Convergence of a finite element discretization for an Arbitrary Lagrangian Eulerian formulation of a fluid-structure interaction problem

Guillaume Legendre, Takéo Takahashi

CONVERGENCE OF A FINITE ELEMENT DISCRETIZATION FOR AN ARBITRARY LAGRANGIAN EULERIAN FORMULATION OF A FLUID-STRUCTURE INTERACTION PROBLEM *

Guillaume Legendre[1] and Takéo Takahashi[2]

Abstract. In this paper, we propose a numerical scheme to compute the motion of a two-dimensional rigid body in a viscous fluid, modeled by the Navier-Stokes equations. Our method combines a finite element approximations and the use of the method of characteristics to solve an Arbitrary Lagrangian Eulerian formulation of the problem. We derive error estimates for this scheme which imply its convergence.

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1. Introduction

The present work aims at proposing and analyzing a Lagrange-Galerkin scheme for the numerical solution of an Arbitrary Lagrangian Eulerian (ALE) formulation of a fluid-rigid solid interaction problem. While the Lagrange-Galerkin technique has been used for years for the numerical treatment of convection diffusion equations like the Navier-Stokes equations (see, for instance, [1, 23, 27]), it was more recently introduced in the context of ALE formulations of free surface or two-fluid flow problems [8, 12, 19] and fluid-structure interaction problems [20, 21].

The system we consider is composed of a viscous homogeneous fluid and a rigid solid, both contained in a bounded domain $\mathcal{O}$ of $\mathbb{R}^2$ with regular boundary $\partial\mathcal{O}$. At the initial time, the rigid body is assumed to occupy a regular open connected subset $\mathcal{S}$ of $\mathcal{O}$, surrounded by the fluid filling the domain $\mathcal{F} = \mathcal{O} \setminus \overline{\mathcal{S}}$. For the sake of simplicity and without loss of generality, we shall suppose that the center of mass of $\mathcal{S}$ is located at the origin. The domain occupied by the rigid body at each instant $t > 0$ is then defined by

$$\mathcal{S}(\boldsymbol{\zeta}(t), \theta(t)) = \left\{ \boldsymbol{\zeta}(t) + \mathbf{R}_{\theta(t)}\boldsymbol{x},\ \boldsymbol{x} \in \mathcal{S} \right\},$$

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[1] Centro de Modelamiento Matemático - FONDAP, UMI 2807 CNRS-Universidad de Chile, Casilla 170 – Correo 3, Santiago, Chile. E-mail: glegendre@dim.uchile.cl
Current address: CEREMADE, UMR CNRS 7534, Université de Paris-Dauphine, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France.

[2] Institut de Mathématiques Élie Cartan de Nancy, Universités de Nancy-CNRS-INRIA, BP 239, 54506 Vandœuvre-lès-Nancy cedex, France. E-mail: takeo.takahashi@iecn.u-nancy.fr

where $\boldsymbol{\zeta}(t)$ and $\mathbf{R}_{\theta(t)}$ are respectively the position of the center of mass and the orientation of the rigid body at time t ($\mathbf{R}_\theta$ being the matrix of rotation of angle θ). The fluid then occupies the domain $\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)) = \mathcal{O} \setminus \overline{\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t))}$.

The fluid flow is assumed to be incompressible and modeled by the classical Navier-Stokes equations, the motion of the rigid body being governed by Newton's laws. As a consequence, the following system of partial and ordinary differential equations describes the evolution of the coupled system

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} - \nu\,\Delta\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} \text{ in } \mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)),\ t\in[0,T], \tag{1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \text{ in } \mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)),\ t\in[0,T], \tag{2}$$

$$\boldsymbol{u} = \mathbf{0} \text{ on } \partial\mathcal{O},\ t\in[0,T], \tag{3}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{\zeta}'(t) + \theta'(t)(\boldsymbol{x}-\boldsymbol{\zeta}(t))^\perp,\ \boldsymbol{x}\in\partial\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t)),\ t\in[0,T], \tag{4}$$

$$M\,\boldsymbol{\zeta}''(t) = -\int_{\partial\mathcal{S}(\boldsymbol{\zeta}(t),\theta)(t)} \boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n}\,\mathrm{d}\Gamma + \int_{\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t))} \rho_s\,\boldsymbol{f}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x},\ t\in[0,T], \tag{5}$$

$$I\,\theta''(t) = -\int_{\partial\mathcal{S}(\boldsymbol{\zeta}(t),\theta)(t)} \boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n}\cdot(\boldsymbol{x}-\boldsymbol{\zeta}(t))^\perp\,\mathrm{d}\Gamma + \int_{\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t))} \rho_s\,\boldsymbol{f}(\boldsymbol{x},t)\cdot(\boldsymbol{x}-\boldsymbol{\zeta}(t))^\perp\,\mathrm{d}\boldsymbol{x},\ t\in[0,T], \tag{6}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}^{(0)} \text{ in } \mathcal{F}, \tag{7}$$

$$\boldsymbol{\zeta}(0) = \mathbf{0},\ \boldsymbol{\zeta}'(0) = \boldsymbol{\zeta}^{(1)}\in\mathbb{R}^2,\ \theta(0) = 0,\ \theta'(0) = \theta^{(1)}\in\mathbb{R}. \tag{8}$$

In the above equations, the unknowns are the Eulerian velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ and the pressure field $p(\boldsymbol{x},t)$ in the fluid, the position $\boldsymbol{\zeta}(t)$ of the center of mass and the angle of rotation $\theta(t)$ of the rigid body. To simplify, we assume that the density of the homogeneous fluid is equal to unity and that the density of the rigid body is a positive constant, denoted by ρ_s. The positive scalar ν denotes the viscosity of the fluid and M and I are respectively the mass and the moment of inertia of the solid. The relations between M, I and ρ_s are given by

$$M = \int_{\mathcal{S}} \rho_s\,\mathrm{d}\boldsymbol{x}, \quad I = \int_{\mathcal{S}} \rho_s\,|\boldsymbol{x}|^2\,\mathrm{d}\boldsymbol{x}.$$

The stress tensor $\boldsymbol{\sigma}$ is defined by

$$\boldsymbol{\sigma}(\boldsymbol{u},p) = -p\,\mathbf{Id} + 2\nu\,\mathbf{D}(\boldsymbol{u}),$$

where $\mathbf{Id}$ is the identity tensor and $\mathbf{D}(\boldsymbol{u})$ is the strain tensor given by

$$\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} +^t \boldsymbol{\nabla}\boldsymbol{u}\right).$$

Finally, the field $\boldsymbol{f}(\boldsymbol{x},t)$ represents the density (per mass unit) of forces applied to the system, $\boldsymbol{n}$ is the unit normal vector to the boundary of the rigid body $\partial\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t))$, pointing to the interior of the solid and, for any $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have denoted by $\boldsymbol{x}^\perp$ the vector $\boldsymbol{x}^\perp = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$.

The well-posedness of this type of problem has been the subject of a large number of papers (see for instance [28] and the references therein). We aim here at approximating strong solutions of the above system.

Concerning the numerical solution of such fluid-solid interaction problems, while several different approaches have been introduced in the literature, some being based on an ALE formulation [8,16,20,21], a fictitious domain formulation [13] or a Lagrange-Galerkin method [25], only a few have actually received a rigorous analysis of their numerical properties. In this area, let us cite the paper of Grandmont *et al.* [14] for proofs of convergence of time decoupling algorithms used to solve an ALE formulation of a one-dimensional fluid-structure interaction problem. More recently, the convergence of a numerical scheme based on a Lagrange-Galerkin method, using

a fixed mesh, has been established in [25]. Also of interest, since the present work involves a finite element approximation for solving an ALE formulation, are the paper of Gastaldi [11], which focuses on the derivation of a priori estimates in space and time in the case of an advection-diffusion equation in a moving two-dimensional domain, and the proof of convergence in [26] of a scheme based on an ALE formulation, a mixed finite element discretization in space and an implicit Euler scheme in time, for the non-steady Stokes equations in a two-dimensional, non-cylindrical domain. However, up to our knowledge, there is no convergence result for the numerical approximation of system (1)-(8) using an ALE method.

Our main result, stated in Theorem 3.5, asserts that the solution to a Lagrange-Galerkin discretization scheme of an ALE formulation of system (1)-(8) converges towards the exact solution of the problem, provided some assumptions on the regularity of this exact solution, on the finite element mesh and on the discrete time step. We stress that we prove here the convergence of a numerical scheme for a free boundary problem, and that we managed to get rid of the hypothesis of a one-dimensional model [14] or of the hypothesis of the equality of the fluid and solid densities [25] which were present in above mentoned works. Let us emphasize that, if a number of issues encountered in establishing this result can be circumvented with techniques already existing in the literature for non-moving domains (we will rely in particular on a paper by Achdou and Guermond [1]), the bulk of the difficulties arises from the coupling condition (4) on the solid boundary. To be more precise, the treatment of this condition at the discrete level brings up some nontrivial questions from the point of view of the finite element approximation on non-conforming meshes. It indeed appears that most of the methods introduced to deal with the discrepancy between the exact and approximate domain prove to be ineffective when applied to a finite element semi-discretization of problem (1)-(8). The solution we propose is based on a particular approximation of the fluid domain, a drawback being that the order of the convergence rate of the method is then confined, whatever the regularity of the exact solution and the order of the finite element approximation.

An outline of the paper is the following. A characteristics-ALE weak formulation of problem (1)-(8) is presented in Section 2. Section 3 deals with the discretization scheme and states the main result of the paper. Section 4 introduces a change of variables which is an essential tool for comparing the exact and approximate solutions. Sections 5 and 6 are devoted to the derivation of error estimates. In the remaining Section 7, the proof of the convergence result is established.

2. A characteristics-ALE formulation of the problem

2.1. **Hypotheses**

In what follows, we shall assume that the data satisfy

$$\begin{aligned} &\boldsymbol{f} \in \mathrm{C}([0,T],\mathrm{H}^1(\mathcal{O})^2),\ \boldsymbol{u}^{(0)} \in \mathrm{H}^1(\mathcal{F})^2,\ \operatorname{div}\boldsymbol{u}^{(0)} = 0 \text{ in } \mathcal{F},\\ &\boldsymbol{u}^{(0)}(\boldsymbol{x}) = \boldsymbol{\zeta}^{(1)} + \theta^{(1)}\boldsymbol{x}^{\perp},\ \forall \boldsymbol{x} \in \partial\mathcal{S},\ \text{and } \boldsymbol{u}^{(0)} = \boldsymbol{0} \text{ on } \partial\mathcal{O}. \end{aligned} \tag{9}$$

We moreover suppose that

$$\operatorname{dist}(\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t)),\partial\mathcal{O}) > 0,\ \forall t \in [0,T]. \tag{10}$$

Owing to the result in [28], we have the following regularity for the solution to problem (1)-(8):

$$\begin{aligned} \boldsymbol{u} \in \mathrm{L}^2(0,T;\mathrm{H}^2(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)))^2) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)))^2) \cap \mathrm{C}([0,T];\mathrm{H}^1(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)))^2),\\ p \in \mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)))),\ \boldsymbol{\zeta} \in \mathrm{H}^2(0,T;\mathbb{R}^2),\ \theta \in \mathrm{H}^2(0,T;\mathbb{R}). \end{aligned} \tag{11}$$

2.2. **Weak formulation of the problem**

For any $\boldsymbol{\zeta} \in \mathcal{O}$ and $\theta \in \mathbb{R}$ such that $\operatorname{dist}(\mathcal{S}(\boldsymbol{\zeta},\theta),\partial\mathcal{O}) > 0$, we introduce the functional spaces

$$\mathcal{V}(\boldsymbol{\zeta},\theta) = \left\{(\boldsymbol{v},\boldsymbol{\xi}_{\boldsymbol{v}},\omega_{\boldsymbol{v}}) \in \mathrm{H}^1(\mathcal{F}(\boldsymbol{\zeta},\theta))^2 \times \mathbb{R}^3\ ;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\mathcal{O} \text{ and } \boldsymbol{v} = \boldsymbol{\xi}_{\boldsymbol{v}} + \omega_{\boldsymbol{v}}(\boldsymbol{x}-\boldsymbol{\zeta})^{\perp} \text{ on } \partial\mathcal{S}(\boldsymbol{\zeta},\theta)\right\}$$

and

$$\mathcal{Q}(\boldsymbol{\zeta},\theta)=\mathrm{L}^2_0(\mathcal{F}(\boldsymbol{\zeta},\theta))=\left\{q\in\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta},\theta))\ ;\ \int_{\mathcal{F}(\boldsymbol{\zeta},\theta)}q(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0\right\}.$$

In the sequel, we denote by $\boldsymbol{\xi}=\boldsymbol{\zeta}'$ and $\omega=\theta'$ the translational and angular velocities of the rigid body, and also use the notations

$$\boldsymbol{f_M}(t)=\int_{\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t))}\rho_s\,\boldsymbol{f}(\cdot,t)\,\mathrm{d}\boldsymbol{x}\ \text{and}\ f_I(t)=\int_{\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t))}\rho_s\,\boldsymbol{f}(\boldsymbol{x},t)\cdot(\boldsymbol{x}-\boldsymbol{\zeta}(t))^{\perp}\,\mathrm{d}\boldsymbol{x},\ \forall t\in[0,T]. \tag{12}$$

One can easily check (see [13,16,20]) that the strong solution of (1)-(8) satisfies the following mixed variational formulation: *find* $(\boldsymbol{u},\boldsymbol{\zeta},\theta,p)$ *verifying* (7), (8), (11), *and, for almost every* t *in* $(0,T)$,

$$\int_{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))}\left(\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}\right)\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}+M\,\boldsymbol{\xi}'\cdot\boldsymbol{\xi}_{\boldsymbol{v}}+I\,\omega'\omega_{\boldsymbol{v}}+2\nu\int_{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))}\mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}-\int_{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))}p\ \mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$$
$$=\int_{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))}\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}+\boldsymbol{f_M}\cdot\boldsymbol{\xi}_{\boldsymbol{v}}+f_I\,\omega_{\boldsymbol{v}},\ \forall\,(\boldsymbol{v},\boldsymbol{\xi}_{\boldsymbol{v}},\omega_{\boldsymbol{v}})\in\mathcal{V}(\boldsymbol{\zeta}(t),\theta(t)), \tag{13}$$

$$-\int_{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))}q\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}=0,\ \forall q\in\mathcal{Q}(\boldsymbol{\zeta}(t),\theta(t)). \tag{14}$$

In conjunction with this weak formulation of the problem, a feature of the numerical scheme we use is the method of characteristics for the treatment of the nonlinear convection term in the Navier-Stokes equations. It is well known (see, for instance, [23]) that the material derivative in the flow $\boldsymbol{u}$ can be written as a total derivative

$$\left(\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}\right)(\boldsymbol{x},t)=\frac{d}{dt}\left[\boldsymbol{u}(\boldsymbol{\mathcal{C}}(t;s,\boldsymbol{x}),t)\right]_{|_{s=t}}, \tag{15}$$

by using the characteristic function $\boldsymbol{\mathcal{C}}$, which, for all $\boldsymbol{x}$ in $\mathcal{F}(\boldsymbol{\zeta}(s),\theta(s))$, is solution to the initial value problem

$$\begin{cases}\dfrac{\partial\boldsymbol{\mathcal{C}}}{\partial t}(t;s,\boldsymbol{x})=\boldsymbol{u}(\boldsymbol{\mathcal{C}}(t;s,\boldsymbol{x}),t),\\ \boldsymbol{\mathcal{C}}(s;s,\boldsymbol{x})=\boldsymbol{x}.\end{cases} \tag{16}$$

However, these characteristics are defined over the moving domain $\mathcal{F}(\boldsymbol{\zeta}(s),\theta(s))$ which complicates their effective computation in a discrete setting. The idea introduced by Maury in [20, 21] consists of adapting this method to an ALE framework. We address the specifics of this combination in the next subsections.

2.3. **Domain velocity and ALE mapping**

A very popular technique for the simulation of fluid-structure interaction problems since its introduction at the beginning of the eighties [7,17], the Arbitrary Lagrangian Eulerian (ALE) formulation combines advantages of both Lagrangian and Eulerian formalisms by introducing a domain velocity which makes possible for the space discretization mesh to follow the motion of the fluid domain. Such a velocity can be defined quite arbitrarily, as long as it satisfies a compatibility condition, with respect to the fluid velocity, on the boundary of the domain [16, 19, 20]. This being done, one is then able to construct a transformation linking any point of a reference configuration to a point of the current configuration, simply by using the characteristic curves associated to the domain velocity.

Choosing the fluid domain at the initial time as the frame of reference, we introduce a family of ALE mappings $\boldsymbol{\mathcal{A}}(t;0,\cdot)$, which, at each t in $[0,T]$, maps $\mathcal{F}$ into $\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))$. At each instant t in $(0,T)$, it is assumed that the application $\boldsymbol{\mathcal{A}}(t;0,\cdot)$ is an homeomorphism, that is, $\boldsymbol{\mathcal{A}}(t;0,\cdot)\in\mathrm{C}^0(\overline{\mathcal{F}})^2$ is invertible with continuous inverse

$\boldsymbol{\mathcal{A}}(t;0,\cdot)^{-1} \in \mathrm{C}^0(\overline{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))})^2$, and that the application $t \mapsto \boldsymbol{\mathcal{A}}(t;0,\boldsymbol{x})$, for all $\boldsymbol{x}$ in $\mathcal{F}$, is differentiable almost everywhere in $[0,T]$. The domain velocity $\boldsymbol{w}$ is defined by

$$\boldsymbol{w}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{\mathcal{A}}}{\partial t}\left(t;0,\boldsymbol{\mathcal{A}}(t;0,\cdot)^{-1}(\boldsymbol{x})\right),\ \forall \boldsymbol{x} \in \mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)), \tag{17}$$

and the ALE mapping between two time levels s and t in $[0,T]$ is given by

$$\boldsymbol{\mathcal{A}}(t;s,\cdot) = \boldsymbol{\mathcal{A}}(t;0,\cdot) \circ \boldsymbol{\mathcal{A}}(s;0,\cdot)^{-1}. \tag{18}$$

We easily check that the application $t \mapsto \boldsymbol{\mathcal{A}}(t;s,\boldsymbol{x})$, $\forall \boldsymbol{x} \in \mathcal{F}(\boldsymbol{\zeta}(s),\theta(s))$, is solution to the initial value problem

$$\begin{cases} \dfrac{\partial \boldsymbol{\mathcal{A}}}{\partial t}(t;s,\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{\mathcal{A}}(t;s,\boldsymbol{x}),t) \\ \boldsymbol{\mathcal{A}}(s;s,\boldsymbol{x}) = \boldsymbol{x}. \end{cases} \tag{19}$$

Since we use the transformations $\boldsymbol{\mathcal{A}}(t;s,\cdot)$ in the sequel, it is important to ensure that these transformations are compatible with the functional spaces involved in the weak formulation (13) of the problem. We obtain this compatibility by adding some regularity properties to the ALE mapping. Let us first recall the following classical proposition (see [15, pp. 19–20] and [10]).

Proposition 2.1. *Assume that the ALE mapping $\boldsymbol{\mathcal{A}}(t;0,\cdot)$ satisfies, for each t in $(0,T)$, the following conditions:*

$$\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)) = \boldsymbol{\mathcal{A}}(t;0,\mathcal{F}) \text{ is bounded and the boundary } \partial\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)) \text{ is Lipschitz continuous}, \tag{20}$$

$$\boldsymbol{\mathcal{A}}(t;0,\cdot) \in \mathrm{W}^{1,\infty}(\mathcal{F})^2,\ \boldsymbol{\mathcal{A}}(t;0,\cdot)^{-1} \in \mathrm{W}^{1,\infty}(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)))^2. \tag{21}$$

Then, a function v belongs to $\mathrm{H}^1(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)))$ if and only if $\hat{v} = v \circ \boldsymbol{\mathcal{A}}(t;0,\cdot)$ belongs to $\mathrm{H}^1(\mathcal{F})$.

As recalled in the references [10, 11], there exist several techniques in the literature to construct a mapping satisfying the above assumptions. We opt for the approach followed in [9, 11], in which the reference domain is viewed as an elastic solid being deformed into the current domain. This leads us to solve a linear elasticity problem: *for all t in $(0,T)$, find $d(\cdot,t)$ satisfying*

$$\begin{cases} -\Delta \boldsymbol{d}(\cdot,t) - \lambda\, \boldsymbol{\nabla} \operatorname{div} \boldsymbol{d}(\cdot,t) = \boldsymbol{0} \text{ in } \mathcal{F}, \\ \boldsymbol{d}(\boldsymbol{x},t) = \boldsymbol{\zeta}(t) + \mathbf{R}_{\theta(t)}\boldsymbol{x} - \boldsymbol{x} \text{ on } \partial\mathcal{S}, \\ \boldsymbol{d}(\cdot,t) = \boldsymbol{0} \text{ on } \partial\mathcal{O}, \end{cases} \tag{22}$$

where λ is an arbitrary positive constant. Existence, uniqueness and regularity issues for solutions of this type of system have been extensively studied and it is known (see, for instance, [5, 11]) that, for all $r \geqslant 2$,

$$\|\boldsymbol{d}(\cdot,t)\|_{W^{2,r}(\mathcal{F})^2} \leqslant C\ (|\boldsymbol{\zeta}(t)| + |\theta(t)|),\ \forall t \in (0,T). \tag{23}$$

The ALE mapping is then defined by

$$\boldsymbol{\mathcal{A}}(t;0,\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{d}(\boldsymbol{x},t),\ \forall \boldsymbol{x} \in \mathcal{F}, \tag{24}$$

and we have the following result.

