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Quantisations of piecewise affine maps on the torus and their quantum limits

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Abstract

For general quantum systems the semiclassical behaviour of eigenfunctions in relation to the ergodic properties of the underlying classical system is quite difficult to understand. The Wignerfunctions of eigenstates converge weakly to invariant measures of the classical system, the so called quantum limits, and one would like to understand which invariant measures can occur that way, thereby classifying the semiclassical behaviour of eigenfunctions.

We introduce a class of maps on the torus for whose quantisations we can understand the set of quantum limits in great detail. In particular we can construct examples of ergodic maps which have singular ergodic measures as quantum limits, and examples of non-ergodic maps where arbitrary convex combinations of absolutely continuous ergodic measures can occur as quantum limits.

The maps we quantise are obtained by cutting and stacking.

1 Introduction

The correspondence principle in quantum mechanics states that in the semiclassical limit $\hbar \rightarrow 0$ classical mechanics emerges and governs quantum mechanical quantities for small de Broglie wavelength. One manifestation of this principle is that the Wignerfunctions of eigenfunctions converge weakly to invariant probability measures on phase space, the so called quantum limits. It is one of the big open problems in the field to classify the set of quantum limits, and it is in general not known

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which invariant measures can occur as quantum limits. In particular the case that the classical system is ergodic has attracted a lot of attention. In this case the celebrated quantum ergodicity theorem, [Šni74, Zel87, CdV85], states that almost all eigenfunctions have the ergodic Liouville measure as quantum limit, and one would like to know if in fact all eigenfunctions converge to the Liouville measure, i.e., if quantum unique ergodicity holds or if there are exceptions. Possible candidates for exceptions would be quantum limits concentrated on periodic orbits, a phenomenon called strong scarring. Another very interesting case is when the classical system is of mixed type, i.e., the phase space has several invariant components of positive measure, or, if there exist several invariant measures which are continuous relative to Liouville measure. Here the question is to what extent the quantum mechanical system respects the splitting of the classical system into invariant components, i.e., is a typical quantum limit ergodic, or can every convex combination of invariant measures appear as a quantum limit.

There has been recently considerable progress in some of these questions. For the cat map it was shown that quantum unique ergodicity does not hold, in [FNDB03] a sequence of eigenfunctions was constructed whose quantum limit is a convex combination of the Liouville measure and an atomic measure supported on a periodic orbit. It was furthermore shown that the orbit can carry at most 1/2 of the total mass of the measure [BDB03, FN04]. The eigenvalues of the cat map have large multiplicities and this behaviour depends on the choice of the basis of eigenfunctions, in [KR00] it was shown that for a so called Hecke basis of eigenfunctions quantum unique ergodicity actually holds.

On compact Riemannian manifolds of negative curvature quantum unique ergodicity was conjectured in [RS94] and for arithmetic manifolds it was recently proved by Lindenstrauss for Hecke bases of eigenfunctions, [Lin06]. The non-arithmetic case is still open, but in [Ana06, AN06] the authors succeeded in proving lower bounds for the entropy of quantum limits on manifolds of negative curvature.

In this paper we introduce a class of model systems for which the set of quantum limits can be determined very precisely. This work was motivated by a paper of Marklof and Rudnick where they gave an example of a quantum ergodic map which one can prove to be quantum uniquely ergodic [MR00]. They mention that there are no examples known where a quantum ergodic map is not quantum uniquely ergodic. The purpose of our work was to provide such examples, in fact examples which are quite close in nature to those considered by Marklof and Rudnick. The map they considered was a skew product map of the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ of the form:

$$F : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + 2q \\ f(q) \end{pmatrix} \pmod{1}$$

where $f(p)$ is an irrational rotation of the circle \mathbf{R}/\mathbf{Z} . In this article we consider skew products of the same form for other functions $f(p)$. In particular since a circle rotation is an interval exchange transformation (IET for short) on two intervals, one of the examples we will consider for $f(p)$ are interval exchanges on more intervals. There are examples known of IETs which are not uniquely ergodic. A consequence of our main result is that if $f(p)$ is an IET which is not uniquely ergodic then each of

the invariant measures of F of the form Lebesgue measure cross an invariant measure (with the exception of finitely supported ones) of $f(p)$ is a quantum limit.

The plan of the paper is as follows. In Section 2 we give a quick review of quantisation of maps on the torus, and introduce the maps we study and their quantisation. In particular we prove Egorov's theorem for these maps. In Section 3 we turn our attention to quantum limits, we first give a general proof of quantum ergodicity for maps with singularities, and then show that for our particular class of maps the quantum limits can be understood purely in terms of the orbits of discretisations of the classical map. Then, in Section 4, we finally come to our main result. We first review the cutting and stacking construction to obtain maps and then show how it can be combined with discretisations to get a detailed understanding of quantum limits. In Theorem 4 we summarise our main findings. Finally in the last two sections we discuss two examples and give some conclusions.

2 Quantisation

We give a short summary of the quantisation of maps on the torus, for more details and background we refer to [DEG03, DB01].

The Hilbert space: For $(p, q) \in \mathbf{R}^2$ we introduce the phase space translation operator

$$\mathbf{T}(p, q) := e^{-\frac{i}{\hbar}(q\hat{p}-p\hat{q})} ,$$

where $\hat{p}\psi(x) := \frac{\hbar}{i}\psi'(x)$ and $\hat{q}\psi(x) := x\psi(x)$ for $\psi \in \mathcal{S}(\mathbf{R})$, are the momentum and position operators, respectively. These operators are unitary on $L^2(\mathbf{R})$ and satisfy for $(p, q), (p', q') \in \mathbf{R}^2$

$$\mathbf{T}(p + p', q + q') = e^{-\frac{i}{2\hbar}(qp' - pq')} \mathbf{T}(p, q) \mathbf{T}(p', q') \quad (1)$$

and they provide therefore a unitary irreducible representation of the Heisenberg group on $L^2(\mathbf{R})$.

The state space of the classical map is obtained from \mathbf{R}^2 by identifying integer translates which gives the two torus

$$\mathbf{T}^2 = \mathbf{R}^2 / \mathbf{Z}^2 .$$

By mimicking this procedure the quantum mechanical state space is defined to be the space of distributions on \mathbf{R} which satisfy

$$\mathbf{T}(1, 0)\psi = \psi , \quad \mathbf{T}(0, 1)\psi = \psi .$$

One finds that these two conditions can only be fulfilled (for $\psi \neq const.$) if Planck's constant meets the condition

$$\frac{1}{2\pi\hbar} = N \quad (2)$$

where N is a positive integer. The allowed states then turn out to be distributions of the form

$$\psi(x) = \frac{1}{\sqrt{N}} \sum_{Q \in \mathbf{Z}} \Psi(Q) \delta\left(x - \frac{Q}{N}\right) \quad (3)$$

with $\Psi(Q)$ a complex number satisfying

$$\Psi(Q + N) = \Psi(Q) .$$

So the $\Psi(Q)$ are functions on $\mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$ and the space of these functions will be denoted by \mathcal{H}_N , it is N -dimensional and through the coefficients $\Psi(Q), Q = 0, 1, \dots, N-1$ it can be identified with \mathbf{C}^N . There is a map $S_N : \mathcal{S}(\mathbf{R}) \rightarrow \mathcal{H}_N$ defined by

$$S_N \psi := \sum_{n, m \in \mathbf{Z}} (-1)^{Nnm} \mathbb{T}(n, m) \psi$$

which is onto. If we equip \mathcal{H}_N furthermore with the inner product

$$\langle \psi, \phi \rangle_N := \frac{1}{N} \sum_{Q \in \mathbf{Z}_N} \Psi^*(Q) \Phi(Q)$$

then \mathcal{H}_N is a Hilbert space and S_N is an isometry.

Observables: In classical mechanics observables on the torus are given by functions on \mathbf{T}^2 , these can be expanded into Fourier series

$$a = \sum_{n \in \mathbf{Z}^2} \hat{a}_n e(-\omega(z, n)) .$$

where $z = (q, p) \in \mathbf{T}^2$, $\omega(z, n) = qn_2 - pn_1$ and $\hat{a}_n := \int_{\mathbf{T}^2} a(z) e(\omega(z, n)) dz$ denotes the n -th Fourier coefficient. We use here and in the following the notation $e(x) = e^{2\pi i x}$ and $e_N(x) := e^{\frac{2\pi i}{N} x}$. These observables can be quantised by replacing $e(-\omega(z, n))$ by the translation operator

$$\mathbb{T}_N(n) := \mathbb{T}\left(\frac{n_1}{N}, \frac{n_2}{N}\right)$$

which acts on \mathcal{H}_N . This is called Weyl quantisation, to a classical observable $a \in C^\infty(\mathbf{T}^2)$ a corresponding quantum observable is defined by

$$\text{Op}_N[a] := \sum_{n \in \mathbf{Z}^2} \hat{a}_n \mathbb{T}_N(n) , \quad (4)$$

which is an operator on \mathcal{H}_N . For example, if a depends only on q then the corresponding operator is just multiplication with a ,

$$\text{Op}_N[a] \psi(q) = a(q) \psi(q) , \quad (5)$$

and in terms of the coefficients $\Psi(Q)$ the action of $\text{Op}_N[a]$ is given by $\Psi(Q) \mapsto a(Q/N) \Psi(Q)$.

