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Geometric Invariant Theory and Generalized Eigenvalue Problem

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Abstract

Let G be a connected reductive subgroup of a complex connected reductive group \hat{G} . Fix maximal tori and Borel subgroups of G and \hat{G} . Consider the cone $\mathcal{LR}(G, \hat{G})$ generated by the pairs $(\nu, \hat{\nu})$ of dominant characters such that V_ν is a submodule of $V_{\hat{\nu}}$ (with usual notation). Here we give a minimal set of inequalities describing $\mathcal{LR}(G, \hat{G})$ as a part of the dominant chamber.

In way, we obtain results about the faces of the Dolgachev-Hu's G -ample cone and variations of this cone.

1 Introduction

In this article, we are mainly interested in the faces of the G -ample cone as defined by Dolgachev-Hu in [DH98]. In this introduction we start by the end: we firstly explain the applications to the generalized Horn's problem.

Let G be a connected reductive subgroup of a connected reductive group \hat{G} both defined over an algebraic closed field \mathbb{K} of characteristic zero. We consider the following question:

What irreducible representations of G appear in a given irreducible representation of \hat{G} ?

Once maximal tori $(T \subset \hat{T})$ and Borel subgroups $(B \supset T$ and $\hat{B} \supset \hat{T})$ fixed, the question is to understand the set $\text{LR}(G, \hat{G})$ of pairs $(\nu, \hat{\nu})$ of dominant character of $T \times \hat{T}$ such that the G -module associated to ν can be

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G -equivariantly embedded in the \hat{G} -module associated to $\hat{\nu}$. By a result of M. Brion and F. Knop (see [É92]), $\text{LR}(G, \hat{G})$ is a finitely generated submonoid of the character group of $T \times \hat{T}$. Our purpose is to study the linear inequalities satisfied by this monoid. More precisely, we consider the convex cone $\mathcal{LR}(G, \hat{G})$ generated by $\text{LR}(G, \hat{G})$: this cone is characterized by finitely many inequalities. It occurs that the interior of $\mathcal{LR}(G, \hat{G})$ is non empty if and only if no non trivial connected normal subgroup of G is normal in \hat{G} : for simplicity, we assume, from now on that $\mathcal{LR}(G, \hat{G})$ has non empty interior. Our first result is a list of inequalities which characterize $\mathcal{LR}(G, \hat{G})$ as a part of the dominant cone. Such a list was already obtained in [BS00]. Whereas the Berenstein-Sjamaar's list is redundant, our list is proved to be irredundant.

To make our statements more precise, we introduce notation. Consider the natural paring $\langle \cdot, \cdot \rangle$ between one parameter subgroups and characters of tori T or \hat{T} . Let W (resp. \hat{W}) denote the Weyl group of T (resp. \hat{T}). If λ is a one parameter subgroup of T (or so of \hat{T}), we denote by W_λ (resp. \hat{W}_λ) the stabilizer of λ for the natural action of the Weyl group on the set of one parameter subgroups. For $w \in W/W_\lambda$ and $\hat{w} \in \hat{W}/\hat{W}_\lambda$, we consider the following linear form on the character group $X(T \times \hat{T})$ of $T \times \hat{T}$:

$$\varphi_{\lambda, w, \hat{w}} : (\nu, \hat{\nu}) \mapsto \langle \hat{w}\lambda, \hat{\nu} \rangle + \langle w\lambda, \nu \rangle.$$

In fact, all the inequalities in Theorem A below have the following form $\varphi_{\lambda, w, \hat{w}} \geq 0$. We need some notation to explain which triples (λ, w, \hat{w}) appear.

Let $P(\lambda)$ denote the usual parabolic subgroup of G associated to λ (see Section 3.7). The cohomology group $H^*(G/P(\lambda), \mathbb{Z})$ is freely generated by the Schubert classes $[\Lambda_w]$ parametrized by the elements $w \in W/W_\lambda$. We will consider $\hat{G}/\hat{P}(\lambda)$, $[\Lambda_{\hat{w}}]$ as above but with \hat{G} in place of G . Consider also the canonical G -equivariant immersion $\iota : G/P(\lambda) \rightarrow \hat{G}/\hat{P}(\lambda)$; and the corresponding morphism in cohomology ι^* .

Let \mathfrak{g} and $\hat{\mathfrak{g}}$ denote the Lie algebras of G and \hat{G} . Let ρ , $\hat{\rho}$ and $\hat{\rho}^\lambda$ be the half-sum of positive roots of G , \hat{G} and the centralizer \hat{G}^λ of λ in \hat{G} . Let $\lambda_1, \dots, \lambda_n$ be the set of indivisible dominant one parameter subgroups of T orthogonal to an hyperplane generated by weights of T in $\hat{\mathfrak{g}}/\mathfrak{g}$.

