



## **$tt^*$ -geometry and pluriharmonic maps**

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# $tt^*$ -geometry and pluriharmonic maps

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## Abstract

In this paper we use the real differential geometric definition of a metric (an unimodular oriented metric)  $tt^*$ -bundle of Cortés and the author [CS] to define a map  $\Phi$  from the space of metric (unimodular oriented metric)  $tt^*$ -bundles of rank  $r$  over a complex manifold  $M$  to the space of pluriharmonic maps from  $M$  to  $GL(r)/O(p, q)$  (respectively  $SL(r)/SO(p, q)$ ), where  $(p, q)$  is the signature of the metric. In the sequel the image of the map  $\Phi$  is characterized. It follows, that in signature  $(r, 0)$  the image of  $\Phi$  is the whole space of pluriharmonic maps. This generalizes a result of Dubrovin [D].

**Keywords:**  $tt^*$ -geometry and  $tt^*$ -bundles, pluriharmonic maps, pseudo-Riemannian symmetric spaces.

**MSC(2000):** 58E20, 53C43.

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## 1 Introduction

$tt^*$ -geometry is a geometry, which has its origin in physics. Around 1990 physicist began to study topological-field-theories and their moduli-spaces, in particular N=2-supersymmetric-field-theories. A special geometric structure called topological-anti-topological fusion was found and studied (see for example [CV] and [D]). A definition of  $tt^*$ -geometry on abstract vector bundles was formulated in [Her] and [Sch1]. The former  $tt^*$ -geometries are included in this version by choosing  $TM^c$  respectively  $T^{1,0}M$  as the bundle in the abstract version. Mathematically this geometry can be considered as a generalization of variations of Hodge-structures (VHS), as it was done in a paper of Hertling [Her]. From his results follows, that a special Kähler-manifold gives a  $tt^*$ -bundle. A definition in terms of real differential geometry was given in [CS] and used to give another proof of this result not using the methods of VHS. A further interesting class of solutions are harmonic bundles first introduced by Simpson [Sim]. These solutions are considered in [Her] and [Sch2, Sch4].

A result of Dubrovin [D] associates to every  $tt^*$ -geometry with positive definite metric a pluriharmonic map to  $GL(r)/O(r)$  where  $r$  is the dimension of the base-manifold and vice-versa to every such map a  $tt^*$ -geometry. This result was proven by the author in his ‘Diplomarbeit’ [Sch1] for the case of a  $tt^*$ -geometry on an abstract vector bundle and is presented here in a more general context. The explicit form of this map in the special Kähler case, which implies its pluriharmonicity, was given in [CS]. In this context indefinite metrics can occur. This is the motivation to generalize the above result to the case of  $tt^*$ -bundles carrying indefinite metrics. In [Sch4] we applied the above result to harmonic bundles with hermitian metric of arbitrary signature and obtained a generalization of the correspondence between harmonic bundles over a compact Kähler manifold  $X$  of complex dimension  $n$  and harmonic maps from  $X$  to  $GL(n, \mathbb{C})/U(n)$ .

May we illustrate now the main results: In theorem 2 we show, that a metric  $tt^*$ -bundle with a metric of signature  $(p, q)$  over a complex manifold  $(M, J)$  gives rise to a pluriharmonic map  $f$  from  $M$  to  $GL(r)/O(p, q)$  being admissible in the following sense

**Definition 1** *Let  $(M, J)$  be a complex manifold and  $G/K$  a locally Riemannian symmetric space with associated Cartan-decomposition  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ . A map  $f : (M, J) \rightarrow G/K$  is said to be **admissible**, if the linear extension of its differential maps  $T_x^{1,0}M$  (respectively  $T_x^{0,1}M$ ) to an Abelian subspace of  $\mathfrak{p}^c$  for all  $x \in M$ .*

Conversely an admissible pluriharmonic map  $f$  from  $M$  to  $GL(r)/O(p, q)$  gives rise to a metric  $tt^*$ -bundle as is shown in theorem 3. In other words we could say, that our construction defines a map  $\Phi$  from the space of metric  $tt^*$ -bundles of rank  $r$  over a complex manifold  $(M, J)$  to the space of pluriharmonic maps from  $M$  to  $GL(r)/O(p, q)$ . The image of the map  $\Phi$  is characterized to be the admissible pluriharmonic maps from  $M$  to  $GL(r)/O(p, q)$ . The case of a metric  $tt^*$ -bundle of rank  $r$  with metric of signature  $(r, 0)$  follows from this theorem, since in this case the pluriharmonic are shown to be admissible using a result of Sampson [Sam]. It remains the question, if these all these pluriharmonic maps are admissible or if there are some counter-examples, which we do not know yet. The described results are also proven for unimodular oriented metric  $tt^*$ -bundles. Here the target space of the pluriharmonic maps is  $SL(r)/SO(p, q)$ .

We hope this approach enables a broader readership to understand this result relating physical/algebro-geometrical objects with well-known differential-geometric objects.

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## 2 $tt^*$ -bundles

For the convenience of the reader we recall the definition of a  $tt^*$ -bundle given in [CS]:

**Definition 2** A  $tt^*$ -bundle  $(E, D, S)$  over a complex manifold  $(M, J)$  is a real vector bundle  $E \rightarrow M$  endowed with a connection  $D$  and a section  $S \in \Gamma(T^*M \otimes \text{End } E)$  satisfying the  $tt^*$ -equation

$$R^\theta = 0 \quad \text{for all } \theta \in \mathbb{R}, \quad (2.1)$$

where  $R^\theta$  is the curvature tensor of the connection  $D^\theta$  defined by

$$D_X^\theta := D_X + \cos(\theta)S_X + \sin(\theta)S_{JX} \quad \text{for all } X \in TM. \quad (2.2)$$

A metric  $tt^*$ -bundle  $(E, D, S, g)$  is a  $tt^*$ -bundle  $(E, D, S)$  endowed with a possibly indefinite  $D$ -parallel fiber metric  $g$  such that  $S$  is  $g$ -symmetric, i.e. for all  $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \text{for all } X, Y, Z \in T_p M. \quad (2.3)$$

An unimodular metric  $tt^*$ -bundle  $(E, D, S, g)$  is a metric  $tt^*$ -bundle  $(E, D, S, g)$  such that  $\text{tr } S_X = 0$  for all  $X \in TM$ . An oriented unimodular metric  $tt^*$ -bundle  $(E, D, S, g, \text{or})$  is an unimodular metric  $tt^*$ -bundle endowed with an orientation or of the bundle  $E$ .