Lemma 2.2. *Let us assume that*

$$\|\boldsymbol{\zeta}\|_{\mathrm{L}^\infty(0,T)^2} + \|\theta\|_{\mathrm{L}^\infty(0,T)} \leqslant c_0, \tag{25}$$

with c_0 a small enough constant. Then, for all t in $(0,T)$, the mapping $\boldsymbol{\mathcal{A}}(t;0,\cdot)$ is a diffeomorphism from $\mathcal{F}$ onto $\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))$. Moreover, it satisfies assumptions (20) and (21).

Proof. We first extend the mapping $\boldsymbol{\mathcal{A}}(t;0,\cdot)$ to the whole of $\mathbb{R}^2$ by setting

$$\boldsymbol{d}(\boldsymbol{x},t)=\boldsymbol{\zeta}(t)+\mathbf{R}_{\theta(t)}\boldsymbol{x}-\boldsymbol{x},\forall \boldsymbol{x}\in\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t)),\ \text{and}\ \boldsymbol{d}(\cdot,t)=\mathbf{0}\ \text{in}\ \mathbb{R}^2\setminus\mathcal{O}.$$

Using the inequality (23) and assuming that the constant c_0 appearing in (25) is small enough, we infer that $\boldsymbol{d}(\cdot,t)$ is a contraction. This implies the invertibility of the mapping defined by (24) from $\mathbb{R}^2$ onto $\mathbb{R}^2$. Since it is clear that

$$\boldsymbol{\mathcal{A}}(t;0,\mathbb{R}^2\setminus\mathcal{O})=\mathbb{R}^2\setminus\mathcal{O}\ \text{and}\ \boldsymbol{\mathcal{A}}(t;0,\mathcal{S})=\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t)),$$

we have proved the assertion. □

Remark 2.3. Assumption (25) is important and supposed to hold hereafter. It expresses the fact the displacement of the rigid solid is not too large. This restriction cannot be avoided when using an ALE formulation. Indeed, as described below, the principle of this method is to modify the mesh, according to a discrete ALE mapping, in order to follow the solid in its movement. In order to preserve the needed properties of the space discretization (regularity and quasi-uniformity of the mesh triangulation, for instance), we have to assume that these displacements are small enough (see [20] for more details).

Remark 2.4. In the proof of Lemma 2.2, we have extended $\boldsymbol{\mathcal{A}}(t;0,\cdot)$ to $\mathbb{R}^2$ and proved that it is a diffeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$. In the sequel, we identify this mapping with its extension. Notice that its extension has also the form (24) with $\boldsymbol{d}(\cdot,t)$ small enough. More precisely, we always consider that the constant c_0 in (25) is such that

$$\|\boldsymbol{d}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}^{1,\infty}(\mathcal{O})^2)}\leqslant\frac{1}{4}. \tag{26}$$

2.4. **Characteristics-ALE formulation**

The introduction of the ALE mapping allows us to introduce a new characteristic function, which involves a fixed spatial domain, and will thus be more manageable from a discrete point of view. The importance of such a mapping comes from the fact that we can rewrite the material derivative in the flow as a total derivative (see (15)). Let $\boldsymbol{\mathcal{B}}$ be a characteristic function such that

$$\boldsymbol{\mathcal{C}}(t;s,\boldsymbol{x})=\boldsymbol{\mathcal{A}}(t;s,\boldsymbol{\mathcal{B}}(t;s,\boldsymbol{x})),\ \forall\boldsymbol{x}\in\mathcal{F}(\boldsymbol{\zeta}(s),\theta(s)). \tag{27}$$

For all t and s in $(0,T)$, we have that the application

$$\boldsymbol{\mathcal{B}}(t;s,\cdot)\ :\ \mathcal{F}(\boldsymbol{\zeta}(s),\theta(s))\to\mathcal{F}(\boldsymbol{\zeta}(s),\theta(s))$$

is a diffeomorphism satisfying, for all $\boldsymbol{x}$ in $\mathcal{F}(\boldsymbol{\zeta}(s),\theta(s))$, the initial value problem

$$\begin{cases}\dfrac{\partial\boldsymbol{\mathcal{B}}}{\partial t}(t;s,\boldsymbol{x})=(\overline{\boldsymbol{u}}-\overline{\boldsymbol{w}})(\boldsymbol{\mathcal{B}}(t;s,\boldsymbol{x}),t),\\ \boldsymbol{\mathcal{B}}(s;s,\boldsymbol{x})=\boldsymbol{x},\end{cases} \tag{28}$$

where the functions $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{w}}$ are respectively defined by

$$\overline{\boldsymbol{u}}(\boldsymbol{x},t)=\left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}(t;s,\boldsymbol{x})\right]^{-1}\boldsymbol{u}(\boldsymbol{\mathcal{A}}(t;s,\boldsymbol{x}),t)\ \text{and}\ \overline{\boldsymbol{w}}(\boldsymbol{x},t)=\left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}(t;s,\boldsymbol{x})\right]^{-1}\boldsymbol{w}(\boldsymbol{\mathcal{A}}(t;s,\boldsymbol{x}),t), \tag{29}$$

for all $\boldsymbol{x}$ in $\mathcal{F}(\boldsymbol{\zeta}(s),\theta(s))$ and t in $(0,T)$.

Remark 2.5. By extending the velocity field $\boldsymbol{u}(\cdot,t)$ to $\mathbb{R}^2$ by the formulas

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{\xi}(t)+\omega(t)\left(\boldsymbol{x}-\boldsymbol{\zeta}(t)\right)^{\perp},\ \forall\boldsymbol{x}\in\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t)),\ \forall t\in[0,T],$$

and

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{0},\ \forall \boldsymbol{x}\in\mathbb{R}^2\setminus\mathcal{O},\ \forall t\in[0,T],$$

the unique solution $\boldsymbol{\mathcal{C}}(\cdot;s,\boldsymbol{x})$ of the initial value problem (16) exists for any $\boldsymbol{x}$ in $\mathbb{R}^2$. Owing to Remark 2.4, the ALE mapping $\boldsymbol{\mathcal{A}}(t;s,\cdot)$ is now defined in $\mathbb{R}^2$ and, consequently, so is the domain velocity $\boldsymbol{w}(\cdot,t)$. Considering these extensions, problem (28) thus actually defines a diffeomorphism $\boldsymbol{\mathcal{B}}(t;s,\cdot)$ from $\mathbb{R}^2$ onto $\mathbb{R}^2$.

The expressions (15) and (27) are finally substituted into the system (13)-(14) to yield the following equivalent weak formulation of problem (1)-(8): *for almost every t in $(0,T)$, find $(\boldsymbol{u},\boldsymbol{\zeta},\theta,p)$ such that $(\boldsymbol{u}(\cdot,t),\boldsymbol{\xi}(t),\omega(t))\in\mathcal{V}(\boldsymbol{\zeta}(t),\theta(t))$ and $p(\cdot,t)\in\mathcal{Q}(\boldsymbol{\zeta}(t),\theta(t))$ are solution to*

$$\begin{aligned}\int_{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))}\frac{d}{dt}\left[\boldsymbol{u}\left(\boldsymbol{\mathcal{A}}\left(t;\cdot,\boldsymbol{\mathcal{B}}(t;\cdot,\boldsymbol{x})\right),t\right)\right](t)\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}+M\,\boldsymbol{\xi}'(t)\cdot\boldsymbol{\xi}_{\boldsymbol{v}}+I\,\omega'(t)\,\omega_{\boldsymbol{v}}&\\ +2\nu\int_{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))}\mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}-\int_{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))}p\ \mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}&\\ =\int_{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))}\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}+\boldsymbol{f_M}(t)\cdot\boldsymbol{\xi}_{\boldsymbol{v}}+f_I(t)\,\omega_{\boldsymbol{v}},\quad\forall\,(\boldsymbol{v},\boldsymbol{\xi}_{\boldsymbol{v}},\omega_{\boldsymbol{v}})\in\mathcal{V}(\boldsymbol{\zeta}(t),\theta(t)),&\quad(30)\\ -\int_{\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))}q\ \mathrm{div}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}=0,\quad\forall q\in\mathcal{Q}(\boldsymbol{\zeta}(t),\theta(t)).&\quad(31)\end{aligned}$$

3. Discretization of the problem and convergence result

This section is concerned with the description of the discrete scheme we propose for computing an approximation of the solution to the variational problem (30)-(31). While clearly inspired from the method introduced by Maury in [20, 21] to simulate the motion of 2D rigid particles in a viscous incompressible fluid, our scheme differs on two main points. First, the discrete domain velocity is derived from its associated discrete ALE mapping in a different way. Second, for the needs of the error analysis in the convergence study, the mesh of the fluid domain must satisfy some special, non-standard features which are absent from references [20, 21].

In the sequel, we suppose that $\mathcal{O}$ is the interior of a convex polygon. This assumption is not essential, but it allows to make simpler the forthcoming finite element analysis, already cumbersome due to the conditions on the moving boundary, while guaranteeing the expected regularity for the solution of the problem. The more general case of a domain $\mathcal{O}$ with a curved boundary $\partial\mathcal{O}$ can be dealt with by using the classical techniques presented in [4, 6] for instance.

3.1. Discrete scheme

For N in $\mathbb{N}^*$, we introduce a partition of the time interval $[0,T]$ by denoting $\delta t=T/N$ and $t^k=k\delta t$, $k=0,\ldots,N$. The quantities $\boldsymbol{u}_h^k$, p_h^k, $\boldsymbol{\zeta}_h^k$, θ_h^k, $\boldsymbol{\xi}_h^k$ and ω_h^k are then the respective approximations of $\boldsymbol{u}(\cdot,t^k)$, $p(\cdot,t^k)$, $\boldsymbol{\zeta}(t^k)$, $\theta(t^k)$, $\boldsymbol{\xi}(t^k)$ and $\omega(t^k)$.

3.1.1. *Initialization*

At the initial time, we consider an approximation $\mathcal{F}_h^0$ of the fluid domain $\mathcal{F}$, which is the union of straight triangles, with mesh size h, of a triangulation $\mathscr{T}_h^0$ and satisfies the property

$$\mathcal{F}_h^0\subset\mathcal{F}.\tag{32}$$

Since $\mathcal{F}$ is not convex, even if $\mathcal{O}$ is convex, the hypothesis (32) is in no way standard. It implies in particular that $\partial\mathcal{F}_h^0$ is a piecewise linear continuous curve whose boundary nodes are not necessarily on $\partial\mathcal{F}$. A way to

construct such a mesh is the following. We first build a quasi-uniform triangulation $\widetilde{\mathscr{T}_h^0}$ of the whole domain $\mathcal{O}$. We then define $\mathcal{H}_h$, the union of all triangles in $\widetilde{\mathscr{T}_h^0}$ such that their three vertices are contained in $\overline{\mathcal{G}_h}$, with

$$\mathcal{G}_h = \bigcup_{\substack{K\in\widetilde{\mathscr{T}_h^0}\\ \mathring{K}\cap\mathring{\mathcal{S}}\neq\emptyset}} K,$$

and divide the triangles into three categories as follows (see Figure 1):

- $\mathscr{T}_1$ is the subset of $\widetilde{\mathscr{T}_h^0}$ formed by all triangles $K \in \widetilde{\mathscr{T}_h^0}$ such that $\overline{K} \subset \mathcal{S}$,
- $\mathscr{T}_2$ is the subset formed by all triangles $K \in \widetilde{\mathscr{T}_h^0}\backslash\mathscr{T}_1$ such that $\overline{K} \subset \overline{\mathcal{H}_h}$,
- $\mathscr{T}_3 = \widetilde{\mathscr{T}_h^0}\backslash(\mathscr{T}_1 \cup \mathscr{T}_2)$.

We finally set $\mathscr{T}_h^0 = \mathscr{T}_3$ and

$$\mathcal{F}_h^0 = \bigcup_{K\in\mathscr{T}_h^0} K, \quad \mathcal{S}_h^0 = \mathcal{O} \setminus \overline{\mathcal{F}_h^0} = \bigcup_{K\in\widetilde{\mathscr{T}_h^0}\setminus\mathscr{T}_h^0} K,$$

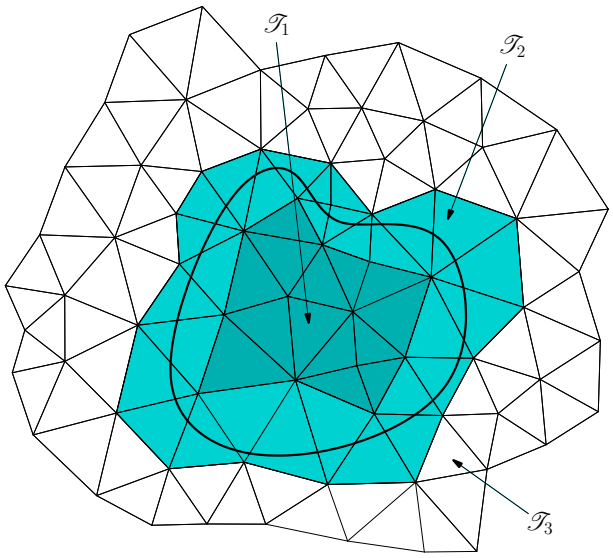

FIGURE 1. Detail of the discretization mesh with the position of the rigid solid and the three categories of triangles.

In this paper, we assume that we obtain the mesh of the fluid by using the above construction[1]. Notice that the discrete domain verifies by construction

$$\text{dist}\left(\mathcal{F}_h^0, \mathcal{F}\right) < Ch. \tag{33}$$

[1] It will also be the process repeated whenever a remeshing is needed (i.e., when the quality of the mesh degrades too much due to its movement). Of course, this step, while common in practical applications of the method (see reference [20], in which the domain is said to be remeshed every five or ten time steps in actual computations), cannot be taken into account in the study of convergence of the scheme and we assume that the mesh remains regular enough during the whole course of its use, which is consistent with assumption (25).

We next define the finite element space

$$\mathcal{P}_h^0 = \left\{ \gamma \in \mathrm{C}^0(\overline{\mathcal{F}_h^0})^2 \ ; \ \gamma_{|_K} \in \mathbb{P}_1(K), \ \forall K \in \mathscr{T}_h^0 \right\},$$

where $\mathbb{P}_1(K)$ denotes the set of affine functions on K, over the triangulation of the domain $\mathcal{F}_h^0$, and its analogous over the triangulation of $\mathcal{O}$,

$$\widetilde{\mathcal{P}}_h^0 = \left\{ \gamma \in \mathrm{C}^0(\overline{\mathcal{O}})^2 \ ; \ \gamma_{|_K} \in \mathbb{P}_1(K), \ \forall K \in \widetilde{\mathscr{T}}_h^0 \right\}.$$

We finally take $\boldsymbol{\zeta}_h^0 = \mathbf{0}$, $\theta_h^0 = 0$ and we obtain the initial approximate velocity $(\boldsymbol{u}_h^0, \boldsymbol{\xi}_h^0, \omega_h^0)$ by first extending $\boldsymbol{u}^{(0)}$ using the rigid velocity formula

$$\boldsymbol{u}^{(0)}(\boldsymbol{x}) = \boldsymbol{\zeta}^{(1)} + \theta^{(1)} \boldsymbol{x}^\perp, \ \forall \boldsymbol{x} \in \mathcal{S},$$

then by considering $\boldsymbol{u}_h^0$ as the projection of this extended field on $\left(\widetilde{\mathcal{P}}_h^0\right)^2$, and finally by setting

$$\boldsymbol{\xi}_h^0 = \int_{\mathcal{S}_h^0} \rho_s \, \boldsymbol{u}_h^0 \, \mathrm{d}\boldsymbol{x} \text{ and } \omega_h^0 = \int_{\mathcal{S}_h^0} \rho_s \, \boldsymbol{u}_h^0(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{\zeta}_h^0)^\perp \, \mathrm{d}\boldsymbol{x}.$$

3.1.2. *Computation of the new domain*

Let us suppose that the quantities $\boldsymbol{\zeta}_h^k$, θ_h^k, $\boldsymbol{\xi}_h^k$ and ω_h^k are known for some k in $\{0, \ldots, N-1\}$. We approximate the position of the center of mass and the orientation of the rigid body at instant t^{k+1} by the relations

$$\boldsymbol{\zeta}_h^{k+1} = \boldsymbol{\zeta}_h^k + (\delta t) \, \boldsymbol{\xi}_h^k \text{ and } \theta_h^{k+1} = \theta_h^k + (\delta t) \, \omega_h^k. \tag{34}$$

The approximations of the domains occupied respectively by the solid and the fluid at instant t^{k+1} are then

$$\mathcal{S}_h^{k+1} = \left\{ \boldsymbol{\zeta}_h^{k+1} + \mathbf{R}_{\theta_h^{k+1} - \theta_h^k}(\boldsymbol{x} - \boldsymbol{\zeta}_h^k), \ \boldsymbol{x} \in \mathcal{S}_h^k \right\} \text{ and } \mathcal{F}_h^{k+1} = \mathcal{O} \setminus \overline{\mathcal{S}_h^{k+1}}. \tag{35}$$

3.1.3. *Computation of the ALE mapping and of the characteristic function*

The finite element approximation at time t^{k+1} of the ALE mapping, denoted $\boldsymbol{\mathcal{A}}_h^{k+1}$, is defined by the relation

$$\boldsymbol{\mathcal{A}}_h^{k+1}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{d}_h^{k+1}(\boldsymbol{x}), \ \forall \boldsymbol{x} \in \mathcal{F}_h^0, \tag{36}$$

where the field $\boldsymbol{d}_h^{k+1} \in \left(\mathcal{P}_h^0\right)^2$ is uniquely determined by

$$\boldsymbol{d}_h^{k+1}(\boldsymbol{x}) = \boldsymbol{\zeta}_h^{k+1} + \mathbf{R}_{\theta_h^{k+1}} \boldsymbol{x} - \boldsymbol{x}, \ \forall \boldsymbol{x} \in \partial\mathcal{S}_h^0, \text{ and } \boldsymbol{d}_h^{k+1} = \mathbf{0} \text{ on } \partial\mathcal{O}, \tag{37}$$

and

$$\int_{\mathcal{F}_h^0} \boldsymbol{\nabla} \boldsymbol{d}_h^{k+1} : \boldsymbol{\nabla} \widetilde{\boldsymbol{d}}_h \, \mathrm{d}\boldsymbol{x} + \lambda \int_{\mathcal{F}_h^0} (\operatorname{div} \boldsymbol{d}_h^{k+1})(\operatorname{div} \widetilde{\boldsymbol{d}}_h) \, \mathrm{d}\boldsymbol{x} = 0, \ \forall \widetilde{\boldsymbol{d}}_h \in \left(\mathcal{P}_h^0\right)^2 \text{ s.t. } \widetilde{\boldsymbol{d}}_h = \mathbf{0} \text{ on } \partial\mathcal{F}_h^0. \tag{38}$$

Remark 3.1. The problem (37)-(38) is well-posed, but we do not know if relation (36) defines an invertible mapping. Let us consider the field $\widetilde{\boldsymbol{d}}_h^{k+1}$ solution to

$$\begin{cases} -\Delta \widetilde{\boldsymbol{d}}_h^{k+1} - \lambda \boldsymbol{\nabla} \operatorname{div} \widetilde{\boldsymbol{d}}_h^{k+1} = \mathbf{0} \text{ in } \mathcal{F}, \\ \widetilde{\boldsymbol{d}}_h^{k+1}(\boldsymbol{x}) = \boldsymbol{\zeta}_h^{k+1} + \mathbf{R}_{\theta_h^{k+1}} \boldsymbol{x} - \boldsymbol{x}, \ \forall \boldsymbol{x} \in \partial\mathcal{S}, \\ \widetilde{\boldsymbol{d}}_h^{k+1} = \mathbf{0} \text{ on } \partial\mathcal{O}. \end{cases} \tag{39}$$

Inequality (4.26) in [11] (see also [22, 24]) yields the following estimate

$$\left\|\widetilde{\boldsymbol{d}}_h^{k+1} - \boldsymbol{d}_h^{k+1}\right\|_{\mathrm{W}^{1,\infty}(\mathcal{F}_h^0)^2} \leqslant Ch|\log h| \left\|\widetilde{\boldsymbol{d}}_h^{k+1}\right\|_{\mathrm{W}^{2,\infty}(\mathcal{F}_h^0)^2}.$$

Since we have, by continuity of the solution of (39) with respect to the data,

$$\left\|\widetilde{\boldsymbol{d}}_h^{k+1}\right\|_{\mathrm{W}^{2,\infty}(\mathcal{F}_h^0)^2} \leqslant C\left(|\boldsymbol{\zeta}_h^{k+1}| + |\theta_h^{k+1}|\right),$$

we get

$$\left\|\boldsymbol{d}_h^{k+1}\right\|_{\mathrm{W}^{1,\infty}(\mathcal{F}_h^0)^2} \leqslant C\left(|\boldsymbol{\zeta}_h^{k+1}| + |\theta_h^{k+1}|\right), \tag{40}$$

which implies, as in the continuous case and if the quantity $|\boldsymbol{\zeta}_h^{k+1}| + |\theta_h^{k+1}|$ is small enough, that the mapping $\boldsymbol{\mathcal{A}}_h^{k+1}$ is invertible. Setting

$$\boldsymbol{\mathcal{A}}_h^{k+1,k} = \boldsymbol{\mathcal{A}}_h^k \circ \left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}, \tag{41}$$

we have in particular that $\boldsymbol{\mathcal{A}}_h^{k+1,k}(\mathcal{F}_h^{k+1}) = \mathcal{F}_h^k$.
To end this remark, notice that the mapping $\boldsymbol{\mathcal{A}}_h^{k+1}$ can be easily extended to a diffeomorphism of $\mathbb{R}^2$ by setting

$$\boldsymbol{d}_h^{k+1}(\boldsymbol{x}) = \boldsymbol{\zeta}_h^{k+1} + \mathbf{R}_{\theta_h^{k+1}}\boldsymbol{x} - \boldsymbol{x},\ \forall \boldsymbol{x} \in \mathcal{S}_h^0, \text{ and } \boldsymbol{d}_h^{k+1} = \mathbf{0} \text{ on } \mathbb{R}^2 \setminus \mathcal{O},$$

and that we will identify $\boldsymbol{\mathcal{A}}_h^{k+1}$ with its extension without any change in the notation in what follows.