The trace of a Weyl operator can be expressed in terms of the symbol, from (4) follows easily, see [DEG03, DB01], that for $a \in C^\infty(\mathbf{T}^2)$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{Tr} \text{Op}_N[a] = \int_{\mathbf{T}^2} a dz . \quad (6)$$

Quantisation of a map: Let

$$F : \mathbf{T}^2 \rightarrow \mathbf{T}^2$$

be a volume preserving map. One calls a sequence of unitary operators $U_N : \mathcal{H}_N \rightarrow \mathcal{H}_N$, $N \in \mathbf{N}$, a quantisation of the map F if the correspondence principle holds, i.e., if for sufficiently nice functions a one has

$$U_N^* \text{Op}_N[a] U_N \sim \text{Op}_N[a \circ F] , \quad (7)$$

for $N \rightarrow \infty$. If this relation holds, it is often called Egorov's theorem and it means that in the semiclassical limit, i.e., for $N \rightarrow \infty$, quantum evolution of observable approaches the classical time evolution.

Let us now turn to the specific class of maps we want to quantise. They are given by

$$F : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + 2q \\ f(q) \end{pmatrix} \pmod{1} \quad (8)$$

where $f : [0, 1] \rightarrow [0, 1]$ is a piecewise affine map given by a cutting and stacking construction which we will describe in detail in Section 4. For the construction of the quantisation we only need the property that the singularity set $\mathbb{S} \subset [0, 1]$ is nowhere dense. In order to quantise this map we proceed similar to the construction in [MR00], i.e., use a sequence of approximations to f . Consider the discretized interval

$$D_N := \{Q/N ; Q \in \{0, 1, 2, \dots, N-1\}\} , \quad (9)$$

i.e., the support of the Hilbert space elements (3). For each $N \in \mathbf{N}$ we will call a map $f_N : D_N \rightarrow D_N$ an approximation of f if it is close to f in a certain sense which we will now explain. Since f is not assumed to be continuous we do not approximate it uniformly in the supremum norm. Let $f(\mathbb{S}) := \{q : \exists q_0 \in \mathbb{S} \text{ such that } q = \lim_{q' \rightarrow q_0} f(q')\}$. We measure the difference between f and an approximation f_N only away from the set $f(\mathbb{S})$. Let us call the relevant set

$$I_\varepsilon := \{q \in [0, 1]; \text{dist}(q, f(\mathbb{S})) \geq \varepsilon\}.$$

In the construction of f_N in Section 4 we will choose a sequence ε_N with $\lim_{N \rightarrow \infty} \varepsilon_N = 0$. For any fixed ε_N the relevant measure for the quality of the approximation will be

$$\delta_N := \delta_N(\varepsilon_N) := \sup_{Q/N \in I_{\varepsilon_N}} |f_N(Q/N) - f(Q/N)| . \quad (10)$$

Any approximation f_N then defines via (8) an approximation F_N of F .

The quantisation of F is now defined to be the sequence of unitary operators

$$U_N \Psi(Q) = e_N(-(\hat{f}_N^{-1}(Q))^2) \Psi(\hat{f}_N^{-1}(Q)) , \quad (11)$$

where $\hat{f}_N(Q) := N f_N(Q/N)$ denotes the map induced by f_N on $\mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$. This is indeed a unitary operator on \mathcal{H}_N , with its adjoint given by

$$U_N^* \Psi(Q) = e_N(Q^2) \Psi(\hat{f}_N(Q)) .$$

That this sequence of operators U_N is really a quantisation of the map F is the content of the Egorov theorem (7) which we will now prove. In our case we have to be careful at the singularities of the map. The singularities of f and F can be naturally identified, thus without confusion we can denote by \mathbb{S} the set of singularities of F as well. By $C_{\mathbb{S}}^{\infty}(\mathbf{T}^2)$ we denote the space of functions in $C^{\infty}(\mathbf{T}^2)$ which vanish in a neighbourhood of $F(\mathbb{S})$. We then find

Theorem 1. *For any $a \in C^{\infty}(\mathbf{T}^2)$ we have*

$$U_N^* \text{Op}_N[a] U_N = \text{Op}_N[a \circ F_N] ,$$

and for any $a \in C_{\mathbb{S}}^{\infty}(\mathbf{T}^2)$ there are constants $C(a), \varepsilon_0(a) > 0$ such that for $\varepsilon_N < \varepsilon_0(a)$

$$\|U_N^* \text{Op}_N[a] U_N - \text{Op}_N[a \circ F]\| \leq C(a) \delta_N .$$

Proof. The map F and its quantisation U_N can be decomposed into a product of two simpler maps and operators. Namely, with

$$F^{(1)}(p, q) = (p + 2q, q) , \quad \text{and} \quad F^{(2)}(p, q) = (p, f(q))$$

we have

$$F = F^{(2)} \circ F^{(1)} .$$

These maps can be quantised separately as

$$U_N^{(1)} \Psi(Q) := e_N(-Q^2) \Psi(Q) , \quad \text{and} \quad U_N^{(2)} \Psi(Q) := \Psi(\hat{f}_N^{-1}(Q)) ,$$

where f_N denotes a discretisation of f on the Heisenberg lattice. We then have

$$U_N = U_N^{(2)} U_N^{(1)} ,$$

and therefore it is sufficient to study the conjugation of an Weyl operator for the two operators separately. In the case of $U_N^{(1)}$ it is well known that Egorov's theorem is exactly fulfilled

$$U_N^{(1)*} \text{Op}_N[a] U_N^{(1)} = \text{Op}_N[a \circ F^{(1)}] ,$$

see [MR00]. For the study of the second operator we use that

$$\text{Op}_N[a] U_N^{(2)} - U_N^{(2)} \text{Op}_N[a] = 0$$

for any observable which is constant in q , therefore we can restrict ourselves in the following to the case that a is constant in p . But then $\text{Op}_N[a]$ is just multiplication with a , and we obtain

$$U_N^{(2)*} \text{Op}_N[a] U_N^{(2)} \psi(q) = a(f_N(q)) \psi(q) .$$

For general observables we therefore obtain

$$U_N^{(2)*} \text{Op}_N[a] U_N^{(2)} = \text{Op}_N[a \circ F^{(2)}] ,$$

and this proves the first part of the theorem.

For the second part we have to estimate

$$\|\mathrm{Op}_N[a \circ F] - \mathrm{Op}_N[a \circ F_N]\|$$

and this can again be reduced to the case that a depends only on q , and then $\mathrm{Op}_N[a \circ F] - \mathrm{Op}_N[a \circ F_N]$ is the multiplication operator with $a(f_N(q)) - a(f(q))$. Since we have for a $b \in C^\infty(\mathbf{T}^2)$ which depends only on q by (5) that

$$\|\mathrm{Op}_N[b]\| = \sup_{q \in D_N} |b(q)| ,$$

we obtain

$$\|\mathrm{Op}_N[a \circ f] - \mathrm{Op}_N[a \circ f_N]\| = \sup_{q \in D_N} |(a \circ f_N - a \circ f)(q)| .$$

But for the right hand side we obtain by using (10) and that $a \equiv 0$ in a neighbourhood of $f\mathbb{S}$

$$\begin{aligned} \sup_{q \in D_N} |(a \circ f_N - a \circ f)(q)| &\leq \sup_{Q/N \in I_{\varepsilon_N}} |(a \circ f_N - a \circ f)(Q/N)| \\ &\quad + \sup_{Q/N \in I \setminus I_{\varepsilon_N}} |(a \circ f_N - a \circ f)(Q/N)| \\ &\leq C(a)\delta_N . \end{aligned}$$

since the second term on the right hand side is 0 if ε_N is small enough. \square

So if we can choose our approximations f_N in a way that $\varepsilon_N \rightarrow 0$ and $\delta_N \rightarrow 0$ for $N \rightarrow \infty$, then the sequence of unitary operators U_N reproduces the classical map F in the semiclassical limit $N \rightarrow \infty$, and so the correspondence principle holds.