In the case of moduli spaces of topological quantum field theories [CV, D] and the moduli spaces of singularities [Her], the complexified  $tt^*$ -bundle  $E^\mathbb{C}$  is identified with  $T^{1,0}M$  and the metric  $g$  is positive definite. The case  $E = TM$ , and hence  $E^\mathbb{C} = T^{1,0}M + T^{0,1}M$  includes special complex and special Kähler manifolds, as was proven in [CS] and follows from [Her].

**Remark 1**

1) If  $(E, D, S)$  is a  $tt^*$ -bundle then  $(E, D, S^\theta)$  is a  $tt^*$ -bundle for all  $\theta \in \mathbb{R}$ , where

$$S^\theta := D^\theta - D = (\cos \theta)S + (\sin \theta)S_J. \quad (2.4)$$

The same remark applies to metric  $tt^*$ -bundles.

2) Notice that an oriented unimodular metric  $tt^*$ -bundle  $(E, D, S, g, \text{or})$  carries a canonical metric volume element  $\nu \in \Gamma(\wedge^r E^*)$ ,  $r = \text{rk } E$ , determined by  $g$  and  $\text{or}$ , which is  $D^\theta$  parallel for all  $\theta \in \mathbb{R}$ .

The following proposition characterizes  $tt^*$ -bundles  $(E, D, S)$  in form of explicit equations for  $D$  and  $S$ . These equations are important in the later calculations

**Proposition 1** *Let  $E$  be a real vector bundle over a complex manifold  $(M, J)$  endowed with a connection  $D$  and a section  $S \in \Gamma(T^*M \otimes \text{End } E)$ .*

*Then  $(E, D, S)$  is a  $tt^*$ -bundle if and only if  $D$  and  $S$  satisfy the following equations:*

$$R^D + S \wedge S = 0, \quad S \wedge S \text{ is of type } (1,1), \quad d^D S = 0 \quad \text{and} \quad d^D S_J = 0. \quad (2.5)$$

*Proof:* As the attentive reader may observe, it is easier to show this proposition after complexifying  $TM$ . But since one idea of the paper was to formulate these results in real differential geometry, we give a real version of the proof.

To prove the proposition, we have to compute the curvature of  $D^\theta$ .

Let  $X, Y \in \Gamma(TM)$  arbitrary:

$$\begin{aligned} R_{X,Y}^\theta &= R_{X,Y}^D + [D_X, \cos(\theta)S_Y + \sin(\theta)S_{JY}] + [\cos(\theta)S_X + \sin(\theta)S_{JX}, D_Y] \\ &\quad + [\cos(\theta)S_X + \sin(\theta)S_{JX}, \cos(\theta)S_Y + \sin(\theta)S_{JY}] - \cos(\theta)S_{[X,Y]} - \sin(\theta)S_{J[X,Y]} \\ &= R_{X,Y}^D + \sin^2(\theta)[S_{JX}, S_{JY}] + \cos^2(\theta)[S_X, S_Y] + \cos(\theta)\sin(\theta)([S_X, S_{JY}] + [S_{JX}, S_Y]) \\ &\quad + \cos(\theta)([D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}) + \sin(\theta)([S_{JX}, D_Y] + [D_X, S_{JY}] - S_{J[X,Y]}). \end{aligned}$$

We now recall the Fourier-expansion of

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2} \cos(2\theta) \quad \text{and} \quad \sin^2(\theta) = \frac{1}{2} - \frac{1}{2} \cos(2\theta)$$

to find

$$\begin{aligned} R_{X,Y}^\theta &= R_{X,Y}^D + \frac{1}{2}([S_X, S_Y] + [S_{JX}, S_{JY}]) + \cos(\theta)([D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}) \\ &\quad + \sin(\theta)([S_{JX}, D_Y] + [D_X, S_{JY}] - S_{J[X,Y]}) + \frac{1}{2} \cos(2\theta)([S_X, S_Y] - [S_{JX}, S_{JY}]) \\ &\quad + \frac{1}{2} \sin(2\theta)([S_X, S_{JY}] + [S_{JX}, S_Y]). \end{aligned}$$

Taking Fourier-coefficients yields

$$\begin{aligned} 0 &= R_{X,Y}^D + \frac{1}{2}([S_X, S_Y] + [S_{JX}, S_{JY}]), \\ 0 &= [S_X, S_Y] - [S_{JX}, S_{JY}], \quad 0 = [S_X, S_{JY}] + [S_{JX}, S_Y], \\ 0 &= [D_X, S_Y] + [S_X, D_Y] - S_{[X,Y]}, \quad 0 = [S_{JX}, D_Y] + [D_X, S_{JY}] - S_{J[X,Y]} \end{aligned}$$

and equivalently

$$R_{X,Y}^D + [S_X, S_Y] = 0, \quad S \wedge S(X, Y) = [S_X, S_Y] = [S_{JX}, S_{JY}], \quad d^D S = 0 \quad \text{and} \quad d^D S_J = 0.$$

□

### 3 Pluriharmonic maps

In this section we recall the notion of pluriharmonic maps and explain some properties of pluriharmonic maps to  $S(p, q) := GL(r)/O(p, q)$  where  $O(p, q)$  is the pseudo-orthogonal group of signature  $(p, q)$  respectively  $S^1(p, q) := SL(r)/SO(p, q)$ , which are needed later to prove the main theorem.

In order to uniform the formulation of the paper we introduce the following notions:

$$\begin{aligned} G_0(r) &= GL(r), \quad G_1(r) = SL(r), \quad \mathfrak{g}_0 = \mathfrak{gl}(r), \quad \mathfrak{g}_1 = \mathfrak{sl}(r), \\ K_0(p, q) &= O(p, q), \quad K_1(p, q) = SO(p, q), \quad \mathfrak{k}_0 = \mathfrak{k}_1 = \mathfrak{so}(p, q), \\ S^0(p, q) &= S(p, q). \end{aligned}$$

These objects are also written with an index  $i \in \{0, 1\}$ .