In order to define the approximate domain velocity $\boldsymbol{w}_h^k : \mathcal{F}_h^k \to \mathbb{R}^2$, we first introduce the following linear interpolations in time of the approximate center of mass and orientation of the rigid solid:

$$\theta_h(t) = \left(\frac{t^{k+1} - t}{\delta t}\right)\theta_h^k + \left(\frac{t - t^k}{\delta t}\right)\theta_h^{k+1},\ \forall t \in [t^k, t^{k+1}],$$

$$\boldsymbol{\zeta}_h(t) = \left(\frac{t^{k+1} - t}{\delta t}\right)\boldsymbol{\zeta}_h^k + \left(\frac{t - t^k}{\delta t}\right)\boldsymbol{\zeta}_h^{k+1},\ \forall t \in [t^k, t^{k+1}],$$

and of the discrete ALE mapping:

$$\boldsymbol{\mathcal{A}}_h(\boldsymbol{x}, t) = \left(\frac{t^{k+1} - t}{\delta t}\right)\boldsymbol{\mathcal{A}}_h^k(\boldsymbol{x}) + \left(\frac{t - t^k}{\delta t}\right)\boldsymbol{\mathcal{A}}_h^{k+1}(\boldsymbol{x}),\ \forall t \in [t^k, t^{k+1}],\ \forall \boldsymbol{x} \in \mathcal{F}_h^0. \tag{42}$$

Using (36) and (40), we infer that $\boldsymbol{\mathcal{A}}_h(\cdot, t)$ is invertible from $\mathcal{F}_h^0$ onto $\mathcal{F}_h(t) = \boldsymbol{\mathcal{A}}_h(\mathcal{F}_h^0, t)$ and set

$$\boldsymbol{w}_h(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{\mathcal{A}}_h}{\partial t}\left(t, \boldsymbol{\mathcal{A}}_h(\cdot, t)^{-1}(\boldsymbol{x})\right),\ \forall \boldsymbol{x} \in \mathcal{F}_h(t). \tag{43}$$

Introducing the finite element space

$$\mathcal{P}_h^k = \left\{\gamma \in \mathrm{H}^1(\mathcal{F}_h^k)\ ;\ \gamma_{|_K} \in \mathbb{P}_1(K),\ \forall K \in \mathscr{T}_h^k\right\},$$

we then define the approximate domain velocity at time t^k, $\boldsymbol{w}_h^k \in \left(\mathcal{P}_h^k\right)^2$, as

$$\boldsymbol{w}_h^k = \lim_{t \to t^k,\ t > t^k} \boldsymbol{w}_h(\cdot, t). \tag{44}$$

Remark 3.2. We easily see that

$$\boldsymbol{w}_h(\boldsymbol{x},t) = \boldsymbol{\xi}_h^k + \omega_h^k(\boldsymbol{x}-\boldsymbol{\zeta}_h(t))^\perp,\ \forall \boldsymbol{x}\in\partial\mathcal{S}_h(t),\ \text{and}\ \boldsymbol{w}_h(\cdot,t)=\boldsymbol{0}\ \text{on}\ \partial\mathcal{O},\ t\in(t^k,t^{k+1}),$$

which in turn implies

$$\boldsymbol{w}_h^k(\boldsymbol{x}) = \boldsymbol{\xi}_h^k + \omega_h^k(\boldsymbol{x}-\boldsymbol{\zeta}_{\boldsymbol{h}}^{\boldsymbol{k}})^\perp,\ \forall \boldsymbol{x}\in\partial\mathcal{S}_h^k,\ \text{and}\ \boldsymbol{w}_h^k=\boldsymbol{0}\ \text{on}\ \partial\mathcal{O}.$$

We next consider the approximate characteristic function $\boldsymbol{\mathcal{B}}_h$ which, for any $\boldsymbol{x}$ in $\mathcal{F}_h^{k+1}$, is solution to

$$\left\{\begin{array}{l} \dfrac{\partial \boldsymbol{\mathcal{B}}_h}{\partial t}(t;t^{k+1},\boldsymbol{x}) = (\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k}} - \overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k}})(\boldsymbol{\mathcal{B}}_h(t;t^{k+1},\boldsymbol{x})), \\ \boldsymbol{\mathcal{B}}_h(t^{k+1};t^{k+1},\boldsymbol{x}) = \boldsymbol{x}, \end{array}\right. \tag{45}$$

where

$$\overline{\boldsymbol{u}}_h^k(\boldsymbol{x}) = \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{x})\right]^{-1}\boldsymbol{u}_h^k(\boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{x}))\ \text{and}\ \overline{\boldsymbol{w}}_h^k(\boldsymbol{x}) = \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{x})\right]^{-1}\boldsymbol{w}_h^k(\boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{x})),\ \forall \boldsymbol{x}\in\mathcal{F}_h^{k+1}, \tag{46}$$

and denote $\boldsymbol{\mathcal{B}}_h^k = \boldsymbol{\mathcal{B}}_h(t^k;t^{k+1},\cdot)$.

Remark 3.3. The discrete ALE mapping having been extended to the whole of $\mathbb{R}^2$ (see Remark 3.1), it suffices to extend the discrete velocity field $\boldsymbol{u}_h^k$ in a similar way to what has been done in the continuous case to define the characteristic mapping $\boldsymbol{\mathcal{B}}_h^k$ in $\mathbb{R}^2$ using problem (45).

3.1.4. *Calculation of the new velocity and pressure*

The triangulation $\mathscr{T}_h^{k+1}$ of the new domain $\mathcal{F}_h^{k+1}$ is obtained as the image of the triangulation $\mathscr{T}_h^k$ at the previous step by the ALE application $\boldsymbol{\mathcal{A}}_h^{k,k+1}$. Likewise, the triangulation of $\mathcal{S}_h^{k+1}$ is given by $\widetilde{\mathscr{T}}_h^{k+1}\backslash\mathscr{T}_h^{k+1}$, where the triangulation $\widetilde{\mathscr{T}}_h^{k+1}$ is obtained as the image of the triangulation $\widetilde{\mathscr{T}}_h^k$ at the previous step by the ALE application $\boldsymbol{\mathcal{A}}_h^{k,k+1}$. Defining the finite element spaces

$$\begin{aligned}\mathcal{V}_h^{k+1} = \Big\{ (\boldsymbol{v}_h^{k+1}, \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}}, \omega_{\boldsymbol{v}_h^{k+1}}) \in \mathrm{C}^0(\overline{\mathcal{F}_h^{k+1}})^2\times\mathbb{R}^3\ ;\ {\boldsymbol{v}_h^{k+1}}_{|_K} \in [\mathbb{P}_1(K)\oplus\langle\lambda_1\lambda_2\lambda_3\rangle]^2,\ \forall K\in\mathscr{T}_h^{k+1},\\ \boldsymbol{v}_h^{k+1}=\boldsymbol{0}\ \text{on}\ \partial\mathcal{O}\ \text{and}\ \boldsymbol{v}_h^{k+1}(\boldsymbol{x}) = \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + \omega_{\boldsymbol{v}_h^{k+1}}(\boldsymbol{x}-\boldsymbol{\zeta}_h^{k+1})^\perp,\ \forall \boldsymbol{x}\in\partial\mathcal{S}_h^{k+1}\Big\},\end{aligned}$$

with $\{\lambda_i\}_{i=1,2,3}$ the set of barycentric coordinates (with respect to the vertices of K), and

$$\mathcal{Q}_h^{k+1} = \left\{ q_h^{k+1}\in \mathrm{L}^2_0(\mathcal{F}_h^{k+1})\cap\mathrm{C}^0(\overline{\mathcal{F}_h^{k+1}})\ ;\ {q_h^{k+1}}_{|_K}\in\mathbb{P}_1(K),\ \forall K\in\mathscr{T}_h^{k+1}\right\},$$

the discrete velocity and pressure at instant t^{k+1} are then obtained as solution of a discrete generalized Stokes problem: *find* $(\boldsymbol{u}_h^{k+1},\boldsymbol{\xi}_h^{k+1},\omega_h^{k+1})\in\mathcal{V}_h^{k+1}$ *and* $p_h^{k+1}\in\mathcal{Q}_h^{k+1}$ *such that*

$$\begin{aligned}\int_{\mathcal{F}_h^{k+1}}\left(\frac{\boldsymbol{u}_h^{k+1}-\boldsymbol{u}_h^k\circ\boldsymbol{\mathcal{A}}_h^{k+1,k}\circ\boldsymbol{\mathcal{B}}_h^k}{\delta t}\right)\cdot\boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} + M\frac{\boldsymbol{\xi}_h^{k+1}-\boldsymbol{\xi}_h^k}{\delta t}\cdot\boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + I\frac{\omega_h^{k+1}-\omega_h^k}{\delta t}\omega_{\boldsymbol{v}_h^{k+1}}\\ + 2\nu\int_{\mathcal{F}_h^{k+1}}\mathbf{D}(\boldsymbol{u}_h^{k+1}):\mathbf{D}(\boldsymbol{v}_h^{k+1})\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{F}_h^{k+1}} p_h^{k+1}\ \mathrm{div}\,\boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x}\\ = \int_{\mathcal{F}_h^{k+1}}\boldsymbol{f}_h^{k+1}\cdot\boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} + \boldsymbol{f}_{h,\boldsymbol{M}}^{k+1}\cdot\boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + f_{h,I}^{k+1}\omega_{\boldsymbol{v}_h^{k+1}},\quad \forall(\boldsymbol{v}_h^{k+1},\boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}},\omega_{\boldsymbol{v}_h^{k+1}})\in\mathcal{V}_h^{k+1},\end{aligned} \tag{47}$$

$$-\int_{\mathcal{F}_h^{k+1}} q_h^{k+1}\ \mathrm{div}\,\boldsymbol{u}_h^{k+1}\,\mathrm{d}\boldsymbol{x}=0,\quad \forall q_h^{k+1}\in\mathcal{Q}_h^{k+1},\tag{48}$$

where $\boldsymbol{f}_h^{k+1}$ denotes the projection of $\boldsymbol{f}(\cdot,t^{k+1})$ on $\left(\widetilde{\mathcal{P}}_h^{k+1}\right)^2$, with

$$\widetilde{\mathcal{P}}_h^{k+1}=\left\{\gamma\in\mathrm{C}^0(\overline{\mathcal{O}})^2\ ;\ \gamma_{|_K}\in\mathbb{P}_1(K),\ \forall K\in\widetilde{\mathscr{T}}_h^k\right\},$$

and

$$\boldsymbol{f}_{h,M}^{k+1}=\int_{\mathcal{S}_h^{k+1}}\rho_s\,\boldsymbol{f}_h^{k+1}\,\mathrm{d}\boldsymbol{x}\ \text{and}\ f_{h,I}^{k+1}=\int_{\mathcal{S}_h^{k+1}}\rho_s\,\boldsymbol{f}_h^{k+1}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{\zeta}_h^{k+1})^\perp\,\mathrm{d}\boldsymbol{x}.$$

Remark 3.4. It is worth noticing that the discrete mixed problem (47)-(48) is well-posed for any $k\in\{0,\dots,N\}$. Indeed, by adapting the proof of Lemma 4.3 in [25], one can prove a discrete inf-sup condition, that is, there exists a positive constant β_k, independent of h, such that

$$\inf_{\boldsymbol{v}_h\in\mathcal{V}_h^k}\sup_{q_h\in\mathcal{Q}_h^k}\frac{\int_{\mathcal{F}_h^k}q_h\ \mathrm{div}\,\boldsymbol{v}_h\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{v}_h\|_{\mathrm{H}^1(\mathcal{F}_h^k)^2}\|q_h\|_{\mathrm{L}^2(\mathcal{F}_h^k)}}\geqslant\beta_k.$$

Using hypothesis (25), with c_0 small enough, we can also prove that β_k is greater than a positive constant β^* independent of k.

3.2. **Statement of the main result**

Let us recall the hypotheses made so far. We assumed that the domain $\mathcal{O}$ is the interior of a convex polygon and that there is no contact between the solid and the boundary $\partial\mathcal{O}$, a condition expressed by (10), and that the data verify the regularity and compatibility conditions (9). We shall now assume that the solution to problem (1)-(8) is smoother than the regularity given in (11) by making the additional regularity hypotheses:

$$\begin{gathered}\boldsymbol{u}\in\mathrm{C}([0,T];\mathrm{H}^2(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)))^2)\cap\mathrm{H}^1(0,T;\mathrm{H}^1(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)))^2),\ \frac{d^2\boldsymbol{u}}{dt^2}\in\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)))^2),\\ p\in\mathrm{C}([0,T];\mathrm{H}^1(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t)))),\ \boldsymbol{\zeta}\in\mathrm{H}^3(0,T)^2,\ \omega\in\mathrm{H}^2(0,T).\end{gathered}\tag{49}$$

The main result of the present paper is the following.

Theorem 3.5. *Assume that there exists two positive constants c_s and C_s such that*

$$c_s\,h\leqslant\delta t\leqslant C_s\,h^{1/2}.$$

Under the hypotheses (10), (11), (25), (49) *and the assumptions on the space discretization, there exist some positive constants C, $\overline{h}$ and κ, depending on neither h nor δt, such that, for all h in $(0,\overline{h})$, δt in $(0,\kappa)$ and k in $\{0,\dots,N\}$, we have*

$$|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|\leqslant C(\delta t+h^{1/2})$$

and

$$\|\boldsymbol{u}(\boldsymbol{\mathcal{A}}(t^k;0,\cdot),t^k)-\boldsymbol{u}_h^k\circ\boldsymbol{\mathcal{A}}_h^k\|_{\mathrm{L}^2(\mathcal{F})^2}+|\boldsymbol{\xi}(t^k)-\boldsymbol{\xi}_h^k|+|\omega(t^k)-\omega_h^k|\leqslant C(\delta t+h^{1/2}).$$

Remark 3.6. In the above result, the ALE mappings appear in the error estimates for the velocity fields since the exact and approximate velocities are not defined *a priori* on the same domain at instant t^k. Of course, we could alternatively use the extensions of these fields given in the previous sections to state a similar result without the ALE mappings and with quantities defined in the whole domain $\mathcal{O}$.

4. Change of variables

To prove Theorem 3.5, one needs to compare the exact solution, defined at instant t^k in the domain $\mathcal{F}(\boldsymbol{\zeta}(t^k),\theta(t^k))$, with the approximate one, defined at the same moment in the approximate domain $\mathcal{F}_h^k$. This is accomplished with the help of a change of variables similar to the one featured in [18] and subsequently used in [25] for an identical purpose. A description of this transformation and some of its properties are briefly recalled in the next subsections.

4.1. **Construction of the change of variables**

In this subsection, we introduce a family of changes of variable which, for any k in $\{0,\dots,N\}$, transforms a function defined on $\mathcal{F}(\boldsymbol{\zeta}(t^k),\theta(t^k))$ into a function defined on $\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k)$.
First, hypotheses (10) and (11) imply the existence of $\eta>0$ such that

$$\operatorname{dist}\left(\mathcal{S}(\boldsymbol{\zeta}(t),\theta(t)),\partial\mathcal{O}\right)>\eta,\ \forall t\in[0,T].$$

We then assume[2] that

$$|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|\leqslant C(\delta t+h^{1/2}), \tag{50}$$

for some positive constant C independent of h and k. As a consequence, for δt and h both small enough, and for all λ in $[0,1]$, we verify that

$$\lambda\boldsymbol{\zeta}(t^k)+(1-\lambda)\boldsymbol{\zeta}_h^k+\mathbf{R}_{\lambda(\theta(t^k)-\theta_h^k)}\left(\mathcal{S}(\boldsymbol{\zeta}_h^k,\theta_h^k)-\boldsymbol{\zeta}_h^k\right)\subset\mathcal{O},$$

and

$$\operatorname{dist}\left(\lambda\boldsymbol{\zeta}(t^k)+(1-\lambda)\boldsymbol{\zeta}_h^k+\mathbf{R}_{\lambda(\theta(t^k)-\theta_h^k)}\left(\mathcal{S}(\boldsymbol{\zeta}_h^k,\theta_h^k)-\boldsymbol{\zeta}_h^k\right),\partial\mathcal{O}\right)>\frac{\eta}{2}.$$

Let $\chi\in\mathrm{C}^\infty(\mathbb{R}^2,\mathbb{R})$ be a function with compact support contained in $\mathcal{O}$, such that, for all $\boldsymbol{x}$ in $\mathcal{O}$,

$$\chi(\boldsymbol{x})=\begin{cases}1 & \text{if } \operatorname{dist}(\boldsymbol{x},\partial\mathcal{O})>\dfrac{\eta}{2},\\ 0 & \text{if } \operatorname{dist}(\boldsymbol{x},\partial\mathcal{O})<\dfrac{\eta}{4}.\end{cases}$$

Defining the following smooth functions

$$\tau^k(\boldsymbol{x},\lambda)=(\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k)\cdot\boldsymbol{x}^\perp+(\theta(t^k)-\theta_h^k)\left(\frac{|\boldsymbol{x}|^2}{2}-(\lambda\,\boldsymbol{\zeta}(t^k)+(1-\lambda)\,\boldsymbol{\zeta}_h^k)\cdot\boldsymbol{x}\right)$$

and

$$\boldsymbol{\Lambda}^k(\boldsymbol{x},\lambda)=\boldsymbol{\nabla}\left(\chi(\boldsymbol{x})\tau^k(\boldsymbol{x},\lambda)\right)^\perp,$$

for all $\boldsymbol{x}$ in $\mathcal{O}$, we check that, for all λ in $[0,1]$,

$$\boldsymbol{\Lambda}^k(\boldsymbol{x},\lambda)=\begin{cases}\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k+(\theta(t^k)-\theta_h^k)(\boldsymbol{x}-\lambda\boldsymbol{\zeta}(t^k)-(1-\lambda)\boldsymbol{\zeta}_h^k)^\perp & \text{if } \operatorname{dist}(\boldsymbol{x},\partial\mathcal{O})>\dfrac{\eta}{2},\\ \mathbf{0} & \text{if } \operatorname{dist}(\boldsymbol{x},\partial\mathcal{O})<\dfrac{\eta}{4},\end{cases}$$

and

$$\operatorname{div}\boldsymbol{\Lambda}^k(\boldsymbol{x},\lambda)=0. \tag{51}$$

[2]This assumption will be proved later by induction.

Considering the following initial value problem

$$\begin{cases} \dfrac{\partial \boldsymbol{\psi}}{\partial \lambda}(\boldsymbol{y},\lambda) = \boldsymbol{\Lambda}^k\left(\boldsymbol{\psi}(\boldsymbol{y},\lambda),\lambda\right), \\ \boldsymbol{\psi}(\boldsymbol{y},0) = \boldsymbol{y},\ \boldsymbol{y} \in \mathcal{O}, \end{cases}$$

we can show that the application

$$\mathbf{X}^k(\boldsymbol{y}) = \boldsymbol{\psi}(\boldsymbol{y},1),\ \forall \boldsymbol{y} \in \mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k), \tag{52}$$

is a diffeomorphism which maps $\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k)$ onto $\mathcal{F}(\boldsymbol{\zeta}(t^k),\theta(t^k))$ and satisfies

$$\mathbf{X}^k(\boldsymbol{y}) = \boldsymbol{\zeta}(t^k) + \mathbf{R}_{\theta(t^k)-\theta_h^k}(\boldsymbol{y}-\boldsymbol{\zeta}_h^k) \tag{53}$$

for any $\boldsymbol{y}$ in a neighborhood of $\partial\mathcal{S}(\boldsymbol{\zeta}_h^k,\theta_h^k)$.