We can now state one of our main results:

Theorem A *We assume that no ideal of \mathfrak{g} is an ideal of $\hat{\mathfrak{g}}$. Then, $\mathcal{LR}(G, \hat{G})$ has non empty interior in $X(T \times \hat{T}) \otimes \mathbb{Q}$.*

A dominant weight $(\nu, \hat{\nu})$ belongs to $\mathcal{LR}(G, \hat{G})$ if and only if for all $i = 1, \dots, n$ and for all pair of Schubert classes $([\Lambda_w], [\Lambda_{\hat{w}}])$ of $G/P(\lambda_i)$ and

$\hat{G}/\hat{P}(\lambda_i)$ associated to a pair $(w, \hat{w}) \in W/W_{\lambda_i} \times \hat{W}_{\lambda_i}$ such that

- (i) $\iota^*([\Lambda_{\hat{w}}]).[\Lambda_w] = [\text{pt}] \in H^*(G/P(\lambda), \mathbb{Z})$, and
- (ii) $\langle w\lambda_i, \rho \rangle + \langle \hat{w}\hat{\lambda}_i, \hat{\rho} \rangle = \langle \lambda_i, \rho \rangle + \langle \lambda_i, 2\hat{\rho}^{\lambda_i} - \hat{\rho} \rangle$,

we have

$$\langle w\lambda_i, \nu \rangle + \langle \hat{w}\lambda_i, \hat{\nu} \rangle \geq 0. \quad (1)$$

In [BS00], Berenstein and Sjamaar showed that $(\nu, \hat{\nu})$ belongs to $\mathcal{LR}(G, \hat{G})$ if and only if $\langle w\lambda_i, \nu \rangle + \langle \hat{w}\lambda_i, \hat{\nu} \rangle \geq 0$ for all $i = 1, \dots, n$ and for all pair of Schubert classes $([\Lambda_w], [\Lambda_{\hat{w}}])$ such that $\iota^*([\Lambda_{\hat{w}}]).[\Lambda_w] = d \cdot [\text{pt}]$ for some positive integer d . In the case when G is diagonally embedded in G^s , Kapovich-Leeb-Millson already proved that one may assume that $d = 1$. Again in the case when G is diagonally embedded in G^s , Belkale and Kumar obtained the same inequalities as in Theorem A in [BK06].

In some sense, our second main result asserts that Theorem A is optimal:

Theorem B *In Theorem A, Inequalities (1) are pairwise distinct and no one can be omitted.*

This result was known in some particular cases. Indeed, Knutson, Tao and Woodward shown in [KTW04] the case when $G = \text{SL}_n$ is diagonally embedded in $\text{SL}_n \times \text{SL}_n$ using combinatorial tools. Using the interpretation of the Littlewood-Richardson coefficients as structure coefficient of the cohomology ring of the Grassmann varieties, Belkale made a geometric proof of Knutson-Tao-Woodward's result (see [Bel07]). Using explicit calculation with the help of a computer, Kapovich, Kumar and Millson proves the case when $G = \text{SO}(8)$ is diagonally embedded in $G \times G$ in [KKM06]. Our proof is different and uses Geometric Invariant Theory.

Let $([\Lambda_w], [\Lambda_{\hat{w}}])$ be a pair of Schubert classes in $G/P(\lambda_i)$ such that $\iota^*([\Lambda_{\hat{w}}]).[\Lambda_w] = d \cdot [\text{pt}] \in H^*(G/P(\lambda), \mathbb{Z})$ for some positive integer d . By [BS00], the set of $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$ such that $\langle w\lambda_i, \nu \rangle + \langle \hat{w}\lambda_i, \hat{\nu} \rangle = 0$ is a face of $\mathcal{LR}(G, \hat{G})$. Theorem B shows that if (w, \hat{w}) does not satisfy Conditions (i) and (ii) of Theorem A then the codimension of this face is greater than one. We can improve this statement:

Theorem C *Let us fix a λ_i . Let $([\Lambda_w], [\Lambda_{\hat{w}}])$ be a pair of Schubert varieties in $G/P(\lambda_i)$ such that $\iota^*([\Lambda_{\hat{w}}]).[\Lambda_w] = d \cdot [\text{pt}] \in H^*(G/P(\lambda), \mathbb{Z})$ for some*

positive integer d . We assume that there exists a pair of dominant weights $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$ such that $\langle w\lambda_i, \nu \rangle + \langle \hat{w}\lambda_i, \hat{\nu} \rangle = 0$ and such that ν is strictly dominant.

Then, $d = 1$ and $\langle w\lambda, \rho \rangle + \langle \hat{w}\hat{\lambda}, \hat{\rho} \rangle = \langle \lambda, \rho \rangle + \langle \lambda, 2\hat{\rho}^\lambda - \hat{\rho} \rangle$.

In particular, the set of $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$ such that $\langle w\lambda_i, \nu \rangle + \langle \hat{w}\lambda_i, \hat{\nu} \rangle = 0$ is a face of codimension one.

Theorems A and B can be thought as a description of the faces of codimension one of $\mathcal{LR}(G, \hat{G})$ which intersect the interior of the dominant chamber. In Theorem 9 below, we study the smaller faces of $\mathcal{LR}(G, \hat{G})$. To avoid some notation in the introduction, we only state our results about these faces in the case when G is diagonally embedded in G^s for an integer $s \geq 2$; that is, in the case of decomposition of the tensor product (see Section 8.3 for a more general result).

If α is a simple root of G , ω_{α^\vee} denote the corresponding fundamental weight. If I is a set of simple roots, $P(I)$ denote the standard parabolic subgroup associated to I and W_I its Weyl group. In [BK06], Belkale and Kumar defined a new product \odot_0 on the cohomology groups $H^*(G/P, \mathbb{Z})$.

Theorem D *We assume that G is semisimple diagonally embedded in $\hat{G} = G^s$ for an integer $s \geq 2$.*

- (i) *Let I be a set of d simple roots and $(w_1, \dots, w_{s+1}) \in (W/W_I)^{s+1}$ such that $[\Lambda_{w_1}] \odot_0 \dots \odot_0 [\Lambda_{w_{s+1}}] = [\Lambda_e] \in H^*(G/P(I), \mathbb{Z})$. Then, the set of $(\nu_1, \dots, \nu_{s+1}) \in \mathcal{LR}(G, G^s)$ such that*

$$\forall \alpha \in I \quad \sum_i \langle \omega_{\alpha^\vee}, w_i^{-1} \nu_i \rangle = 0,$$

is a face of codimension d of $\mathcal{LR}(G, G^s)$.

