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## LOCALIZATION OF INJECTIVE MODULES OVER PRÜFER DOMAINS

FRANÇOIS COUCHOT

ABSTRACT. It is proved that  $S^{-1}G$  is injective if  $G$  is an injective module of finite Goldie dimension over a reduced arithmetic ring  $R$ , for each multiplicative subset  $S$ . Moreover, if  $R$  is a Prüfer domain of finite character then localizations of injective modules are injective too.

If  $R$  is a noetherian or hereditary ring, it is well known that localizations of injective  $R$ -modules are injective. By [1, Corollary 8] this property holds if  $R$  is a h-local Prüfer domain. However [1, Example 1] shows that this result is not generally true. Moreover, by [2, Theorem 25] there exist a coherent domain  $R$ , a multiplicative subset  $S$  and an injective module  $G$  such that  $S^{-1}G$  is not injective.

The aim of this paper is to study localizations of injective modules over arithmetic rings. We deduce from [1, Theorem 3] the two following results: any localization of an injective module of finite Goldie dimension over a reduced arithmetic ring is injective (Theorem 1) and any localization of an injective module over a Prüfer domain of finite character is injective (Theorem 3).

In this paper all rings are associative and commutative with unity and all modules are unital. A module is said to be **uniserial** if its submodules are linearly ordered by inclusion. A ring  $R$  is a **valuation ring** if it is uniserial as  $R$ -module and  $R$  is **arithmetic** if  $R_P$  is a valuation ring for every maximal ideal  $P$ . An arithmetic domain  $R$  is said to be **Prüfer**. We say that a module  $M$  is of **Goldie dimension**  $n$  if and only if its injective hull  $E(M)$  is a direct sum of  $n$  indecomposable injective modules. We say that a domain  $R$  is of **finite character** if every non-zero element is contained in finitely many maximal ideals.

**Theorem 1.** *Let  $R$  be a reduced arithmetic ring. Then, for each injective module  $G$  of finite Goldie dimension,  $S^{-1}G$  is injective for every multiplicative subset  $S$  of  $R$ .*

**Proof.**  $G$  is a finite direct sum of indecomposable injective modules. So, we may assume that  $G$  is indecomposable. Let  $L = \{r \in R \mid \exists 0 \neq x \in G \text{ such that } rx = 0\}$ . Since  $G$  is a uniform module,  $L$  is a prime ideal of  $R$ . Moreover  $G$  is a module over  $R_L$ . If  $S'$  is the multiplicative subset of  $R_L$  generated by  $S$ , then  $L' := R_L \setminus S'$  is a prime ideal of  $R_L$  and  $S^{-1}G = G_{L'}$ . Since  $R_L$  is a valuation domain, we conclude that  $S^{-1}G$  is injective by [1, Theorem 3].  $\square$

**Lemma 2.** *Let  $R$  be a Prüfer domain of finite character. For each maximal ideal  $P$ , let  $F_{(P)}$  be an injective  $R_P$ -module and let  $F = \prod_{P \in \text{Max } R} F_{(P)}$ . Then  $S^{-1}F$  is injective for every multiplicative subset  $S$  of  $R$ .*

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**Proof.** Let  $T_{(P)}$  be the torsion submodule of  $F_{(P)}$ , let  $G_{(P)} = F_{(P)}/T_{(P)}$ , let  $T = \prod_{P \in \text{Max } R} T_{(P)}$  and let  $G = \prod_{P \in \text{Max } R} G_{(P)}$ . Then  $G$  is torsion-free and  $F \cong T \oplus G$ . It is obvious that  $S^{-1}G$  is injective. Let  $T' = \bigoplus_{P \in \text{Max } R} T_{(P)}$ . Since  $R$  has finite character, it is easy to check that  $T'$  is the torsion submodule of  $T$ . So,  $T'$  is injective and  $S^{-1}(T/T')$  is injective. For each maximal ideal  $P$ ,  $S^{-1}T_{(P)}$  is injective by [1, Theorem 3]. Since  $S^{-1}T'$  is the torsion submodule of  $\prod_{P \in \text{Max } R} S^{-1}T_{(P)}$ , we successively deduce the injectivity of  $S^{-1}T'$  and  $S^{-1}T$ .  $\square$

**Theorem 3.** *Let  $R$  be a Prüfer domain of finite character. Then, for each injective module  $G$ ,  $S^{-1}G$  is injective for every multiplicative subset  $S$  of  $R$ .*

**Proof.** Let  $E = \prod_{P \in \text{Max } R} E_R(R/P)$  and let  $F = \text{Hom}_R(\text{Hom}_R(G, E), E)$ . Then  $E$  is an injective cogenerator and  $G$  is isomorphic to a summand of  $F$ . Since  $R$  is coherent,  $\text{Hom}_R(G, E)$  is flat by [3, Theorem XIII.6.4(b)]. Thus  $F$  is injective. We put  $F_{(P)} = \text{Hom}_R(\text{Hom}_R(G, E), E_R(R/P))$ . Then  $F_{(P)}$  is an injective  $R_P$ -module and  $F \cong \prod_{P \in \text{Max } R} F_{(P)}$ . By Lemma 2  $S^{-1}F$  is injective. We conclude that  $S^{-1}G$  is injective too.  $\square$

**Corollary 4.** *Let  $R$  be a semilocal Prüfer domain. Then, for each injective module  $G$ ,  $S^{-1}G$  is injective for every multiplicative subset  $S$  of  $R$ .*

The following example shows that the finite character is not a necessary condition in order that localizations of injective modules at multiplicative subsets are still injective.

**Example 5.** Let  $R$  be the ring defined in [4, Example 39]. Then  $R$  is a Prüfer domain which is not of finite character. But, since  $R$  is the union of a countable family of principal ideal subrings, it is easy to check that, for any multiplicative subset  $S$ ,  $R$  satisfies [2, Condition 14]. So, for each injective module  $G$ ,  $S^{-1}G$  is injective by [2, Theorem 15].

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