**Definition 3** *Let  $(M, J)$  be a complex manifold and  $(N, h)$  a pseudo-Riemannian manifold with Levi-Civita connection  $\nabla^h$ ,  $D$  a connection on  $M$  which satisfies*

$$D_{JY}X = JD_YX \tag{3.1}$$

*for all vector fields which satisfy  $\mathcal{L}_X J = 0$  (i.e. for which  $X - iJX$  is holomorphic) and  $\nabla$  the connection on  $T^*M \otimes f^*TN$  which is induced by  $D$  and  $\nabla^h$ .*

*A map  $f : M \rightarrow N$  is pluriharmonic if and only if it satisfies the equation*

$$\nabla'' \partial f = 0, \tag{3.2}$$

*where  $\partial f = df^{1,0} \in \Gamma(\bigwedge^{1,0} T^*M \otimes_{\mathbb{C}} (TN)^{\mathbb{C}})$  is the  $(1, 0)$ -component of  $d^c f$  and  $\nabla''$  is the  $(0, 1)$ -component of  $\nabla = \nabla' + \nabla''$ .*

*Equivalently one regards  $\alpha = \nabla d\phi \in \Gamma(T^*M \otimes T^*M \otimes \phi^*TN)$ .*

*Then  $\phi$  is pluriharmonic if and only if*

$$\alpha(X, Y) + \alpha(JX, JY) = 0$$

*for all  $X, Y \in TM$ .*

#### Remark 2

1. *Note, that  $f$  is pluriharmonic iff  $f$  restricted to every holomorphic curve is harmonic. In fact, this gives a definition of pluriharmonic maps, which is independent of the choosen connections. For a short discussion of this the reader is referred to [CS].*
2. *Any complex manifold  $(M, J)$  admits a torsion free complex connection  $D$  (Complex means  $DJ = 0$ .) and consequently a connection satisfying (3.1). In the rest of the paper, we want therefore suppose, that the connection on  $(M, J)$  is also torsion free.*

Let  $Sym_{p,q}^i(\mathbb{R}^r)$  be the symmetric  $r \times r$  matrices in  $G_i(r)$  of signature  $(p, q)$ . These define pseudo-scalar-products of same signature by  $\langle \cdot, \cdot \rangle_A = \langle A \cdot, \cdot \rangle_{\mathbb{R}^r}$ , where  $\langle \cdot, \cdot \rangle_{\mathbb{R}^r}$  is the Euclidean scalar-product. The natural action of an element  $g \in G_i(r)$  is given by  $\langle g^{-1} \cdot, g^{-1} \cdot \rangle_A = \langle (g^{-1})^t A g^{-1} \cdot, \cdot \rangle_{\mathbb{R}^r}$ . This gives us an action of  $G_i(r)$   $A \mapsto (g^{-1})^t A g^{-1}$  on  $Sym_{p,q}^i(\mathbb{R}^r)$  which we use to identify  $Sym_{p,q}^i(\mathbb{R}^r)$  with  $S^i(p, q)$  in the following

**Proposition 2** *Let  $\Psi^i$  be the canonical map  $\Psi^i : S^i(p, q) \xrightarrow{\sim} \text{Sym}_{p,q}^i(\mathbb{R}^r) \subset G_i(r)$  where  $G_i(r)$  carries the pseudo-Riemannian Ad-invariant trace-form. Then  $\Psi^i$  is a totally-geodesic immersion and a map  $f$  from a complex manifold  $(M, J)$  to  $S^i(p, q)$  is pluriharmonic, iff the map  $\Psi^i \circ f : M \rightarrow G^i(r)$  is pluriharmonic.*

*Proof:* The proof is done by expressing the map  $\Psi^i$  in terms of the well-known Cartan-immersion. For further information see for example [Hel], [CE], [GHL], [KN].

- 1) First we study the identification  $S^i(p, q) \xrightarrow{\sim} \text{Sym}_{p,q}^i(\mathbb{R}^r)$ .  
By Sylvesters theorem  $G_i(r)$  operates on  $\text{Sym}_{p,q}^i(\mathbb{R}^r)$  via

$$G_i(r) \times \text{Sym}_{p,q}^i(\mathbb{R}^r) \rightarrow \text{Sym}_{p,q}^i(\mathbb{R}^r), \quad (g, B) \mapsto g \cdot B := (g^{-1})^t B g^{-1}.$$

The stabilisator of the point  $I_{p,q} = \text{diag}(\mathbb{1}_p, -\mathbb{1}_q)$  is  $K_i(p, q)$  and the above action is transitive by Sylvesters theorem. Therefore by the orbit-stabilizer theorem (compare Gallot, Hulin, Lafontaine [GHL] 1.100) we obtain a diffeomorphism  $\Psi^i : S^i(p, q) \xrightarrow{\sim} \text{Sym}_{p,q}^i(\mathbb{R}^r)$ ,  $g K_i(p, q) \mapsto g \cdot I_{p,q} = (g^{-1})^t I_{p,q} g^{-1}$ .

- 2) We recall some results about symmetric spaces (see: [CE]). Let  $G$  be a Lie-group and  $\sigma : G \rightarrow G$  a group-homomorphism with  $\sigma^2 = \text{Id}_G$ . Let  $K$  denote the subgroup  $K = G^\sigma = \{g \mid \sigma(g) = g\}$ . The Lie-algebra  $\mathfrak{g}$  of  $G$  decomposes in  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  with  $d\sigma_{\text{Id}_G}(\mathfrak{h}) = \mathfrak{h}$ ,  $d\sigma_{\text{Id}_G}(\mathfrak{p}) = -\mathfrak{p}$ . And we have the following information: The map  $\phi : G/K \rightarrow G$  with  $\phi : [gK] \mapsto g\sigma(g^{-1})$  defines a totally geodesic immersion called the Cartan-immersion.

We want to utilize this:

Therefore we define  $\sigma : G_i(r) \rightarrow G_i(r)$ ,  $g \mapsto (g^{-1})^\dagger$  where  $g^\dagger = I_{p,q} g^t I_{p,q}$  is the adjoint with respect to the pseudo-scalar product  $\langle \cdot, \cdot \rangle_{I_{p,q}} = \langle \cdot, I_{p,q} \cdot \rangle_{\mathbb{R}^n}$ .