4.2. **Transformed system**

The purpose of the change of variables defined by the mapping $\mathbf{X}^k$ is to be applied to the exact solution of problem (1)-(8). We thus introduce the following quantities

$$\mathbf{U}^k(\boldsymbol{y}) = J_{\mathbf{Y}^k}(\mathbf{X}^k(\boldsymbol{y}))\boldsymbol{u}(\mathbf{X}^k(\boldsymbol{y}),t^k),\ P^k(\boldsymbol{y}) = p(\mathbf{X}^k(\boldsymbol{y}),t^k),\ \forall \boldsymbol{y} \in \mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k),$$

$$\boldsymbol{\Xi}^k = \mathbf{R}_{\theta_h^k-\theta(t^k)}\boldsymbol{\xi}(t^k) \text{ and } \Omega^k = \omega(t^k),$$

where we have denoted by $\mathbf{Y}^k$ the inverse of $\mathbf{X}^k$ and by $J_{\mathbf{Y}^k}$ the jacobian matrix of $\mathbf{Y}^k$:

$$J_{\mathbf{Y}^k} = \left(\frac{\partial {Y^k}_i}{\partial x_j}\right)_{1\leqslant i,j\leqslant 2}.$$

Also needed will be the transformed characteristic function

$$\boldsymbol{\mathcal{C}}^k = \mathbf{Y}^k \circ \boldsymbol{\mathcal{C}}(t^k;t^{k+1},\cdot) \circ \mathbf{X}^{k+1}. \tag{54}$$

Note that, using a Liouville lemma (see [2, pp. 251], for instance) and property (51), we verify that

$$\operatorname{div} \mathbf{U}^k = 0 \text{ in } \mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k), \tag{55}$$

and also, using notably (53),

$$\mathbf{U}^k(\boldsymbol{y}) = \boldsymbol{\Xi}^k + \Omega^k(\boldsymbol{y}-\boldsymbol{\zeta}_h^k)^\perp,\ \forall \boldsymbol{y} \in \partial\mathcal{S}(\boldsymbol{\zeta}_h^k,\theta_h^k), \text{ and } \mathbf{U}^k = \mathbf{0} \text{ on } \partial\mathcal{O}.$$

Finally, in order to write down the expressions of $\Delta\boldsymbol{u}$ and $\boldsymbol{\nabla} p$ after the change of variables, we define (see [18])

$$[L^k\mathbf{U}^k]_i = \sum_{j,l=1}^{2} \frac{\partial}{\partial y_j}\left(g^{jl}\frac{\partial {\mathrm{U}^k}_i}{\partial y_l}\right) + 2\sum_{j,l,m=1}^{2} g^{lm}\Gamma_{jl}^i\frac{\partial {\mathrm{U}^k}_j}{\partial y_m} + \sum_{j,l,m=1}^{2}\left(\frac{\partial}{\partial y_l}(g^{lm}\Gamma_{jm}^i) + \sum_{n=1}^{2} g^{lm}\Gamma_{jm}^n\Gamma_{ln}^i\right){\mathrm{U}^k}_j, \tag{56}$$

$$[\boldsymbol{G}^k P^k]_i = \sum_{j=1}^{2} g^{ij}\frac{\partial P^k}{\partial y_j}, \tag{57}$$

$$g^{ij} = \sum_{l=1}^{2}\frac{\partial {\mathrm{Y}^k}_i}{\partial x_l}\frac{\partial {\mathrm{Y}^k}_j}{\partial x_l}, \quad g_{ij} = \sum_{l=1}^{2}\frac{\partial {\mathrm{X}^k}_i}{\partial y_l}\frac{\partial {\mathrm{X}^k}_j}{\partial y_l}, \tag{58}$$

$$\Gamma_{ij}^{l} = \frac{1}{2}\sum_{m=1}^{2} g^{lm}\left(\frac{\partial g_{im}}{\partial y_j} + \frac{\partial g_{jm}}{\partial y_i} + \frac{\partial g_{ij}}{\partial y_m}\right), \tag{59}$$

4.3. **Properties**

We end this section by stating a series of estimates satisfied by the change of variables. Their proofs can obtained by using the papers [28] and [25] where the same kind of estimates are derived. All the constants C appearing below and in the sequel denote positive constants which are independent of δt, h and k.

Lemma 4.1. *The functions $\mathbf{X}^k$, defined by (52), and its inverse $\mathbf{Y}^k$ satisfy the following inequalities*

$$\|\mathbf{X}^k\|_{\mathrm{L}^\infty(\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k))^2} \leqslant C,\ \|\mathbf{Y}^k\|_{\mathrm{L}^\infty(\mathcal{F}(\boldsymbol{\zeta}(t^k),\theta(t^k)))^2} \leqslant C,$$

$$\|\mathbf{X}^k - \mathbf{Id}\|_{\mathrm{L}^\infty(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right), \quad \|\mathbf{Y}^k - \mathbf{Id}\|_{\mathrm{L}^\infty(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right),$$

$$\|J_{\mathbf{X}^k} - \mathbf{Id}\|_{\mathrm{L}^\infty(\mathcal{O})^4} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right), \quad \|J_{\mathbf{Y}^k} - \mathbf{Id}\|_{\mathrm{L}^\infty(\mathcal{O})^4} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right),$$

$$\left\|\frac{\partial^2 \mathbf{X}^k}{\partial y_i \partial y_j}\right\|_{\mathrm{L}^\infty(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right), \quad \left\|\frac{\partial^2 \mathbf{Y}^k}{\partial x_i \partial x_j}\right\|_{\mathrm{L}^\infty(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right),$$
$$\forall i,j \in \{1,2\}.$$

Lemma 4.2. *Assume that $(\mathbf{U}^k, P^k) \in \mathrm{H}^2(\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k))^2 \times \mathrm{H}^1(\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k))$. We have*

$$\|\nu\,(L^k - \Delta)\mathbf{U}^k\|_{\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k))^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right)\|\mathbf{U}^k\|_{\mathrm{H}^2(\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k))^2},$$

and

$$\|(\boldsymbol{\nabla} - \boldsymbol{G}^k)\mathrm{P}^k\|_{\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k))^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right)\|\mathrm{P}^k\|_{\mathrm{H}^1(\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k))}.$$

Lemma 4.3. *The function $\mathbf{X}^k$, defined by (52), and its inverse $\mathbf{Y}^k$ satisfy the following inequalities*

$$\|\mathbf{X}^{k+1} - \mathbf{X}^k\|_{\mathrm{L}^\infty(\mathcal{O})^2} \leqslant C(\delta t)\left(|\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k|\right) + C(\delta t)^2\left(\|\boldsymbol{\xi}'\|_{\mathrm{L}^\infty(0,T)^2} + \|\omega'\|_{\mathrm{L}^\infty(0,T)}\right),$$

$$\|\mathbf{Y}^{k+1} - \mathbf{Y}^k\|_{\mathrm{L}^\infty(\mathcal{O})^2} \leqslant C(\delta t)\left(|\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k|\right) + C(\delta t)^2\left(\|\boldsymbol{\xi}'\|_{\mathrm{L}^\infty(0,T)^2} + \|\omega'\|_{\mathrm{L}^\infty(0,T)}\right),$$

$$\|J_{\mathbf{X}^{k+1}} - J_{\mathbf{X}^k}\|_{\mathrm{L}^\infty(\mathcal{O})^4} \leqslant C(\delta t)\left(|\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k|\right) + C(\delta t)^2\left(\|\boldsymbol{\xi}'\|_{\mathrm{L}^\infty(0,T)^2} + \|\omega'\|_{\mathrm{L}^\infty(0,T)}\right).$$

5. Error bounds on the ALE mappings

We need to establish a number of preliminary results related to the ALE mappings. We assume, as in the previous section, that (50) holds and that the steps δt and h are small enough, so that the preceding results remain valid. For sake of simplicity, we assume in the estimates that δt and h are smaller than unity.

We first complete Lemma 2.2 by giving additional results on the continuous ALE mapping. By differentiating problem (22) with respect to time, we get that $\frac{\partial \boldsymbol{d}}{\partial t}(\cdot,t)$ is solution of the system

$$\left\{\begin{array}{l} -\Delta\left(\dfrac{\partial \boldsymbol{d}}{\partial t}\right)(\cdot,t) - \lambda\,\boldsymbol{\nabla}\,\mathrm{div}\left(\dfrac{\partial \boldsymbol{d}}{\partial t}\right)(\cdot,t) = \boldsymbol{0} \text{ in } \mathcal{F}, \\ \left(\dfrac{\partial \boldsymbol{d}}{\partial t}\right)(\boldsymbol{x},t) = \boldsymbol{\xi}(t) + \omega(t)\,\mathbf{R}_{\theta(t)}\boldsymbol{x}^{\perp},\ \forall \boldsymbol{x} \in \partial\mathcal{S}, \\ \left(\dfrac{\partial \boldsymbol{d}}{\partial t}\right)(\cdot,t) = \boldsymbol{0} \text{ on } \partial\mathcal{O}. \end{array}\right. \tag{60}$$

In particular, the following estimate holds (see, for instance, [5, 11])

$$\left\|\frac{\partial \boldsymbol{d}}{\partial t}(\cdot,t)\right\|_{W^{2,\infty}(\mathcal{F})^2} \leqslant C\ \left(|\boldsymbol{\xi}(t)| + |\omega(t)|\right),\ \forall t \in (0,T).$$

Using the regularity assumptions (9), we conclude that

$$\boldsymbol{d},\ \boldsymbol{\mathcal{A}}(\cdot;0,\cdot) \in \mathrm{W}^{2,\infty}(0,T;\mathrm{W}^{2,\infty}(\mathcal{F}))^2. \tag{61}$$

We also have, owing to definition (17),

$$\boldsymbol{w} \in \mathrm{W}^{1,\infty}(0,T;\mathrm{W}^{2,\infty}(\mathcal{F}))^2, \tag{62}$$

and, from the proof of Lemma 2.2,

$$(\boldsymbol{x},t) \mapsto \left[\boldsymbol{\mathcal{A}}(t;0,\cdot)\right]^{-1}(\boldsymbol{x}) \in \mathrm{W}^{2,\infty}(0,T;\mathrm{W}^{2,\infty}(\mathcal{F}(\boldsymbol{\zeta}(t),\theta(t))))^2. \tag{63}$$

We now state a result on the difference between the mapping $\boldsymbol{\mathcal{A}}(t^k;0,\cdot)$ and its discrete counterpart $\boldsymbol{\mathcal{A}}_h^k$. Let us recall that both of these mappings have been extended (see Remarks 2.4 and 3.1) to the whole of $\mathbb{R}^2$.

Lemma 5.1. *For all $k \in \{0,\ldots,N\}$, there exists a positive constant C independent of h and k such that*

$$\left\|\boldsymbol{\mathcal{A}}(t^k;0,\cdot) - \boldsymbol{\mathcal{A}}_h^k\right\|_{\mathrm{L}^\infty(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + h\right)$$

and

$$\left\|\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}(t^k;0,\cdot) - \boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^k\right\|_{\mathrm{L}^2(\mathcal{O})^4} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + \left|\theta(t^k) - \theta_h^k\right| + h^{1/2}\right).$$

Proof. From definitions (24) and (36), we have, for all $\boldsymbol{x}$ in $\mathcal{O}$,

$$\boldsymbol{\mathcal{A}}(t^k;0,\boldsymbol{x}) - \boldsymbol{\mathcal{A}}_h^k(\boldsymbol{x}) = \boldsymbol{d}(\boldsymbol{x},t^k) - \boldsymbol{d}_h^k(\boldsymbol{x})$$

and

$$\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}(t^k;0,\boldsymbol{x}) - \boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^k(\boldsymbol{x}) = \boldsymbol{\nabla}\boldsymbol{d}(\boldsymbol{x},t^k) - \boldsymbol{\nabla}\boldsymbol{d}_h^k(\boldsymbol{x}).$$

On the one hand, the error estimates (4.26) and (4.25) in [11] respectively yield

$$\|\widetilde{\boldsymbol{d}}_h^k - \boldsymbol{d}_h^k\|_{\mathrm{L}^\infty(\mathcal{F}_h^0)^2} \leqslant Ch^2|\log h|\left(|\boldsymbol{\zeta}_h^k| + |\theta_h^k|\right)$$

and

$$\|\boldsymbol{\nabla}\widetilde{\boldsymbol{d}}_h^k - \boldsymbol{\nabla}\boldsymbol{d}_h^k\|_{\mathrm{L}^2(\mathcal{F}_h^0)^4} \leqslant Ch\left(|\boldsymbol{\zeta}_h^k| + |\theta_h^k|\right),$$

and, on the other hand, it holds

$$\|\boldsymbol{d}(\cdot,t^k)-\widetilde{\boldsymbol{d}}_h^k\|_{\mathrm{L}^\infty(\mathcal{O})^2}\leqslant C\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|\right)$$

and

$$\|\boldsymbol{\nabla}\boldsymbol{d}(\cdot,t^k)-\boldsymbol{\nabla}\widetilde{\boldsymbol{d}}_h^k\|_{\mathrm{L}^2(\mathcal{O})^4}\leqslant C\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|\right).$$

We infer from these relations and the triangle inequality that

$$\|\boldsymbol{d}(\cdot,t^k)-\boldsymbol{d}_h^k\|_{\mathrm{L}^\infty(\mathcal{F}_h^0)^2}\leqslant C\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|+h^2|\log h|\right),$$

and

$$\|\boldsymbol{\nabla}\boldsymbol{d}(\cdot,t^k)-\boldsymbol{\nabla}\boldsymbol{d}_h^k\|_{\mathrm{L}^2(\mathcal{F}_h^0)^4}\leqslant C\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|+h\right).$$

Then, given the form of the extensions of the ALE mappings (see Remarks 2.4 and 3.1), we easily prove the following estimates

$$\|\boldsymbol{d}(\cdot,t^k)-\boldsymbol{d}_h^k\|_{\mathrm{L}^\infty(\mathcal{S})^2}\leqslant C\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|\right),$$
$$\|\boldsymbol{\nabla}\boldsymbol{d}(\cdot,t^k)-\boldsymbol{\nabla}\boldsymbol{d}_h^k\|_{\mathrm{L}^2(\mathcal{S})^4}\leqslant C|\theta(t^k)-\theta_h^k|.$$

Finally, owing to the mean value theorem and property (33), it follows

$$\|\boldsymbol{d}(\cdot,t^k)-\boldsymbol{d}_h^k\|_{\mathrm{L}^\infty(\mathcal{F}\setminus\mathcal{F}_h^0)^2}\leqslant C\left(h\,\|\boldsymbol{d}(\cdot,t^k)\|_{\mathrm{W}^{1,\infty}(\mathcal{F})^2}+|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|\right)$$

and, using directly (33),

$$\|\boldsymbol{\nabla}\boldsymbol{d}(\cdot,t^k)-\boldsymbol{\nabla}\boldsymbol{d}_h^k\|_{\mathrm{L}^2(\mathcal{F}\setminus\mathcal{F}_h^0)^4}\leqslant Ch^{1/2}\left(\|\boldsymbol{d}(\cdot,t^k)\|_{\mathrm{W}^{1,\infty}(\mathcal{F})^2}+|\mathbf{R}_{\theta_h^k}-\mathbf{Id}|\right),$$

which ends the proof of the lemma. □

It is also natural to compare the discrete mapping $\boldsymbol{\mathcal{A}}_h^k$ with the diffeomorphism

$$\widehat{\boldsymbol{\mathcal{A}}}^k=\mathbf{Y}^k\circ\boldsymbol{\mathcal{A}}(t^k;0,\cdot)\tag{64}$$

which verifies $\widehat{\boldsymbol{\mathcal{A}}}^k(\mathcal{F})=\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k)$. Notice that, from assumptions (9) and Lemma 4.1, we have

$$\left\|\widehat{\boldsymbol{\mathcal{A}}}^k\right\|_{\mathrm{W}^{2,\infty}(\mathcal{O})^2}\leqslant C,\tag{65}$$

for all h and k. Using (63) and Lemma 4.1, we can also check that

$$\left\|\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1}\right\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2}\leqslant C,\tag{66}$$

for all h and k.

Lemma 5.2. *For all $k\in\{0,\dots,N\}$, there exists a constant C independent of h and k such that*

$$\left\|\widehat{\boldsymbol{\mathcal{A}}}^k-\boldsymbol{\mathcal{A}}_h^k\right\|_{\mathrm{L}^\infty(\mathcal{O})^2}\leqslant C\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|+h\right)$$

and

$$\left\|\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1}-\left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1}\right\|_{\mathrm{L}^\infty(\mathcal{O})^2}\leqslant C\left(|\boldsymbol{\zeta}(t_k)-\boldsymbol{\zeta}_h^k|+|\theta(t_k)-\theta_h^k|+h\right).$$

Proof. The first assertion is an easy consequence of Lemmata 4.1 and 5.1. To prove the second one, we write

$$\begin{aligned}\left\|\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1}-\left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1}\right\|_{\mathrm{L}^\infty(\mathcal{O})^2} &= \left\|\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1}\circ\boldsymbol{\mathcal{A}}_h^k-\left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1}\circ\boldsymbol{\mathcal{A}}_h^k\right\|_{\mathrm{L}^\infty(\mathcal{O})^2}\\ &= \left\|\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1}\circ\boldsymbol{\mathcal{A}}_h^k-\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1}\circ\widehat{\boldsymbol{\mathcal{A}}}^k\right\|_{\mathrm{L}^\infty(\mathcal{O})^2}.\end{aligned}$$

Consequently, using the mean value theorem and (66), we find

$$\left\|\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1}-\left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1}\right\|_{\mathrm{L}^\infty(\mathcal{O})^2}\leqslant C\left\|\widehat{\boldsymbol{\mathcal{A}}}^k-\boldsymbol{\mathcal{A}}_h^k\right\|_{\mathrm{L}^\infty(\mathcal{O})^2}.$$

The above inequality and the first statement of the lemma imply the result. □

We now recall a classical result (see [1]).

Lemma 5.3. *Assume that $\boldsymbol{Z}_i : \mathcal{O}\to\mathcal{O}$ $(i=1,2)$ are two diffeomorphisms such that, for all μ in $[0,1]$,*

$$\mu\,\boldsymbol{Z}_1+(1-\mu)\,\boldsymbol{Z}_2 : \mathcal{O}\to\mathcal{O}$$

is a diffeomorphism with Jacobian J_μ satisfying

$$J_\mu\geqslant c_1, \tag{67}$$

with c_1 a positive constant. Then, for any η in $\mathrm{H}^1(\mathcal{O})$, we have

$$\|\eta\circ\boldsymbol{Z}_1-\eta\circ\boldsymbol{Z}_2\|_{\mathrm{L}^1(\mathcal{O})}\leqslant\frac{1}{c_1}\,\|\boldsymbol{\nabla}\eta\|_{\mathrm{L}^2(\mathcal{O})^2}\|\boldsymbol{Z}_1-\boldsymbol{Z}_2\|_{\mathrm{L}^2(\mathcal{O})^2}.$$

From the above lemma, we infer the following.

Lemma 5.4. *Assume that the steps δt and h are small enough. Then, for all k in $\{0,\dots,N\}$ and μ in $[0,1]$, the application*

$$\mu\,\widehat{\boldsymbol{\mathcal{A}}}^k+(1-\mu)\,\boldsymbol{\mathcal{A}}_h^k : \mathcal{O}\to\mathcal{O}$$

is a diffeomorphism with Jacobian satisfying inequality (67)*. The same property holds for the family of mappings*

$$\mu\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1}+(1-\mu)\left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1} : \mathcal{O}\to\mathcal{O},$$

with μ in $[0,1]$.

Proof. Let us recall that the constant c_0 in (25) has been chosen in such a way that the bound (26) is satisfied. We first deduce from estimate (40) and hypothesis (50) that, for all k in $\{0,\dots,N\}$,

$$\|\boldsymbol{d}_h^k\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2}\leqslant C(c_0+\delta t+h^{1/2}).$$

Then, choosing c_0, $\overline{h}$ and κ small enough, we obtain that, for all h in $(0,\overline{h})$ and δt in $(0,\kappa)$,

$$\|\boldsymbol{d}_h^k\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2}\leqslant\frac{1}{4}. \tag{68}$$

This implies, for all k in $\{0,\dots,N\}$ and μ in $[0,1]$,

$$\|\mu\,\boldsymbol{d}(\cdot,t^k)+(1-\mu)\,\boldsymbol{d}_h^k\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2}\leqslant\frac{1}{4},$$

which yields the first result (see the proof of Lemma 2.2 for details).
Next, using again the bounds (26) and (68), we infer that, for all k in $\{0,\dots,N\}$,

$$\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1} = \mathbf{Id} + \check{\boldsymbol{d}}^k,\ \left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1} = \mathbf{Id} + \check{\boldsymbol{d}}_h^k,$$

with

$$\|\check{\boldsymbol{d}}^k\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2} \leqslant \frac{1}{2} \text{ and } \|\check{\boldsymbol{d}}_h^k\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2} \leqslant \frac{1}{2},$$

so that, for all μ in $[0,1]$, the mapping $\mu\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1} + (1-\mu)\left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1}$ is a diffeomorphism of $\mathcal{O}$ onto itself with Jacobian satisfying (67). □

We are now in position to prove the following result.

Lemma 5.5. *For all k in $\{0,\dots,N-1\}$, there exists a positive constant C independent of h and k such that*

$$\left\| \left[\boldsymbol{\nabla}\left(\widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1} - \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k+1,k}\right]^{-1} \right\|_{\mathrm{L}^2(\mathcal{O})^4} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + h^{1/2}\right).$$

Proof. Differentiating, we have respectively

$$\left[\boldsymbol{\nabla}\left(\widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1} = \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]\ \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^k\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1}$$

and

$$\left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k+1,k}\right]^{-1} = \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k+1}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]\ \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]^{-1}.$$

We use this to rewrite the following difference as the sum of four terms:

$$\left[\boldsymbol{\nabla}\left(\widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1} - \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k+1,k}\right]^{-1} = \sum_{i=1}^{4} R_i, \tag{69}$$

where

$$R_1 = \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]\ \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^k\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1} - \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]\ \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^k\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1},$$

$$R_2 = \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]\ \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^k\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1} - \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k+1}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]\ \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^k\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1},$$

$$R_3 = \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k+1}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]\left[\left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^k\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1} - \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^k\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]^{-1}\right],$$

$$R_4 = \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k+1}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]\left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]^{-1} - \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k+1}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]\left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]^{-1}.$$

Using the bounds (65) and (66), we infer from Lemma 5.4 that

$$\|R_1\|_{\mathrm{L}^2(\mathcal{O})^4} \leqslant C\left\|\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1} - \left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right\|_{\mathrm{L}^2(\mathcal{O})^2}.$$

Combining the above inequality with Lemma 5.2 then yields

$$\|R_1\|_{\mathrm{L}^2(\mathcal{O})^4} \leqslant C\left(|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + |\theta(t_k) - \theta_h^k| + h\right).$$

Next, using Lemmata 5.1 and 5.4, we obtain

$$\|R_2\|_{\mathrm{L}^2(\mathcal{O})^4} \leqslant C\left(|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + |\theta(t_k) - \theta_h^k| + h^{1/2}\right).$$

To estimate R_3, we first write

$$\left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k}\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1} - \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]^{-1} = \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k}\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]^{-1}\left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right) - \boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k}\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right)\right]\left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k}\left(\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right)\right]^{-1}.$$

Consequently, using Lemmata 5.2, 5.3 and 5.4, we get

$$\|R_3\|_{\mathrm{L}^2(\mathcal{O})^4} \leqslant C\left\|\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1} - \left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + |\theta(t_k) - \theta_h^k| + h\right).$$

Finally, using the relation

$$\left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k}\right]^{-1} - \left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k}\right]^{-1} = \left[\boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k}\right]^{-1}\left(\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k} - \boldsymbol{\nabla}\widehat{\boldsymbol{\mathcal{A}}}^{k}\right)\left[\boldsymbol{\nabla}\boldsymbol{\mathcal{A}}_h^{k}\right]^{-1},$$

in conjunction with Lemmata 5.1, 5.3 and 5.4, it can be shown that

$$\|R_4\|_{\mathrm{L}^2(\mathcal{O})^4} \leqslant C\left(|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + |\theta(t_k) - \theta_h^k| + h^{1/2}\right),$$

which ends the proof of the lemma. □

We close this subsection with an error estimate for the discrete domain velocity, which is derived from the approximate ALE mapping. Let us denote

$$\mathbf{W}^k(\boldsymbol{y}) = J_{\mathbf{Y}^k}(\mathbf{X}^k(\boldsymbol{y}))\boldsymbol{w}(\mathbf{X}^k(\boldsymbol{y}), t^k).$$

We have the following result.