Definition 1. A sequence of operators U_N for which $\delta_N(\varepsilon_N)$ and ε_N tend to 0 for $N \rightarrow \infty$ will be called a **proper quantisation** of F

The restriction on the support of the classical observables is necessary in order that $a \circ F_N$ and $a \circ F$ are smooth for N large enough. For a general a the composition $a \circ F$ is discontinuous which causes problems with the Weyl quantisation. Theorem 1 is not valid without the assumption on the singularities. This is shown by the following counter-example.

Proposition 1. Let $s \in \mathbb{S}$, $a(q) \in C^\infty(\mathbf{T}^2)$ depend only on q with $a(s) \neq 0$ and let $g_s(q) := \sqrt{N}e^{-(q-s)^2/N}$ be a Gaussian centred at s and $\psi_s := S_N f_s$ be its projection to \mathcal{H}_N . Then there exists a constant C such that

$$\lim_{N \rightarrow \infty} \|(U_N^* \mathrm{Op}_N[a]U_N - \mathrm{Op}_N[a \circ F])\psi_s\| \geq C|a(s)|\|\psi_s\|$$

Proof. Since a depends only on q the operator $U_N^* \mathrm{Op}_N[a]U_N - \mathrm{Op}_N[a \circ F]$ is given by multiplication with $a \circ f_N - a \circ f$, and we have

$$\|(U_N^* \mathrm{Op}_N[a]U_N - \mathrm{Op}_N[a \circ F])\psi_s\|^2 = \frac{1}{N} \sum_{n=1}^N |(a \circ f_N - a \circ f)(n/N)|^2 |\Psi_s(n/N)|^2 , \tag{12}$$

where $\Psi_s(q) = \sum_{m \in \mathbf{Z}} g_s(q - m)$. Since f is discontinuous at s there exists an $\varepsilon > 0$ and a $C > 0$ such that

$$|(a \circ f_N - a \circ f)(q)| \geq C|a(s)|, \quad \text{for } q \in [s - \varepsilon, s + \varepsilon]. \quad (13)$$

If we use now that g_s is exponentially concentrated around $q = s$, which in particular implies

$$\Psi_s(q) = g_s(q) + O(e^{-c/N}), \quad \text{for } q \in [s - \varepsilon, s + \varepsilon], \quad (14)$$

we obtain

$$\begin{aligned} & \frac{1}{N} \sum_{n=1}^N |(a \circ f_N - a \circ f)(n/N)|^2 |\Psi_s(n/N)|^2 \\ &= \frac{1}{N} \sum_{|n/N - s| \leq \varepsilon} |(a \circ f_N - a \circ f)(n/N)|^2 |\Psi_s(n/N)|^2 \\ & \quad + \frac{1}{N} \sum_{|n/N - s| > \varepsilon} |(a \circ f_N - a \circ f)(n/N)|^2 |\Psi_s(n/N)|^2 \\ & \geq C^2 |a(s)|^2 \|\psi_s\|^2 + O(e^{-c/N}), \end{aligned} \quad (15)$$

where we have furthermore used $\|\psi_s\|^2 = \frac{1}{N} \sum_{|n/N - s| \leq \varepsilon} |\Psi_s(n/N)|^2 + O(e^{-c/N})$. \square

We want to close this section with some comments about the underlying motivation for the specific quantisation assumptions on the neighbourhood of the singularities. Classically the singularities act like points with infinite local expansion rate respectively Lyapunov exponent. Therefore any perturbation in a small neighbourhood of the singularity set gives rise to an error which becomes unbounded if the perturbation approaches the singularity set. Since the quantised maps are a specific kind of perturbation it is natural to leave the allowed error big for points close to the singularity set.

3 Quantum limits and orbits

We will now discuss the implications of the Theorem 1 for the eigenfunctions of the quantised map. We will denote a orthonormal basis of eigenfunctions of U_N by ψ_k^N , $k = 1, \dots, N$,

$$U_N \psi_k^N = e_N(\theta_k^N) \psi_k^N,$$

where we use the notation $e_N(x) = e^{\frac{2\pi i}{N}x}$ and θ_k^N are the eigenphases. Each eigenfunction defines a linear map on the algebra of observables

$$\text{Op}_N[a] \mapsto \langle \psi_k^N, \text{Op}_N[a] \psi_k^N \rangle$$

and the leading term for $N \rightarrow \infty$ depends only on the principal symbol $\sigma(a)$. The limit points of the sequence of all these maps defined by the eigenfunctions define measures on the set of classical observables and are called quantum limits (see, e.g., [MR00]). To put it more explicitly, a measure ν on \mathbf{T}^2 is called a quantum limit of

the system defined by the U_N if there exist a sequence of eigenfunctions $\{\psi_{k_j}^{N_j}\}_{j \in \mathbf{N}}$ such that

$$\int_{\mathbf{T}^2} a(z) \, d\nu = \lim_{j \rightarrow \infty} \langle \psi_{k_j}^{N_j}, \text{Op}_N[a] \psi_{k_j}^{N_j} \rangle .$$

One of the major goals in quantum chaos, and more generally in semiclassical analysis, is to determine all quantum limits that can occur and the relative density of the corresponding subsequences of eigenfunctions. We say that a subsequence of eigenfunctions $\{\psi_{k_j}^{N_j}\}_{j \in \mathbf{N}}$ has density $\alpha \in [0, 1]$ if

$$\lim_{N \rightarrow \infty} \frac{\#\{k_j : N_j = N\}}{N} = \alpha , \quad (16)$$

provided that the limit exists.

Egorov's theorem usually implies that all quantum limits are invariant measures for the classical map. In our case the same is true, but we have to be careful at the singularities. If the set of singularities \mathbb{S} is nowhere dense then the space $C_{\mathbb{S}}^{\infty}(\mathbf{T}^2)$ in Theorem 1 is large enough so that as an immediate consequence we have:

Corollary 1. *Let us denote by $\mathcal{M}_{\text{inv}}(F)$ the convex set of F -invariant probability measures on \mathbf{T}^2 , and by $\mathcal{M}_{\text{qlim}}(U_N)$ the set of quantum limits μ of U_N with $\mu(\mathbb{S}) = 0$, then*

$$\mathcal{M}_{\text{qlim}}(U_N) \subset \mathcal{M}_{\text{inv}}(F)$$

So we only have to look at invariant measures as candidates for quantum limits. In the simplest case that there is only one invariant probability measure, i.e., that the system is uniquely ergodic, all eigenfunctions must converge to this measure, and we have the so called unique quantum ergodicity. This was the situation in the example of Marklof and Rudnick, [MR00].

We will study now the relationship between properties of quantum limits and the density of subsequences of eigenfunctions converging to them more closely. Our first result gives an upper bound on the density.

Theorem 2. *Let U_N be a proper quantisation of F and let μ be a quantum limit of U_N with support $\Sigma \subset \mathbf{T}^2$. Then any sequence of eigenfunctions which converge to μ has at most density $\mu_{\mathbf{T}^2}(\Sigma)$, where $\mu_{\mathbf{T}^2}$ is the Lebesgue measure on \mathbf{T}^2 .*

Proof. Let $a_{\varepsilon} \in C^{\infty}(\mathbf{T}^2)$, $\varepsilon \in (0, 1]$, be a sequence satisfying $a_{\varepsilon}|_{\Sigma} = 1$ and $\lim_{\varepsilon \rightarrow 0} a_{\varepsilon}(z) = 0$ for all $z \in \mathbf{T}^2 \setminus \Sigma$, i.e., a sequence approximating the characteristic function of Σ . If $\mathcal{F} = \{\psi_{k_j}^{N_j}\}_{j \in \mathbf{N}}$ is a sequence of eigenfunctions with μ as quantum limit, and $\mathcal{F}_N := \{\psi_{k_j}^{N_j} ; N_j = N\}$, then

$$\lim_{j \rightarrow \infty} \langle \psi_{k_j}^{N_j}, \text{Op}_N[a_{\varepsilon}] \psi_{k_j}^{N_j} \rangle = 1 \quad (17)$$

and therefore

$$\begin{aligned} \alpha(\mathcal{F}) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\psi \in \mathcal{F}_N} \langle \psi, \text{Op}_N[a_{\varepsilon}] \psi \rangle \\ &\leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \langle \psi_k^N, \text{Op}_N[a_{\varepsilon}] \psi_k^N \rangle = \int_{\mathbf{T}^2} a_{\varepsilon} \, d\mu_{\mathbf{T}^2} , \end{aligned} \quad (18)$$

where we have used (6). We now take the limit $\varepsilon \rightarrow 0$ and the theorem follows. \square

Since $\mu_{\mathbf{T}^2}(\mathbb{S}) = 0$ it follows in particular that a possible sequence of eigenfunctions converging to a quantum limit concentrated on \mathbb{S} must have density 0. This result is as well interesting for non-ergodic maps, because it gives an upper bound on the number of eigenfunctions whose quantum limits are supported on an invariant subset Σ of \mathbf{T}^2 by the volume of Σ .