- (ii) *Any face of $\mathcal{LR}(G, G^s)$ intersecting the interior of the dominant chamber of G^{s+1} is obtained in this way.*

If there is a lot of literature on the faces of codimension one, I do not know any other result about smaller faces even in the case of Horn's conjecture that is for $\mathrm{SL}_n \subset \mathrm{SL}_n \times \mathrm{SL}_n$. Moreover, Theorem D gives an application of Belkale-Kumar's product \odot_0 for any G/P whereas in [BK06] only the case when P is maximal is used.

Let us explain the starting point of the proofs of Theorems A to D. Consider the variety $X = G/B \times G/\hat{B}$ endowed with the diagonal action of G . To

any character $(\nu, \hat{\nu})$ of $T \times \hat{T}$, one associates a G -linearized line bundle $\mathcal{L}_{(\nu, \hat{\nu})}$ on X such that $H^0(X, \mathcal{L}_{(\nu, \hat{\nu})}) = V_{\hat{\nu}} \otimes V_{\nu}$. Hence, $(\nu, \hat{\nu})$ belongs to $\mathcal{LR}(G, \hat{G})$ if and only if a positive power of $\mathcal{L}_{(\nu, \hat{\nu})}$ admit a non zero G -invariant section. If $\mathcal{L}_{(\nu, \hat{\nu})}$ is ample this is equivalent to saying that X admit semistable points for $\mathcal{L}_{(\nu, \hat{\nu})}$. So, one can use classical results of Geometric Invariant Theory as Hilbert-Mumford's Theorem and Luna's Slice Etale Theorem. This method was already used in [Kly98, BS00, BK06].

We obtain results in the following more general context. Consider a connected reductive group G acting on a normal projective variety X . To any G -linearized line bundle \mathcal{L} on X we associate the following open subset $X^{\text{ss}}(\mathcal{L})$ of X :

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \sigma \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\}.$$

The points of $X^{\text{ss}}(\mathcal{L})$ are said to be *semistable* for \mathcal{L} . Note that if \mathcal{L} is not ample, this notion of semistability is not the standard one. In particular, the quotient $\pi_{\mathcal{L}} : X^{\text{ss}}(\mathcal{L}) \rightarrow X^{\text{ss}}(\mathcal{L})//G$ is a good quotient, if \mathcal{L} is ample; but not in general. In this context, we ask for:

What are the \mathcal{L} 's with non empty set $X^{\text{ss}}(\mathcal{L})$?

Let us fix a freely finitely generated subgroup Λ of the group $\text{Pic}^G(X)$ of G -linearized line bundles on X . Let $\Lambda_{\mathbb{Q}}$ denote the \mathbb{Q} -vector space containing Λ as a lattice. Consider the convex cones $\mathcal{TC}_{\Lambda}^G(X)$ (resp. $\mathcal{AC}_{\Lambda}^G(X)$) generated in $\Lambda_{\mathbb{Q}}$ by the \mathcal{L} 's (resp. the ample \mathcal{L} 's) in Λ which have non zero G -invariant sections. By [DH98] (see also [Res00]), $\mathcal{AC}_{\Lambda}^G(X)$ is a closed convex rational polyhedral cone in the dominant cone of $\Lambda_{\mathbb{Q}}$. We are interested in the faces of $\mathcal{AC}_{\Lambda}^G(X)$ and $\mathcal{TC}_{\Lambda}^G(X)$.

We are now going to define the notion of well covering pair which will play a central role in this work. Let λ be a one parameter subgroup of G and C be an irreducible component of its fix points. Consider $C^+ = \{x \in X \mid \lim_{t \rightarrow 0} \lambda(t)x \in C\}$ and the natural G -equivariant map $\eta : G \times_{P(\lambda)} C^+ \rightarrow X$. The pair (C, λ) is said to be *well covering* if η induces an isomorphism over an open subset of X intersecting C . The pair is said to be *dominant* if η is dominant.

For $\mathcal{L} \in \text{Pic}^G(X)$, we denote by $\mu^{\mathcal{L}}(C, \lambda)$ the integer giving the action of λ on the restriction of \mathcal{L} on C . We now state a first description of $\mathcal{AC}_{\Lambda}^G(X)$:

Theorem E *An ample G -linearized line bundle $\mathcal{L} \in \Lambda$ belongs to $\mathcal{AC}_\Lambda^G(X)$ if and only if for all well covering pair (C, λ) with a dominant one parameter subgroup λ of T we have $\mu^{\mathcal{L}}(C, \lambda) \leq 0$.*

Let (C, λ) be a well covering pair. Theorem E shows that the set of $\mathcal{L} \in \mathcal{AC}_\Lambda^G(X)$ such that $\mu^{\mathcal{L}}(C, \lambda) = 0$ is a face of $\mathcal{AC}_\Lambda^G(X)$. We obtained several relations between the faces of $\mathcal{AC}_\Lambda^G(X)$ and pairs (C, λ) . Firstly, for \mathcal{L} in the boundary of $\mathcal{AC}_\Lambda^G(X)$, we describes the quotient variety $X^{\text{ss}}(\mathcal{L})//G$ in terms of covering pair:

Theorem F *We assume that $X = G/B \times Y$, for a normal projective G -variety Y . Let (C, λ) be a dominant pair and \mathcal{L} be an ample G -linearized line bundle on X such that $\mu^{\mathcal{L}}(C, \lambda) = 0$. Consider the action of the centralizer G^λ of λ in G on C and the associated set of semistable points $C^{\text{ss}}(\mathcal{L}, G^\lambda)$. We assume that $X^{\text{ss}}(\mathcal{L})$ is not empty.*

Then,

- (i) $X^{\text{ss}}(\mathcal{L})//G$ is isomorphic to $C^{\text{ss}}(\mathcal{L}, G^\lambda)//G^\lambda$; and,
- (ii) (C, λ) is well covering.