$\sigma$  is obviously a homomorphism and an involution with  $G_i(r)^\sigma = K_i(p, q)$ . By a direct calculation one gets  $d\sigma_{\text{Id}_G} = -h^\dagger$  and hence

$$\begin{aligned} \mathfrak{h} &= \{h \in \mathfrak{gl}(r) \mid h^\dagger = -h\} = \mathfrak{o}(p, q) = \mathfrak{so}(p, q), \\ \mathfrak{p} &= \{h \in \mathfrak{gl}(r) \mid h^\dagger = h\} =: \text{sym}^i(p, q). \end{aligned}$$

Thus we end up with

$$\phi : S^i(p, q) \rightarrow G_i(r), \tag{3.3}$$

$$g \mapsto g\sigma(g^{-1}) = gg^\dagger = gI_{p,q}g^tI_{p,q} = R_{I_{p,q}} \circ \Psi^i \circ \Lambda(g). \tag{3.4}$$

Here  $R_h$  is the right multiplication by  $h$  and  $\Lambda$  is the map  $\Lambda : G_i \rightarrow G_i$ ,  $h \mapsto (h^{-1})^t$ . Both maps are isometries of the invariant metric. Hence  $\Psi^i$  is a totally-geodesic immersion.

- 3) Pluriharmonicity is independent of the connection satisfying (3.1) chosen on  $M$ . Therefore we can take it torsion free (see remark 2). We calculate the tensor

$$\nabla df(X, Y) = \nabla_X^N(df(Y)) - df(D_X Y).$$

for holomorphic vector fields  $X, Y$ . The (1,1)-part of the second term vanishes for holomorphic  $X, Y$ , since

$$D_X Y + D_{JX} JY = D_X Y + JD_{JX} Y = D_X Y + J^2 D_X Y = 0.$$

Hence we have only to regard the Levi-Civita-connections on  $G_i$  and  $G_i/K_i = S^i(p, q)$ . Let  $X, Y \in \Gamma(TM)$  holomorphic and calculate:

$$\nabla_X^{G_i} d(\Psi^i \circ f)(Y) = \nabla_X^{G_i} d\Psi^i(df(Y)) = \nabla_X^{G_i} \Psi_*^i(df(Y)) = \Psi_*^i(\nabla_X^{G_i/K_i} df(Y)) + II(X, Y)$$

where  $II$  is the second fundamental-form which vanishes, as the immersion is totally geodesic. This implies with the notation  $\alpha^{G_i} = \nabla^{G_i} d(\Psi^i \circ f)$  and  $\alpha^{G_i/K_i} = \nabla^{G_i/K_i} df$

$$\begin{aligned} \alpha^{G_i}(X, Y) + \alpha^{G_i}(JX, JY) &= \nabla_X^{G_i} d(\Psi^i \circ f)(Y) + \nabla_{JX}^{G_i} d(\Psi^i \circ f)(JY) \\ &= \Psi_*^i \left( \nabla_X^{G_i/K_i} df(Y) + \nabla_{JX}^{G_i/K_i} df(JY) \right) \\ &= \Psi_*^i \left( \alpha^{G_i/K_i}(X, Y) + \alpha^{G_i/K_i}(JX, JY) \right). \end{aligned}$$

Since  $\Psi^i$  is an immersion, the left side is zero iff the right is and the proof is finished.  $\square$

**Remark 3** (compare [CS])

Above we have identified  $G_i(r)/K_i(p, q)$  with  $\text{Sym}_{p,q}^i(\mathbb{R}^r)$  via  $\Psi^i$ .

Let us choose  $o = eK_i(p, q)$  as base point and suppose that  $\Psi^i$  is chosen to map  $o$  to  $I = I_{p,q}$ . By construction  $\Psi^i$  is  $G_i(r)$ -equivariant. We identify the tangent-space  $T_S \text{Sym}_{p,q}^i(\mathbb{R}^r)$  at  $S \in \text{Sym}_{p,q}^i(\mathbb{R}^r)$  with the (ambient) vector space of symmetric matrices:

$$T_S \text{Sym}_{p,q}^i(\mathbb{R}^r) = \text{Sym}^i(\mathbb{R}^r) := \{A \in \mathfrak{g}_i(r) | A^t = A\}. \quad (3.5)$$

For  $\Psi^i(\tilde{S}) = S$ , the tangent space  $T_{\tilde{S}} S^i(p, q)$  is canonically identified with the vector space of  $S$ -symmetric matrices:

$$T_{\tilde{S}} S^i(p, q) = \text{sym}^i(S) := \{A \in \mathfrak{g}_i(r) | AS = SA^t\}. \quad (3.6)$$

Note that  $\text{sym}^i(I_{p,q}) = \text{sym}^i(p, q)$ .

**Proposition 3** The differential of  $\varphi^i := (\Psi^i)^{-1}$  at  $S \in \text{Sym}_{p,q}^i(\mathbb{R}^r)$  is given by

$$\text{Sym}^i(\mathbb{R}^r) \ni X \mapsto -\frac{1}{2} S^{-1} X \in S^{-1} \text{Sym}^i(\mathbb{R}^r) = \text{sym}^i(S). \quad (3.7)$$

Using this proposition we relate now the differentials

$$df_x : T_x M \rightarrow \text{Sym}^i(\mathbb{R}^r) \quad (3.8)$$

of a map  $f : M \rightarrow \text{Sym}_{p,q}^i(\mathbb{R}^r)$  at  $x \in M$  and

$$d\tilde{f}_x : T_x M \rightarrow \text{sym}^i(f(x)) \quad (3.9)$$

a map  $\tilde{f} = \varphi \circ f : M \rightarrow S^i(p, q)$ :  $d\tilde{f}_x = d\varphi df_x = -\frac{1}{2} f(x)^{-1} df_x$ .