In case that the system is ergodic, we can actually determine the quantum limit of most eigenfunctions.

Theorem 3. *Let U_N be a proper quantisation of F and assume that $\mu_{\mathbf{T}^2}$ is ergodic. Then there exists a subsequence of eigenfunctions of density one which converges to $\mu_{\mathbf{T}^2}$.*

This is the usual quantum ergodicity result, but our proof differs from the standard one (see e.g. [DEG03]) in that we rely on the convexity definition of ergodicity, this is more convenient when dealing with maps with singularities as has been observed in [GL93]. Recall that $\mu_{\mathbf{T}^2}$ is ergodic if it is extremal in the convex set of invariant probability measures, i.e., if $\mu_{\mathbf{T}^2} = \alpha\mu_1 + (1 - \alpha)\mu_2$ with $\mu_2 \neq \mu_{\mathbf{T}^2}$ then $\alpha = 1$ and $\mu_1 = \mu_{\mathbf{T}^2}$.

Proof. The existence of a subsequence $\mathcal{F} = \{\psi_{k_j}^{N_j}\}_{j \in \mathbf{N}}$ of density one of eigenfunctions with quantum limit $\mu_{\mathbf{T}^2}$ is equivalent to

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N |\langle \psi_k^N, \text{Op}_N[a] \psi_k^N \rangle - \bar{a}| = 0, \quad (19)$$

see [DEG03]. We first observe that by ergodicity every subsequence $\mathcal{F} = \{\psi_{k_j}^{N_j}\}_{j \in \mathbf{N}}$ of positive density satisfies

$$\lim_{N \rightarrow \infty} \frac{1}{|\mathcal{F}_N|} \sum_{\psi \in \mathcal{F}_N} \langle \psi, \text{Op}_N[a] \psi \rangle = \bar{a}. \quad (20)$$

To see this we consider the sequence

$$a_N := \frac{1}{|\mathcal{F}_N|} \sum_{\psi \in \mathcal{F}_N} \langle \psi, \text{Op}_N[a] \psi \rangle, \quad (21)$$

this is a bounded sequence since $\text{Op}_N[a]$ is bounded, and therefore there exists a convergent subsequence $\{a_{N_j}\}_{j \in \mathbf{N}}$. Now using (6) we have with a convergent subse-

quence

$$\begin{aligned}
\bar{a} &= \lim_{j \rightarrow \infty} \frac{1}{N_j} \sum_{k=1}^{N_j} \langle \psi_k^{N_j}, \text{Op}_N[a] \psi_k^{N_j} \rangle \\
&= \lim_{j \rightarrow \infty} \frac{|\mathcal{F}_{N_j}|}{N_j} \frac{1}{|\mathcal{F}_{N_j}|} \sum_{\psi \in \mathcal{F}_{N_j}} \langle \psi, \text{Op}_N[a] \psi \rangle \\
&\quad + \lim_{j \rightarrow \infty} \frac{N_j - |\mathcal{F}_{N_j}|}{N_j} \frac{1}{N_j - |\mathcal{F}_{N_j}|} \sum_{\psi_k^{N_j} \in \mathcal{H}_{N_j} \setminus \mathcal{F}_{N_j}} \langle \psi_k^{N_j}, \text{Op}_N[a] \psi_k^{N_j} \rangle \\
&= \alpha(\mathcal{F}) \int a \, d\mu_1 + (1 - \alpha(\mathcal{F})) \int a \, d\mu_2
\end{aligned} \tag{22}$$

where μ_1 and μ_2 are invariant measures defined by

$$\lim_{j \rightarrow \infty} \frac{1}{|\mathcal{F}_{N_j}|} \sum_{\psi \in \mathcal{F}_{N_j}} \langle \psi, \text{Op}_N[a] \psi \rangle = \int a \, d\mu_1 \tag{23}$$

$$\lim_{j \rightarrow \infty} \frac{1}{N_j - |\mathcal{F}_{N_j}|} \sum_{\psi_k^{N_j} \in \mathcal{H}_{N_j} \setminus \mathcal{F}_{N_j}} \langle \psi_k^{N_j}, \text{Op}_N[a] \psi_k^{N_j} \rangle = \int a \, d\mu_2 \tag{24}$$

These two measures exist by the assumption that the subsequence $\{a_{N_j}\}_{j \in \mathbf{N}}$ was convergent, and they are invariant by Theorem 1. But equation (22) can be rewritten as

$$\mu = \alpha \mu_1 + (1 - \alpha) \mu_2 \tag{25}$$

and if μ is ergodic and $\alpha \neq 0$ this is only possible if $\mu_1 = \mu$, and this proves that

$$\lim_{j \rightarrow \infty} a_{N_j} = \bar{a} . \tag{26}$$

Since this holds for every convergent subsequence of $\{a_N\}_{N \in \mathbf{N}}$ \bar{a} is the only limit point and (20) follows.

Now assume that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N |\langle \psi_k^N, \text{Op}_N[a] \psi_k^N \rangle - \bar{a}| = C > 0 , \tag{27}$$

then there must either exists a subsequence $\{k_j\}_{j \in \mathbf{N}}$ of positive density with

$$\langle \psi_{k_j}^{N_j}, \text{Op}_N[a] \psi_{k_j}^{N_j} \rangle - \bar{a} \geq C/2 \tag{28}$$

or one with

$$\langle \psi_{k_j}^{N_j}, \text{Op}_N[a] \psi_{k_j}^{N_j} \rangle - \bar{a} \leq -C/2 . \tag{29}$$

But the mean value of the sequence $\langle \psi_{k_j}^{N_j}, \text{Op}_N[a] \psi_{k_j}^{N_j} \rangle - \bar{a}$ must tend to 0 by (20) and so we have a contradiction if $C \neq 0$. \square

The previous results, Corollary 1, Theorem 2 and Theorem 3, are quite general, they are valid for all quantised maps which satisfy Egorov's theorem. We now turn to a more concrete study of the eigenfunctions for the specific quantum maps (11). Our aim is to show that the quantum limits are determined by the spatial distribution of the periodic orbits of the discretisation of the classical map. The eigenvalue equation

$$U_N \psi = e_N(\theta) \psi$$

leads to the following recursion equation for ψ

$$\Psi(\hat{f}_N(Q)) = e_N(\theta - Q^2) \Psi(Q) . \quad (30)$$

From this recursion relation we obtain

$$|\Psi(\hat{f}_N(Q))|^2 = |\Psi(Q)|^2 ,$$

and this implies that the probability densities in position space defined by the eigenfunctions are invariant under the map f_N . In order to determine these densities it is therefore sufficient to determine the spatial distribution of the orbits of f_N .

For the further investigation we note that each periodic orbit of f_N carries at least one eigenfunction. And we can determine the eigenfunctions and eigenvalues more explicitly, let \mathcal{O} be an periodic orbit of period $|\mathcal{O}| = K$, then the recursion relation (30) gives

$$\Psi(Q) = e_N \left(K\theta - \sum_{k=0}^{K-1} [\hat{f}_N^k(Q)]^2 \right) \Psi(Q) .$$

So if ψ should be an eigenfunction with eigenvalue $e_N(\theta)$ we get the condition

$$K\theta - \sum_{k=0}^{K-1} [\hat{f}_N^k(Q)]^2 = Nm$$

with $m \in \{0, 1, \dots, K-1\}$. This determines the eigenvalues, and then the corresponding eigenfunctions follow from the recursion relation and the normalisation condition. Summarising we get:

Proposition 2. *Let \mathcal{O} be an orbit of period $K = |\mathcal{O}|$ of f_N , then there exists K orthogonal eigenfunctions of U_N with support \mathcal{O} . The eigenphases are given by*

$$\theta_k = S_{\mathcal{O}} + \frac{N}{K} k$$

with $k \in \{0, 1, \dots, K-1\}$ and

$$S_{\mathcal{O}} := \frac{1}{K} \sum_{k'=0}^{K-1} [\hat{f}_N^{k'}(Q)]^2 ,$$

and a normalised eigenfunction corresponding to θ_k is given by

$$\Psi_k(\hat{f}_N^{k'}(Q_0)) = e_N \left(k'\theta_k - \sum_{m=0}^{k'} [\hat{f}_N^m(Q_0)]^2 \right) \left(\frac{N}{K} \right)^{1/2}$$

where $Q_0 \in \mathcal{O}$ is an arbitrary point on the orbit and $k' \in \{0, 1, \dots, K-1\}$.