Lemma 5.6. *For all k in $\{0,\dots,N\}$, there exists a positive constant C independent of h and k such that*

$$\|\mathbf{W}^k - \boldsymbol{w}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| + \delta t + h\right).$$

Proof. From definition (17) of the domain velocity and identity (24) defining the ALE mapping, it follows that

$$\boldsymbol{w}(\boldsymbol{\mathcal{A}}(t^{k+1};0,\cdot),t) = \frac{\partial \boldsymbol{\mathcal{A}}}{\partial t}(t;0,\cdot) = \frac{\partial \boldsymbol{d}}{\partial t}(\cdot,t) \text{ in } \mathcal{F},$$

and $\frac{\partial \boldsymbol{d}}{\partial t}(\cdot,t)$ is solution of (60).
On the other hand, combining definitions (42), (43) and (44), we obtain that

$$\boldsymbol{w}_h^k \circ \boldsymbol{\mathcal{A}}_h^k = \frac{\boldsymbol{\mathcal{A}}_h^{k+1} - \boldsymbol{\mathcal{A}}_h^k}{\delta t}.$$

Moreover, using (36), (37) and (38), we get that the field

$$\boldsymbol{\Upsilon}_h^k = \frac{\boldsymbol{\mathcal{A}}_h^{k+1} - \boldsymbol{\mathcal{A}}_h^k}{\delta t}$$

satisfies

$$\boldsymbol{\Upsilon}_h^k(\boldsymbol{x}) = \boldsymbol{\xi}_h^{k+1} + \omega_h^k \mathbf{R}_{\theta_h^k}\boldsymbol{x}^\perp + (\delta t)\,\boldsymbol{e}_{\boldsymbol{\Upsilon}_h^k}(\boldsymbol{x}),\ \forall \boldsymbol{x} \in \partial\mathcal{S}_h^0, \text{ and } \boldsymbol{\Upsilon}_h^k = \mathbf{0} \text{ on } \partial\mathcal{O},$$

where

$$\boldsymbol{e}_{\boldsymbol{\Upsilon}_h^k}(\boldsymbol{x}) = -(\omega_h^k)^2 \int_0^1 (1-s)\,\mathbf{R}_{s(\delta t)\omega_h^k}\boldsymbol{x}\,\mathrm{d}s,$$

and

$$\int_{\mathcal{F}_h^0} \boldsymbol{\nabla}\boldsymbol{\Upsilon}_h^k : \boldsymbol{\nabla}\widetilde{\boldsymbol{d}}_h\,\mathrm{d}\boldsymbol{x} + \lambda\int_{\mathcal{F}_h^0} (\operatorname{div}\boldsymbol{\Upsilon}_h^k)(\operatorname{div}\widetilde{\boldsymbol{d}}_h)\,\mathrm{d}\boldsymbol{x} = 0,\ \forall \widetilde{\boldsymbol{d}}_h \in \left(\mathcal{P}_h^0\right)^2 \text{ s.t. } \widetilde{\boldsymbol{d}}_h = \mathbf{0} \text{ on } \partial\mathcal{F}_h^0.$$

Then, using inequality (4.26) in reference [11] and proceeding as in Lemma 5.1, we deduce that

$$\left\|\frac{\partial \boldsymbol{\mathcal{A}}}{\partial t}(t^k;0,\cdot) - \frac{\boldsymbol{\mathcal{A}}_h^{k+1} - \boldsymbol{\mathcal{A}}_h^k}{\delta t}\right\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| + \delta t + h\right). \tag{70}$$

Owing to the definition of $\mathbf{W}^k$, we can write

$$\begin{aligned}\mathbf{W}^k - \boldsymbol{w}_h^k = J_{\mathbf{Y}^k}(\mathbf{X}^k)\boldsymbol{w}(\mathbf{X}^k,t^k) - \boldsymbol{w}_h^k = \left(J_{\mathbf{Y}^k}(\mathbf{X}^k) - \mathbf{Id}\right)\boldsymbol{w}(\mathbf{X}^k,t^k) + \frac{\partial \boldsymbol{d}}{\partial t}\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1},t^k\right) - \frac{\partial \boldsymbol{d}}{\partial t}\left(\left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1},t^k\right)\\ + \frac{\partial \boldsymbol{d}}{\partial t}\left(\left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1},t^k\right) - \frac{\boldsymbol{\mathcal{A}}_h^{k+1} - \boldsymbol{\mathcal{A}}_h^k}{\delta t} \circ \left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1}.\end{aligned}$$

From assumptions (9) and Lemma 4.1, we get

$$\left\|\left(J_{\mathbf{Y}^k}(\mathbf{X}^k) - \mathbf{Id}\right)\boldsymbol{w}(\mathbf{X}^k,t^k)\right\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right).$$

From the regularity of $\boldsymbol{d}$ and Lemma 5.2, we deduce that

$$\left\|\frac{\partial \boldsymbol{d}}{\partial t}\left(\left(\widehat{\boldsymbol{\mathcal{A}}}^k\right)^{-1},t^k\right) - \frac{\partial \boldsymbol{d}}{\partial t}\left(\left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1},t^k\right)\right\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + h\right).$$

Finally, from inequality (70) and Lemma 5.4, we obtain

$$\left\| \frac{\partial \boldsymbol{d}}{\partial t}\left(\left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1}, t^k\right) - \frac{\boldsymbol{\mathcal{A}}_h^{k+1} - \boldsymbol{\mathcal{A}}_h^k}{\delta t} \circ \left(\boldsymbol{\mathcal{A}}_h^k\right)^{-1} \right\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant C\left(|\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| + \delta t + h\right).$$

The last four relations yield the result. □

Lemma 5.7. *For all k in $\{0, \ldots, N-1\}$, there exists a positive constant C independent of h and k such that*

$$\left\| \widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1} - \boldsymbol{\mathcal{A}}_h^k \circ \left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1} \right\|_{\mathrm{L}^2(\mathcal{O})^2}$$
$$\leqslant C(\delta t)\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| + \delta t + h\right).$$

Proof. We have

$$\begin{aligned}
\widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1} - \boldsymbol{\mathcal{A}}_h^k \circ \left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1} &= (\delta t)\left(\frac{\widehat{\boldsymbol{\mathcal{A}}}^k - \widehat{\boldsymbol{\mathcal{A}}}^{k+1}}{\delta t} \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1} - \frac{\boldsymbol{\mathcal{A}}_h^k - \boldsymbol{\mathcal{A}}_h^{k+1}}{\delta t} \circ \left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right) \\
&= (\delta t)\left(\frac{\boldsymbol{\mathcal{A}}_h^{k+1} - \boldsymbol{\mathcal{A}}_h^k}{\delta t} \circ \left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1} - \frac{\boldsymbol{\mathcal{A}}^k - \boldsymbol{\mathcal{A}}^{k+1}}{\delta t} \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\right) \\
&\quad + (\delta t)\frac{\mathbf{Y}^k - \mathbf{Y}^{k+1}}{\delta t} \circ \boldsymbol{\mathcal{A}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1} \\
&\quad + (\delta t)\left(\mathbf{Y}^k - \mathrm{Id}\right) \circ \frac{\boldsymbol{\mathcal{A}}^k - \boldsymbol{\mathcal{A}}^{k+1}}{\delta t} \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}.
\end{aligned}$$

Owing to (49), the mapping $\boldsymbol{\mathcal{A}}(\cdot; 0, \cdot)$ belongs to $\mathrm{W}^{2,\infty}(0, T; \mathrm{L}^2(\mathcal{O}))$ and we thus can write

$$\frac{\boldsymbol{\mathcal{A}}^k - \boldsymbol{\mathcal{A}}^{k+1}}{\delta t} \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1} = \frac{\partial \boldsymbol{\mathcal{A}}}{\partial t}(t^k; 0, \cdot) \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1} + \boldsymbol{e}_{\partial \boldsymbol{\mathcal{A}}}^k,$$

with

$$\|\boldsymbol{e}_{\partial \boldsymbol{\mathcal{A}}}^k\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant C(\delta t).$$

Consequently, using (61), (65) and Lemma 5.4, we deduce that

$$\begin{aligned}
\left\| \widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1} - \boldsymbol{\mathcal{A}}_h^k \circ \left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1} \right\|_{\mathrm{L}^2(\mathcal{O})^2} &\leqslant C(\delta t)\Bigg(\left\| \frac{\partial \boldsymbol{\mathcal{A}}}{\partial t}(t^k; 0, \cdot) - \frac{\boldsymbol{\mathcal{A}}_h^{k+1} - \boldsymbol{\mathcal{A}}_h^k}{\delta t} \right\|_{\mathrm{L}^2(\mathcal{O})^2} \\
&\quad + \left\| \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1} - \left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1} \right\|_{\mathrm{L}^2(\mathcal{O})^2} + \left\| \frac{\mathbf{Y}^k - \mathbf{Y}^{k+1}}{\delta t} \right\|_{\mathrm{L}^2(\mathcal{O})^2} + \|\mathbf{Y}^k - \mathbf{Id}\|_{\mathrm{L}^2(\mathcal{O})^2} + \delta t \Bigg).
\end{aligned}$$

The result is obtained by combining the above inequality with the estimate (70) and those in Lemmata 4.1, 4.3 and 5.2. □

6. Error bounds on the characteristics

In the forthcoming proof of the main result, we will also need some estimates on the difference $\boldsymbol{\mathcal{C}}^k - \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k$ between the exact and approximated characteristics. We assume, as in the previous sections, that (50) holds

and that the steps δt and h are small enough, so that the preceding results remain valid. We continue as well to suppose that δt and h are smaller than unity.

Let us recall once more that the various mappings are now defined in $\mathbb{R}^2$. In the sequel, the diffeomorphisms $\mathbf{X}^k$ and $\mathbf{Y}^k$ are extended to $\mathbb{R}^2$ by setting $\mathbf{X}^k = \mathbf{Y}^k = \mathbf{Id}$ in $\mathbb{R}^2 \setminus \mathcal{O}$. We assume moreover that there exists ε small enough such that following relations hold:

$$(\delta t) \left\| \mathbf{Y}^{k+1} \circ (\overline{\boldsymbol{u}} - \overline{\boldsymbol{w}}) \circ \mathbf{X}^{k+1} \right\|_{\mathrm{L}^\infty(t^k, t^{k+1}; \mathrm{W}^{1,\infty}(\mathcal{O}))^2} \leqslant \varepsilon \tag{71}$$

and

$$(\delta t) \left\| \overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k}} - \overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k}} \right\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2} \leqslant \varepsilon, \tag{72}$$

where, for all t in $[t^k, t^{k+1}]$ and $\boldsymbol{x}$ in $\mathcal{F}(\boldsymbol{\zeta}(t^{k+1}), \theta(t^{k+1}))$, the functions $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{w}}$ are defined by

$$\overline{\boldsymbol{u}}(\boldsymbol{x}, t) = \left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}(t; t^{k+1}, \boldsymbol{x})\right]^{-1} \boldsymbol{u}(\boldsymbol{\mathcal{A}}(t; t^{k+1}, \boldsymbol{x}), t), \quad \overline{\boldsymbol{w}}(\boldsymbol{x}, t) = \left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}(t; t^{k+1}, \boldsymbol{x})\right]^{-1} \boldsymbol{w}(\boldsymbol{\mathcal{A}}(t; t^{k+1}, \boldsymbol{x}), t). \tag{73}$$

These assumptions are stability results to be proven in the proof of Theorem 3.5 by induction, and the positive constant ε will be fixed below. The main result of this subsection is the

Theorem 6.1. *For all k in $\{0, \ldots, N-1\}$, the error on the characteristics can be estimated by*

$$\|\boldsymbol{\mathcal{C}}^k - \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant C(\delta t) \left(\|\mathbf{U}^k - \boldsymbol{u}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + \delta t + h^{1/2} \right), \tag{74}$$

where C is a positive constant which is independent of h and δt.

To prove Theorem 6.1, we have to derive some auxiliary results. First, let us consider the mapping

$$\widehat{\boldsymbol{\mathcal{B}}}(t; t^{k+1}, \cdot) = \mathbf{Y}^{k+1} \circ \boldsymbol{\mathcal{B}}(t; t^{k+1}, \cdot) \circ \mathbf{X}^{k+1}, \ \forall t \in [t^k, t^{k+1}],$$

and denote

$$\widehat{\boldsymbol{\mathcal{B}}}^k = \widehat{\boldsymbol{\mathcal{B}}}(t^k; t^{k+1}, \cdot). \tag{75}$$

From problem (28), we know that $\widehat{\boldsymbol{\mathcal{B}}}(t; t^{k+1}, \cdot)$ satisfies, for any $\boldsymbol{x}$ in $\mathcal{F}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})$, the initial value problem

$$\begin{cases} \dfrac{\partial \widehat{\boldsymbol{\mathcal{B}}}}{\partial t}(t; t^{k+1}, \boldsymbol{x}) = \mathbf{Y}^{k+1} \circ (\overline{\boldsymbol{u}} - \overline{\boldsymbol{w}})(\cdot, t) \circ \mathbf{X}^{k+1}(\widehat{\boldsymbol{\mathcal{B}}}(t; t^{k+1}, \boldsymbol{x}), t), \\ \widehat{\boldsymbol{\mathcal{B}}}(t^{k+1}; t^{k+1}, \boldsymbol{x}) = \boldsymbol{x}, \end{cases} \tag{76}$$

in which the functions $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{w}}$ are defined by (73). Using the bounds (71) and (72) $f$or ε small enough , we can show that, for any μ in $[0, 1]$, the mapping $\mu\, \widehat{\boldsymbol{\mathcal{B}}}^k + (1 - \mu)\, \boldsymbol{\mathcal{B}}_h^k$ is a diffeomorphism with Jacobian satisfying (67) for some positive constant c_1.

We next establish the following result.

Lemma 6.2. *Consider the functions $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{w}}$ given by* (73)*, and $\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k}}$ and $\overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k}}$ given by* (46)*. Then, we have, for all k in $\{0, \ldots, N-1\}$,*

$$\int_{t^k}^{t^{k+1}} \left\| \mathbf{Y}^{k+1} \circ (\overline{\boldsymbol{u}} - \overline{\boldsymbol{w}})(\cdot, t) \circ \mathbf{X}^{k+1} - (\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k}} - \overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k}}) \right\|_{\mathrm{L}^2(\mathcal{O})^2} \mathrm{d}t \\ \leqslant C\,(\delta t) \left(\|\mathbf{U}^k - \boldsymbol{u}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + \delta t + h^{1/2} \right).$$

Proof. From the definitions (46) and (73), it follows that, for any $\boldsymbol{y}$ in $\mathcal{O}$ and t in $[t^k, t^{k+1}]$,

$$\begin{aligned}\mathbf{Y}^{k+1} \circ (\overline{\boldsymbol{u}} - \overline{\boldsymbol{w}})(\cdot,t) \circ \mathbf{X}^{k+1}(\boldsymbol{y}) - (\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k}} - \overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k}})(\boldsymbol{y}) \\ = \mathbf{Y}^{k+1} \left(\left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}(t; t^{k+1}, \mathbf{X}^{k+1}(\boldsymbol{y})) \right]^{-1} (\boldsymbol{u} - \boldsymbol{w}) \left(\boldsymbol{\mathcal{A}}(t; t^{k+1}, \mathbf{X}^{k+1}(\boldsymbol{y})), t \right) \right) \\ - \left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{y}) \right]^{-1} (\boldsymbol{u}_h^k - \boldsymbol{w}_h^k)(\boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{y})).\end{aligned}$$

The right hand side of the above identity is then decomposed into a sum of five terms:

$$\mathbf{Y}^{k+1} \circ (\overline{\boldsymbol{u}} - \overline{\boldsymbol{w}})(\cdot,t) \circ \mathbf{X}^{k+1}(\boldsymbol{y}) - (\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k}} - \overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k}})(\boldsymbol{y}) = \sum_{i=1}^{5} \boldsymbol{\mathcal{E}}_i(\boldsymbol{y}),$$

where

$$\boldsymbol{\mathcal{E}}_1(\boldsymbol{y}) = \left[\mathbf{Y}^{k+1} - \mathbf{Id} \right] \left(\left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}(t; t^{k+1}, \mathbf{X}^{k+1}(\boldsymbol{y})) \right]^{-1} (\boldsymbol{u} - \boldsymbol{w}) \left(\boldsymbol{\mathcal{A}}(t; t^{k+1}, \mathbf{X}^{k+1}(\boldsymbol{y})), t \right) \right),$$

$$\begin{aligned}\boldsymbol{\mathcal{E}}_2(\boldsymbol{y}) = \left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}(t; t^{k+1}, \mathbf{X}^{k+1}(\boldsymbol{y})) \right]^{-1} (\boldsymbol{u} - \boldsymbol{w}) \left(\boldsymbol{\mathcal{A}}(t; t^{k+1}, \mathbf{X}^{k+1}(\boldsymbol{y})), t \right) \\ - \left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}(t^k; t^{k+1}, \mathbf{X}^{k+1}(\boldsymbol{y})) \right]^{-1} (\boldsymbol{u} - \boldsymbol{w}) \left(\boldsymbol{\mathcal{A}}(t^k; t^{k+1}, \mathbf{X}^{k+1}(\boldsymbol{y})), t^k \right),\end{aligned}$$

$$\boldsymbol{\mathcal{E}}_3(\boldsymbol{y}) = \left(\left[\boldsymbol{\nabla} \widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} (\boldsymbol{y}) \right]^{-1} - \left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{y}) \right]^{-1} \right) (\mathbf{U}^k - \mathbf{W}^k) \left(\widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} (\boldsymbol{y}) \right),$$

$$\boldsymbol{\mathcal{E}}_4(\boldsymbol{y}) = \left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{y}) \right]^{-1} \left((\mathbf{U}^k - \mathbf{W}^k) \left(\widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} (\boldsymbol{y}) \right) - (\mathbf{U}^k - \mathbf{W}^k)(\boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{y})) \right),$$

$$\boldsymbol{\mathcal{E}}_5(\boldsymbol{y}) = \left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{y}) \right]^{-1} \left((\mathbf{U}^k - \mathbf{W}^k) - \left(\boldsymbol{u}_h^k - \boldsymbol{w}_h^k \right) \right) (\boldsymbol{\mathcal{A}}_h^{k+1,k}(\boldsymbol{y})).$$

First, we deduce from Lemma 4.1 that

$$\int_{t^k}^{t^{k+1}} \|\boldsymbol{\mathcal{E}}_1\|_{\mathrm{L}^2(\mathcal{O})^2} \, \mathrm{d}t \leqslant C(\delta t) \|\mathbf{Y}^{k+1} - \mathbf{Id}\|_{\mathrm{L}^\infty(\mathcal{O})^2} \leqslant C(\delta t) \left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| \right).$$

We then infer from (49), (61), (62) and Lemma 4.1 that the family of mappings

$$(\boldsymbol{y}, t) \mapsto \left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}(t; t^{k+1}, \mathbf{X}^{k+1}(\boldsymbol{y})) \right]^{-1} (\boldsymbol{u} - \boldsymbol{w}) \left(\boldsymbol{\mathcal{A}}(t; t^{k+1}, \mathbf{X}^{k+1}(\boldsymbol{y})), t \right)$$

is bounded in $\mathrm{W}^{1,\infty}(0, T; \mathrm{L}^\infty(\mathcal{O}))$ with respect to k and we thus find

$$\int_{t^k}^{t^{k+1}} \|\boldsymbol{\mathcal{E}}_2\|_{\mathrm{L}^2(\mathcal{O})^2} \, \mathrm{d}t \leqslant C \, (\delta t)^2.$$

For the third term, the use of (49), (62), (65) and Lemmata 4.1 and 5.5 yield that

$$\int_{t^k}^{t^{k+1}} \|\boldsymbol{\mathcal{E}}_3\|_{\mathrm{L}^2(\mathcal{O})^2} \, \mathrm{d}t \leqslant C(\delta t) \left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + h^{1/2} \right).$$

Using assumption (25), we can show that $\left\| \left[\boldsymbol{\nabla} \boldsymbol{\mathcal{A}}_h^{k+1,k} \right]^{-1} \right\|_{\mathrm{L}^\infty(\mathcal{O})^4}$ is uniformly bounded with respect to k and h (see the proof of Lemma 5.4 for details). In addition, using the regularity hypothesis, we obtain that $\|\nabla(\mathbf{U}^k - \mathbf{W}^k)\|_{\mathrm{L}^\infty(\mathcal{O})^4}$ is also uniformly bounded with respect to k and h. Therefore, we have

$$\int_{t^k}^{t^{k+1}} \|\boldsymbol{\mathcal{E}}_4\|_{\mathrm{L}^2(\mathcal{O})^2} \, \mathrm{d}t \leqslant C(\delta t) \left\| \widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} - \boldsymbol{\mathcal{A}}_h^{k+1,k} \right\|_{\mathrm{L}^\infty(\mathcal{O})^2},$$

which in turn implies

$$\int_{t^k}^{t^{k+1}} \|\boldsymbol{\mathcal{E}}_4\|_{\mathrm{L}^2(\mathcal{O})^2} \, \mathrm{d}t \leqslant C(\delta t) \left(\left\| \widehat{\boldsymbol{\mathcal{A}}}^k - \boldsymbol{\mathcal{A}}_h^k \right\|_{\mathrm{L}^\infty(\mathcal{O})^2} + \left\| \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} - \left(\boldsymbol{\mathcal{A}}_h^{k+1} \right)^{-1} \right\|_{\mathrm{L}^\infty(\mathcal{O})^2} \right).$$

