'} + 1 \right\} + \sum_{B' \in \mathcal{P}_{2q-1}} + 1 \\
&= \sum_{\substack{B \in \mathcal{P}_{2q+1} \\ 2q \notin B, 2q-1 \notin B}} + \sum_{\substack{B \in \mathcal{P}_{2q+1} \\ 2q \notin B, 2q-1 \in B}} + \sum_{\substack{B \in \mathcal{P}_{2q+1} \\ 2q \in B, 2q-1 \notin B}} + \sum_{\substack{B \in \mathcal{P}_{2q+1} \\ 2q \in B, 2q-1 \in B}} + 1 \\
&= \sum_{B \in \mathcal{P}_{2q+1}} + 1 = P_\sigma(k_0, \dots, k_{2q}) + 1. \quad \square
\end{aligned}$$

**Proposition 3. 37** *Let  $\sigma = \sigma_0 \cdots \sigma_{s-1}$  be a decomposition of  $\sigma$  into simple sequences and let  $A_0, \dots, A_{s-1}$  be the branches of the dual graph, then*

- i)  $\Delta(X_\sigma) = \prod_{i=0}^{s-1} \Delta(X_{\sigma_i}) = \prod_{i=0}^{s-1} \det A_i$ ,
- ii)  $P_\sigma = \prod_{i=0}^{s-1} (P_{\sigma_i} + 1) - 1 = (\prod_{i=0}^{s-1} \det A_i) - 1$ .

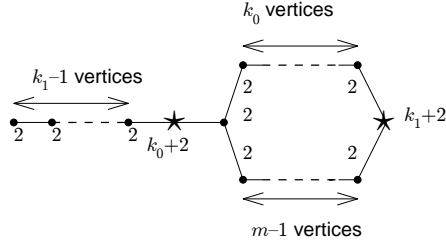
(notice that different polynomials depend on different indeterminates).

We shall call the invariant  $\Delta(X_\sigma)$  the **branch invariant**.

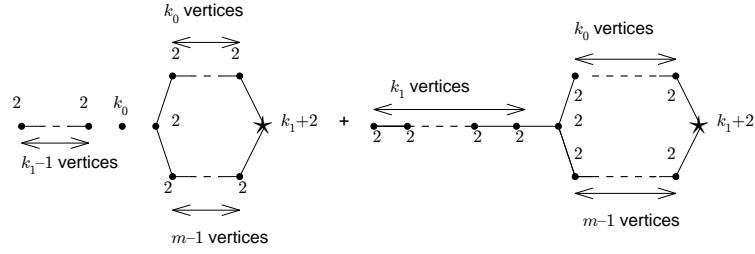
## 4 Proof of the main theorem

The aim is to compute the discriminant of the quadratic form using the family of polynomials previously introduces. We work with matrices  $M(\sigma)$ , hence with

positive definite matrices. The weights in the dual graph are positive. The vertices with weight 2 are represented by a bullet, the vertices with weight  $\geq 3$  are represented by a star. The idea is to develop the determinant splitting it into pieces which have a geometrical meaning. For example, consider  $M = M(s_{k_0} r_m s_{k_1})$ . Its dual graph is



It splits into



which corresponds to the development of the determinant along the  $(k_1 - 1)$ -th column by the splitting

$$\begin{pmatrix} \vdots \\ -1 \\ k_0 + 2 \\ -1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ 0 \\ k_0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ -1 \\ 2 \\ -1 \\ \vdots \end{pmatrix},$$

$\det M =$

$$\begin{aligned}
& \begin{vmatrix} 2 & -1 & & & & & & & & & & \\ -1 & \ddots & \ddots & & & & & & & & & \\ & \ddots & 2 & -1 & & & & & & & & \\ & & -1 & k_0+2 & -1 & & & & & & & \\ & & & -1 & 2 & -1 & 0 & \dots & \dots & 0 & -1 & \\ & & & & -1 & \ddots & \ddots & & & & 0 & \\ & & & & 0 & \ddots & 2 & \ddots & & & \vdots & \\ & & & & \vdots & & \ddots & k_1+2 & \ddots & & \vdots & \\ & & & & \vdots & & & \ddots & 2 & \ddots & 0 & \\ & & & & 0 & & & \ddots & \ddots & -1 & \\ -1 & 0 & \dots & \dots & 0 & -1 & 2 & & & & & \end{vmatrix} \begin{matrix} 0 \\ k_1-1 \\ k_1 \\ 0 \\ k_0+k_1+m-1 \end{matrix} \\
& = k_0 \begin{vmatrix} 2 & -1 & & & & & & & & & & \\ -1 & \ddots & \ddots & & & & & & & & & \\ & \ddots & \ddots & -1 & & & & & & & & \\ & & -1 & 2 & & & & & & & & \end{vmatrix} \begin{matrix} 0 \\ k_1-2 \end{matrix} \begin{vmatrix} 2 & -1 & & & & & & & & & & \\ -1 & \ddots & \ddots & & & & & & & & & \\ & \ddots & 2 & \ddots & & & & & & & & \\ & & \ddots & k_1+2 & \ddots & & & & & & & \\ & & & \ddots & 2 & \ddots & & & & & & \\ & & & & \ddots & \ddots & -1 & & & & & \\ & & & & & -1 & 2 & & & & & \\ -1 & & & & & & & & & & & \end{vmatrix} \begin{matrix} -1 \\ k_1 \\ k_0+k_1+m-1 \end{matrix} \\
& + \begin{vmatrix} 2 & -1 & & & & & & & & & & \\ -1 & \ddots & \ddots & & & & & & & & & \\ & \ddots & 2 & -1 & & & & & & & & \\ & & -1 & 2 & -1 & & & & & & & \\ & & & -1 & 2 & -1 & 0 & \dots & \dots & 0 & -1 & \\ & & & & -1 & \ddots & \ddots & & & & 0 & \\ & & & & 0 & \ddots & 2 & \ddots & & & \vdots & \\ & & & & \vdots & & \ddots & k_1+2 & \ddots & & \vdots & \\ & & & & \vdots & & & \ddots & 2 & \ddots & 0 & \\ & & & & 0 & & & \ddots & \ddots & -1 & \\ -1 & 0 & \dots & \dots & 0 & -1 & 2 & & & & & \end{vmatrix} \begin{matrix} k_1-1 \\ k_1 \\ k_0+k_1+m-1 \end{matrix}
\end{aligned}$$