The quantum lattice D_N of N points is a disjoint union of all periodic orbits of f_N , and on each of these orbits are as many eigenfunctions concentrated as the orbit is long. But that means that the orbits determine the quantum limits and the relative density of the corresponding sequence of eigenfunctions.

To each periodic orbit \mathcal{O} we can associate a probability measure on $[0, 1]$

$$\delta_{\mathcal{O}}(q) := \frac{1}{|\mathcal{O}|} \sum_{Q \in \mathcal{O}} \delta\left(q - \frac{Q}{N}\right) \quad (31)$$

which is invariant under f_N .

Corollary 2. *Let $\mathcal{O}_j^{(N)}$, $j = 1, \dots, J_N$ be the periodic orbits of f_N and let $\delta_j^{(N)}$ be the corresponding probability measures (31). Assume that there is an invariant measure ν of f and a sequence of periodic orbits $\{\mathcal{O}_{j_k}^{(N_k)}\}_{k \in \mathbf{N}}$ such that*

$$\lim_{k \rightarrow \infty} \delta_{j_k}^{(N_k)} = \nu,$$

then ν is a quantum limit of U_N .

In all our examples the sequence (N_k) contains all natural numbers. Thus there are two possible definitions of the density of a sequence of periodic orbits, $\mathcal{G} = \{\mathcal{O}_{j_k}^{(N_k)}\}_{k \in \mathbf{N}}$,

$$\alpha(\mathcal{G}) = \lim_{N \rightarrow \infty} \frac{|\mathcal{O}_{j_N}^{(N)}|}{N} \quad \text{or} \quad \beta(\mathcal{G}) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k: N_k = N} |\mathcal{O}_{j_k}^{(N_k)}|,$$

whenever the limit exists, which we will call the α -density or β -density of \mathcal{G} , respectively.

This corollary suggests that the set of quantum limits coincides with the set of limits points of the sequence of orbit measures $\delta_j^{(N)}$ and that moreover the relative densities of the convergent subsequences coincide too. This is true if there are no multiplicities in the eigenvalues. If there are eigenvalues of multiplicity larger than one, then the eigenspace can mix the contribution of the different orbits. But even in this case there always exists a choice of a basis of eigenfunctions corresponding to the orbit measures $\delta_j^{(N)}$. Notice that in this case the β -density of the sequence of orbits coincides with the density of the corresponding sequence of eigenfunctions defined in (16).

4 Cutting and Stacking constructions

Cutting and stacking is a popular method in ergodic theory to construct maps on the interval which are isomorphic models of arbitrary measure preserving dynamical systems. The construction gives a piecewise isometric mapping on the interval with Lebesgue measure as an invariant measure. One can also think of this transformation as a countable interval exchange transformation.

The final mapping will be defined only Lebesgue almost everywhere. None the less we can use this model to study certain other invariant measures which are well behaved with respect to the cutting and stacking construction.

For a very readable introduction into cutting and stacking see the recent book by Shields [Shi96] or the old book by Friedman [Fri70]. We will now give a short description of the basic construction scheme and the relevant definitions.

A stack (or column) S is a finite family of enumerated disjoint intervals $\{I_j\}_{j=1}^{h(S)}$ where $h(S)$ is called the height of S . The I_j are subintervals of $[0, 1]$ of equal length which is called the width of S . There are two possible conventions: either one can take all the intervals to be open, or all the intervals to be closed on the left and open on the right. Which convention we choose is not important for this paper. The intervals I_1 and $I_{h(S)}$ are called the bottom and top of S respectively. We define a transformation f_S as follows, if the point $x \in I_j$ is not on the top of S and not on the boundary of I_j then it gets mapped to the point directly above it (see Figure 1(a)). Since I_{j+1} and I_j have equal width, f_S is simply the canonical identification map between I_j and I_{j+1} . Interpreted in $[0, 1]$ this means that $f_S : I_j \rightarrow I_{j+1}$ such that $x \rightarrow x + \partial^- I_{j+1} - \partial^- I_j$ where ∂^- denotes the left boundary point of an interval. The construction clearly defines f^{-1} on all I_j except at the bottom.

A stack family \mathcal{S} is a finite or countable set of stacks $\{S_i\} = \{\{I_j^i\}_{j=1}^{h(S_i)}\}$ such that all I_j^i are disjoint and $\cup I_j^i = [0, 1]$. In this paper we will work only with finite stack families. On \mathcal{S} one defines a transformation $f_{\mathcal{S}}$ by $f_{\mathcal{S}}|_{S_i} = f_{S_i}$ except on the collection of top intervals.

Given two stacks S_i and S_j , $i \neq j$ of equal width one can define a new stack S' by stacking S_j on S_i that is

$$\begin{aligned} S' &= \{I'_k\}_{k=1}^{h(S_i)+h(S_j)} \\ I'_k &= I_k^i \text{ for } k \leq h(S_i) \text{ and } I'_k = I_{k-h(S_i)}^j \text{ for } k > h(S_i). \end{aligned}$$

Correspondingly one gets a new transformation $f_{S'}$ which agrees with f_{S_i} and f_{S_j} except on $I_{h(S_i)}^i$, where f_{S_i} was not defined before.

It remains to define the cutting of stacks. A cutting of a stack $S = \{I_k\}$ is a splitting of S into two (or more) disjoint stacks S_1 and S_2 with intervals $\{I_k^1\}$ and $\{I_k^2\}$ such that

$$I_k^1 \cup I_k^2 = I_k \text{ and } \partial^- I_k^1 < \partial^+ I_k^1 = \partial^- I_k^2 < \partial^+ I_k^2 \quad \forall k$$

that is I_k^1 is always the left component of the partition of I_k into I_k^1 and I_k^2 (Figure 1(b)). The definition of $f_{\{S_1, S_2\}}$ is as above. Multiple cutting of S is defined analogously.

A stack family $\mathcal{S}(n)$ is obtained from a stack family $\mathcal{S}(n-1)$ by cutting and stacking, if each $S_i(n)$ from $\mathcal{S}(n)$ can be obtained by successive cuttings and stackings of stacks from $\mathcal{S}(n-1)$. By construction $f_{\mathcal{S}(n)}$ is an extension of $f_{\mathcal{S}(n-1)}$. If one has a sequence $\{\mathcal{S}(n)\}_{n \geq 1}$ of stack families such that each $\mathcal{S}(n)$ is obtained from $\mathcal{S}(n-1)$ by cutting and stacking and furthermore

$$\lim_{n \rightarrow \infty} \sum_{S_k(n) \in \mathcal{S}(n)} \text{width}(S_k(n)) = 0, \quad (32)$$

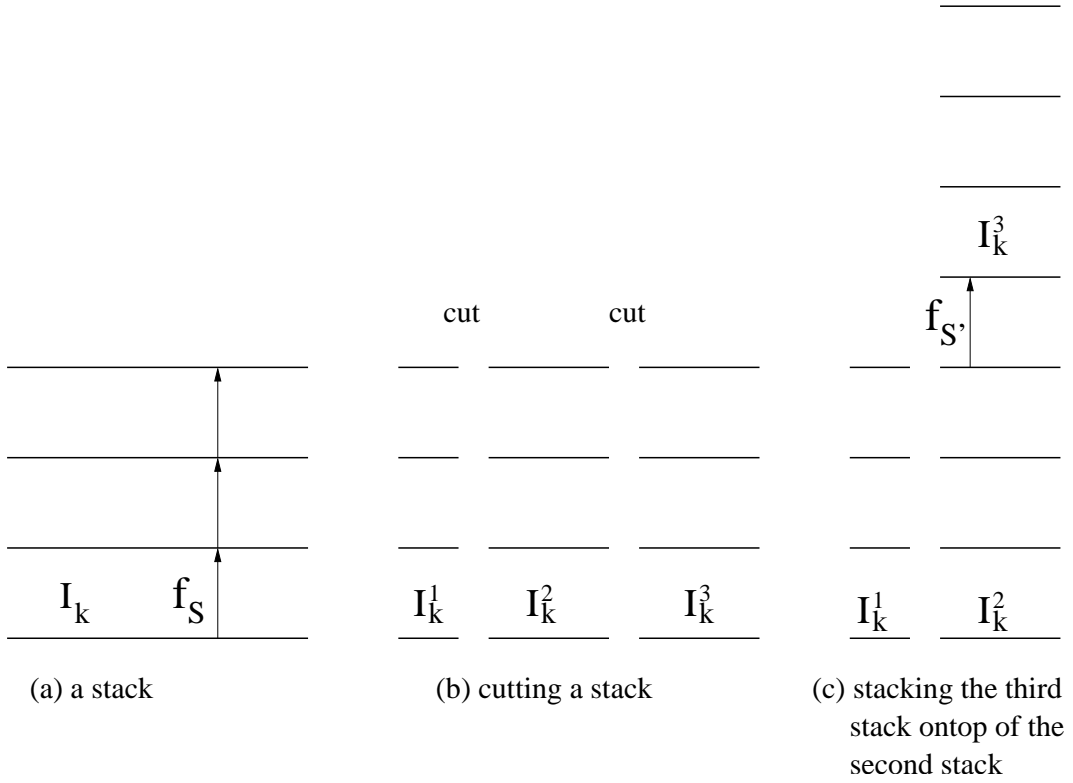


Figure 1: The cutting and stacking construction. In a given stack (a) the mapping f_S is defined, except at the top interval. In (b) the stack is cut into three substacks, and in (c) the third substack is stacked onto the second one. This gives an extension $f_{S'}$ of the map f_S which was not defined on the top of substack two before.

then $\lim f_{S(n)} = f$ is an invertible transformation on $[0, 1]$ defined everywhere except at a set of zero Lebesgue measure. Note that f is always aperiodic.