Hence, using Lemma 5.2, we obtain

$$\int_{t^k}^{t^{k+1}} \|\boldsymbol{\mathcal{E}}_4\|_{\mathrm{L}^2(\mathcal{O})^2} \, \mathrm{d}t \leqslant C(\delta t) \left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + h \right).$$

Finally, using a change of variables, we get

$$\int_{t^k}^{t^{k+1}} \|\boldsymbol{\mathcal{E}}_5\|_{\mathrm{L}^2(\mathcal{O})^2} \, \mathrm{d}t \leqslant C(\delta t) \left(\|\mathbf{U}^k - \boldsymbol{u}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} + \|\mathbf{W}^k - \boldsymbol{w}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} \right)$$

and infer from Lemma 5.6 that

$$\int_{t^k}^{t^{k+1}} \|\boldsymbol{\mathcal{E}}_5\|_{\mathrm{L}^2(\mathcal{O})^2} \, \mathrm{d}t \leqslant C(\delta t) \left(\|\mathbf{U}^k - \boldsymbol{u}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + \delta t + h \right).$$

Collecting all these bounds then yields the desired estimate. □

Proof of Theorem 6.1. From definition (54), we have

$$\boldsymbol{\mathcal{C}}^k - \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k = \mathbf{Y}^k \circ \boldsymbol{\mathcal{A}}(t^k; t^{k+1}, \cdot) \circ \boldsymbol{\mathcal{B}}(t^k; t^{k+1}, \cdot) \circ \mathbf{X}^{k+1} - \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k,$$

and it follows from (64) and (75) that

$$\begin{aligned} \boldsymbol{\mathcal{C}}^k - \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k &= \mathbf{Y}^k \circ \boldsymbol{\mathcal{A}}(t^k; t^{k+1}, \cdot) \circ \mathbf{X}^{k+1} \circ \mathbf{Y}^{k+1} \circ \boldsymbol{\mathcal{B}}(t^k; t^{k+1}, \cdot) \circ \mathbf{X}^{k+1} - \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k \\ &= \widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} \circ \widehat{\boldsymbol{\mathcal{B}}}^k - \boldsymbol{\mathcal{A}}_h^k \circ \left(\boldsymbol{\mathcal{A}}_h^{k+1} \right)^{-1} \circ \boldsymbol{\mathcal{B}}_h^k. \end{aligned}$$

The right-hand side of the above equality is then broken into two contributions:

$$\begin{aligned} \boldsymbol{\mathcal{C}}^k - \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k &= \widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} \circ \widehat{\boldsymbol{\mathcal{B}}}^k - \widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} \circ \boldsymbol{\mathcal{B}}_h^k \\ &\quad + \widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} \circ \boldsymbol{\mathcal{B}}_h^k - \boldsymbol{\mathcal{A}}_h^k \circ \left(\boldsymbol{\mathcal{A}}_h^{k+1} \right)^{-1} \circ \boldsymbol{\mathcal{B}}_h^k. \end{aligned} \tag{77}$$

Using (65) and (66), we have that

$$\left\| \widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} \circ \widehat{\boldsymbol{\mathcal{B}}}^k - \widehat{\boldsymbol{\mathcal{A}}}^k \circ \left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1} \right)^{-1} \circ \boldsymbol{\mathcal{B}}_h^k \right\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant C \|\widehat{\boldsymbol{\mathcal{B}}}^k - \boldsymbol{\mathcal{B}}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2}. \tag{78}$$

Additionally, taking the difference between problems (76) and (45), we find that, for all t in $[t^k, t^{k+1}]$,

$$\begin{aligned}\left\|\widehat{\boldsymbol{\mathcal{B}}}(t;t^{k+1},\cdot)-\boldsymbol{\mathcal{B}}_h(t;t^{k+1},\cdot)\right\|_{\mathrm{L}^2(\mathcal{O})^2}&\\ \leqslant\int_t^{t^{k+1}}\left\|\boldsymbol{\nabla}\left(\mathbf{Y}^{k+1}\circ(\overline{\boldsymbol{u}}-\overline{\boldsymbol{w}})(\cdot,s)\circ\mathbf{X}^{k+1}\right)\right\|_{\mathrm{L}^\infty(\mathcal{O})^4}&\left\|\widehat{\boldsymbol{\mathcal{B}}}(s;t^{k+1},\cdot)-\boldsymbol{\mathcal{B}}_h(s;t^{k+1},\cdot)\right\|_{\mathrm{L}^2(\mathcal{O})^2}\,\mathrm{d}s\\ +\int_t^{t^{k+1}}\left\|\left(\mathbf{Y}^{k+1}\circ(\overline{\boldsymbol{u}}-\overline{\boldsymbol{w}})(\cdot,s)\circ\mathbf{X}^{k+1}-(\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k}}-\overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k}})\right)\circ\boldsymbol{\mathcal{B}}_h(s;t^{k+1},\cdot)\right\|_{\mathrm{L}^2(\mathcal{O})^2}&\,\mathrm{d}s.\end{aligned}$$

Combining the above inequality with Grönwall's inequality and the fact that the jacobian of $\boldsymbol{\mathcal{B}}_h$ is bounded (owing to assumption (72)), we get

$$\left\|\widehat{\boldsymbol{\mathcal{B}}}(t;t^{k+1},\cdot)-\boldsymbol{\mathcal{B}}_h(t;t^{k+1},\cdot)\right\|_{\mathrm{L}^2(\mathcal{O})^2}\leqslant C\int_{t^k}^{t^{k+1}}\left\|\mathbf{Y}^{k+1}\circ(\overline{\boldsymbol{u}}-\overline{\boldsymbol{w}})(\cdot,s)\circ\mathbf{X}^{k+1}-(\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k}}-\overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k}})\right\|_{\mathrm{L}^2(\mathcal{O})^2}\,\mathrm{d}s.$$

We finally deduce that

$$\left\|\widehat{\boldsymbol{\mathcal{B}}}(t^k;t^{k+1},\cdot)-\boldsymbol{\mathcal{B}}_h^k\right\|_{\mathrm{L}^2(\mathcal{O})^2}\leqslant C(\delta t)\left(\|\mathbf{U}^k-\boldsymbol{u}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2}+|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|+\delta t+h^{1/2}\right),\tag{79}$$

using Lemma 6.2.

We now turn our attention on the second term in the right-hand side of (77). Since the jacobian of $\boldsymbol{\mathcal{B}}_h^k$ is bounded, we get

$$\left\|\widehat{\boldsymbol{\mathcal{A}}}^k\circ\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\circ\boldsymbol{\mathcal{B}}_h^k-\boldsymbol{\mathcal{A}}_h^k\circ\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\circ\boldsymbol{\mathcal{B}}_h^k\right\|_{\mathrm{L}^2(\mathcal{O})^2}\leqslant C\left\|\widehat{\boldsymbol{\mathcal{A}}}^k\circ\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}-\boldsymbol{\mathcal{A}}_h^k\circ\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\right\|_{\mathrm{L}^2(\mathcal{O})^2}.$$

Given the bound obtained in Lemma 5.7, we have

$$\begin{aligned}\left\|\widehat{\boldsymbol{\mathcal{A}}}^k\circ\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\circ\boldsymbol{\mathcal{B}}_h^k-\boldsymbol{\mathcal{A}}_h^k\circ\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\circ\boldsymbol{\mathcal{B}}_h^k\right\|_{\mathrm{L}^2(\mathcal{O})^2}&\leqslant\\ C(\delta t)\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|+|\boldsymbol{\xi}(t^k)-\boldsymbol{\xi}_h^k|+|\omega(t^k)-\omega_h^k|+\delta t+h\right).&\end{aligned}\tag{80}$$

Collecting (77), (78), (79) and (80) therefore yields estimate (74). □

Lemma 6.3. *Assume that h is small enough. Then, for all k in $\{0,\dots,N-1\}$ and μ in $[0,1]$, the mapping*

$$\mu\,\boldsymbol{\mathcal{C}}^k+(1-\mu)\,\boldsymbol{\mathcal{A}}_h^{k+1,k}\circ\boldsymbol{\mathcal{B}}_h^k\,:\,\mathcal{O}\to\mathcal{O}$$

is a diffeomorphism with Jacobian satisfying (67) *for some positive constant c_1.*

Proof. It suffices to write

$$\mu\,\boldsymbol{\mathcal{C}}^k+(1-\mu)\,\boldsymbol{\mathcal{A}}_h^{k+1,k}\circ\boldsymbol{\mathcal{B}}_h^k=\mu\,\widehat{\boldsymbol{\mathcal{A}}}^k\circ\left(\widehat{\boldsymbol{\mathcal{A}}}^{k+1}\right)^{-1}\circ\widehat{\boldsymbol{\mathcal{B}}}^k+(1-\mu)\,\boldsymbol{\mathcal{A}}_h^k\circ\left(\boldsymbol{\mathcal{A}}_h^{k+1}\right)^{-1}\circ\boldsymbol{\mathcal{B}}_h^k,$$

and follow the proof of Lemma 5.4 using assumptions (71) and (72). □

Lemma 6.4. *For all k in $\{0,\dots,N\}$ and μ in $[0,1]$, there exists a constant C independent of h and k such that*

$$\|\boldsymbol{\mathcal{C}}^k-\mathbf{Id}\|_{\mathrm{L}^2(\mathcal{O})^2}\leqslant C(\delta t)\left(|\theta(t^k)-\theta_h^k|+|\boldsymbol{\xi}(t^k)-\boldsymbol{\xi}_h^k|+|\omega(t^k)-\omega_h^k|\right)+C(\delta t).\tag{81}$$

Proof. Let us define, for all t in $[t^k, t^{k+1}]$,

$$\widehat{\boldsymbol{\mathcal{C}}}(t; t^{k+1}, \cdot) = \mathbf{Y}^k \circ \boldsymbol{\mathcal{C}}(t; t^{k+1}, \cdot) \circ \mathbf{X}^{k+1},$$

so that $\widehat{\boldsymbol{\mathcal{C}}}(t^k; t^{k+1}, \cdot) = \boldsymbol{\mathcal{C}}^k$. We then deduce from problem (16) that, for all $\boldsymbol{x}$ in $\mathcal{O}$, the function $\widehat{\boldsymbol{\mathcal{C}}}(\cdot; t^{k+1}, \boldsymbol{x})$ satisfies the following initial value problem

$$\begin{cases} \dfrac{\partial \widehat{\boldsymbol{\mathcal{C}}}}{\partial t}(t; t^{k+1}, \boldsymbol{x}) = J_{\mathbf{Y}^k}(\mathbf{X}^k(\widehat{\boldsymbol{\mathcal{C}}}(t; t^{k+1}, \boldsymbol{x}))) \, \boldsymbol{u}(\mathbf{X}^k(\widehat{\boldsymbol{\mathcal{C}}}(t; t^{k+1}, \boldsymbol{x})), t), \\ \widehat{\boldsymbol{\mathcal{C}}}(t^{k+1}; t^{k+1}, \boldsymbol{x}) = \mathbf{Y}^k \circ \mathbf{X}^{k+1}(\boldsymbol{x}). \end{cases}$$

Consequently, we get from a Taylor-Lagrange inequality combined with the regularity assumption (49) and with Lemma 4.1 that

$$\|\boldsymbol{\mathcal{C}}^k - \mathbf{Id}\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant \|\mathbf{Y}^k \circ \mathbf{X}^{k+1} - \mathbf{Id}\|_{\mathrm{L}^2(\mathcal{O})^2} + C(\delta t),$$

from which the result is inferred using Lemma 4.3. □

7. Proof of the main result

As previously, we assume that δt and h are smaller than unity in order to simplify the estimates.

7.1. **Consistency analysis**

We now show that the exact strong solution of (1)-(8) satisfies at each instant t^k a perturbed system of semidiscretized equations which leads to a weak formulation similar to problem (47)-(48), and estimate the perturbations.

Consider $(\boldsymbol{u}, \boldsymbol{\zeta}, \theta, p)$ the solution to (1)-(8) and assume that it satisfies (49). For any k in $\{0, \ldots, N-1\}$, we have that

$$\frac{\boldsymbol{u}(\cdot, t^{k+1}) - \boldsymbol{u}(\boldsymbol{\mathcal{C}}(t^k; t^{k+1}, \cdot), t^k)}{\delta t} - \nu \Delta \boldsymbol{u}(\cdot, t^{k+1}) + \boldsymbol{\nabla} p(\cdot, t^{k+1}) = \boldsymbol{f}(\cdot, t^{k+1}) + \boldsymbol{e}^k_{d_t \boldsymbol{u}} \text{ in } \mathcal{F}(\boldsymbol{\zeta}(t^{k+1}), \theta(t^{k+1})), \quad (82)$$

$$M \frac{\boldsymbol{\xi}(t^{k+1}) - \boldsymbol{\xi}(t^k)}{\delta t} = - \int_{\partial \mathcal{S}(\boldsymbol{\zeta}(t^{k+1}), \theta(t^{k+1}))} \boldsymbol{\sigma}(\boldsymbol{u}, p) \boldsymbol{n}(\boldsymbol{x}, t^{k+1}) \, \mathrm{d}\Gamma + \boldsymbol{f}_M(t^{k+1}) + \boldsymbol{e}^k_{\boldsymbol{\xi}}, \quad (83)$$

$$I \frac{\omega(t^{k+1}) - \omega(t^k)}{\delta t} = - \int_{\partial \mathcal{S}(\boldsymbol{\zeta}(t^{k+1}), \theta(t^{k+1}))} (\boldsymbol{x} - \boldsymbol{\zeta}_h^{k+1})^\perp \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p) \boldsymbol{n}(\boldsymbol{x}, t^{k+1}) \, \mathrm{d}\Gamma + f_I(t^{k+1}) + e^k_\omega, \quad (84)$$

where

$$\boldsymbol{e}^k_{d_t \boldsymbol{u}} = \frac{\boldsymbol{u}(\cdot, t^{k+1}) - \boldsymbol{u}(\boldsymbol{\mathcal{C}}(t^k; t^{k+1}, \cdot), t^k)}{\delta t} - \frac{d}{dt} \left[\boldsymbol{u} \circ \boldsymbol{\mathcal{C}} \right] (\cdot, t^{k+1})$$

and

$$\boldsymbol{e}^k_{\boldsymbol{\xi}} = \frac{\boldsymbol{\xi}(t^{k+1}) - \boldsymbol{\xi}(t^k)}{\delta t} - \boldsymbol{\xi}'(t^{k+1}), \; e^k_\omega = \frac{\omega(t^{k+1}) - \omega(t^k)}{\delta t} - \omega'(t^{k+1}).$$

Using the regularity assumptions (49) and the Taylor-Lagrange inequality, we get the following consistency estimates.

Lemma 7.1. *There exists a positive constant C independent of h and k such that the quantities $\boldsymbol{e}^k_{d_t \boldsymbol{u}}$, $\boldsymbol{e}^k_{\boldsymbol{\xi}}$ and $\boldsymbol{e}^k_\omega$ defined above satisfy the following inequalities*

$$\|\boldsymbol{e}^k_{d_t \boldsymbol{u}}\|_{\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}(t^{k+1}), \theta(t^{k+1})))^2} \leqslant C(\delta t)^{1/2} \left\| \frac{d^2}{dt^2} \left[\boldsymbol{u} \circ \boldsymbol{\mathcal{C}} \right] \right\|_{\mathrm{L}^2(t^k, t^{k+1}; \mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}(t), \theta(t))))^2}, \quad (85)$$

$$|\boldsymbol{e}^k_{\boldsymbol{\xi}}| \leqslant C(\delta t)^{1/2} \|\boldsymbol{\xi}''\|_{\mathrm{L}^2(t^k, t^{k+1})^2} \quad \textit{and} \quad |e^k_\omega| \leqslant C(\delta t)^{1/2} \|\omega''\|_{\mathrm{L}^2(t^k, t^{k+1})}. \quad (86)$$

The change of variable introduced in Section 4 allows us to transform the equations (82) to (84), set at instant t^{k+1} in a domain in which the rigid body has $\boldsymbol{\zeta}(t^{k+1})$ as the position for its center of mass and $\theta(t^{k+1})$ as its orientation, into a system of equations set in a domain where the solid is centered on $\boldsymbol{\zeta}_h^{k+1}$ with orientation θ_h^{k+1}. This step will permit the subsequent comparison between the exact and approximate solution of the problem. We suppose that (50) holds so we can consider the change of variables $\mathbf{X}^{k+1}$ and keeping with the other notations introduced in Section 4, we get that the couple $(\mathbf{U}^{k+1}, P^{k+1})$ satisfies

$$\frac{\mathbf{U}^{k+1} - (J_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1})(J_{\mathbf{X}^k} \circ \boldsymbol{\mathcal{C}}^k)(\mathbf{U}^k \circ \boldsymbol{\mathcal{C}}^k)}{\delta t} - \nu\,[L^{k+1}\mathbf{U}^{k+1}] + [\boldsymbol{G}^{k+1}P^{k+1}] = \mathbf{F}^{k+1} + \mathbf{E}^k_{d_t\boldsymbol{u}} \text{ in } \mathcal{F}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1}), \tag{87}$$

where

$$\mathbf{F}^{k+1}(\boldsymbol{y}) = J_{\mathbf{Y}^{k+1}}(\mathbf{X}^{k+1}(\boldsymbol{y}))\boldsymbol{f}(\mathbf{X}^{k+1}(\boldsymbol{y}), t^{k+1}),\ \mathbf{E}^k_{d_t\boldsymbol{u}}(\boldsymbol{y}) = J_{\mathbf{Y}^{k+1}}(\mathbf{X}^{k+1}(\boldsymbol{y}))\,\boldsymbol{e}^k_{d_t\boldsymbol{u}}(\mathbf{X}^{k+1}(\boldsymbol{y})),$$

and, from property (55),

$$\operatorname{div}\mathbf{U}^{k+1} = 0 \text{ in } \mathcal{F}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1}). \tag{88}$$

The transformed equations for the rigid solid are obtained using property (53), and we have

$$M\frac{\boldsymbol{\Xi}^{k+1} - \boldsymbol{\Xi}^k}{\delta t} = -\int_{\partial\mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \boldsymbol{\sigma}(\mathbf{U}^{k+1}, P^{k+1})\boldsymbol{n}\,\mathrm{d}\Gamma + \mathbf{F}_M^{k+1} + \mathbf{E}^k_{\boldsymbol{\xi}}, \tag{89}$$

$$I\frac{\Omega^{k+1} - \Omega^k}{\delta t} = -\int_{\partial\mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} (\boldsymbol{x} - \boldsymbol{\zeta}_h^{k+1})^\perp \cdot \boldsymbol{\sigma}(\mathbf{U}^{k+1}, P^{k+1})\boldsymbol{n}\,\mathrm{d}\Gamma + \mathrm{F}_I^{k+1} + \mathrm{E}^k_\omega, \tag{90}$$

where $\mathbf{F}_M^{k+1} = \mathbf{R}_{\theta_h^{k+1} - \theta(t^{k+1})}\boldsymbol{f}_M(t^{k+1})$, $\mathrm{F}_I^{k+1} = f_I^{k+1}$, $\mathbf{E}^k_{\boldsymbol{\xi}} = \mathbf{R}_{\theta_h^k - \theta(t^k)}\boldsymbol{e}^k_{\boldsymbol{\xi}}$ and $\mathrm{E}^k_\omega = e^k_\omega$.