In the sequel we shall associate to  $M$  a family of matrices obtained by this type of development. The easy cases are those of a chain and of a cycle with all diagonal entries equal to 2:

**Lemma 4. 38** *Let  $\delta_m$  and  $\Delta_m$  be the determinants of order  $m \geq 1$  defined by*

$$\delta_1 = 2, \quad \delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}, \quad \Delta_1 = 0, \quad \Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}$$

$$\delta_m = \begin{vmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{vmatrix}, \quad \Delta_m = \begin{vmatrix} 2 & -1 & & & -1 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{vmatrix}, \quad m \geq 3,$$

then

$$\delta_m = m + 1 \quad \text{and} \quad \Delta_m = 0.$$

**Proof:** left to the reader. □

**Lemma 4. 39** *Let  $N = (n_{ij})_{0 \leq i, j \leq p-1}$  be a matrix of order  $p \geq 2$ , of the form*

$$N = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & \ddots & \ddots & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & -1 & \\ & & & & -1 & n_{mm} & \dots & n_{m,p-1} \\ & & & & & \vdots & & \vdots \\ & & & & & n_{p-1,m} & \dots & n_{p-1,p-1} \end{pmatrix} \begin{matrix} 0 \\ \\ \\ m-1 \\ m \\ \\ p-1 \end{matrix},$$

where  $m \leq p-2$ . For  $J \subset \{0, \dots, p-1\}$ , we denote by  $N_J = (n_{ij})_{i,j \in J}$  the submatrix whose entries depend on indices in  $J$ . Then

$$\det N = (m+1) \det N_{\{m, \dots, p-1\}} - m \det N_{\{m+1, \dots, p-1\}}.$$

**Proof:** The result is trivial if  $m = 0$ . If  $m \geq 1$ , development along the first column yields with induction hypothesis

$$\begin{aligned} \det N &= 2 \det N_{\{1, \dots, p-1\}} - \det N_{\{2, \dots, p-1\}} \\ &= 2 \left( m \det N_{\{m, \dots, p-1\}} - (m-1) \det N_{\{m+1, \dots, p-1\}} \right) \\ &\quad - \left( (m-1) \det N_{\{m, \dots, p-1\}} - (m-2) \det N_{\{m+1, \dots, p-1\}} \right) \\ &= (m+1) \det N_{\{m, \dots, p-1\}} - m \det N_{\{m+1, \dots, p-1\}}. \end{aligned}$$

□

## 4.1 Expression of determinants by polynomials

**Notations 4. 40** Let  $N \geq 0$  and  $\rho \geq 0$  be integers such that  $\rho = 1$  if  $N = 0$  and  $\rho \leq N$  if  $N \geq 1$ .

Let  $M = M(\sigma)$  where  $\sigma = \sigma_0 \cdots \sigma_{N+\rho-1} = (a_0, \dots, a_{n-1})$  contains  $N$  singular sequences  $s_{k_i}$ ,  $i = 0, \dots, N-1$  and  $\rho$  regular sequences  $r_{m_j}$ ,  $j = 0, \dots, \rho-1$ . Let

$$n = \sum_{i=0}^{N-1} k_i + \sum_{j=0}^{\rho-1} m_j$$

be the order of  $M = (m_{ij})_{0 \leq i, j \leq n-1}$  or the number of vertices of the associated dual graph.

We denote by  $\mathcal{C}$  the set of subsets  $J \subset \{0, \dots, n-1\}$  which satisfy the following condition

$$(C) \quad \begin{cases} \text{let } 0 \leq l \leq N + \rho - 1, \text{ and } \sigma_l = (a_r, \dots, a_s). \\ \text{If } \alpha \text{ satisfies } r+1 \leq \alpha \leq s \text{ and } \alpha \in J \\ \text{then for all } \beta \text{ such that } r+1 \leq \beta \leq s, \text{ we have } \beta \in J. \end{cases}$$

Splitting the graph into some pieces or changing the weights of some vertices, we associate to  $M$  a family  $\mathcal{M}$  of matrices in the following way:

For  $J \in \mathcal{C}$ , let  $K_J$  defined by

$$K_J = \{j \in J \mid m_{jj} > 2\}.$$

For  $K \subset K_J$ , denote by  $M_J^K$  the matrix

$$M_J^K := (m'_{ij})_{i, j \in J}$$

where

$$\begin{cases} m'_{kk} &= 2 & \text{if } k \in K \\ m'_{ij} &= m_{ij} & \text{in other cases} \end{cases}$$

The family  $\mathcal{M}$  is

$$\mathcal{M} = \{M_J^K \mid J \in \mathcal{C}, K \subset K_J\}.$$

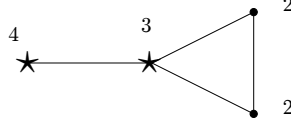
Now, for a fixed matrix  $M_J^K$ , we consider

- a partition  $J = J' \cup J''$  of  $J$ , where  $J'$  (resp.  $J''$ ) is the subset of indices of vertices of the cycle (resp. of the branches), and
- another partition of  $J'$  and of  $J''$  depending on  $K$ , composed of subsets of the following two types:
  - (1) single elements  $\{i\}$  such that  $m_{ii} > 2$ ,
  - (2) when elements of type (1) is removed, connected components of vertices  $j$  the weight of which is  $m_{jj} = 2$