Actually, Theorem F is a key point in the proof of Theorem C.

We already noticed that to any dominant pair, one can associate a face (eventually empty) of $\mathcal{AC}_\Lambda^G(X)$. Conversely, we have:

Theorem G *Let \mathcal{F} be a face of $\mathcal{AC}_\Lambda^G(X)$.*

Then, there exists a well covering pair (C, λ) such that \mathcal{F} is the set of the \mathcal{L} 's in $\mathcal{AC}_\Lambda^G(X)$ such that $\mu^{\mathcal{L}}(C, \lambda) = 0$.

We now assume that $\Lambda \otimes \mathbb{Q} = \text{Pic}^G(X) \otimes \mathbb{Q}$. Notice that if $\text{Pic}(X)$ is not finitely generated it can be replaced by the Neron-Severi group (see [DH98]). We denote $\mathcal{TC}_\Lambda^G(X)$ by $\mathcal{TC}^G(X)$. We are interested in a kind of converse of Theorem G. So, we fix a well covering pair (C, λ) .

Theorem H *With above notation, consider the linear map ρ induced by the restriction:*

$$\rho : \text{Pic}^G(X) \otimes \mathbb{Q} \longrightarrow \text{Pic}^{G^\lambda}(C) \otimes \mathbb{Q}.$$

Then, the subspace of $\text{Pic}^G(X) \otimes \mathbb{Q}$ spanned by the $\mathcal{L} \in \mathcal{TC}^G(X)$ such that $\mu^{\mathcal{L}}(C, \lambda) = 0$ is the pullback by ρ of the intersection of the image of ρ and $\mathcal{TC}^{G^\lambda}(C)$.

By induction, Theorem H will allow us to prove Theorem B. Indeed, it will be used to prove that the set of $\mathcal{L} \in \mathcal{LR}(G, \hat{G})$ which realize a given equality in Theorem A is of codimension one.

In this article, we are mainly interested in the cones $\mathcal{LR}(G, \hat{G})$ which are examples of cones $\mathcal{SAC}_\Lambda^G(X)$. Other interesting examples will be studied elsewhere. For example, if Y is any G -variety endowed with an G -linearized line bundle \mathcal{L} the moment polytope $P(Y, \mathcal{L})$ is an affine section of a cone $\mathcal{SAC}_\Lambda^G(X)$. These polytopes are studied from symplectic point of view (see [Ver96]) or from an algebro-geometric point of view in [Bri99, Man97].

In Section 2, we fix notation about parabolic fiber products and prove a useful result on their linearized Picard group. Section 3 is concerned by the Hilbert-Mumford's numerical criterion of semistability. In Section 4, we recall some useful results about the Bialinicki-Birula's cells. Section 5 recalls a useful consequence of Luna's Slice Etale Theorem. In Section 6, we introduce the notion of well covering pair and prove their first relations with the G -cones. In Section 7, our general results about the faces of the G -cones are obtained. In Section 8, we apply our results to the cone $\mathcal{LR}(G, \hat{G})$ essentially by making more explicit the notion of well covering pair.

Convention. The ground field \mathbb{K} is assumed to be algebraic closed of characteristic zero. The notation introduced in the environments “**Notation.**” are fixed for all the sequence of the article.

2 Preliminaries on parabolic fiber products

Notation. Let \mathbb{K}^* denote the multiplicative group of \mathbb{K} . If G is an affine algebraic group, $X(G)$ denotes the group of characters of G ; that is, of algebraic group homomorphisms from G on \mathbb{K}^* . If G acts algebraically on a variety X , X is said to be a G -variety. As in [MFK94], we denote by $\text{Pic}^G(X)$ the group of G -linearized line bundles on X . If $\mathcal{L} \in \text{Pic}^G(X)$, $H^0(X, \mathcal{L})$ denotes the G -module of regular sections of \mathcal{L} .

In this section we collect some useful properties of the fiber product. Let us fix a reductive group G and a parabolic subgroup P of G .

2.1 Construction

Let Y be a P -variety. Consider over $G \times Y$ the action of $G \times P$ given by the formula (with obvious notation):

$$(g, p) \cdot (g', y) = (gg'p^{-1}, py).$$

Since the quotient map $G \rightarrow G/P$ is a Zariski-locally trivial principal P -bundle; one can easily construct a quotient $G \times_P Y$ of $G \times Y$ by the action of $\{e\} \times P$. The action of $G \times \{e\}$ induces an action of G on $G \times_P Y$. Moreover, the first projection $G \times Y \rightarrow G$ induces a G -equivariant map $G \times_P Y \rightarrow G/P$ which is a locally trivial fibration with fiber Y .