One can interpret the 1-form  $A = -2d\tilde{f} = f^{-1}df$  with values in  $\mathfrak{g}_i(r)$  as connection form on the vector bundle  $E = M \times \mathbb{R}^r$ . We note, that the definition of  $A$  is the pure gauge. This means, that  $A$  is gauge-equivalent to  $A' = 0$ , as for  $A' = 0$  one has  $A = f^{-1}A'f + f^{-1}df = f^{-1}df$ . The curvature vanishes, since it is independent of gauge. Thus we get:



**Proposition 4** *Let  $f : M \rightarrow G_i(r)$  be a  $C^\infty$ -mapping and  $A := f^{-1}df : TM \rightarrow \mathfrak{g}_i(r)$ . Then the curvature of  $A$  vanishes, i.e. for  $X, Y \in \Gamma(TM)$*

$$Y(A_X) - X(A_Y) = A_{[X,Y]} - [A_X, A_Y]. \quad (3.10)$$

In the next proposition we give the equations for pluriharmonic maps from a complex manifold to  $G_i(r)$ .

**Proposition 5** *Let  $(M, J)$  be a complex manifold,  $f : M \rightarrow G_i(r)$  a  $C^\infty$ -map and  $A$  defined as in proposition 4.*

*The pluriharmonicity of  $f$  is equivalent to the equation*

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] + JY(A_{JX}) + \frac{1}{2}[A_{JY}, A_{JX}] = 0, \quad (3.11)$$

*for holomorphic  $X, Y \in \Gamma(TM)$ .*

*Proof:* Again pluriharmonicity of  $f$  does not depend on the connection satisfying (3.1) on  $M$ . Hence the (1,1)-part of the second term of  $\nabla df(X, Y)$  vanishes for holomorphic  $X, Y$ , as in the proof of proposition 2. Therefore we only have to regard the pulled back Levi-Civita connection  $\nabla$  on  $G_i(r)$ .

Let  $X, Y \in \Gamma(TM)$ . To find these equations we write  $df(X)$  and  $df(Y)$  that are sections in  $f^*TG_i(r)$ , as linear combination of left invariant vector fields  $f^*\tilde{E}_{ij} = \tilde{E}_{ij} \circ f$ , with  $\tilde{E}_{ij}(g) = gE_{ij}$ ,  $\forall g \in G_i(r)$  and a basis  $E_{ij}, i, j = 1 \dots r$  of  $\mathfrak{g}_i(r)$ .

In this notation we have

$$df(X) = \sum_{i,j} a_{ij} \tilde{E}_{ij} \circ f = \sum_{i,j} a_{ij} fE_{ij} \text{ and } df(Y) = \sum_{i,j} b_{ij} \tilde{E}_{ij} \circ f = \sum_{i,j} b_{ij} fE_{ij},$$

with functions  $a_{ij}$  and  $b_{ij}$  on  $M$  and further

$$A_X = f^{-1}df(X) = \sum_{i,j} a_{ij} E_{ij} \text{ and } A_Y = f^{-1}df(Y) = \sum_{i,j} b_{ij} E_{ij}.$$

With this information we compute

$$\begin{aligned} (f^*\nabla)_Y df(X) &= (f^*\nabla)_Y \sum_{i,j} a_{ij} \tilde{E}_{ij} \circ f \\ &= \sum_{i,j} Y(a_{ij}) \tilde{E}_{ij} \circ f + \sum_{i,j} a_{ij} (f^*\nabla)_Y \tilde{E}_{ij} \circ f \\ &= \sum_{i,j} Y(a_{ij}) \tilde{E}_{ij} \circ f + \sum_{i,j} a_{ij} \nabla_{df(Y)} \tilde{E}_{ij} \circ f \\ &= \sum_{i,j} Y(a_{ij}) fE_{ij} + \sum_{abij} a_{ij} b_{ab} \underbrace{(\nabla_{\tilde{E}_{ab}} \tilde{E}_{ij}) \circ f}_{\frac{1}{2}f[E_{ab}, E_{ij}]} \\ &= f(Y(A_X) + \frac{1}{2}[A_Y, A_X]). \end{aligned}$$

Therefore the pluriharmonicity is equivalent to the equation

$$Y(A_X) + \frac{1}{2}[A_Y, A_X] + JY(A_{JX}) + \frac{1}{2}[A_{JY}, A_{JX}] = 0$$

for holomorphic  $X, Y$ .  $\square$

Suppose that  $N$  is a locally Riemannian symmetric space with universal cover  $G/K$  with non-compact semi-simple Lie group  $G$ , maximal compact subgroup  $K$  and associated Cartan-decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ . In each point one identifies the tangent space of  $N$  with  $\mathfrak{p}$ . This is unique up to right action of  $K$  and left action of the fundamental group. All relevant structures are preserved by these actions. Therefore, given a  $f : M \rightarrow N$ , we can regard  $df_x(T_x^{1,0}M)$ ,  $x \in M$  as a subspace of  $\mathfrak{p}^c$ . For the ‘complexified’ sectional-curvature of  $N$  holds using the Killing-form  $b$

$$b(R(X, Y)\bar{Y}, \bar{X}) = -b([X, Y], [\bar{Y}, \bar{X}]) \leq 0. \quad (3.12)$$

It is a well-known result of Sampson [Sam], that a harmonic map of a compact complex manifold to a locally symmetric space of non-compact type is pluriharmonic and that its differential sends  $T^{1,0}M$  to an Abelian subspace of  $\mathfrak{p}^c$ . The second claim, that the image of  $T^{1,0}M$  under the differential of a pluriharmonic map is Abelian is true on non-compact manifolds, too. To illustrate this, we are going to prove, that pluriharmonicity implies this property.

**Theorem 1** (compare [Sam]) *Let  $(M, J)$  be a complex manifold and  $N = G/K$  be a locally Riemannian symmetric space as above.*

*Then the complex linear extended differential of a pluriharmonic map  $f : M \rightarrow N$  maps for all  $x \in M$   $T_x^{1,0}M$  (respectively  $T_x^{0,1}M$ ) to an Abelian subspace of  $\mathfrak{p}^c$ .*

*On  $TM$  the differential of a pluriharmonic map  $f : M \rightarrow N$  obeys the equation*

$$[df_x(X), df_x(Y)] = [df_x(JX), df_x(JY)]$$

*with  $X, Y \in T_xM, x \in M$ .*

*Proof:* The strategy is to show the vanishing of the curvature.

