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LOCALIZATION OF INJECTIVE MODULES OVER PRÜFER DOMAINS

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Abstract. It is proved that $S^{-1}G$ is injective if G is an injective module of finite Goldie dimension over a reduced arithmetic ring R, for each multiplicative subset S. Moreover, if R is a Prüfer domain of finite character then localizations of injective modules are injective too.

If R is a noetherian or hereditary ring, it is well known that localizations of injective R-modules are injective. By [1, Corollary 8] this property holds if R is a h-local Prüfer domain. However [1, Example 1] shows that this result is not generally true. Moreover, by [2, Theorem 25] there exist a coherent domain R, a multiplicative subset S and an injective module G such that $S^{-1}G$ is not injective.

The aim of this paper is to study localizations of injective modules over arithmetic rings. We deduce from [1, Theorem 3] the two following results: any localization of an injective module of finite Goldie dimension over a reduced arithmetic ring is injective (Theorem 1) and any localization of an injective module over a Prüfer domain of finite character is injective (Theorem 3).

In this paper all rings are associative and commutative with unity and all modules are unital. A module is said to be **uniserial** if its submodules are linearly ordered by inclusion. A ring R is a **valuation ring** if it is uniserial as R-module and R is **arithmetic** if R_P is a valuation ring for every maximal ideal P. An arithmetic domain R is said to be **Prüfer**. We say that a module M is of **Goldie dimension** n if and only if its injective hull $\mathrm{E}(M)$ is a direct sum of n indecomposable injective modules. We say that a domain R is **of finite character** if every non-zero element is contained in finitely many maximal ideals.

Theorem 1. *Let R be a reduced arithmetic ring. Then, for each injective module G of finite Goldie dimension, $S^{-1}G$ is injective for every multiplicative subset S of R.*

Proof. G is a finite direct sum of indecomposable injective modules. So, we may assume that G is indecomposable. Let $L = \{r \in R \mid \exists 0 \neq x \in G \text{ such that } rx = 0\}$. Since G is a uniform module, L is a prime ideal of R. Moreover G is a module over R_L. If S' is the multiplicative subset of R_L generated by S, then $L' := R_L \setminus S'$ is a prime ideal of R_L and $S^{-1}G = G_{L'}$. Since R_L is a valuation domain, we conclude that $S^{-1}G$ is injective by [1, Theorem 3]. □

Lemma 2. *Let R be a Prüfer domain of finite character. For each maximal ideal P, let $F_{(P)}$ be an injective R_P-module and let $F = \prod_{P\in \mathrm{Max}\ R} F_{(P)}$. Then $S^{-1}F$ is injective for every multiplicative subset S of R.*

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Proof. Let $T_{(P)}$ be the torsion submodule of $F_{(P)}$, let $G_{(P)} = F_{(P)}/T_{(P)}$, let $T = \prod_{P\in\mathrm{Max}\ R} T_{(P)}$ and let $G = \prod_{P\in\mathrm{Max}\ R} G_{(P)}$. Then G is torsion-free and $F \cong T \oplus G$. It is obvious that $S^{-1}G$ is injective. Let $T' = \oplus_{P\in\mathrm{Max}\ R} T_{(P)}$. Since R has finite character, it is easy to check that T' is the torsion submodule of T. So, T' is injective and $S^{-1}(T/T')$ is injective. For each maximal ideal P, $S^{-1}T_{(P)}$ is injective by [1, Theorem 3]. Since $S^{-1}T'$ is the torsion submodule of $\prod_{P\in\mathrm{Max}\ R} S^{-1}T_{(P)}$, we successively deduce the injectivity of $S^{-1}T'$ and $S^{-1}T$. □

Theorem 3. *Let R be a Prüfer domain of finite character. Then, for each injective module G, $S^{-1}G$ is injective for every multiplicative subset S of R.*

Proof. Let $E = \prod_{P\in\mathrm{Max}\ R} \mathrm{E}_R(R/P)$ and let $F = \mathrm{Hom}_R(\mathrm{Hom}_R(G,E),E)$. Then E is an injective cogenerator and G is isomorphic to a summand of F. Since R is coherent, $\mathrm{Hom}_R(G,E)$ is flat by [3, Theorem XIII.6.4(b)]. Thus F is injective. We put $F_{(P)} = \mathrm{Hom}_R(\mathrm{Hom}_R(G,E),\mathrm{E}_R(R/P))$. Then $F_{(P)}$ is an injective R_P-module and $F \cong \prod_{P\in\mathrm{Max}\ R} F_{(P)}$. By Lemma 2 $S^{-1}F$ is injective. We conclude that $S^{-1}G$ is injective too. □

Corollary 4. *Let R be a semilocal Prüfer domain. Then, for each injective module G, $S^{-1}G$ is injective for every multiplicative subset S of R.*

The following example shows that the finite character is not a necessary condition in order that localizations of injective modules at multiplicative subsets are still injective.

Example 5. Let R be the ring defined in [4, Example 39]. Then R is a Prüfer domain which is not of finite character. But, since R is the union of a countable family of principal ideal subrings, it is easy to check that, for any multiplicative subset S, R satisfies [2, Condition 14]. So, for each injective module G, $S^{-1}G$ is injective by [2, Theorem 15].

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