The class of a pair $(g, y) \in G \times Y$ in $G \times_P Y$ is denoted by $[g : y]$. If Y is a P -stable locally closed subvariety of a G -variety X , it is well known that the map

$$\begin{aligned} G \times_P Y &\longrightarrow G/P \times X \\ [g : y] &\longmapsto (gP, gy) \end{aligned}$$

is an immersion; its image is the set of the $(gP, x) \in G/P \times X$ such that $g^{-1}x \in Y$.

Let ν be a character of P . If Y is the field \mathbb{K} endowed with the action of P defined by $p \cdot \tau = \nu(p^{-1})\tau$ for all $\tau \in \mathbb{K}$ and $p \in P$, $G \times_P Y$ is a G -linearized line bundle on G/P . We denote by \mathcal{L}_ν this element of $\text{Pic}^G(G/P)$. In fact, the map $X(P) \rightarrow \text{Pic}^G(G/P)$, $\nu \mapsto \mathcal{L}_\nu$ is an isomorphism.

Let B be a Borel subgroup of G contained in P , and T be a maximal torus contained in B . Then, $X(P)$ identifies with a subgroup of $X(T)$ which contains dominant weights. For $\nu \in X(P)$, \mathcal{L}_ν is generated by its sections if and only if it has non zero sections if and only if ν is dominant. Moreover, $H^0(G/P, \mathcal{L}_\nu)$ is the dual of the simple G -module of highest weight ν . For ν dominant, \mathcal{L}_ν is ample if and only if ν cannot be extended to a subgroup of G bigger than P .

2.2 Line bundles

We are now interested in the G -linearized line bundles on $G \times_P Y$.

Lemma 1 *With above notation, we have:*

(i) *The map $\mathcal{L} \mapsto G \times_P \mathcal{L}$ defines a morphism*

$$e : \text{Pic}^P(Y) \longrightarrow \text{Pic}^G(G \times_P Y).$$

- (ii) The map $\iota : Y \longrightarrow G \times_P Y$, $y \longmapsto [e : y]$ is a P -equivariant immersion. We denote by $\iota^* : \text{Pic}^G(G \times_P Y) \longrightarrow \text{Pic}^P(Y)$ the associated restriction homomorphism.
- (iii) The morphisms e and ι^* are the inverse one of each other; in particular, they are isomorphisms.
- (iv) For any $\mathcal{L} \in \text{Pic}^G(G \times_P Y)$, the restriction map from $H^0(G \times_P Y, \mathcal{L})$ to $H^0(Y, \iota^*(\mathcal{L}))$ induces a linear isomorphism

$$H^0(G \times_P Y, \mathcal{L})^G \simeq H^0(Y, \iota^*(\mathcal{L}))^P.$$

Proof. Let \mathcal{M} be a P -linearized line bundle on Y . Since the natural map $G \times \mathcal{M} \longrightarrow G \times_P \mathcal{M}$ is a categorical quotient, we have the following commutative diagram:

$$\begin{array}{ccc} G \times \mathcal{M} & \longrightarrow & G \times_P \mathcal{M} \\ \downarrow & & \downarrow p \\ G \times Y & \longrightarrow & G \times_P Y. \end{array}$$

Since $G \longrightarrow G/P$ is locally trivial, the map p endows $G \times_P \mathcal{M}$ with a structure of line bundle on $G \times_P Y$. Moreover, the action of G on $G \times_P \mathcal{M}$ endows this line bundle with a G -linearization. This proves Assertion (i). The second one is obvious.

By construction, the restriction of $G \times_P \mathcal{M}$ to Y is \mathcal{M} . So, $\iota^* \circ e$ is the identity map. Conversely, let $\mathcal{L} \in \text{Pic}^G(G \times_P Y)$. Then, we have:

$$e \circ \iota^*(\mathcal{L}) \simeq \{(gP, l) \in G/P \times \mathcal{L} : g^{-1}l \in \mathcal{L}|_Y\}.$$

The second projection induces an isomorphism from $e \circ \iota^*(\mathcal{L})$ onto \mathcal{L} . This ends the proof of Assertion (iii).

The map $H^0(G \times_P Y, \mathcal{L})^G \longrightarrow H^0(Y, \iota^*(\mathcal{L}))^P$ is clearly well defined and injective. Let us prove the surjectivity. Let $\sigma \in H^0(Y, \iota^*(\mathcal{L}))^P$. Consider the morphism

$$\begin{aligned} \hat{\sigma} : G \times Y &\longrightarrow G \times_P \mathcal{L} \\ (g, y) &\longmapsto [g : \sigma(y)]. \end{aligned}$$

Since σ is P -invariant, so is $\hat{\sigma}$; and $\hat{\sigma}$ induces a section of $G \times_P \mathcal{L}$ over $G \times_P Y$ which is G -invariant and extends σ . \square

3 Numerical criterion of Hilbert-Mumford

Notation. If x is a point of a G -variety, we will denote by G_x its isotropy subgroup and by $G.x$ its orbit.

We will use classical results in Geometric Invariant Theory (GIT) about the numerical criterion of Hilbert-Mumford. In this section, we present these results and give some useful complements. Let us fix a connected reductive group G acting on an irreducible projective algebraic variety X .