The “partition” of $[0, 1]$ into the intervals of $\mathcal{S}(1)$,¹ the starting object of the construction, gives a natural symbolic dynamics for f . The coding is unique for all points whose infinite orbit is defined.

For convenience, we denote the intervals of $\mathcal{S}(n)$ by $I_j^i(n)$ where i indexes the stack and j the interval in the stack. We consider the set

$$\mathbb{S} = \overline{\left(\cup_{i,n} \partial I_{h(S_i)}^i(n) \right) \cup \left(\cup_{i,n} \partial I_1^i(n) \right)}.$$

The set $\mathbb{S} = \mathbb{S}(f)$ is called the singularity set of the map f . It consists of all the points of discontinuity of f and all the points where the map f or f^{-1} is not defined. The boundary points of the intervals I_j^i which are not top or bottom intervals are not included in this set, the map is defined and continuous on such points! Furthermore $\cup_{i,j,n} \partial I_j^i(n)$ is included in the set $\cup_{k \in \mathbb{Z}} f^k \mathbb{S}$, where $f^k \mathbb{S} := \{x : f^{-k}x \in \mathbb{S}\}$.

¹Here we can ignore the boundary points of the interval since f is not defined on them.

Historically the cutting and stacking construction was invented to represent the dynamics with respect to a single invariant measure as a countable interval exchange transformation with the canonical invariant Lebesgue measure. The construction is universal in the sense that for every measurable dynamical system (M, g, μ) one can explicitly give a cutting and stacking representation $([0, 1], f(g), \mu_L)$ [Shi96]. The following proposition which will be needed for the application of Theorem 3 seemed to be unknown.

Proposition 3. *Let (Σ, σ) be a symbolic dynamical system over a finite or countable alphabet and let μ be a shift-invariant measure such that $\mu([w]) > 0$ for all cylinder-sets corresponding to finite words w appearing in Σ . Then the associated cutting and stacking construction yields a representation $([0, 1], f, \mu_L)$ of (Σ, σ, μ) such that all nonatomic invariant measures ν^* on (Σ, σ) have a corresponding isomorphic invariant measure ν on $([0, 1], f)$ with the property $\nu(\mathbb{S}(f)) = 0$.*

Proof. By definition all finite symbolic words have a representation as orbit segments of the cutting and stacking construction. Furthermore to each nonperiodic symbol sequence corresponds a unique point in $[0, 1]$ whose orbit under f is well defined and commutes with the shift. By the Poincaré recurrence theorem every f -invariant measure is supported on recurrent points. Since the singular orbits, respectively symbolic sequences, are the ones which eventually or asymptotically fall onto the singularity set they do not intersect with the recurrent nonperiodic symbol sequences. Therefore all invariant measures- except the finitely supported ones- give zero measure to the singularity set. \square

4.1 The approximating family

Each f defined by cutting and stacking provides us with a natural approximation family $\{f_{S(n)}\}$ which we will use now to define the approximation mappings on the rational points $D_N = \{\frac{Q}{N} : Q \in \{0, \dots, N-1\}\}$ for the quantisation. Let the points in $G_{i,j}(n, N) := D_N \cap I_j^i(n)$ be enumerated from left to right and let

$$K(N, I_j^i(n)) := \sharp G_{i,j}(n, N) \text{ and } K(N, S_i(n)) := \min_j \{K(N, I_j^i(n))\}.$$

$K(N, S_i(n))$ is just the smallest number of points from the discretisation in an interval in the stack $S_i(n)$. Let $\hat{G}_{i,j}(n, N)$ be the set of the first $K(N, S_i(n))$ points from $G_{i,j}(n, N)$ denoted by $\{x_e^{j,i}\}_{e=1}^{K(N, S_i(n))}$.

We define $f_{N,n}$ first on $\hat{D}_{N,n} = \cup_{i,j} \hat{G}_{i,j}(n, N)$ by setting

$$f_{N,n} x_e^{j,i} = x_e^{j+1,i} \text{ for } j < h_i(n) := h(S_i(n)). \quad (33)$$

We call these the internal orbit segments. Clearly $|f_{N,n} x_e^{j,i} - f x_e^{j,i}| = O(\frac{1}{N})$.

We call each approximation mapping $f_{N,n}$ on D_N whose restriction to $\hat{D}_{N,n}$ is given by the above construction an ergodic approximation.

Let $\check{D}_{N,n}$ be the set of points not in $\hat{D}_{N,n}$ and not in in any of the top intervals $I_{h(S_i)}^i(n)$. For $x \in \check{D}_{N,n}$ let $f_{N,n} x$ be the closest point to fx . Note that $f_{N,n}$ is not

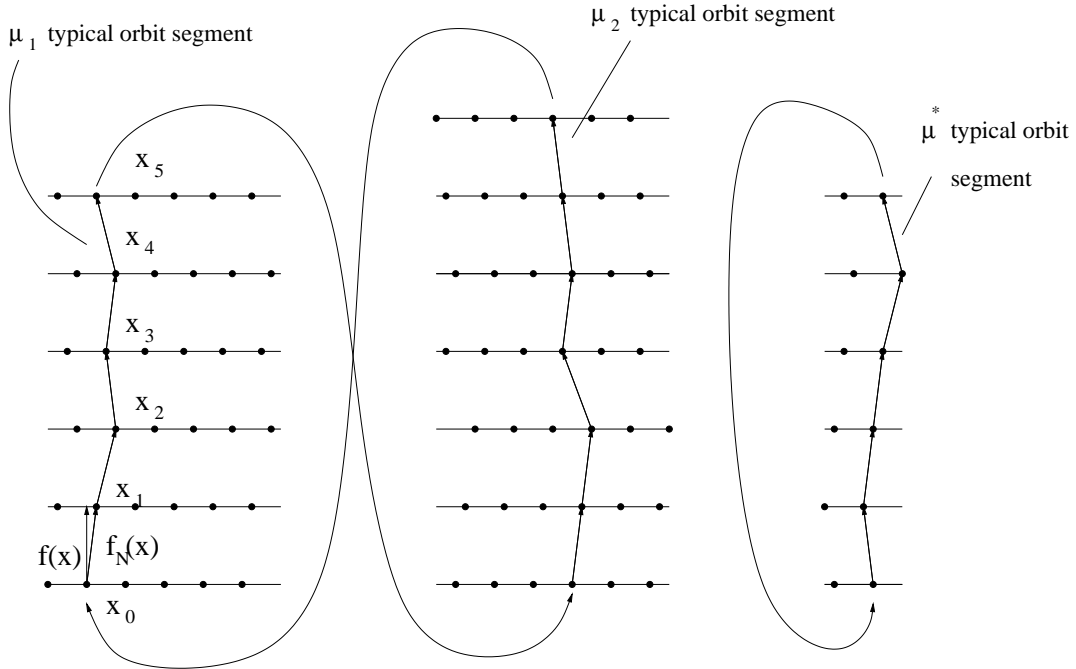


Figure 2: The approximate mapping f_N on the discrete set D_N (denoted by the full dots), and the way the orbit segments from different stacks are concatenated in order to produce a quantum limit of the form $\alpha_1\mu_1 + \alpha_2\mu_2$. A singular limit μ^* is obtained by a stack of small width and where the orbits are concatenated from top to the bottom.