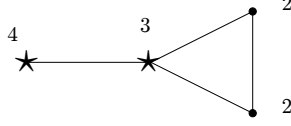
To end, denote by  $\nu_1(M_J^K)$  (resp.  $\nu_2(M_J^K)$ ) the total number of subsets of type (1) (resp. type (2)) in the partition of  $J'$  and  $J''$  and we set

$$\nu(M_J^K) = \nu_1(M_J^K) + \nu_2(M_J^K).$$

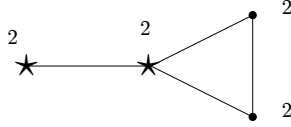
**Examples 4. 41** Let  $M = M(r_1 s_1 s_2) = M(2, 3, 42)$ . Its dual graph is



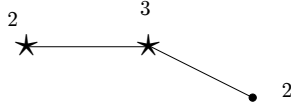
- If  $J = \{0, 1, 2, 3\}$  and  $K = \emptyset$  then  $J' = \{0, 1, 3\}$ ,  $J'' = \{2\}$  and  $\nu(M_J^K) = 3$  and the dual graph of  $M_J^K$  is



- If  $J = \{0, 1, 2, 3\}$  and  $K = \{1, 2\}$  then  $\nu(M_J^K) = 2$  and the dual graph of  $M_J^K$  is



- If  $J = \{1, 2, 3\}$  and  $K = \{2\}$  then  $J' = \{1, 3\}$ ,  $J'' = \{2\}$ ,  $\nu(M_J^K) = 3$ ,



**Lemma 4. 42** Let  $M_J^K \in \mathcal{M}$ ,  $\nu_1 = \nu_1(M_J^K)$  and  $\nu_2 = \nu_2(M_J^K)$ . Then there exists a polynomial

$$Q \in \mathbb{Z}[X_0, \dots, X_{\nu_1-1}, Y_0, \dots, Y_{\nu_2-1}]$$

of degree 1 respectively each variable, such that

$$\det M_J^K = Q(k_{i_0}, \dots, k_{i_{\nu_1-1}}, m_0, \dots, m_{\nu_2-1})$$

where subsets  $\{i_j\}$  are of type (1) and  $m_j$  are the cardinals of the subsets of type (2) which compose the partition of  $J$ .

**Proof:** By induction on  $\nu = \nu_1 + \nu_2 \geq 1$ . We have  $\nu_1 \leq N$  and  $\nu_2 \leq N + \rho$  by condition (C).

If  $\nu = 1$ , either  $\nu_1 = 1$ , i.e. the determinant is of order 1 and the result is clear, either  $\nu_2 = 1$  and the results derives from lemma 38.

If  $\nu \geq 2$ , we may suppose that  $M_J^K = (m'_{ij})$  is irreducible because reducible case is an immediate consequence of the induction hypothesis. Several cases may happen: 1)  $M_J^K$  is a matrix of a cycle: Since  $\nu \geq 2$  there exists an index  $j \in J$  such that  $m'_{jj} = k_{i_j} + 2$ . The decomposition of the  $j$ -th column

$$\begin{pmatrix} \vdots \\ 0 \\ -1 \\ k_{i_j} + 2 \\ -1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ k_{i_j} \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ 0 \\ -1 \\ 2 \\ -1 \\ 0 \\ \vdots \end{pmatrix}$$

yields the relation

$$(\dagger) \quad \det M_J^K = k_{i_j} \det M_{J \setminus \{j\}}^K + \det M_J^{K \cup \{j\}}$$

where  $M_{J \setminus \{j\}}^K$  (resp.  $M_J^{K \cup \{j\}}$ ) is a matrix of a chain (resp. of a cycle). Setting

$$\nu'_i = \nu_i(M_{J \setminus \{j\}}^K), \quad \nu''_i = \nu_i(M_J^{K \cup \{j\}}), \quad i = 1, 2,$$

we have

$$\begin{cases} \nu'_1 = \nu_1 - 1 & \nu'_2 = \nu_2 \\ \nu''_1 = \nu_1 - 1 & \nu''_2 = \nu_2 - 1 \end{cases}$$

By induction hypothesis there exists polynomials

$$Q \in \mathbb{Z}[X_0, \dots, \widehat{X_j}, \dots, X_{\nu_1-1}, Y_0, \dots, Y_{\nu_2-1}]$$

$$R \in \mathbb{Z}[X_0, \dots, \widehat{X_j}, \dots, X_{\nu_1-1}, Y_0, \dots, Y_l, \widehat{Y_{l+1}}, \dots, Y_{\nu_2-1}]$$

such that, by a suitable numbering of the indices

$$\det M_{J \setminus \{j\}}^K = Q(k_{i_0}, \dots, \widehat{k_{i_j}}, \dots, k_{\nu_1-1}, m_0, \dots, m_{\nu_2-1})$$

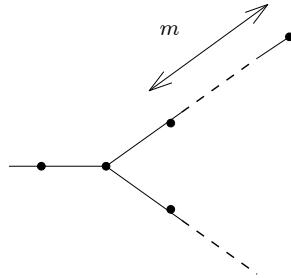
$$\det M_J^{K \cup \{j\}} = R(k_{i_0}, \dots, \widehat{k_{i_j}}, \dots, k_{\nu_1-1}, m_0, \dots, m_{l-1}, m_l + m_{l+1} + 1, m_{l+2}, \dots, m_{\nu_2-1}).$$

We conclude replacing in  $(\dagger)$ .