Let  $X, Y, Z, W \in \Gamma(T^{1,0}M)$  be holomorphic

$$\begin{aligned} R^N(f_*X, f_*Y)f_*\bar{Z} &= R^{f^*\nabla^N}(X, Y)f_*\bar{Z} \\ &= (f^*\nabla^N)_X(f^*\nabla^N)_Yf_*\bar{Z} - (f^*\nabla^N)_Y(f^*\nabla^N)_Xf_*\bar{Z} - (f^*\nabla^N)_{[X, Y]}f_*\bar{Z} \end{aligned}$$

We remark now, that the pluriharmonic equation for holomorphic vector fields depends not on the connection chosen on the manifold  $M$ . Hence it reduces to the equation  $(f^*\nabla^N)_Xf_*\bar{Y} = 0$ , which implies  $R^N(f_*X, f_*Y)f_*\bar{Z} = 0$ . From equation (3.12) we get  $b([f_*X, f_*Y], [f_*\bar{Z}, f_*\bar{W}]) = 0$  and in the end  $[f_*X, f_*Y] = 0$  for all  $X, Y$ .

Let  $Z, W \in \Gamma(T^{1,0}M)$  be of the form  $Z = X - iJX$  and  $W = Y - iJY$  with  $X, Y \in \Gamma(TM)$  and compute  $[f_*Z, f_*W] = [f_*X, f_*Y] - [f_*JX, f_*JY] - i([f_*X, f_*JY] + [f_*JX, f_*Y])$ . Hence we conclude  $[df(X), df(Y)] = [df(JX), df(JY)]$ .  $\square$

**Corollary 1** *Let  $(M, J)$  be a complex manifold,  $f : M \rightarrow \text{Sym}_{r,0}^i(\mathbb{R}^r) \xrightarrow{\iota} G_i(r)$  a pluriharmonic map induced by a pluriharmonic map to  $G_i(r)/K_i(r)$  and  $A$  defined as in proposition 4. If  $f$  is a pluriharmonic map, then the operators  $A$  satisfy for all  $X, Y \in T_x M$ , with  $x \in M$ , the equation  $[A_X, A_Y] = [A_{JX}, A_{JY}]$ .*

*Proof:* First, we apply theorem 1 to  $A = -2d\tilde{f} : M \rightarrow G_1 = SL(r)$ . This yields the corollary for  $G_1 = SL(r)$ .

For  $S^0(p, q) = S(p, q)$  we have the de Rham-decomposition  $S(p, q) = \mathbb{R} \times S^1(p, q)$ , where  $\mathbb{R}$  corresponds to the connected central subgroup  $\mathbb{R}^{>0} = \{\lambda Id | \lambda > 0\} \subset G_0 = GL(r)$ . Hence we have the decomposition of  $\mathfrak{gl}(r) = \mathbb{R} \oplus \mathfrak{sl}(r)$ , where the  $\mathbb{R}$ -factor is central. Therefore we are in the situation to apply the result for  $G_1$ .  $\square$

**Remark 4** *Since the trace-form on  $SL(r)$  is a multiple of the Killing-form and on  $GL(r)$  it corresponds to the metric on the decomposition  $S(p, q) = \mathbb{R} \times S^1(p, q)$ , we can choose the trace-form as metric and obtain the same result as in theorem 1 and corollary 1.*

## 4 $tt^*$ -geometry and pluriharmonic maps

In this section we are going to state and prove the main results. Like in section 3 one regards the mapping  $A = f^{-1}df$  as a map  $A : TM \rightarrow \mathfrak{g}_i(r)$

We now suppose, that the complex manifold  $(M, J)$  is simply connected. Using the same considerations as in [Sch1] the main theorems, theorem 2 and theorem 3, can be extended to non-simply connected manifolds by pulling back the metric  $tt^*$ -bundles to the universal cover of  $M$ . The according pluriharmonic maps have to be replaced by twisted pluriharmonic maps

**Theorem 2** *Let  $(M, J)$  be a simply-connected complex manifold. Let  $(E, D, S, g [, or])$  be a metric [an unimodular oriented metric]  $tt^*$ -bundle where  $E$  has rank  $r$  and  $M$  dimension  $n$ .*

*Then the representation of the metric  $g$  in a  $D^\theta$ -flat frame of  $E$   $f : M \rightarrow \text{Sym}_{p,q}^i(\mathbb{R}^r)$  induces an admiseable pluriharmonic map  $\tilde{f} : M \xrightarrow{f} \text{Sym}_{p,q}^i(\mathbb{R}^r) \xrightarrow{\sim} S^i(p, q)$ , where  $S^i(p, q)$  carries the metric induced by the biinvariant pseudo-Riemannian trace-form on  $\mathfrak{g}_i(r)$ .*

*Let  $s'$  be another  $D^\theta$ -flat frame. Then  $s' = s \cdot U$  for a constant matrix and the pluriharmonic map associated to  $S'$  is  $f' = U^t f U$ .*

**Remark 5** *(see also [CS]) Before proving the theorem we make some remarks on the condition on  $d\tilde{f}$ . Let  $x \in M$  and  $\tilde{f}(x) = uo$ . If  $d\tilde{f}(T_x^{1,0}M)$  consist of commuting matrices, then  $dL_u^{-1}d\tilde{f}(T_x^{1,0}M)$  is commutative, too. This follows from the fact, that*

$$dL_u : T_o S^i(p, q) \rightarrow T_{uo} S^i(p, q) = T_{\tilde{f}(x)} S^i(p, q)$$

*equals*

$$Ad_u : \text{sym}^i(p, q) = \text{sym}^i(I_{p,q}) \rightarrow \text{sym}^i(u \cdot I_{p,q}) = \text{sym}^i(\tilde{f}),$$

*which preserves the Lie-bracket.*

*Proof:* Using remark 1.1) it suffices to prove the case  $\theta = \pi$ .

We first consider a metric  $tt^*$ -bundle  $(E, D, S, g)$ .