necessarily an invertible map, thus the construction implies that $\max_{x \in \hat{D}_{N,n}} |f_{N,n}x - fx| = O(\frac{1}{N})$. It is clear that for fixed n we have

$$\lim_{N \rightarrow \infty} \frac{\#\hat{D}_{N,n}}{\#D_N} = 1. \quad (34)$$

To complete the definition of $f_{N,n}$, it remains to define the mapping of the points $\hat{G}_{i,h_i(n)}(n, N)$ on the tops of the stacks to points $\hat{G}_{i,1}(n, N)$ on the bottoms of the stacks. This will be done in a way to produce periodic orbits which approximately mimic a given invariant measure. Furthermore it remains to link N to a given n to get a good approximation. In essence we have to require that for each fixed stage n construction we have enough discretisation points in each stack. That means that with the increase of N we pass from n to $n+1$ only as we pass a critical threshold value N_n . This can be done already without a precise description of the gluing between top and bottom of the stacks. We need that the approximation family f_N is good enough to apply Theorem 1 for a sequence $\varepsilon_N \rightarrow 0$ such that $\delta_N(\varepsilon_N) \rightarrow 0$. Note that Theorem 1 does not impose any requirements on the rate of convergence. The basic idea is to keep N large enough compared with n such that all intervals $I_j^i(n)$ of $\mathcal{S}(n)$ contain sufficiently many points from D_N . Let $b_n = \min_i \text{width}(S_i(n))$. Choose

a function $n(N)$ going to infinity such that

$$\lim_{N \rightarrow \infty} \min_i K(N, S_i(n(N))) = \infty \quad (35)$$

which is equivalent to require $\frac{1}{b_n} = o(N)$. Furthermore let $\varepsilon_N = \max_i \text{width}(S_i(n))$. Equation (32) implies that ε_N tends to 0 since $n(N)$ tends to infinity. With this choice of ε_N the points where $f_{N,n(N)}$ is not yet defined do not contribute to $\delta_N(\varepsilon_N)$, hence one obtains $\delta_N(\varepsilon_N) = O(\frac{1}{N})$. An approximation family $f_N := f_{N,n(N)}$ for any function $n(N)$ satisfying the above requirements is called proper.

We say that a measure μ appears as a quantum limit if one can find a proper approximating family f_N and associated quantisation U_N such that μ is a quantum limit of U_N . The notion of quantum limit as well as the notion of density were introduced in Corollary 2.

Theorem 4. *a) If μ is an absolute continuous ergodic measure for f , then μ appears as a positive α and β - density quantum limit. Furthermore the quantum limit has full α and β - density if μ is the Lebesgue measure.*

b) If μ is a nonatomic, singular ergodic measure for f then μ appears as a quantum limit. Furthermore the quantum limit must have zero α and β - density.

c) If μ_1 and μ_2 are two absolute continuous ergodic invariant measures, then $\alpha_1\mu_1 + \alpha_2\mu_2$ appears as a positive α and β - density quantum limit for any $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 + \alpha_2 = 1$.

d) If μ_1 and μ_2 are two ergodic measures at least one of which is singular, then $\alpha_1\mu_1 + \alpha_2\mu_2$ appears as a quantum limit for any $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 + \alpha_2 = 1$. Furthermore the quantum limit must have zero α and β - density.

A transformation $f : [0, 1] \rightarrow [0, 1]$ is called a finite rank transformation if one can construct it via cutting and stacking such that the number of stacks $\#\{S_i(n)\}$ in the n -th stack family $\mathcal{S}(n)$ is bounded (independent of n).

Corollary 3. *If f is a finite rank transformation, then every ergodic non atomic invariant measure appears as the quantum limit of any proper ergodic approximation family.*

Proof. For a finite rank transformation, the singularity set \mathbb{S} is a countable set which has only a finite number of points of density. Thus any non atomic invariant measure can not be support on \mathbb{S} . \square

A point x is called μ -typical if $\lim_{m,u \rightarrow \infty} \frac{1}{m+u+1} \sum_{i=-u}^m \delta(f^i x) = \mu$. For the proof of the theorem we need the following simple fact whose proof is omitted since it is immediate from the definition of weak convergence of measures.

Proposition 4. *Fix x be μ -typical. Let $j(m, u)$ and $\varepsilon(j)$ be functions such that $j(m, u) \rightarrow \infty$ for $m, u \rightarrow \infty$ and $\varepsilon(j) \rightarrow 0$ as $j \rightarrow \infty$. Let $\{y_k^{(j)}\}_{k \in \mathbb{Z}}$ be a family of sequences with the property that $\left| y_k^{(j)} - f^k x \right| \leq \varepsilon(j)$ for $-u \leq k \leq m$. Then*

$$\lim_{m,u \rightarrow \infty} \frac{1}{m+u+1} \sum_{k=-u}^m \delta\left(y_k^{(j(m,u))}\right) = \mu.$$

Proof of Theorem 3. For the proof of part b) and part a) for α -density we complete the definition of f_N as follows. Let us complete each internal orbit segment into a periodic orbit by setting $f_N x_e^{h_i(n),i} = x_e^{1,i}$ (compare (33)). For points $x \in D_N \setminus \mathbb{S}$ where the map is not yet defined we have the freedom to map x anywhere, for preciseness define $f_N x$ to be the closest point to $f x$. By Corollary 2 it is enough to show that there is a sequence of periodic orbits on D_N whose point-mass average converge for $N \rightarrow \infty$ to the considered measure μ .

Fix an arbitrary enumeration $\{\mathcal{O}_j^{(N)}\}$ of the periodic orbits on D_N . For μ absolute continuous the set of points $x \in [0, 1]$ with $\lim_{m,u \rightarrow \infty} \frac{1}{m+u+1} \sum_{i=-u}^m \delta(f^i x) = \mu$ has positive Lebesgue measure and in the case μ is the Lebesgue measure it has full measure. Let x be μ -typical and consider for each N the stack $S_i(n(N))$ in which x is placed, where $n(N)$ is a function satisfying the requirements of Equation (35). Let $\mathcal{O}_{j_i}^{(N)}$ be the set of periodic orbits in $\cup_j \tilde{G}_{i,j}(n(N), N)$. They stay $width(S_i(n(N)))$ -close to the orbit segment $\{f^{-m_0} x, \dots, f^{n_0} x\}$ where m_0 and n_0 are the smallest and largest iterates such that $f_{S_i(n(N))}^k x$ is still defined. Note that $m_0 + n_0 + 1 = h(S_i(n(N)))$. Since

$$width(S_i(n(N))) \leq \varepsilon_N := \max_i width(S_i(n(N))) \rightarrow 0 \text{ for } N \rightarrow \infty$$

we can apply the above proposition to the family $\mathcal{O}_{j_i}^{(N)}$ (with fixed l) to conclude that μ is a quantum limit, however this construction has not yet proved the positive density.

To prove the positive density we need a quantified version of the above. Let \mathcal{J}_q be the set of subintervals of $[0, 1]$ with boundary points of the form $\frac{p}{q}$. For the convergence of a sequence to a measure it is clearly enough to check the characteristic function averages with respect to the elements of $\cup_q \mathcal{J}_q$. A stack $S_i(n(N))$ is called $\varepsilon - q$ -good with respect to μ if for all $x \in I_1^i(n(N))$ (i.e. x in the base of the stack $S_i(n(N))$)

$$\mu(J) - \varepsilon \leq \frac{1}{h_i(n(N))} \sum_{i=0}^{h_i(n)} \mathbf{1}_J \left(f^i|_{S_i(n(N))}(x) \right) \leq \mu(J) + \varepsilon \text{ for } \forall J \in \mathcal{J}_q$$

where $h_i(n(N))$ denotes the height of the stack $S_i(n(N))$. Denote the family of such stacks by $\mathcal{G}(n, q, \varepsilon, \mu)$ and by $\mathbb{G}(n, q, \varepsilon, \mu)$ the set of points contained in $\mathcal{G}(n, q, \varepsilon, \mu)$. Clearly one has $\forall q, \varepsilon > 0$

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu_L(\mathbb{G}(n, q, \varepsilon, \mu)) &= \mu_L(x : x \text{ is } \mu\text{-typical}) \text{ and} \\ \lim_{n \rightarrow \infty} \mu(\mathbb{G}(n, q, \varepsilon, \mu)) &= 1. \end{aligned}$$

Let $\varepsilon(n)$ be a sufficiently slowly decreasing function and $q(\varepsilon(n))$ be a sufficiently slowly increasing function that

$$\lim_{n \rightarrow \infty} \mu_L(\mathbb{G}(n, q(n), \varepsilon(n), \mu)) = \mu_L(x : x \text{ is } \mu\text{-typical}). \quad (36)$$

With this new notation we are ready to prove that μ is a quantum limit of positive density. We define the sequence $\mathcal{O}_{j_l}^{(N)}$ of periodic orbits which give rise to the desired quantum limit as follows. For fixed N the set $\mathbb{G}(n(N), q(n(N)), \varepsilon(n(N)), \mu) \cap \hat{D}_{N,n(N)}$ consists of a collection of points of periodic orbits. The sequence $\mathcal{O}_{j_l}^{(N)}$ consists of the set of these orbits. The positive β -density and full density in the case of Lebesgue measure then follows from Equation (36) and 34. To prove part a) for the α -density we only need to modify the map f_N on top of the stacks. This will be done such that the collection of periodic orbits $\mathcal{O}_{j_l}^{(N)}$ becomes just one periodic orbit for each N . This completes the proof of part a).