2)  $M_J^K$  is not the matrix of a cycle: then the dual graph is a part of a cycle or contains bits of branches of  $M$ . In any cases, the dual graph contains a terminal vertex

$$\dots \longrightarrow \star \quad \text{or} \quad \dots \longrightarrow \bullet$$

- If in this chain there is a vertex with weight  $> 2$ , we develop as before,
- If not, all vertices have a weight equal to 2, but since  $\nu \geq 2$ , this chain leads to a bifurcation



Either the vertex of bifurcation has a weight  $> 2$  and we develop as before, either we apply lemma 39 with appropriate numbering of entries of  $M_J^K$ :

$$(\ddagger) \quad \det M_J^K = (m+1) \det M_{J \setminus \{0, \dots, m-1\}}^K - m \det (M_J^K)_{J \setminus \{0, \dots, m\}}.$$

The matrix  $(M_J^K)_{J \setminus \{0, \dots, m\}}$  obtained by deletion of the branch with its root may not be in  $\mathcal{M}$ , however applying once again lemma 39, we obtain a matrix in  $\mathcal{M}$  thanks to the explicit description of  $M$  given by theorem 2.39 in [1]. We apply then induction hypothesis and  $(\ddagger)$ .  $\square$

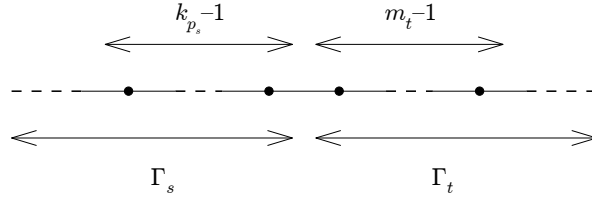
**Lemma 4. 43** *Let  $M = M(\sigma_1 \cdots \sigma_{N+\rho})$  be a matrix satisfying notations 40. Then, there exists a polynomial*

$$Q \in \mathbb{Z}[X_0, \dots, X_{N-1}, Y_0, \dots, Y_{\rho-1}]$$

*of degree at most 2 (resp. 1) relatively  $X_i$ ,  $i = 0, \dots, N-1$  (resp.  $Y_j$ ,  $j = 0, \dots, \rho-1$ ) which satisfies*

$$\det M = Q(k_0, \dots, k_{N-1}, m_0, \dots, m_{\rho-1}).$$

**Proof:** We have  $M = M_{\{0, \dots, n-1\}}^\emptyset \in \mathcal{M}$  and by theorem 9,  $\nu_1 = N$ ,  $\nu_2 \leq N + \rho$  (with  $\rho = \rho(S)$ ). More precisely (with notations of 9), if there exists an integer  $s$  such that  $p_s = 0 \bmod 2$ , we have for  $t = s + 1 \bmod \rho$ , the chain



Let  $\mathfrak{S} = \{t \mid p_s = 0 \bmod 2, \text{ for } s = t - 1\}$ . Then

$$\nu_2 = N + \rho - \text{Card } \mathfrak{S}.$$

Lemma 42 gives a polynomial in  $2N + \rho - \text{Card } \mathfrak{S}$  indeterminates

$$Q \in \mathbb{Z}[X_0, \dots, X_{N-1}, Y_0, \dots, Y_{N-1}, Y_N, \dots, \widehat{Y_{N+t}}, \dots, Y_{N+\rho-1} \mid t \in \mathfrak{S}]$$

such that for suitable indices  $i(t) \leq N-1$ ,

$$\det M(\sigma_0 \cdots \sigma_{N+\rho-1}) = Q(k_0, \dots, k_{N-1}, k_0, \dots, k_{i(t)} + m_t, \dots, k_{N-1}, m_0, \dots, \widehat{m_t}, \dots, m_{\rho-1})$$

Setting  $Y_i = X_i$  for  $i \leq N-1$ ,  $i \neq i(t)$ ,  $t \in \mathfrak{S}$ , and substituting  $X_{i(t)} + m_t$  in  $Y_{i(t)}$  for  $t \in \mathfrak{S}$ , the wished polynomial is obtained.  $\square$

Here is the key lemma for the reduction lemma:

**Lemma 4. 44** *1) Let  $P, Q$  be two polynomials in  $\mathbb{Q}[X_0, \dots, X_{n-1}]$ . Suppose there exists an integer  $N$  such that if  $k_0 \geq N, \dots, k_{n-1} \geq N$  the following equality*

$$P(k_0, \dots, k_{n-1}) = \pm Q(k_0, \dots, k_{n-1})$$

*is fulfilled. Then  $P = Q$  or  $P = -Q$ .*

*2) Let  $P \in \mathbb{Q}[X_0, \dots, X_{n-1}]$  of degree at most 2 relatively to each indeterminate. Suppose that there exists an integer  $N$  such that if  $k_0 \geq N, \dots, k_{n-1} \geq N$ ,  $P(k_0, \dots, k_{n-1})$  is the square of a rational. Then there exists  $Q \in \mathbb{Q}[X_0, \dots, X_{n-1}]$  satisfying*

$$P = Q^2.$$

*In particular, if  $\deg_{X_i} P \leq 1$ ,  $P$  does not depend on  $X_i$ .*

**Proof:** 1) By induction on  $n \geq 1$ .

2) The statement is true for  $n = 1$  without condition on the power by [21]. Then by induction: suppose  $n \geq 2$  and fix  $k_0, \dots, k_{n-2} \geq N$ . Set

$$\begin{aligned} A(X_{n-1}) &= P(k_0, \dots, k_{n-2}, X_{n-1}) \\ &= X_{n-1}^2 P_2(k_0, \dots, k_{n-2}) + X_{n-1} P_1(k_0, \dots, k_{n-2}) + P_0(k_0, \dots, k_{n-2}). \end{aligned}$$

For each  $k_{n-1} \geq N$ ,  $A(k_{n-1})$  is the square of a rational, hence by the case of one variable,  $P_0(k_0, \dots, k_{n-2})$  and  $P_2(k_0, \dots, k_{n-2})$  are squares of rationals. Induction hypothesis shows that there exists polynomials  $Q_0, Q_1 \in \mathbb{Q}[X_0, \dots, X_{n-2}]$  which satisfy

$$P_0 = Q_0^2, \quad \text{and} \quad P_2 = Q_1^2.$$

Replacing, one obtains

$$P_1(k_0, \dots, k_{n-2}) = \pm 2Q_0(k_0, \dots, k_{n-2})Q_1(k_0, \dots, k_{n-2}).$$

By 1), one conclude that

$$P = (X_{n-1}Q_1 \pm Q_0)^2.$$

□

## 4.2 The reduction lemma

In this section we shall prove that the polynomial which computes a determinant depends on the positions of regular sequences in  $\sigma$ , however not on their length.