3.1 Convex geometry

In this subsection, we recall some useful facts about polytopes. Let E be an Euclidean space whose the scalar product is denoted by $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A *polytope* in E is the convex hull of finitely many points. Let \mathcal{P} be a polytope which does not contain 0. The real $\inf_{x \in \mathcal{P}} \|x\|$ is called the *distance from 0 to \mathcal{P}* and denoted by $d(0, \mathcal{P})$. The orthogonal projection theorem asserts that there exists a unique $x_0 \in \mathcal{P}$ such that $d(0, \mathcal{P}) = \|x_0\|$. Moreover,

$$d(0, \mathcal{P}) = \sup_{y \in E - \{0\}} \inf_{x \in \mathcal{P}} \frac{\langle y, x \rangle}{\|y\|},$$

and $x'_0 = \frac{x_0}{\|x_0\|}$ is the unique vector with norm 1 such that

$$d(0, \mathcal{P}) = \inf_{x \in \mathcal{P}} \langle x'_0, x \rangle.$$

In particular, the set of $x \in \mathcal{P}$ such that $\langle x'_0, x \rangle = d(0, \mathcal{P})$ is a face of \mathcal{P} ; we call it *the face viewed from 0* and denote it by $\mathcal{F}(0)$.

We will use the following easy fact:

Lemma 2 *Let r be a positive integer. We consider the map which associate to any element $\underline{x} = (x_1, \dots, x_r)$ of E^r their convex hull $\text{Conv}(x_1, \dots, x_r) = \mathcal{P}(\underline{x})$.*

Then, the set U of $\underline{x} \in E^r$ such that $0 \notin \mathcal{P}(\underline{x})$ is open in E^r . To any such \underline{x} , we associate the set $I(\underline{x})$ of the i 's such that x_i belongs to the face of $\mathcal{P}(\underline{x})$ viewed from 0. Then, there exists an open neighborhood U of \underline{x} such that for any $\underline{x}' \in U$ we have $I(\underline{x}') \subset I(\underline{x})$.

Proof. The only point is that the application which maps \underline{x} to the orthogonal projection of 0 on $\mathcal{P}(\underline{x})$ is continuous. \square

3.2 An Ad Hoc notion of semistability

As in the introduction, for any G -linearized line bundle \mathcal{L} on X , we consider the following set of *semistable points*:

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \sigma \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\}.$$

To precise the acting group, we sometimes denote $X^{\text{ss}}(\mathcal{L})$ by $X^{\text{ss}}(\mathcal{L}, G)$.

The subset $X^{\text{ss}}(\mathcal{L})$ is open and stable by G . A point x which is not semistable is said to be *unstable*; and, we set $X^{\text{us}}(\mathcal{L}) = X - X^{\text{ss}}(\mathcal{L})$.

Remark. Note that this definition of $X^{\text{ss}}(\mathcal{L})$ is NOT standard. Indeed, one usually imposes that the open subset defined by the non vanishing of σ to be affine. This property which is useful to construct a good quotient is automatic if \mathcal{L} is ample but not in general; hence, our definition coincides with the usual one if \mathcal{L} is ample.

If \mathcal{L} is ample, there exists a categorical quotient:

$$\pi : X^{\text{ss}}(\mathcal{L}) \longrightarrow X^{\text{ss}}(\mathcal{L})//G,$$

such that $X^{\text{ss}}(\mathcal{L})//G$ is a projective variety and π is affine. A point $x \in X^{\text{ss}}(\mathcal{L})$ is said to be *stable* if G_x is finite and $G.x$ is closed in $X^{\text{ss}}(\mathcal{L})$. Then, for all stable point x we have $\pi^{-1}(\pi(x)) = G.x$; and the set $X^{\text{s}}(\mathcal{L})$ of stable points is open in X .

The following lemma is easy and well known. It will be very useful here.

Lemma 3 *Let \mathcal{L} be an G -linearized line bundle on X and $x \in X$ be a point semistable for \mathcal{L} .*

Then, the restriction of \mathcal{L} to $G.x$ is of finite order.

Proof. Let us recall that for any $\mathcal{L} \in \text{Pic}^G(G.x)$, the action of G_x on the fiber over x in \mathcal{L} determines a character $\mu^{\mathcal{L}}(x, G_x)$ of G_x . Moreover, the map $\mathcal{L} \mapsto \mu^{\mathcal{L}}(x, G_x)$ is an injective homomorphism.

Now, let \mathcal{L} be a G -linearized line bundle on X such that the character $\mu^{\mathcal{L}}(x, G_x)$ is of infinite order. It remains to prove that x is unstable for \mathcal{L} . Let σ be a G -invariant section of $\mathcal{L}^{\otimes n}$ for some $n > 0$. Then $\sigma(x)$ is a G_x fix point of the fiber in $\mathcal{L}^{\otimes n}$ over x . Since, $n \cdot \mu^{\mathcal{L}}(x, G_x)$ is non trivial, $\sigma(x)$ must be zero. So, x is unstable. \square

3.3 The functions $\mu^\bullet(x, \lambda)$

Let $\mathcal{L} \in \text{Pic}^G(X)$. Let x be a point in X and λ be a one parameter subgroup of G . Since X is complete, $\lim_{t \rightarrow 0} \lambda(t)x$ exists; let z denote this limit. The image of λ fixes z and so the group \mathbb{K}^* acts via λ on the fiber \mathcal{L}_z . This action defines a character of \mathbb{K}^* , that is, an element of \mathbb{Z} denoted by $\mu^\mathcal{L}(x, \lambda)$. One can immediately prove that the numbers $\mu^\mathcal{L}(x, \lambda)$ satisfy the following properties:

- (i) $\mu^\mathcal{L}(g \cdot x, g \cdot \lambda \cdot g^{-1}) = \mu^\mathcal{L}(x, \lambda)$ for any $g \in G$;
- (ii) the map $\mathcal{L} \mapsto \mu^\mathcal{L}(x, \lambda)$ is a homomorphism from $\text{Pic}^G(X)$ to \mathbb{Z} ;
- (iii) for any G -variety Y and for any G -equivariant morphism $f : X \rightarrow Y$, $\mu^{f^*(\mathcal{L})}(x, \lambda) = \mu^\mathcal{L}(f(x), \lambda)$, where $x \in X$ and \mathcal{L} is a G -linearized line bundle on Y .