Let  $s := (s_1, \dots, s_r)$  be a  $D^\pi$ -flat frame of  $E$  (i.e.  $Ds = Ss$ ),  $f$  the matrix  $g(s_k, s_l)$  and further  $S^s$  the matrix-valued one-form representing  $S$  in the frame  $s$ . For  $X \in \Gamma(TM)$  we get:

$$\begin{aligned} X(f) &= Xg(s, s) = g(D_X s, s) + g(s, D_X s) \\ &= g(S_X s, s) + g(s, S_X s) \\ &= 2g(S_X s, s) = 2f \cdot S^s(X) = 2f \cdot S_X^s. \end{aligned}$$

Consequently  $A_X = 2S_X^s$ . We now prove the pluriharmonicity using

$$d^D S(X, Y) = D_X(S_Y) - D_Y(S_X) - S_{[X, Y]} = 0, \quad (4.1)$$

$$d^D S_J(X, Y) = D_X(S_{JY}) - D_Y(S_{JX}) - S_{J[X, Y]} = 0. \quad (4.2)$$

The equation (4.2) implies

$$\begin{aligned} 0 = d^D S_J(JX, Y) &= D_{JX}(S_{JY}) + \underbrace{D_Y(S_X)}_{\stackrel{(4.1)}{=} D_X(S_Y) - S_{[X, Y]}} - S_{J[JX, Y]} \\ &= D_{JX}(S_{JY}) + D_X(S_Y) - S_{[X, Y]} - S_{J[JX, Y]}. \end{aligned}$$

In local holomorphic coordinate fields  $X, Y$  on  $M$  we get in the frame  $s$

$$JX(S_{JY}^s) + X(S_Y^s) + [S_X^s, S_Y^s] + [S_{JX}^s, S_{JY}^s] = 0.$$

Now  $A = 2S^s$  gives equation (3.11) and proves the pluriharmonicity of  $f$ .

Using  $A_X = 2S_X^s = -2d\tilde{f}(X)$ , we find the property of the differential, as  $S \wedge S$  is of type (1,1) using the  $tt^*$ -equations, see proposition 1.

The last statement is obvious.

In the case of an oriented unimodular metric  $tt^*$ -bundle  $(E, D, S, g, or)$  we can take the frame  $s$  to be oriented and of volume 1, with respect to the canonical  $D^\theta$ -parallel- metric volume  $\nu$ . Therefore the map  $f$  takes values in  $Sym_{p,q}^1(\mathbb{R}^r)$  and the above arguments show the rest.  $\square$

**Theorem 3** *Let  $(M, J)$  be a simply-connected complex manifold and put  $E = M \times \mathbb{R}^r$ .*

*Then a pluriharmonic map  $\tilde{f} : M \rightarrow S^i(p, q)$  give rise to a pluriharmonic map  $f : M \xrightarrow{\tilde{f}} S^i(p, q) \xrightarrow{\sim} Sym_{p,q}^i(\mathbb{R}^r) \xhookrightarrow{\iota} G_i(r)$ .*

*If  $\tilde{f}$  is admissible, then the map  $f$  induces a metric  $tt^*$ -bundle [an unimodular oriented metric  $tt^*$ -bundle]  $(E, D = \partial + S, S = -d\tilde{f}, g = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r}[, or])$  on  $M$  where  $\partial$  is the canonical flat connection on  $E$  and  $or$  is the canonical orientation on  $E$ .*

**Remark 6** *We observe, that for Riemannian surfaces  $M = \Sigma$  the condition on the differential holds, since  $T^{1,0}\Sigma$  is one-dimensional.*

*Proof:*

Let  $\tilde{f} : M \rightarrow S^i(p, q)$  be a pluriharmonic map. Then by proposition 2 we know, that  $f : M \rightarrow \text{Sym}_{p,q}^i(\mathbb{R}) \xrightarrow{\iota} G_i(r)$  is pluriharmonic.

Since  $E = M \times \mathbb{R}^r$ , we can regard sections of  $E$  as  $r$ -tuples of  $C^\infty(M, \mathbb{R})$ -functions.

In the spirit of section 3 we regard the one form  $A = -2d\tilde{f} = f^{-1}df$  with values in  $\mathfrak{g}_i(r)$  as a connection on  $E$ . We remind, that the curvature of this connection vanishes (proposition 4).

a) First, we check the conditions on the metric:

**Lemma 1** *The connection  $D$  is compatible with the metric  $g$  and  $S$  is symmetric with respect to  $g$ .*

*Proof:* This is a direct computation with  $X \in \Gamma(TM)$  and  $v, w \in \Gamma(E)$  using the relations  $(*)$   $S = \frac{1}{2}f^{-1}df$ ,  $(**)$   $df_x : T_x M \rightarrow T_{f(x)}\text{Sym}_{p,q}^i(\mathbb{R}^r) = \text{Sym}^i(\mathbb{R}^r)$  (compare remark 3) and  $g = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r} = \langle \cdot, f \cdot \rangle_{\mathbb{R}^r}$  which follows from  $f : M \rightarrow \text{Sym}_{p,q}^i(\mathbb{R}^r)$  :

$$\begin{aligned} X(g(v, w)) &= X(\langle f v, w \rangle_{\mathbb{R}^r}) = \langle X(f)v, w \rangle_{\mathbb{R}^r} + \langle f(\partial_X v), w \rangle_{\mathbb{R}^r} + \langle f v, \partial_X w \rangle_{\mathbb{R}^r} \\ &\stackrel{(**)}{=} \frac{1}{2} \langle X(f)v, w \rangle_{\mathbb{R}^r} + \frac{1}{2} \langle v, X(f)w \rangle_{\mathbb{R}^r} + \langle f(\partial_X v), w \rangle_{\mathbb{R}^r} + \langle f v, \partial_X w \rangle_{\mathbb{R}^r} \\ &= \frac{1}{2} \langle f \cdot f^{-1}(X(f))v, w \rangle_{\mathbb{R}^r} + \frac{1}{2} \langle v, f \cdot f^{-1}(X(f))w \rangle_{\mathbb{R}^r} \\ &\quad + \langle f \partial_X v, w \rangle_{\mathbb{R}^r} + \langle f v, \partial_X w \rangle_{\mathbb{R}^r} \\ &\stackrel{(*),(**)}{=} g(X.v + S_X v, w) + g(v, X.w + S_X w) = g(D_X v, w) + g(v, D_X w). \end{aligned}$$

For  $x \in M$   $df_x$  takes by remark 3 values in  $\text{sym}^i(f(x))$ . This shows that  $S = -d\tilde{f}$  is symmetric with respect to  $g = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r}$ .  $\square$

To finish the proof, we have to check the  $tt^*$ -equations. The second  $tt^*$ -equation

$$[S_X, S_Y] = [S_{JX}, S_{JY}] \quad (4.3)$$

for  $S$  follows from the assumption that the image of  $T^{1,0}M$  under  $d^c \tilde{f}$  is Abelian. In fact, this is equivalent to  $[d\tilde{f}(JX), d\tilde{f}(JY)] = [d\tilde{f}(X), d\tilde{f}(Y)] \forall X, Y \in TM$ .

$$\begin{aligned} d^D S(X, Y) &= [D_X, S_Y] - [D_Y, S_X] - S_{[X, Y]} \\ &= \partial_X(S_Y) - \partial_Y(S_X) + 2[S_X, S_Y] - S_{[X, Y]} = 0 \end{aligned}$$

is equivalent to the vanishing of the curvature of  $A = 2S$  interpreted as a connection on  $E$  (see proposition 4).