**Lemma 4. 45 (Reduction lemma)** *Let  $M = M(\sigma_0 \cdots \sigma_{N+\rho-1})$  be a matrix which fulfils conditions 40. Then, there exists a polynomial  $P_\sigma \in \mathbb{Q}[X_0, \dots, X_{N-1}]$  of degree at most 1 relatively each indeterminate  $X_i$ ,  $i = 0, \dots, N-1$  such that*

$$\det M(\sigma) = P_\sigma(k_0, \dots, k_{N-1})^2.$$

*In particular the determinant of  $M$  does not depend on the length of the regular sequences.*

**Proof:** By lemma 43 there exists  $Q \in \mathbb{Q}[X_0, \dots, X_{N-1}, Y_0, \dots, Y_{\rho-1}]$  such that

$$\det M = Q(k_0, \dots, k_{N-1}, m_0, \dots, m_{\rho-1}),$$

of degree at most 2 in  $X_i$  and at most 1 in  $Y_j$ . The matrix  $-M$  is an intersection matrix hence  $\det M$  is the square of an integer by 7 when  $k_i \geq 1$ ,  $i = 0, \dots, N-1$  and  $m_j \geq 1$ ,  $j = 0, \dots, \rho$ . Then lemma 44 implies the existence of

$$P_\sigma \in \mathbb{Q}[X_0, \dots, X_{N-1}, Y_0, \dots, Y_\rho]$$

which satisfies  $Q = P^2$ . But  $\deg_{Y_j} Q \leq 1$ , therefore  $P$  and  $Q$  do not depend on  $Y_j$ . □

## 4.3 Relation between determinants and polynomials of $\mathfrak{P}$

The next step is to prove that the polynomials  $P_\sigma$  belong in fact in the family  $\mathfrak{P}$  previously defined. We shall apply the characteristic properties of  $\mathfrak{P}$  given in 29. We start with examples.

**Examples 4. 46** 1) Case  $N = 0$ : then  $M = M(\sigma) = M(r_m)$  and  $\det M = 0$ .  
Therefore  $P_\sigma = 0$ .

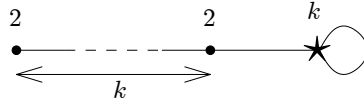
$$2) \text{ Case } N = 1: \text{ If } M = M(s_k) = \begin{pmatrix} k+2 & -1 & & -1 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{pmatrix}, \text{ then}$$

$$\det M = k\delta_{k-1} + \Delta_k = k^2, \quad \text{and} \quad P_\sigma(X) = X.$$

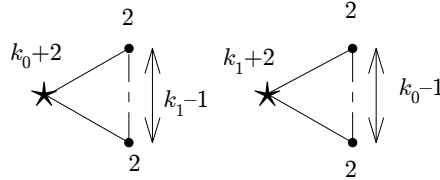
If  $M = M(s_k r_m)$  we have by the reduction lemma 45

$$\det M = \det M(s_k r_1) = \begin{vmatrix} k & & & -1 \\ & 2 & -1 & \\ & -1 & \ddots & \ddots \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{vmatrix} = k\delta_k - \delta_{k-1} = k(k+1) - k = k^2,$$

and  $P_\sigma(X) = X$ .

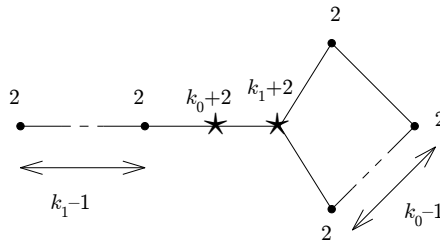


3) Case  $N = 2$ : If  $M = M(s_{k_0} s_{k_1})$  the matrix is reducible and  $\det M = (k_0 k_1)^2$ .



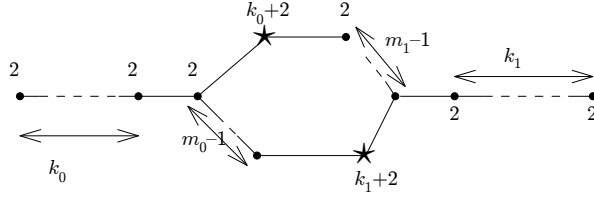
If  $M = M(s_{k_0} r_m s_{k_1})$  we have by 45,

$$\det M = \det M(s_{k_0} r_1 s_{k_1}) = (k_0 k_1 + k_1)^2$$



If  $M = M(s_{k_0} r_{m_0} s_{k_1} r_{m_1})$  we have

$$\det M = \det M(s_{k_0} r_1 s_{k_1} r_1) = (k_0 k_1 + k_0 + k_1)^2.$$



**Proposition 4. 47** Let  $\mathfrak{P}'_N$  be the family of polynomials  $P_\sigma \in \mathbb{Q}[X_0, \dots, X_{N-1}]$  such that

$$\det M(\sigma) = \det M(\sigma_0 \cdots \sigma_{N+\rho-1}) = P_\sigma(k_0, \dots, k_{N-1})^2,$$

with notations of (40).