We observe that, while $\mathcal{V}_h^{k+1}$ is not a subspace of $\mathcal{V}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})$ due to the nonconforming approximation of the domain, any function $(\boldsymbol{v}_h^{k+1}, \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}}, \omega_{\boldsymbol{v}_h^{k+1}})$ of $\mathcal{V}_h^{k+1}$, defined on $\mathcal{F}_h^{k+1}$, can be easily extended to the whole domain $\mathcal{O}$ into a function of $\mathrm{H}_0^1(\mathcal{O})^2 \times \mathbb{R}^3$ by setting

$$\boldsymbol{v}_h^{k+1}(\boldsymbol{x}) = \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + \omega_{\boldsymbol{v}_h^{k+1}}(\boldsymbol{x} - \boldsymbol{\zeta}_h^{k+1})^\perp,\ \forall \boldsymbol{x} \in \mathcal{S}_h^{k+1}.$$

Similarly, any function q_h^{k+1} of $\mathcal{Q}_h^{k+1}$ can be extended into a function of $\mathrm{L}_0^2(\mathcal{O})$ by setting $q_h^{k+1} = 0$ in $\mathcal{S}_h^{k+1}$. We therefore introduce the couple of "extended" discretization spaces, defined over the triangulation $\widetilde{\mathscr{T}}_h^{k+1}$,

$$\hat{\mathcal{V}}_h^{k+1} = \Big\{ \big(\boldsymbol{v}_h, \boldsymbol{\xi}_{\boldsymbol{v}_h}, \omega_{\boldsymbol{v}_h}\big) \in \mathrm{H}_0^1(\mathcal{O})^2 \cap \mathrm{C}^0(\overline{\mathcal{O}})^2 \times \mathbb{R}^3 \ ;\ \boldsymbol{v}_{h|_K} \in [\mathbb{P}_1(K) \oplus \langle \lambda_1\lambda_2\lambda_3 \rangle]^2\,,\ \forall K \in \widetilde{\mathscr{T}}_h^{k+1},$$
$$\boldsymbol{v}_h(\boldsymbol{x}) = \boldsymbol{\xi}_{\boldsymbol{v}_h} + \omega_{\boldsymbol{v}_h}(\boldsymbol{x} - \boldsymbol{\zeta}_h^{k+1})^\perp,\ \forall \boldsymbol{x} \in \mathcal{S}_h^{k+1} \Big\},$$

and $\hat{\mathcal{Q}}_h^{k+1} = \Big\{ q_h \in \mathrm{L}_0^2(\mathcal{O}) \cap \mathrm{C}^0(\overline{\mathcal{O}})\ ;\ q_{h|_K} \in \mathbb{P}_1(K),\ \forall K \in \widetilde{\mathscr{T}}_h^{k+1},\ q_h = 0 \text{ in } \mathcal{S}_h^{k+1} \Big\}$, and their respective continuous counterparts

$$\hat{\mathcal{V}}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1}) = \Big\{ (\boldsymbol{v}, \boldsymbol{\xi}_{\boldsymbol{v}}, \omega_{\boldsymbol{v}}) \in \mathrm{H}_0^1(\mathcal{O})^2 \cap \mathrm{C}^0(\overline{\mathcal{O}})^2 \times \mathbb{R}^3 \ ;\ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\xi}_{\boldsymbol{v}} + \omega_{\boldsymbol{v}}(\boldsymbol{x} - \boldsymbol{\zeta}_h^{k+1})^\perp,\ \forall \boldsymbol{x} \in \mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1}) \Big\},$$

and $\hat{\mathcal{Q}}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1}) = \Big\{ q \in \mathrm{L}_0^2(\mathcal{O}) \cap \mathrm{C}^0(\overline{\mathcal{O}})\ ;\ q = 0 \text{ in } \mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1}) \Big\}$.

Then, multiplying scalarly equation (87) (respectively (88)) by a test function $\boldsymbol{v}_h^{k+1}$ such that $(\boldsymbol{v}_h^{k+1}, \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}}, \omega_{\boldsymbol{v}_h^{k+1}})$ belongs to $\hat{\mathcal{V}}_h^{k+1}$ (respectively by q_h^{k+1} in $\hat{\mathcal{Q}}_h^{k+1}$), integrating over the domain $\mathcal{F}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})$ and using equations

(89) and (90), we obtain that the quadruplet $(\mathbf{U}^{k+1}, \boldsymbol{\Xi}^{k+1}, \Omega^{k+1}, P^{k+1})$ is solution to the following variational problem

$$\begin{aligned}
\int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} \left(\frac{\mathbf{U}^{k+1}-\mathbf{U}^k\circ\boldsymbol{\mathcal{C}}^k}{\delta t}\right)\cdot\boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} + M\frac{\boldsymbol{\Xi}^{k+1}-\boldsymbol{\Xi}^k}{\delta t}\cdot\boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + I\frac{\Omega^{k+1}-\Omega^k}{\delta t}\omega_{\boldsymbol{v}_h^{k+1}} \\
+ 2\nu\int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})}\mathbf{D}(\mathbf{U}^{k+1}):\mathbf{D}(\boldsymbol{v}_h^{k+1})\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} P^{k+1}\ \mathrm{div}\,\boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} \\
= \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})}\mathbf{F}^{k+1}\cdot\boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} + \mathbf{F}_M^{k+1}\cdot\boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + \mathrm{F}_I^{k+1}\omega_{\boldsymbol{v}_h^{k+1}} \\
+ \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})}\left(\mathbf{E}_{d_t\boldsymbol{u}}^k + \mathbf{E}_{\mathrm{ch}}^k\right)\cdot\boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} + \mathbf{E}_{\boldsymbol{\xi}}^k\cdot\boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + \mathrm{E}_\omega^k\,\omega_{\boldsymbol{v}_h^{k+1}},\ \forall(\boldsymbol{v}_h^{k+1},\boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}},\omega_{\boldsymbol{v}_h^{k+1}})\in\hat{\mathcal{V}}_h^{k+1}, \quad (91)
\end{aligned}$$

$$-\int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} q_h^{k+1}\ \mathrm{div}\,\mathbf{U}^{k+1}\,\mathrm{d}\boldsymbol{x} = 0, \quad \forall q_h^{k+1}\in\hat{\mathcal{Q}}_h^{k+1}, \quad (92)$$

with

$$\mathbf{E}_{\mathrm{ch}}^k = \frac{\left[(J_{\mathbf{Y}^{k+1}}\circ\mathbf{X}^{k+1})(J_{\mathbf{X}^k}\circ\boldsymbol{\mathcal{C}}^k)-\mathbf{Id}\right](\mathbf{U}^k\circ\boldsymbol{\mathcal{C}}^k)}{\delta t} + \nu(L^{k+1}-\Delta)\mathbf{U}^{k+1} + (\boldsymbol{G}^{k+1}-\boldsymbol{\nabla})P^{k+1}. \quad (93)$$

We infer from Lemmata 4.1 and 4.2 that

$$\begin{aligned}
\|\mathbf{E}_{\mathrm{ch}}^k\|_{\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1}))^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k| + |\theta(t^k)-\theta_h^k|\right) \\
+ \frac{C}{\delta t}\left(\|J_{\mathbf{X}^{k+1}}-J_{\mathbf{X}^k}\|_{\mathrm{L}^\infty(\mathcal{O})^4} + \|\boldsymbol{\mathcal{C}}^k-\mathbf{Id}\|_{\mathrm{L}^2(\mathcal{O})^2}\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k| + |\theta(t^k)-\theta_h^k|\right)\right).
\end{aligned}$$

Then, using Lemmata 4.3 and 6.4 and the above inequality, we reach

$$\begin{aligned}
\|\mathbf{E}_{\mathrm{ch}}^k\|_{\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1}))^2} \leqslant C\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k| + |\theta(t^k)-\theta_h^k| + |\boldsymbol{\xi}(t^k)-\boldsymbol{\xi}_h^k| + |\omega(t^k)-\omega_h^k|\right) \\
+ C\left(|\theta(t^k)-\theta_h^k| + |\boldsymbol{\xi}(t^k)-\boldsymbol{\xi}_h^k| + |\omega(t^k)-\omega_h^k| + 1\right)\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k| + |\theta(t^k)-\theta_h^k|\right) + C(\delta t). \quad (94)
\end{aligned}$$

We are now in position to proceed with the finite element analysis of problem (47)-(48). We make use of a result from [25] relative to the existence of projectors, which is something made possible due to the particular construction of the domain $\mathcal{F}_h^{k+1}$ (compare Subsection 3.1 of the present paper with Section 4 in [25]). First, let us notice that

$$\begin{aligned}
2\nu\int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})}\mathbf{D}(\mathbf{U}^{k+1}):\mathbf{D}(\boldsymbol{v}_h^{k+1})\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} P^{k+1}\ \mathrm{div}\,\boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} \\
= 2\nu\int_{\mathcal{O}}\mathbf{D}(\mathbf{U}^{k+1}):\mathbf{D}(\boldsymbol{v}_h^{k+1})\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{O}} P^{k+1}\ \mathrm{div}\,\boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x}
\end{aligned}$$

and that

$$-\int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} q_h^{k+1}\ \mathrm{div}\,\mathbf{U}^{k+1}\,\mathrm{d}\boldsymbol{x} = -\int_{\mathcal{O}} q_h^{k+1}\ \mathrm{div}\,\mathbf{U}^{k+1}\,\mathrm{d}\boldsymbol{x}.$$

Adapting the proof of Lemma 4.4 in [25], we can show the following result.

Proposition 7.2. *Suppose that assumption* (25) *holds. Then, for any k in $\{0,\dots,N\}$, there exists a unique quadruplet $(\boldsymbol{V}_h^k, \boldsymbol{\xi}_{\boldsymbol{V}_h^k}, \omega_{\boldsymbol{V}_h^k}, Q_h^k) \in \hat{\mathcal{V}}_h^k \times \hat{\mathcal{Q}}_h^k$ satisfying*

$$2\nu \int_{\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k)} \mathbf{D}(\mathbf{U}^k - \boldsymbol{V}_h^k) : \mathbf{D}(\boldsymbol{v}_h^k)\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k)} \left(P^k - Q_h^k\right)\ \mathrm{div}\,\boldsymbol{v}_h^k\,\mathrm{d}\boldsymbol{x} = 0,$$

$$- \int_{\mathcal{F}(\boldsymbol{\zeta}_h^k,\theta_h^k)} q_h^k\ \mathrm{div}(\mathbf{U}^k - \boldsymbol{V}_h^k)\,\mathrm{d}\boldsymbol{x} = 0,$$

for all $(\boldsymbol{v}_h^k, \boldsymbol{\xi}_{\boldsymbol{v}_h^k}, \omega_{\boldsymbol{v}_h^k}, q_h^k)$ in $\hat{\mathcal{V}}_h^k \times \hat{\mathcal{Q}}_h^k$. Moreover, there exists a positive constant C independent of h and k such that

$$\|\mathbf{U}^k - \boldsymbol{V}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant Ch. \tag{95}$$

Using the above lemma and equations (91) and (92), we deduce that

$$\begin{aligned}
\int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} \left(\frac{\mathbf{U}^{k+1} - \mathbf{U}^k \circ \boldsymbol{\mathcal{C}}^k}{\delta t}\right) \cdot \boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} + M\frac{\boldsymbol{\Xi}^{k+1} - \boldsymbol{\Xi}^k}{\delta t} \cdot \boldsymbol{\xi}_{\boldsymbol{v}_h}^k + I\frac{\Omega^{k+1} - \Omega^k}{\delta t}\omega_{\boldsymbol{v}_h}^k \\
+ 2\nu \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} \mathbf{D}(\boldsymbol{V}_h^{k+1}) : \mathbf{D}(\boldsymbol{v}_h^{k+1})\,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} Q_h^{k+1}\ \mathrm{div}\,\boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} \\
= \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} \mathbf{F}^{k+1} \cdot \boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} + \mathbf{F}_M^{k+1} \cdot \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + \mathrm{F}_I^{k+1}\,\omega_{\boldsymbol{v}_h^{k+1}} \\
+ \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} \left(\mathbf{E}_{d_t\boldsymbol{u}}^k + \mathbf{E}_{\mathrm{ch}}^k\right) \cdot \boldsymbol{v}_h^{k+1}\,\mathrm{d}\boldsymbol{x} + \mathbf{E}_{\boldsymbol{\xi}}^k \cdot \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + \mathrm{E}_\omega^k\,\omega_{\boldsymbol{v}_h^{k+1}},\ \forall (\boldsymbol{v}_h^{k+1}, \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}}, \omega_{\boldsymbol{v}_h^{k+1}}) \in \hat{\mathcal{V}}_h^{k+1},
\end{aligned} \tag{96}$$

$$- \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})} q_h^{k+1}\,\mathrm{div}\,\boldsymbol{V}_h^{k+1}\,\mathrm{d}\boldsymbol{x} = 0,\ \forall q_h^{k+1} \in \hat{\mathcal{Q}}_h^{k+1}. \tag{97}$$

Choosing $\boldsymbol{v}_h^{k+1} = \boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}$ as a test function in both (47) and (96) and subtracting the respective results, we then obtain

$$\begin{aligned}
\frac{1}{\delta t}\int_{\mathcal{F}_h^{k+1}} |\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}|^2\,\mathrm{d}\boldsymbol{x} + \frac{M}{\delta t}|\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}|^2 + \frac{I}{\delta t}|\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}|^2 + 2\nu\int_{\mathcal{F}_h^{k+1}} |\mathbf{D}(\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1})|^2\,\mathrm{d}\boldsymbol{x} \\
= \frac{1}{\delta t}\int_{\mathcal{F}_h^{k+1}} \left(\boldsymbol{V}_h^k \circ \boldsymbol{\mathcal{C}}^k - \boldsymbol{u}_h^k \circ \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k\right) \cdot (\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1})\,\mathrm{d}\boldsymbol{x} + \frac{M}{\delta t}(\boldsymbol{\xi}_{\boldsymbol{V}_h^k} - \boldsymbol{\xi}_h^k) \cdot (\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}) \\
+ \frac{I}{\delta t}(\omega_{\boldsymbol{V}_h^k} - \omega_h^k)(\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}) + \sum_{i=1}^{9} E_i.
\end{aligned} \tag{98}$$

where the terms in the indexed sum are defined as follows

$$E_1 = -\int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})\setminus\mathcal{F}_h^{k+1}} \left(\frac{\mathbf{U}^{k+1}-\mathbf{U}^k\circ\boldsymbol{\mathcal{C}}^k}{\delta t}\right)\cdot(\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1})\,\mathrm{d}\boldsymbol{x}, \tag{99}$$

$$E_2 = -\frac{1}{\delta t}\int_{\mathcal{F}_h^{k+1}}(\mathbf{U}^{k+1}-\boldsymbol{V}_h^{k+1})\cdot(\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1})\,\mathrm{d}\boldsymbol{x}, \tag{100}$$

$$E_3 = \frac{1}{\delta t}\int_{\mathcal{F}_h^{k+1}}(\mathbf{U}^{k}\circ\boldsymbol{\mathcal{C}}^k-\boldsymbol{V}_h^{k}\circ\boldsymbol{\mathcal{C}}^k)\cdot(\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1})\,\mathrm{d}\boldsymbol{x}, \tag{101}$$

$$E_4 = -\frac{M}{\delta t}(\boldsymbol{\Xi}^{k+1}-\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}})\cdot(\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}}-\boldsymbol{\xi}_h^{k+1}), \tag{102}$$

$$E_5 = \frac{M}{\delta t}(\boldsymbol{\Xi}^{k}-\boldsymbol{\xi}_{\boldsymbol{V}_h^{k}})\cdot(\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}}-\boldsymbol{\xi}_h^{k+1}), \tag{103}$$

$$E_6 = -\frac{I}{\delta t}(\Omega^{k+1}-\omega_{\boldsymbol{V}_h^{k+1}})\cdot(\omega_{\boldsymbol{V}_h^{k+1}}-\omega_h^{k+1}), \tag{104}$$

$$E_7 = \frac{I}{\delta t}(\Omega^{k}-\omega_{\boldsymbol{V}_h^{k}})\cdot(\omega_{\boldsymbol{V}_h^{k+1}}-\omega_h^{k+1}), \tag{105}$$

$$\begin{aligned} E_8 = {} & \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})}\mathbf{F}^{k+1}\cdot(\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1})\,\mathrm{d}\boldsymbol{x}-\int_{\mathcal{F}_h^{k+1}}\boldsymbol{f}_h^{k+1}\cdot(\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1})\,\mathrm{d}\boldsymbol{x},\\ & +\left(\mathbf{F}_M^{k+1}-\boldsymbol{f}_{h,\boldsymbol{M}}^{k+1}\right)\cdot(\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}}-\boldsymbol{\xi}_h^{k+1})+(\mathrm{F}_I^{k+1}-f_{h,I}^{k+1})(\omega_{\boldsymbol{V}_h^{k+1}}-\omega_h^{k+1}), \end{aligned} \tag{106}$$

$$E_9 = \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1})}\left(\mathbf{E}_{d_t\boldsymbol{u}}^k+\mathbf{E}_{\mathrm{ch}}^k\right)\cdot(\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1})\,\mathrm{d}\boldsymbol{x}+\mathbf{E}_{\boldsymbol{\xi}}^k\cdot(\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}}-\boldsymbol{\xi}_h^{k+1})+\mathrm{E}_\omega^k\,(\omega_{\boldsymbol{V}_h^{k+1}}-\omega_h^{k+1}). \tag{107}$$

To estimate $|E_1|$, we recall that

$$\frac{\mathbf{U}^{k+1}-\mathbf{U}^k\circ\boldsymbol{\mathcal{C}}^k}{\delta t} = \left[J_{\mathbf{Y}^{k+1}}\circ\mathbf{X}^{k+1}\right]\ \frac{d}{dt}\left[\boldsymbol{u}\circ\boldsymbol{\mathcal{C}}\right](\mathbf{X}^{k+1},t^{k+1})+\mathbf{E}_{d_t\boldsymbol{u}}^k + \frac{\left[(J_{\mathbf{Y}^{k+1}}\circ\mathbf{X}^{k+1})(J_{\mathbf{X}^k}\circ\boldsymbol{\mathcal{C}}^k)-\mathbf{Id}\right](\mathbf{U}^k\circ\boldsymbol{\mathcal{C}}^k)}{\delta t}.$$

The property (33), the regularity hypotheses (49), the bound (94) and Lemmata 4.1, 4.3 and 6.4 can therefore be used to get

$$\begin{aligned} |E_1| \leqslant Ch\Big[& |\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|+|\boldsymbol{\xi}(t^k)-\boldsymbol{\xi}_h^k|+|\omega(t^k)-\omega_h^k|+\delta t+1\\ & +\left(|\theta(t^k)-\theta_h^k|+|\boldsymbol{\xi}(t^k)-\boldsymbol{\xi}_h^k|+|\omega(t^k)-\omega_h^k|\right)\left(|\boldsymbol{\zeta}(t^k)-\boldsymbol{\zeta}_h^k|+|\theta(t^k)-\theta_h^k|\right)\Big]\\ & \left(\|\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^2}+|\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}}-\boldsymbol{\xi}_h^{k+1}|+|\omega_{\boldsymbol{V}_h^{k+1}}-\omega_h^{k+1}|\right). \end{aligned} \tag{108}$$

The terms $|E_i|$, $i=2,\dots,7$, can be bounded using Proposition 7.2, which gives

$$\sum_{i=2}^{7}|E_i| \leqslant C\frac{h}{\delta t}\left(\|\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^2}+|\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}}-\boldsymbol{\xi}_h^{k+1}|+|\omega_{\boldsymbol{V}_h^{k+1}}-\omega_h^{k+1}|\right). \tag{109}$$

We next remark that, for all $(\boldsymbol{v}_h^{k+1}, \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}}, \omega_{\boldsymbol{v}_h^{k+1}})$ in $\hat{\mathcal{V}}_h^{k+1}$,

$$\int_{\mathcal{F}_h^{k+1}} \boldsymbol{f}_h^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x} + \boldsymbol{f}_{h,\boldsymbol{M}}^{k+1} \cdot \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + f_{h,I}^{k+1} \omega_{\boldsymbol{v}_h^{k+1}} = \int_{\mathcal{F}_h^{k+1}} \boldsymbol{f}_h^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{S}_h^{k+1}} \rho_s \, \boldsymbol{f}_h^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x}. \quad (110)$$

Since $\mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1}) \subset \mathcal{S}_h^{k+1}$ by construction, we also have

$$\begin{aligned}\int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \mathbf{F}^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x} + \mathbf{F}_M^{k+1} \cdot \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + \mathrm{F}_I^{k+1} \omega_{\boldsymbol{v}_h^{k+1}} \\ = \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \mathbf{F}^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \rho_s \, \mathbf{F}^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x}, \quad (111)\end{aligned}$$

for all $(\boldsymbol{v}_h^{k+1}, \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}}, \omega_{\boldsymbol{v}_h^{k+1}})$ in $\hat{\mathcal{V}}_h^{k+1}$. Indeed, for any element $(\boldsymbol{v}_h^{k+1}, \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}}, \omega_{\boldsymbol{v}_h^{k+1}})$ of $\hat{\mathcal{V}}_h^{k+1}$, one can write

$$\begin{aligned}\int_{\mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \rho_s \, \mathbf{F}^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x} &= \int_{\mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \rho_s \, \mathbf{F}^{k+1}(\boldsymbol{x}) \cdot \left(\boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + \omega_{\boldsymbol{v}_h^{k+1}} (\boldsymbol{x} - \boldsymbol{\zeta}_h^{k+1})^\perp \right) \mathrm{d}\boldsymbol{x} \\ &= \left(\int_{\mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \rho_s \, \mathbf{F}^{k+1} \,\mathrm{d}\boldsymbol{x} \right) \cdot \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + \left(\int_{\mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \rho_s \, \mathbf{F}^{k+1}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{\zeta}_h^{k+1})^\perp \,\mathrm{d}\boldsymbol{x} \right) \omega_{\boldsymbol{v}_h^{k+1}}.\end{aligned}$$

Using (12), the respective definitions of $\mathbf{F}^{k+1}$, $\mathbf{F}_M^{k+1}$ and F_I^{k+1}, and the property (53) of the change of variables $\mathbf{X}^{k+1}$, the above relation implies (111). Subtracting (110) from (111) then yields

$$\begin{aligned}\int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \mathbf{F}^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{F}_h^{k+1}} \boldsymbol{f}_h^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x} + \left(\mathbf{F}_M^{k+1} - \boldsymbol{f}_{h,M}^{k+1} \right) \cdot \boldsymbol{\xi}_{\boldsymbol{v}_h^{k+1}} + \left(\mathrm{F}_I^{k+1} - f_{h,I}^{k+1} \right) \omega_{\boldsymbol{v}_h^{k+1}} \\ = \int_{\mathcal{F}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \mathbf{F}^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{S}(\boldsymbol{\zeta}_h^{k+1}, \theta_h^{k+1})} \rho_s \, \mathbf{F}^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x} \\ - \int_{\mathcal{F}_h^{k+1}} \boldsymbol{f}_h^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x} - \int_{\mathcal{S}_h^{k+1}} \rho_s \, \boldsymbol{f}_h^{k+1} \cdot \boldsymbol{v}_h^{k+1} \,\mathrm{d}\boldsymbol{x},\end{aligned}$$

and we obtain

$$|E_8| \leqslant Ch \left(|\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}| + |\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}| \right) + C \, \|\mathbf{F}^{k+1} - \boldsymbol{f}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{O})^2} \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{O})^2}. \quad (112)$$