To prove part b) observe that due to Proposition 3 we have $\mu(\mathbb{S}) = 0$ and hence can apply Theorem 1 to get an invariant measure out of the quantum-limit. One considers the set of μ -typical points. From the proof of Proposition 3 follows that the orbit of every μ -typical point does not intersect nor converge to the singularity set \mathbb{S} . Since the set of μ -typical points has zero Lebesgue measure we can obtain only a zero density quantum limit just as in the proof of part a).

To prove part c) one has to modify the construction of the approximating mapping f_N in the following way. Instead of making f_N periodic within each stack $S_i(n(N))$ we want to connect two stacks say $S_i(n(N))$ and $S_j(n(N))$ where the orbit segments in the i -th stack respectively j -th stack are approximately typical for μ_1 respectively μ_2 to get an average of μ_1 and μ_2 .

For $l = 1, 2$ let $A_l(N) := \{i : S_i(n(N)) \in \mathcal{G}(n(N), q(n), \varepsilon(n), \mu_l)\}$. On $\hat{D}_{N,n(N)}$ define f_N as before by $f_N x_e^{j,i} = x_e^{j+1,i}$ for $j < h_i(n)$. For $x_e^{j,i} \in \mathcal{G}(n(N), q(n), \varepsilon(n), \mu_l) \cap \hat{D}_{N,n(N)}$ one has for $\forall J \in \mathcal{J}_{q(n)}$

$$\mu_l(J) - \varepsilon(n) + O\left(\frac{1}{N}\right) \leq \frac{1}{h_i(n)} \sum_{0 \leq k \leq h_i(n)-1} \mathbf{1}_J \left(f_N^k x_e^{1,i} \right) \leq \mu_l(J) + \varepsilon(n) + O\left(\frac{1}{N}\right).$$

Let $\theta_l := \mu_L(x : x \text{ is } \mu_l \text{-typical})$ and note that

$$\frac{\#A_1(N) \cap \hat{D}_{N,n(N)}}{\#A_2(N) \cap \hat{D}_{N,n(N)}} \rightarrow \frac{\theta_1}{\theta_2} \text{ for } N \rightarrow \infty.$$

Thus by gluing all the orbit segments of f_N in the sets $A_1(N) \cap \hat{D}_{N,n(N)}$ and $A_2(N) \cap \hat{D}_{N,n(N)}$ in such a way that they form one periodic orbit we obtain a family of periodic orbits with quantum limit $\alpha_1 \mu_1 + \alpha_2 \mu_2$ where $\alpha_l = \frac{\theta_l}{\theta_1 + \theta_2}$. The α and β -densities are just $\theta_1 + \theta_2$.

It is easy to construct in the same spirit approximation families f_N for any values α_1 and $\alpha_2 = 1 - \alpha_1$. Suppose first that $\alpha_1 < \frac{\theta_1}{\theta_1 + \theta_2}$ and hence $\alpha_2 > \frac{\theta_2}{\theta_1 + \theta_2}$. Take in each stack $S_i(n(N))$ with $i \in A_1(N)$ approximately $\frac{\alpha_1}{\alpha_2}$ of the internal orbit segments. The function $n(N)$ is sufficiently slowly growing (35) to ensure that there are enough points in the discretisation set $\hat{D}_{N,n(N)}$ we can guarantee the convergence to $\frac{\alpha_1}{\alpha_2}$. Gluing these segments together with all the internal orbit segments of $A_2(N)$ yields a single periodic orbit $\mathcal{O}^{(N)}$. The family of periodic orbits $\{\mathcal{O}^{(N)}\}_N$ defines a quantum limit for the measure $\alpha_1 \mu_1 + \alpha_2 \mu_2$ with α and β -density $\frac{\alpha_1}{\alpha_2} \theta_1 + \theta_2 > 0$. The case $\alpha_1 > \frac{\theta_1}{\theta_1 + \theta_2}$ is analogous.

The proof of d) follows immediately by combining the arguments from parts b) and c). \square

5 Examples

5.1 Interval Exchange Maps

Consider a permutation π of $\{1, 2, \dots, n\}$ and a vector $\vec{v} = (v_1, \dots, v_n)$ such that $v_i > 0$ for all i and $\sum_{i=1}^n v_i = 1$. Let $u_0 = 0$, $u_i = v_1 + \dots + v_i$ and $\Delta_i = (u_{i-1}, u_i)$. The interval exchange transformation $T = T_{\pi, \vec{v}}$, $T : [0, 1] \rightarrow [0, 1]$ is the map that is an isometry of each interval Δ_i which rearranges these intervals according to the permutation π .

The Lebesgue measure is always an invariant measure for an IET. A typical IET is uniquely ergodic, however there exist minimal, non uniquely ergodic IETs. The first example of a minimal, non uniquely ergodic IET was given by Keynes and Newton [KN76] and Keane [Kea77]. The number of ergodic invariant measures for a minimal IET on m intervals is bounded by the $\lfloor m/2 \rfloor$. [Kat73, Vee78]. The set of invariant measures always includes absolutely continuous measures but can also include singular measures. It is known that an interval exchange transformation on m intervals is at most of rank m , in particular it is a finite rank transformation (see for example [Fer97]). In fact the typical IET is of rank 1 [Vee84], although we will not use this fact. Thus we can apply Corollary 3 to conclude:

1. any uniquely ergodic IET is quantum uniquely ergodic,
2. any minimal, non uniquely ergodic IET is not quantum uniquely ergodic,
3. any absolutely continuous invariant measures appear as a positive density quantum limit,
4. any singular ergodic invariant measure appears as a zero density quantum limit.

5.2 The full shift

Another example of a cutting and stacking transformation $f_{\mathcal{B}}$ that has μ_L as an ergodic invariant measure and admits further singular measures μ such that $\mu(\mathbb{S}) = 0$ is given by the full shift. Take any cutting and stacking model of the full two-sided shift on two symbols with Bernoulli-measure $p_0 = p_1 = \frac{1}{2}$ (for details of such models we refer to the book [Shi96]). Note that although the full shift has many periodic orbits the cutting and stacking model has none. We remark that one could introduce some periodic orbits at the boundaries of the subintervals but they would all sit or fall at singularity points and hence do not appear as quantum limits, in other words there are no scars in quantised cutting and stacking skew product mappings. By Proposition 3 all other invariant measures of the full shift have no support on the singularity set. Hence we can apply Theorem 3. It is interesting to note, that the fractal-dimensions (box or Hausdorff dimension) of the singularity set are rather

large and that the upper and lower dimensions do not coincide. A straightforward counting argument shows for instance that the upper and lower box dimensions are in the open interval $(\frac{1}{2}, 1)$.

6 Comments and Conclusions

We have shown in this paper that for a rather general class of dynamical systems on the torus the variety of different invariant measures can be recovered as quantum limits of the corresponding proper families of quantised maps. The quantisation scheme here used is based on the one introduced by [MR00]. For a discussion of alternative quantisation procedures and a critical comparison we refer to the recent work [Zel05].

One of the main features in our systems is the presence of singularities. In the quantisation procedure this provides enough freedom to obtain eigenfunctions reflecting the typical orbit structure with respect to any non atomic ergodic measure. It is an interesting question whether our results are still valid in case the classical dynamical system has no singularities. We conjecture that similar statements can be obtained. For this it seems natural to replace the top-bottom gluing scheme in the interval exchange approximating family by cutting and “crossover-concatenation” of touching period orbits.

Concerning the quantisation of flows one might hope that a good understanding of the associated quantised Poincare maps can guide one to a deeper understanding of concrete features of eigenfunctions and spectrum. An natural class of examples to study this questions are polygonal billiards. In the case of rational polygons the associated Poincare maps for the directional flow are interval exchange transformations which can be quantised similar to the quantisation used in this paper. It would be interesting to compare the results obtained that way with the semiclassical properties of the direct flow quantisation via the billiard Hamiltonian.

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