- 1) For any  $N \geq 0$ ,  $\mathfrak{P}_N = \mathfrak{P}'_N$ ,
- 2) Let  $\sigma_{i_j} = s_{k_j}$ ,  $0 \leq i_j \leq N + \rho - 1$ ,  $0 \leq j \leq N - 1$  be the singular sequences in  $\sigma$  and let  $A \subset \mathbb{Z}/N\mathbb{Z}$  be the subset of indices  $j$  such that  $\sigma_{i_j+1}$  is a regular sequence, then

$$P_\sigma = P_A.$$

**Proof:** To prove 1) it is sufficient to check conditions i) to iv) of proposition 29.

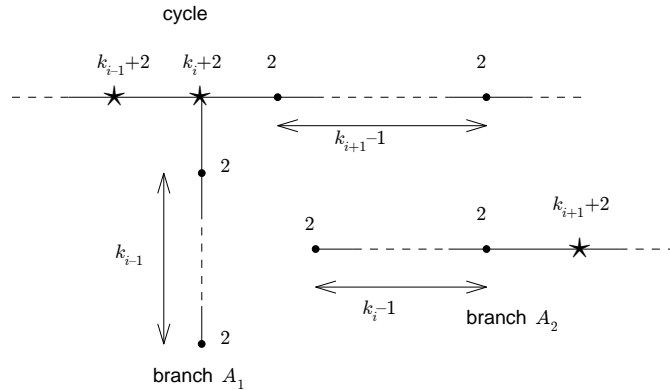
a) Condition i) has been checked in examples 21 and 46. It is not possible to have two adjacent regular sequences, hence there are  $2^N$  ways to insert regular sequences among  $N$  singular sequences, therefore we have ii).

b) We suppose now that  $N \geq 3$ . Let  $A$  be the subset (perhaps empty) of indices  $j$  in  $\mathbb{Z}/N\mathbb{Z}$  such that the singular sequence  $\sigma_{i_j} = s_{k_j}$  is followed by a regular sequence. Let  $\lambda \prod_{j \in J} X_j$ ,  $\lambda \in \mathbb{Q}$ ,  $J \subset \{0, \dots, N - 1\}$  be a monomial of  $P_\sigma$ . We are going to prove that:

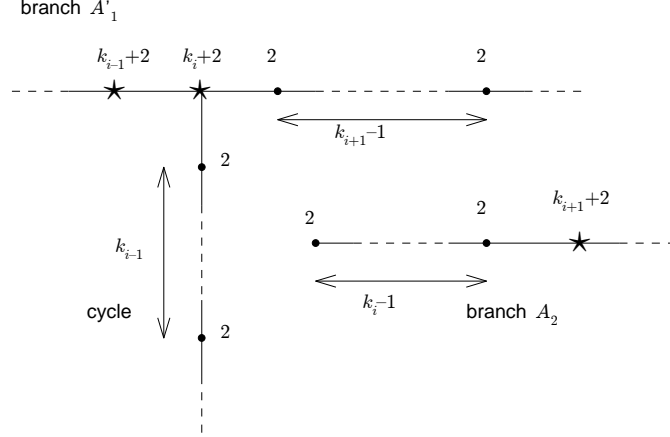
If  $i \notin J$  but  $i - 1 \in J$  and  $i + 1 \in J$ , then  $i \in A$ .

On that purpose, suppose that  $i \notin A$ ,  $i - 1 \in J$  and  $i + 1 \in J$ . Since  $\det M = P_\sigma(k_0, \dots, k_{N-1})^2$ , it is sufficient to show that in the development of  $\det M$ , any term which contains the factor  $(k_{i-1}k_{i+1})^2$  must also contain the factor  $k_i^2$ , or more simply the factor  $k_i$ . In view of the reduction lemma 45, there are two possible cases:

- $\sigma$  contains the sequence  $s_{k_{i-1}} r_1 s_{k_i} s_{k_{i+1}}$ : By theorem 9, the dual graph of  $M$  contains one of the two subgraphs







Notice that to obtain the factor  $(k_{i-1}k_{i+1})^2$  one has to develop the determinant relatively to the branch  $A_2$ , then each term containing the factor  $k_{i+1}^2$  has to contain  $k_i k_{i+1}^2$ .

- $\sigma$  contains the sequence  $s_{k_{i-1}} s_{k_i} s_{k_{i+1}}$ : a similar argument gives the result.

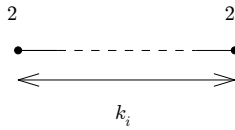
c) Any monomial of  $P_\sigma$  may be written as  $\lambda \prod_{j \notin C} X_j$ ,  $\lambda \in \mathbb{Q}$  ( $C$  is the complement of  $J$ !). Suppose that  $C \neq \emptyset$ . Then, either  $C$  contains an element of  $A$ , either  $C$  doesn't, however by b),  $C$  contains a pair  $\{j, j+1\}$ . We have proved that in all cases  $C$  contains a generating allowed subset, hence we have the second part of iv).

d) In order to see that for each allowed subset  $B \in \mathcal{P}_A$  we have

$$P_\sigma(X_i = 0, i \in B) \in \mathfrak{P}'_N$$

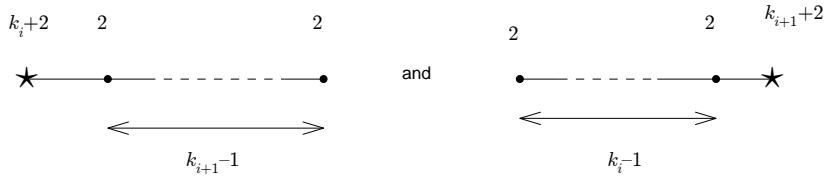
it is sufficient to check this property for generating allowed subsets  $B$ . By theorem 9,

- If  $i \in A$ , then the dual graph of  $M$  contains the subgraph



with  $k_i$  vertices (and not  $k_i - 1$ ). Vanishing of  $k_i$  yields a configuration of a branch  $A_s$  and part of cycle  $\Gamma_s$  whose parity is changed (see 9).