Due to assumptions (9) and Lemma 4.1, we have

$$\|\mathbf{F}^{k+1} - \boldsymbol{f}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant C \left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| \right) + \|\boldsymbol{f}(\cdot, t^{k+1}) - \boldsymbol{f}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{O})^2}. \quad (113)$$

In addition, since $\boldsymbol{f}_h^{k+1}$ is the projection of $\boldsymbol{f}(\cdot, t^{k+1})$ on $\left(\widetilde{\mathcal{P}}_h^0 \right)^2$, it satisfies

$$\|\boldsymbol{f}(\cdot, t^{k+1}) - \boldsymbol{f}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{O})^2} \leqslant Ch. \quad (114)$$

Note that the positive constant C in the above inequality can be chosen in such a way that it does not depend on k, as it is assumed that (25) holds for some constant c_0 small enough. Gathering the bounds (112) to (114), we get

$$|E_8| \leqslant Ch \left(|\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}| + |\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}| \right) + C \left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + h \right) \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{O})^2}. \quad (115)$$

Finally, we infer from Lemma 7.1 and estimate (94) that

$$
\begin{aligned}
|E_9| \leqslant C(\delta t)^{1/2} \Big(& \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1}))^2} + |\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}| + |\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}| \Big) \\
& + C \Bigg(\Big(|\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| \Big) \Big(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| \Big) \\
& + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| + \delta t \Bigg) \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1}))^2}. \quad (116)
\end{aligned}
$$

Collecting the bounds (108), (109), (115) and (116) and identity (98) leads to the following inequality:

$$
\begin{aligned}
& 2 \int_{\mathcal{F}_h^{k+1}} |\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}|^2 \,\mathrm{d}\boldsymbol{x} + M|\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}|^2 + I|\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}|^2 + 4\nu(\delta t) \int_{\mathcal{F}_h^{k+1}} |\mathbf{D}(\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1})|^2 \,\mathrm{d}\boldsymbol{x} \\
& \leqslant 2 \int_{\mathcal{F}_h^{k+1}} \Big(\boldsymbol{V}_h^k \circ \boldsymbol{\mathcal{C}}^k - \boldsymbol{u}_h^k \circ \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k \Big) \cdot (\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}) \,\mathrm{d}\boldsymbol{x} + M|\boldsymbol{\xi}_{\boldsymbol{V}_h^k} - \boldsymbol{\xi}_h^k|^2 + I|\omega_{\boldsymbol{V}_h^k} - \omega_h^k|^2 \\
& \quad + Ch(\delta t) \Bigg(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| + \delta t + 1 \\
& \quad + \Big(|\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| \Big) \Big(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| \Big) \Bigg) \\
& \quad + Ch \Big(\|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^2} + |\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}| + |\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}| \Big) \\
& + Ch(\delta t) \Big(|\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}| + |\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}| \Big) + C(\delta t) \Big(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + h \Big) \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{O})^2} \\
& \quad + C(\delta t)^{3/2} \Big(\|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1}))^2} + |\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}| + |\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}| \Big) \\
& \quad + C(\delta t) \Bigg(\Big(|\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| \Big) \Big(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| \Big) \\
& \quad + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| + \delta t \Bigg) \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1}))^2}. \quad (117)
\end{aligned}
$$

Next, the first term in the right-hand side of (117) can be decomposed as follows

$$
\begin{aligned}
\int_{\mathcal{F}_h^{k+1}} \Big(\boldsymbol{V}_h^k \circ \boldsymbol{\mathcal{C}}^k - \boldsymbol{u}_h^k \circ \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k \Big) \cdot (\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}) \,\mathrm{d}\boldsymbol{x} &= \int_{\mathcal{F}_h^{k+1}} \Big((\boldsymbol{V}_h^k - \boldsymbol{u}_h^k) \circ \boldsymbol{\mathcal{C}}^k \Big) \cdot (\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}) \,\mathrm{d}\boldsymbol{x} \\
& - \int_{\mathcal{F}_h^{k+1}} \Big((\boldsymbol{V}_h^k - \boldsymbol{u}_h^k) \circ \boldsymbol{\mathcal{C}}^k - (\boldsymbol{V}_h^k - \boldsymbol{u}_h^k) \circ \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k \Big) \cdot (\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}) \,\mathrm{d}\boldsymbol{x} \\
& + \int_{\mathcal{F}_h^{k+1}} \Big(\boldsymbol{V}_h^k \circ \boldsymbol{\mathcal{C}}^k - \boldsymbol{V}_h^k \circ \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k \Big) \cdot (\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}) \,\mathrm{d}\boldsymbol{x}. \quad (118)
\end{aligned}
$$

Denoting respectively by I_1, I_2 and I_3 the three integrals in the right-hand side above, we first easily check that

$$
|I_1| \leqslant \frac{1}{2} \left(\int_{\mathcal{F}_h^{k+1}} |\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{F}_h^k} |\boldsymbol{V}_h^k - \boldsymbol{u}_h^k|^2 \,\mathrm{d}\boldsymbol{x} \right). \quad (119)
$$

Then, arguing as in [1], we have

$$
|I_2| \leqslant \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^\infty(\mathcal{F}_h^{k+1})^2} \left\| (\boldsymbol{V}_h^k - \boldsymbol{u}_h^k) \circ \boldsymbol{\mathcal{C}}^k - (\boldsymbol{V}_h^k - \boldsymbol{u}_h^k) \circ \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k \right\|_{\mathrm{L}^1(\mathcal{F}_h^{k+1})^2},
$$

which yields, using an inverse inequality (see, for instance, [3]) and Lemmata 5.3 and 6.3,

$$|I_2| \leqslant C\left(\log\left(\frac{1}{h}\right)\right)^{1/2} \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{H}^1(\mathcal{F}_h^{k+1})^2} \|\boldsymbol{\nabla}(\boldsymbol{V}_h^k - \boldsymbol{u}_h^k)\|_{\mathrm{L}^2(\mathcal{O})^4} \|\boldsymbol{\mathcal{C}}^k - \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2}.$$

From the Poincaré and Korn inequalities, we get

$$|I_2| \leqslant C\left(\log\left(\frac{1}{h}\right)\right)^{1/2} \|\mathbf{D}(\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1})\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^4} \|\mathbf{D}(\boldsymbol{V}_h^k - \boldsymbol{u}_h^k)\|_{\mathrm{L}^2(\mathcal{F}_h^k)^4} \|\boldsymbol{\mathcal{C}}^k - \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2}.$$

Combining the above inequality with Theorem 6.1, we reach

$$|I_2| \leqslant C(\delta t)\left(\log\left(\frac{1}{h}\right)\right)^{1/2} \|\mathbf{D}(\boldsymbol{V}_h^k - \boldsymbol{u}_h^k)\|_{\mathrm{L}^2(\mathcal{F}_h^k)^4} \|\mathbf{D}(\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1})\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^4} \left(\|\mathbf{U}^k - \boldsymbol{u}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + \delta t + h^{1/2}\right). \quad (120)$$

The last term can be treated as follows

$$|I_3| \leqslant \|\boldsymbol{\mathcal{C}}^k - \boldsymbol{\mathcal{A}}_h^{k+1,k} \circ \boldsymbol{\mathcal{B}}_h^k\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^2} \|\boldsymbol{V}_h^k\|_{\mathrm{W}^{1,\infty}(\mathcal{F}_h^k)^2} \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^2},$$

so that finally

$$|I_3| \leqslant C(\delta t)\|\boldsymbol{V}_h^k\|_{\mathrm{W}^{1,\infty}(\mathcal{F}_h^k)^2} \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^2} \left(\|\mathbf{U}^k - \boldsymbol{u}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + \delta t + h^{1/2}\right). \quad (121)$$

Gathering (95), (117), (118), (119), (120) and (121), we obtain that

$$\begin{aligned}
&\int_{\mathcal{F}_h^{k+1}} |\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}|^2 \,\mathrm{d}\boldsymbol{x} + M|\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}|^2 + I|\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}|^2 + 4\nu(\delta t)\int_{\mathcal{F}_h^{k+1}} |\mathbf{D}(\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1})|^2 \,\mathrm{d}\boldsymbol{x} \\
&\qquad \leqslant \int_{\mathcal{F}_h^k} |\boldsymbol{V}_h^k - \boldsymbol{u}_h^k|^2 \,\mathrm{d}\boldsymbol{x} + M|\boldsymbol{\xi}_{\boldsymbol{V}_h^k} - \boldsymbol{\xi}_h^k|^2 + I|\omega_{\boldsymbol{V}_h^k} - \omega_h^k|^2 \\
&+ C(\delta t)\left(\left(\log\left(\frac{1}{h}\right)\right)^{1/2} \|\mathbf{D}(\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1})\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^4} \|\mathbf{D}(\boldsymbol{V}_h^k - \boldsymbol{u}_h^k)\|_{\mathrm{L}^2(\mathcal{F}_h^k)^4} + \|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^2}\right) \\
&\qquad \left(\|\mathbf{V}^k - \boldsymbol{u}_h^k\|_{\mathrm{L}^2(\mathcal{O})^2} + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + \delta t + h^{1/2}\right) \\
&\quad + C\left((\delta t)^{3/2} + h(\delta t) + h\right)\left(|\boldsymbol{\xi}_{\boldsymbol{V}_h^{k+1}} - \boldsymbol{\xi}_h^{k+1}| + |\omega_{\boldsymbol{V}_h^{k+1}} - \omega_h^{k+1}|\right) + Ch\,\|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^2} \\
&+ C(\delta t)\|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{F}(\boldsymbol{\zeta}_h^{k+1},\theta_h^{k+1}))^2}\Big[|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| + (\delta t)^{1/2} \\
&\qquad + \left(|\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k|\right)\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right)\Big] \\
&\qquad + C(\delta t)\|\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1}\|_{\mathrm{L}^2(\mathcal{O})^2}\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + h\right) \\
&\quad + Ch(\delta t)\Big[|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k| + \delta t + 1 \\
&\qquad + \left(|\theta(t^k) - \theta_h^k| + |\boldsymbol{\xi}(t^k) - \boldsymbol{\xi}_h^k| + |\omega(t^k) - \omega_h^k|\right)\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right)\Big]. \quad (122)
\end{aligned}$$

Denoting, for all k in $\{0,\dots,N\}$,

$$\mathcal{N}^k = \|\boldsymbol{V}_h^k - \boldsymbol{u}_h^k\|^2_{\mathrm{L}^2(\mathcal{F}_h^k)^2} + M|\boldsymbol{\xi}_{\boldsymbol{V}_h^k} - \boldsymbol{\xi}_h^k|^2 + I|\omega_{\boldsymbol{V}_h^k} - \omega_h^k|^2,$$

and using that $h \leqslant 1$, $\delta \leqslant 1$, we infer from inequality (122) that

$$\begin{aligned}
\mathcal{N}^{k+1} + 4\nu(\delta t)\|\mathbf{D}(\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1})\|^2_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^4} &\leqslant \mathcal{N}^k \\
&+ C(\delta t)\left(\log\left(\frac{1}{h}\right)\right)^{1/2}\|\mathbf{D}(\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1})\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^4}\|\mathbf{D}(\boldsymbol{V}_h^{k} - \boldsymbol{u}_h^{k})\|_{\mathrm{L}^2(\mathcal{F}_h^{k})^4} \\
&\quad\left(\mathcal{N}^k + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + \delta t + h^{1/2}\right) \\
&\quad + C((\delta t)^{3/2} + (\delta t)h^{1/2}))\mathcal{N}^{k+1} \\
&+ C(\delta t)\mathcal{N}^{k+1}\left(\mathcal{N}^k + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + \left(\mathcal{N}^k + |\theta(t^k) - \theta_h^k|\right)\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right)\right) \\
&+ Ch(\delta t)\left(\mathcal{N}^k + |\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| + 1 + \left(\mathcal{N}^k + |\theta(t^k) - \theta_h^k|\right)\left(|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k|\right)\right). \quad (123)
\end{aligned}$$

7.2. **Conclusion**

We are finally in a position to demonstrate the main result of the paper.

Proof of Theorem 3.5. Hereafter, the constants do not depend on δt, h, nor on the time index k.

First, from the definitions of $\boldsymbol{u}_h^0$, $\boldsymbol{\zeta}_h^0$, θ_h^0, $\boldsymbol{\xi}_h^0$ and ω_h^0 (see Subsection 3.1.1) and the assumption (9) on the regularity of the initial datum $\boldsymbol{u}^{(0)}$, we have

$$\mathcal{N}^0 + 2\nu(\delta t)\|\mathbf{D}(\boldsymbol{V}_h^0 - \boldsymbol{u}_h^0)\|^2_{\mathrm{L}^2(\mathcal{F}_h^0)^4} \leqslant C_0(\delta t + h^{1/2}),$$

for some positive constant C_0.

Second, we proceed by induction on k to show that the bounds (71) and (72) and the following induction relations

$$\mathcal{N}^k + 2\nu(\delta t)\|\mathbf{D}(\boldsymbol{V}_h^k - \boldsymbol{u}_h^k)\|^2_{\mathrm{L}^2(\mathcal{F}_h^k)^4} \leqslant C_k(\delta t + h^{1/2}), \quad (124)$$

$$|\boldsymbol{\zeta}(t^k) - \boldsymbol{\zeta}_h^k| + |\theta(t^k) - \theta_h^k| \leqslant c_k(\delta t + h^{1/2}) \quad (125)$$

hold, with

$$C_k = \frac{C_0}{\left(1 - 4C(\delta t)\right)^k} + 3C(\delta t)\sum_{j=0}^{k-1}\frac{1}{\left(1 - 4C(\delta t)\right)^j} \text{ and } c_k = (\delta t)\sum_{j=0}^{k-1}\left(C_j + C^r\right),$$

where C is the constant appearing in (123) and $C^r = \|\boldsymbol{\xi}'\|_{\mathrm{L}^\infty(0,T)^2} + \|\omega'\|_{\mathrm{L}^\infty(0,T)}$.

Notice that, since $c_s h \leqslant \delta t$, there exist two positive constants C_∞ and c_∞, independent of δt and h, such that, for all k in $\{0,\dots,N\}$,

$$C_k \leqslant C_\infty \text{ and } c_k \leqslant c_\infty. \quad (126)$$

The first step is to show that the statement is true when $k = 0$. In that case, it remains only to prove that (71) and (72) are satisfied, that is

$$(\delta t)\,\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{w}}\|_{\mathrm{L}^\infty(0,t^1;\mathrm{W}^{1,\infty}(\mathcal{O}))^2} \leqslant \varepsilon$$

and

$$(\delta t)\,\left\|\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\mathbf{0}} - \overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\mathbf{0}}\right\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2} \leqslant \varepsilon,$$

where, for all t in $[0,t^1]$ and $\boldsymbol{x}$ in $\mathcal{F}(\boldsymbol{\zeta}(t^1),\theta(t^1))$, the functions $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{w}}$ are defined by taking s equal to t^1 in (29), and where $\overline{\boldsymbol{u}}_h^0$ and $\overline{\boldsymbol{w}}_h^0$ are given by (46).
The first bound being only a condition on the time step, we check the second one by using an inverse inequality:

$$(\delta t)\|\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{0}}-\overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2}\leqslant(\delta t)\left(\log\left(\frac{1}{h}\right)\right)^{1/2}\|\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{0}}-\overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}\|_{\mathrm{H}^1(\mathcal{O})^2}.$$

Then, from assumption (25) and the properties of the mapping $\boldsymbol{\mathcal{A}}_h^0$, we get that

$$(\delta t)\|\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{0}}-\overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{0}}\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2}\leqslant C(\delta t)\left(\log\left(\frac{1}{h}\right)\right)^{1/2}.$$

The above relation and the fact that $\delta t\leqslant C_s h^{1/2}$ thus imply the result for $k=0$.

The second step of the proof is to assume that the relations (71), (72), (124) and (125) hold for some $k\geqslant 0$, and subsequently show they are also true for $k+1$.

The relation (71) is simply a consequence of Lemma 4.1 and results (61) to (63) if δt is chosen small enough. Next, from regularity assumptions (49), we deduce

$$|\boldsymbol{\zeta}(t^{k+1})-\boldsymbol{\zeta}_h^{k+1}|+|\theta(t^{k+1})-\theta_h^{k+1}|\leqslant(c_k+(C_k+C^r)(\delta t))\,(\delta t+h^{1/2}),\tag{127}$$

and (125) thus holds. Then, using inequality (123), we infer that

$$\begin{aligned}\mathcal{N}^{k+1}+4\nu(\delta t)\|\mathbf{D}(\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1})\|^2_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^4}\,\mathrm{d}\boldsymbol{x}&\leqslant\mathcal{N}^k\\
+C(\delta t)\left(\log\left(\frac{1}{h}\right)\right)^{1/2}&((c_k+C_k+1)(\delta t+h^{1/2}))\|\mathbf{D}(\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1})\|_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^4}\|\mathbf{D}(\boldsymbol{V}_h^{k}-\boldsymbol{u}_h^{k})\|_{\mathrm{L}^2(\mathcal{F}_h^{k})^4}\\
&+C\mathcal{N}^{k+1}\left((\delta t)^{3/2}+(\delta t)h^{1/2}+(\delta t)(c_k+C_k)(\delta t+h^{1/2})((c_k+C_k)(\delta t+h^{1/2})+1)\right)\\
&+Ch(\delta t)\left((c_k+C_k)(\delta t+h^{1/2})((c_k+C_k)(\delta t+h^{1/2})+1)+1\right).\end{aligned}\tag{128}$$

Using that $\delta t\leqslant C_s h^{1/2}$, we have

$$C\left(\log\left(\frac{1}{h}\right)\right)^{1/2}(c_k+C_k+1)(\delta t+h^{1/2})<2\nu$$

so relation (128) and a Young inequality imply

$$\begin{aligned}\mathcal{N}^{k+1}+2\nu(\delta t)\|\mathbf{D}(\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1})\|^2_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^4}&\leqslant\mathcal{N}^k+2\nu(\delta t)\|\mathbf{D}(\boldsymbol{V}_h^{k}-\boldsymbol{u}_h^{k})\|^2_{\mathrm{L}^2(\mathcal{F}_h^{k})^4}\\
&+C\mathcal{N}^{k+1}\left((\delta t)^{3/2}+\delta t(h^{1/2})+(\delta t)(c_k+C_k)(\delta t+h^{1/2})((c_k+C_k)(\delta t+h^{1/2})+1)\right)\\
&+Ch(\delta t)\left((c_k+C_k)(\delta t+h^{1/2})((c_k+C_k)(\delta t+h^{1/2})+1)+1\right),\end{aligned}$$

and, owing to (124),

$$\begin{aligned}\mathcal{N}^{k+1}+2\nu(\delta t)\|\mathbf{D}(\boldsymbol{V}_h^{k+1}-\boldsymbol{u}_h^{k+1})\|^2_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^4}&\leqslant C_k(\delta t+h^{1/2})\\
&+C\mathcal{N}^{k+1}\left((\delta t)^{3/2}+\delta t(h^{1/2})+(\delta t)(c_k+C_k)(\delta t+h^{1/2})((c_k+C_k)(\delta t+h^{1/2})+1)\right)\\
&+Ch(\delta t)\left((c_k+C_k)(\delta t+h^{1/2})((c_k+C_k)(\delta t+h^{1/2})+1)+1\right).\end{aligned}\tag{129}$$

Taking into account the fact that

$$h \leqslant 1,\ \delta t \leqslant 1 \text{ and } (c_\infty + C_\infty)(\delta t + h^{1/2}) \leqslant 1,$$

we infer

$$\mathcal{N}^{k+1} + 2\nu(\delta t)\|\mathbf{D}(\boldsymbol{V}_h^{k+1} - \boldsymbol{u}_h^{k+1})\|^2_{\mathrm{L}^2(\mathcal{F}_h^{k+1})^4} \leqslant \frac{C_k}{1-4C(\delta t)}(\delta t + h^{1/2}) + \frac{3C(\delta t)h}{1-4C(\delta t)},$$

which clearly implies

$$C_{k+1} = \frac{C_k}{1-4C(\delta t)} + \frac{3C(\delta t)}{1-4C(\delta t)}.$$

It remains to prove (72) holds. We have

$$(\delta t)\|\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k+1}} - \overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k+1}}\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2} \leqslant (\delta t)\left(\|\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k+1}} - \boldsymbol{V}_h^{k+1}\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2} + \|\overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k+1}} - \boldsymbol{W}_h^{k+1}\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2} + \|\boldsymbol{V}_h^{k+1}\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2} + \|\boldsymbol{W}_h^{k+1}\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2}\right),$$

which yields, proceeding as in the first part of the proof and using an inverse inequality,

$$(\delta t)\|\overline{\boldsymbol{u}}_{\boldsymbol{h}}^{\boldsymbol{k+1}} - \overline{\boldsymbol{w}}_{\boldsymbol{h}}^{\boldsymbol{k+1}}\|_{\mathrm{W}^{1,\infty}(\mathcal{O})^2} \leqslant C(\delta t)\left(\delta t + h^{1/2} + \left(\log\left(\frac{1}{h}\right)\right)^{1/2}\right).$$

Taking the time step δt small enough thus gives the result.

The two estimates of the theorem finally derive from (124) and (125). Indeed, combining these bounds with (95), (126) and the results of Lemma 4.1, we have

$$\|\boldsymbol{u}(\cdot, t^k) - \boldsymbol{u}_h^k\|_{\mathrm{L}^2(\mathcal{O}^2)} \leqslant C(\delta t + h^{1/2}).$$

Given the regularity hypotheses (49), the error bounds on $\boldsymbol{\zeta}$ and θ obtained above and Lemma 5.1, the proof is complete. □

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