A less direct property of the function $\mu^\mathcal{L}(x, \lambda)$ is

Proposition 1 *Let \mathcal{L} , x , λ and z be as above. Let \tilde{x} be a non zero point in the fiber in \mathcal{L} over x . We embed X in \mathcal{L} by the zero section. Then, we have*

- (i) if $\mu^\mathcal{L}(x, \lambda) > 0$, then $\lambda(t)\tilde{x}$ tends to z when $t \rightarrow 0$;
- (ii) if $\mu^\mathcal{L}(x, \lambda) = 0$, then $\lambda(t)\tilde{x}$ tends to a non zero point \tilde{z} in the fiber in \mathcal{L} over z when $t \rightarrow 0$;
- (iii) if $\mu^\mathcal{L}(x, \lambda) < 0$, then $\lambda(t)\tilde{x}$ has no limit in \mathcal{L} when $t \rightarrow 0$.

Proof. Set $V = \{\lambda(t).x \mid t \in \mathbb{K}^*\} \cup \{z\}$: it is a locally closed subvariety of X stable by the action of \mathbb{K}^* via λ . Moreover, z is the unique closed orbit of \mathbb{K}^* in V . So, [Res00, Lemma 7] implies that $\text{Pic}^{\mathbb{K}^*}(V)$ is isomorphic to $X(\mathbb{K}^*)$; and finally that for all $\mathcal{L} \in \text{Pic}^G(X)$ the restriction \mathcal{L} to V is the trivial line bundle endowed with the action of \mathbb{K}^* given by $\mu^\mathcal{L}(x, \lambda)$. The proposition follows immediately. \square

The numbers $\mu^\mathcal{L}(x, \lambda)$ are used in [MFK94] to give a numerical criterion for stability with respect to an ample G -linearized line bundle \mathcal{L} :

$$\begin{aligned} x \in X^{\text{ss}}(\mathcal{L}) &\iff \mu^\mathcal{L}(x, \lambda) \leq 0 \text{ for all one parameter subgroup } \lambda, \\ x \in X^{\text{s}}(\mathcal{L}) &\iff \mu^\mathcal{L}(x, \lambda) < 0 \text{ for all non trivial } \lambda. \end{aligned}$$

A line bundle \mathcal{L} over X is said to be *semiample* if a positive power of \mathcal{L} is base point free. With our notion of semistability, Hilbert-Mumford's theorem admits the following direct generalization:

Lemma 4 *Let \mathcal{L} be a G -linearized line bundle over X and x a point in X . Then,*

- (i) *if x is semistable for \mathcal{L} , $\mu^{\mathcal{L}}(x, \lambda) \leq 0$ for any one parameter subgroup λ of G ;*
- (ii) *for a one parameter subgroup λ of G , if x is semistable for \mathcal{L} and $\mu^{\mathcal{L}}(x, \lambda) = 0$, then $\lim_{t \rightarrow 0} \lambda(t)x$ is semistable for \mathcal{L} ;*
- (iii) *if in addition \mathcal{L} is semiample, x is semistable for \mathcal{L} if and only if $\mu^{\mathcal{L}}(x, \lambda) \leq 0$ for any one parameter subgroup λ of G .*

Proof. Assume that x is semistable for \mathcal{L} and consider a G -invariant section σ of $\mathcal{L}^{\otimes n}$ which does not vanish at x . Since $\lambda(t)\sigma(x) = \sigma(\lambda(t)x)$ tends to $\sigma(z)$ when $t \rightarrow 0$, Proposition 1 shows that $\mu^{\mathcal{L}}(x, \lambda) \leq 0$. If in addition $\mu^{\mathcal{L}}(x, \lambda) = 0$, Proposition 1 shows that $\sigma(z)$ is non zero; and so that z is semistable for \mathcal{L} . This proves the two first assertions.

Assume now that \mathcal{L} is semiample. Let n be a positive integer such that $\mathcal{L}^{\otimes n}$ is base point free. Let V denote the dual of the space of global sections of $\mathcal{L}^{\otimes n}$: V is a finite dimensional G -module. Moreover, the usual map $\phi : X \rightarrow \mathbb{P}(V)$ is G -equivariant. Let Y denote the image of ϕ and \mathcal{M} denote the restriction of $\mathcal{O}(1)$ to Y .

Then $\mathcal{L}^{\otimes n}$ is the pullback of \mathcal{M} by ϕ . Since X is projective, ϕ induces isomorphisms from $H^0(Y, \mathcal{M}^{\otimes k})$ onto $H^0(X, \mathcal{L}^{\otimes nk})$ (for all k). So, $X^{\text{ss}}(\mathcal{L}) = \phi^{-1}(Y^{\text{ss}}(\mathcal{M}))$. We deduce the last assertion of the lemma by applying Hilbert-Mumford's criterion to Y and \mathcal{M} and Property (iii) of the functions $\mu^{\bullet}(x, \lambda)$. \square

Remark.