- If  $\{i, i+1\}$  is generating allowed pair, the dual graph contains the subgraphs



Vanishing of  $k_i$  and  $k_{i+1}$  yields the graph of  $M(\sigma')$ , where  $\sigma'$  is obtained from  $\sigma$  deleting the sequences  $s_{k_i}$  and  $s_{k_{i+1}}$ .

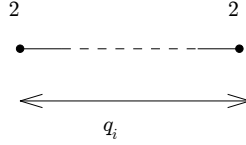
e) To end we have, on one hand, to compute the homogeneous parts of degrees  $N$  and, on second hand, to compute the homogeneous part of degree  $N - 1$  of  $P_\sigma$  to check that  $P_\sigma = P_A$  thanks to proposition 22. By reduction lemma,  $\deg_{X_i} P_\sigma \leq 1$ , hence if we show that  $P_\sigma$  contains the monomial  $\prod_{i=0}^{N-1} X_i$ . It is then necessarily its homogeneous part of highest degree.

If  $A = \emptyset$ , the dual graph contains one or two cycles without branches. To obtain in the development of  $\det M$  the term  $(\prod_{i=0}^{N-1} k_i)^2$ , it is sufficient to develop successively relatively each vertex of weight  $> 2$ . By b),  $P_\sigma$  contains no monomials of degree  $N - 1$ , which gives the result in this case.

If  $A \neq \emptyset$ , we may suppose by reduction lemma, that all regular sequences are equal to  $r_1$ . By theorem 9, all roots of the branches have weight  $> 2$ . If we develop successively relatively to each column corresponding to a vertex of weight  $> 2$ , we obtain:

$$\det M(\sigma) = \prod_{i=0}^{N-1} k_i \det B + \sum_{i=0}^{N-1} \prod_{j \neq i} k_j \det B_i \mod (k_0, \dots, k_{N-1})^{2N-2},$$

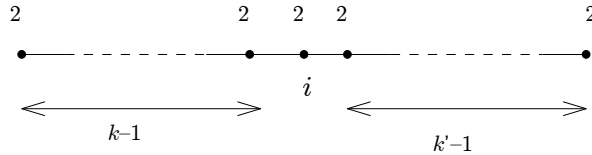
where the dual graph of  $B$  is obtained from the one of  $M(\sigma)$  by deletion of all the vertices of weight  $> 2$ , and the dual graph of  $B_i$  by deletion all the vertices of weight  $> 2$  but the one of  $k_i + 2$  and setting the weight of the latter equal to 2. Now the graph of  $B$  is composed of connected components which are chains of the form



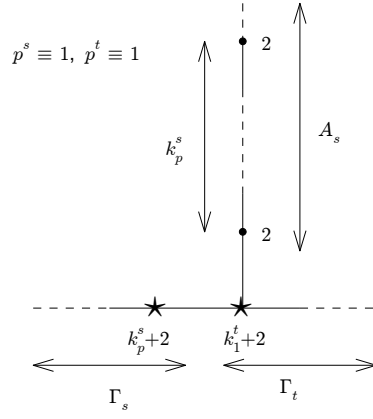
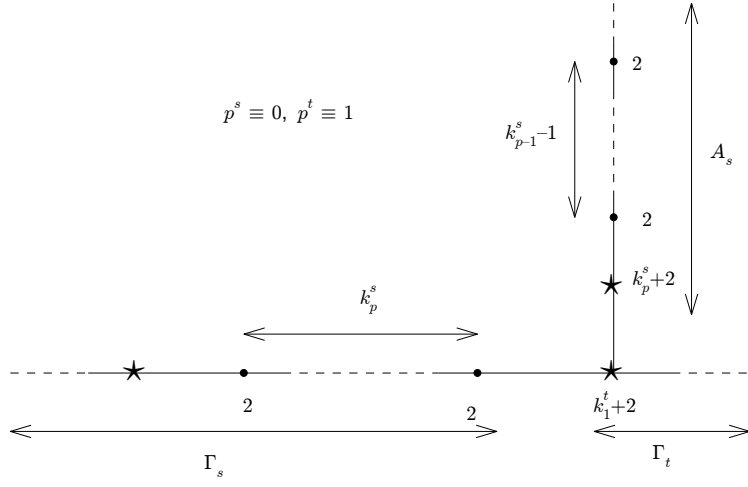
where  $q_i = k_i - 1$  (resp.  $q_i = k_i$ ) if the sequence which follows  $s_{k_i}$  is singular (resp. regular). Therefore the contribution of this term is

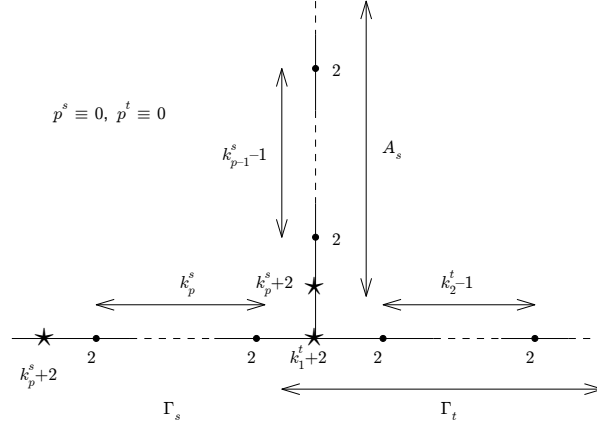
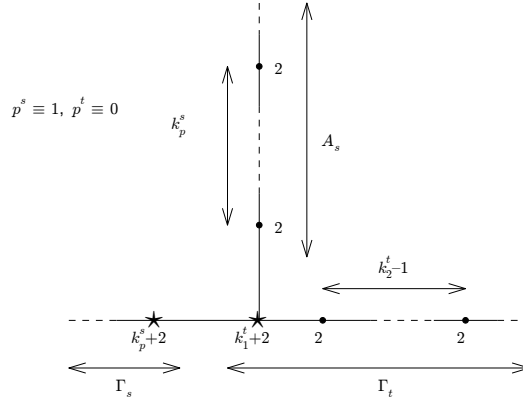
$$\prod_{i \notin A} k_i \prod_{i \in A} (k_i + 1) = \prod_{0 \leq i \leq N-1} k_i + \sum_{i \in A} \prod_{j \neq i} k_j \mod (k_0, \dots, k_{N-1})^{N-2}.$$