Finally one has for holomorphic coordinate fields  $X, Y \in \Gamma(TM)$

$$\begin{aligned} d^D S_J(JX, Y) &= [D_{JX}, S_{JY}] + [D_Y, S_X] = [\partial_{JX} + S_{JX}, S_{JY}] + [\partial_Y + S_Y, S_X] \\ &= \partial_{JX}(S_{JY}) + \partial_Y(S_X) + [S_{JX}, S_{JY}] - [S_X, S_Y] \\ &\stackrel{(4.3)}{=} \frac{1}{2} (\partial_{JX}(A_{JY}) + \partial_Y(A_X)) \stackrel{(3.10)}{=} \frac{1}{2} (\partial_{JX}(A_{JY}) + \partial_X(A_Y) + [A_X, A_Y]) \\ &\stackrel{(4.3)}{=} \frac{1}{2} \left( \partial_{JX}(A_{JY}) + \partial_X(A_Y) + \frac{1}{2}[A_X, A_Y] + \frac{1}{2}[A_{JX}, A_{JY}] \right) \stackrel{(3.11)}{=} 0. \end{aligned}$$

This shows the vanishing of the tensor  $d^D S_J$ .

It remains to show the curvature equation for  $D$ . We observe, that  $D + S = \partial + A$  and that  $A$  is flat, to find

$$0 = R_{X,Y}^{D+S} = R_{X,Y}^D + d^D S(X,Y) + [S_X, S_Y] \stackrel{d^D S=0}{=} R_{X,Y}^D + [S_X, S_Y].$$

- b) With the same proof as in part a) we get a metric  $tt^*$ -bundle. The orientation is given by the orientation of  $E = M \times \mathbb{R}^r$ .

It remains to check the condition on the trace of  $S$ . This property is clear, since in this case  $d\tilde{f}_x$  takes values in  $\text{sym}^1(f(x))$  for all  $x \in M$ .  $\square$

We want to emphasize the last result in the positive definite case:

**Theorem 4** *Let  $(M, J)$  be a simply-connected complex manifold and put  $E = M \times \mathbb{R}^r$ . Then a pluriharmonic map  $\tilde{f} : M \rightarrow S^i(r, 0)$  is admissible. Moreover, it induces a second pluriharmonic map  $f : M \xrightarrow{\tilde{f}} S^i(r, 0) \xrightarrow{\sim} \text{Sym}_{r,0}^i(\mathbb{R}^r) \xrightarrow{\iota} G_i(r)$  and a metric  $tt^*$ -bundle  $(E, D = \partial + S, S = -d\tilde{f}, g = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r} [, or])$  on  $M$  where  $\partial$  is the canonical flat connection on  $E$  and  $or$  is the canonical orientation of  $E$ .*

*Proof:* In the case of signature  $(r, 0)$  corollary 1 implies that for all  $x \in M$  the image of  $d\tilde{f}_x$  is Abelian and the differential of any pluriharmonic map  $\tilde{f} : M \rightarrow S(r, 0)$  is admissible as required in theorem 3.  $\square$

In the situation of theorem 3 the two constructions are inverse in the following sense:

**Proposition 6**

1. *Let  $(E, D, S, g [, or])$  be a metric [an unimodular oriented metric]  $tt^*$ -bundle on a complex manifold  $(M, J)$  and let  $\tilde{f}$  be the associated pluriharmonic map constructed to a  $D^\theta$ -flat frame  $s$  in theorem 2. Then  $\tilde{f}$  is admissible and the metric [unimodular oriented metric]  $tt^*$ -bundle  $(M \times \mathbb{R}^r, \tilde{D} = \partial + \tilde{S}, \tilde{S}, \tilde{g}, [, or])$  associated to  $\tilde{f}$  in theorem 3 is the representation of  $(E, D, S, g [, or])$  in the frame  $s$ .*
2. *Given a pluriharmonic map  $\tilde{f}$  from a complex manifold  $(M, J)$  to  $S^i(p, q)$ , then one obtains via theorem 3 a metric [an unimodular oriented metric]  $tt^*$ -bundle  $(M \times \mathbb{R}^r, D, S, g [, or])$ . The pluriharmonic map associated to this metric  $tt^*$ -bundle is conjugated to the map  $\tilde{f}$  by a constant matrix in  $G_i(r)$ .*

*Proof:* Using again remark 1.1) we can set  $\theta = \pi$ .

1. The maps  $f, \tilde{f}$  and the metric  $\tilde{g} = \langle f \cdot, \cdot \rangle_{\mathbb{R}^r}$  express the metric  $g$  in the frame  $s$ . In the computations of theorem 2 and with theorem 3 one finds  $2\tilde{S} = A = f^{-1}df = 2S^s$ . From  $0 = D^\pi s = Ds - Ss$  we obtain that the connection  $D$  in the frame  $s$  is just  $\partial + S^s = \partial + \frac{A}{2} = \partial + \tilde{S} = \tilde{D}$ .
2. To find the pluriharmonic map associated to  $(M \times \mathbb{R}^r, D, S, g [, or])$  we have to express the metric  $g$  in a  $D^\pi$ -flat frame  $s$ . But  $D^\pi = \partial + \frac{A}{2} - \frac{A}{2} = \partial$ . Hence we can take  $s$  as the standard-basis of  $\mathbb{R}^r$  and we get  $\tilde{f}$ . Every other basis gives a conjugated result.  $\square$

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