It remains to compute  $\det B_i$ : By lemma 42,  $\det B_i$  is a polynomial of degree at most  $N$  and we have to determine when this degree is precisely  $N$ . On that purpose, suppose that the index  $i$  corresponds to a vertex between two chains of vertices of weight 2, that is to say we have a subgraph



By lemma 38 the determinant of this connected component is  $k + k'$ , hence of degree 1 and  $\det B_i$  will be of degree at most  $N - 1$ . Therefore we are only interested in vertices which are the root of a branch or linked to a root. By theorem 9, for  $\rho(S) \geq 1$  and  $t = s + 1 \mod \rho(S)$  there are four possible situations:

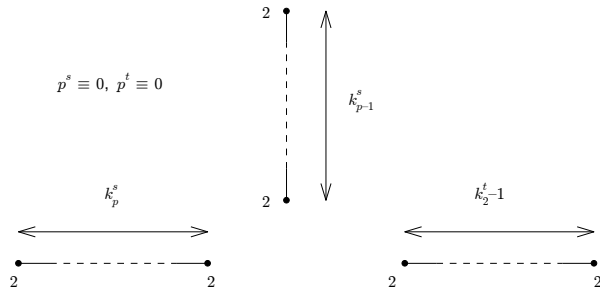
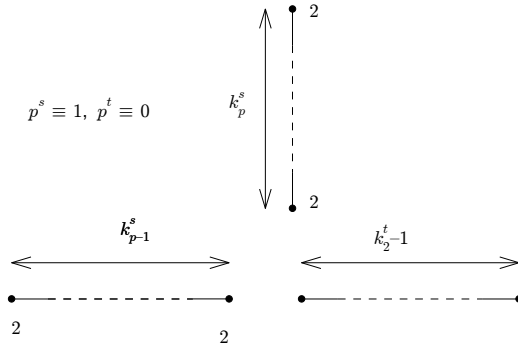
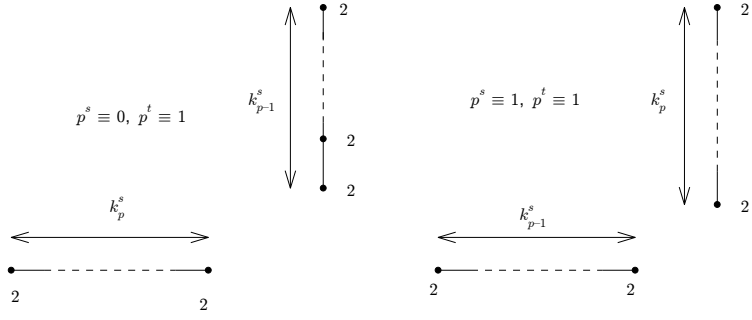




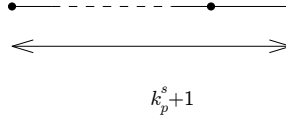
We see that the two only involved vertices are those of weight  $k_p^s + 2$  and  $k_1^t + 2$ .

- If  $s_{k_i}$  is followed by a regular sequence, i.e.  $s_{k_i} = s_{k_p^s}$  or ( $s_{k_i} = s_{k_1^t}$  and  $p^t = 1$ ):

In the first case the graph of  $B_i$  contains one of the subgraphs

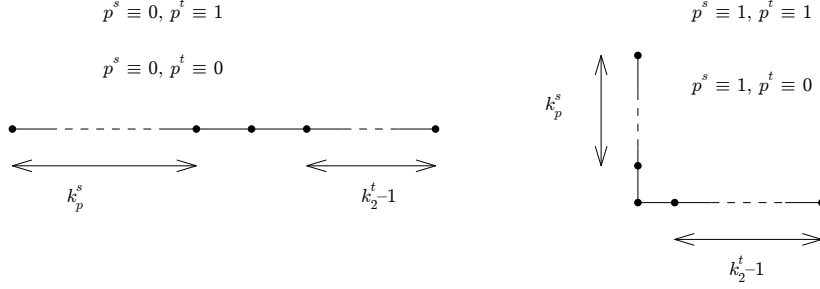


If  $p^t = 1$  and any  $p^s$  we have as connected component the subgraph



in all these cases  $\deg B_i = N$  with contribution  $\prod_{i=0}^{N-1} k_i$ .

- If  $s_{k_i}$  is followed by a singular sequence, i.e.  $s_{k_i} = s_{k_1^t}$  and (if  $p^t \equiv 1$  then  $p^t \geq 3$ ): the dual graph contains the following connected components



In all these cases,  $\deg \det B_i = N - 1$ .

Finally, we have

$$\begin{aligned}
 P_\sigma(k_0, \dots, k_{N-1})^2 &= \det M(\sigma) \\
 &= \prod_{i=0}^{N-1} k_i \left( \prod_{i=0}^{N-1} k_i + \sum_{i \in A} \prod_{j \neq i} k_j \right) + \sum_{i \in A} \prod_{j \neq i} k_j \prod_{i=0}^{N-1} k_i \\
 &= \left( \prod_{i=0}^{N-1} k_i \right)^2 + 2 \sum_{i \in A} \prod_{j \neq i} k_j \prod_{i=0}^{N-1} k_i \\
 &\quad \text{mod } (k_0, \dots, k_{N-1})^{2N-2}
 \end{aligned}$$

and  $P_\sigma = P_A$  by proposition 22, 3) as wanted.  $\square$

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