- (i) If \mathcal{L} is ample, Assertion (ii) of Lemma 4 is Lemma 3 in [Res00].
- (ii) The proof of Assertion (iii) shows that a lot of properties of semistability for an ample line bundle are also available for semiample line bundles (see Propositions 2 and 3, and Theorems 1 and 2 below).

3.4 Definition of the functions $M^{\bullet}(x)$

Notation. Let Γ be any affine algebraic group. The neutral component of Γ is denoted by Γ° . Let $Y(\Gamma)$ denote the set of one parameter subgroups of Γ ; that is, of group homomorphisms from \mathbb{K}^* in Γ . Note that if Γ° is a torus, $Y(\Gamma)$ is a group.

If Λ is an abelian group, we denote by $\Lambda_{\mathbb{Q}}$ (resp. $\Lambda_{\mathbb{R}}$) the tensor product of Λ with \mathbb{Q} (resp. \mathbb{R}) over \mathbb{Z} .

Let T be a maximal torus of G . The Weyl group W of T acts linearly on $Y(T)_{\mathbb{R}}$. Since W is finite, there exists a W -invariant Euclidean norm (defined over \mathbb{Q}) $\|\cdot\|$ on $Y(T)_{\mathbb{R}}$. On the other hand, if $\lambda \in Y(G)$ there exists $g \in G$ such that $g \cdot \lambda \cdot g^{-1} \in Y(T)$. Moreover, if two elements of $Y(T)$ are conjugate by an element of G , then they are by an element of the normalizer of T (see [MFK94, Lemma 2.8]). This allows to define the norm of λ by $\|\lambda\| = \|g \cdot \lambda \cdot g^{-1}\|$.

Let $\mathcal{L} \in \text{Pic}^G(X)$. One can now introduce the following notation:

$$\bar{\mu}^{\mathcal{L}}(x, \lambda) = \frac{\mu^{\mathcal{L}}(x, \lambda)}{\|\lambda\|}, \quad M^{\mathcal{L}}(x) = \sup_{\lambda \in Y(G)} \bar{\mu}^{\mathcal{L}}(x, \lambda).$$

In fact, we will see in Corollary 1 that $M^{\mathcal{L}}(x)$ is finite.

3.5 $M^{\bullet}(x)$ for a torus action

Notation. If Y is a variety, and Z is a part of Y , the closure of Z in Y will be denote by \bar{Z} . If Γ is a group acting on Y , Y^{Γ} denote the set of fix point of Γ in Y .

If V is a finite dimensional vector space, and v is a non zero vector in V , $[v]$ denote the class of v in the projective space $\mathbb{P}(V)$. If V is a Γ -module, and χ is a character of Γ , we denote by V_{χ} the set of $v \in V$ such that $g.v = \chi(g)v$ for all $g \in \Gamma$.

In this subsection we assume that $G = T$ is a torus. Let z be a point of X fixed by T . The action of T on the fiber \mathcal{L}_z over the point z in the T -linearized line bundle \mathcal{L} define a character $\mu^{\mathcal{L}}(z, T)$ of T ; we obtain a morphism

$$\mu^{\bullet}(z, T) : \text{Pic}^T(X) \longrightarrow X(T).$$

For any point x in X , we denote by $\mathcal{P}_T^{\mathcal{L}}(x)$ the convex hull in $X(T)_{\mathbb{R}}$ of the characters $-\mu^{\mathcal{L}}(z, T)$ for $z \in \overline{T.x}^T$.

The following proposition is an adaptation of a result of L. Ness and gives a pleasant interpretation of the number $M^{\mathcal{L}}(x)$:

Proposition 2 *Let \mathcal{L} be a semiample T -linearized line bundle on X . With the above notation, we have:*

- (i) The point x is unstable if and only if 0 does not belong to $\mathcal{P}_T^{\mathcal{L}}(x)$. In this case, $M^{\mathcal{L}}(x)$ is the distance from 0 to $\mathcal{P}_T^{\mathcal{L}}(x)$.
- (ii) If x is semistable, the opposite of $M^{\mathcal{L}}(x)$ is the distance from 0 to the boundary of $\mathcal{P}_T^{\mathcal{L}}(x)$.
- (iii) There exists $\lambda \in Y(T)$ such that $\bar{\mu}^{\mathcal{L}}(x, \lambda) = M^{\mathcal{L}}(x)$. If moreover λ is indivisible, we call it an adapted one parameter subgroup for x .
- (iv) If x is unstable, there exists a unique adapted one parameter subgroup for x .

Proof. Since \mathcal{L} is semiample, there exist a positive integer n , a T -module V , and a T -equivariant morphism $\phi : X \rightarrow \mathbb{P}(V)$ such that $\mathcal{L}^{\otimes n} = \phi^*(\mathcal{O}(1))$. Since $\mu^\bullet(z, T)$ is a morphism, we have: $\mathcal{P}_T^{\mathcal{L}^{\otimes n}}(x) = n\mathcal{P}_T^{\mathcal{L}}(x)$ for all x . Moreover, $\bar{\mu}^{\mathcal{L}^{\otimes n}}(x, \lambda) = n\bar{\mu}^{\mathcal{L}}(x, \lambda)$, for all x and λ ; so, $M^{\mathcal{L}^{\otimes n}}(x) = nM^{\mathcal{L}}(x)$. As a consequence, it is sufficient to prove the proposition for $\mathcal{L}^{\otimes n}$; in other words, we may assume that $n = 1$.

Let us recall that:

$$V = \bigoplus_{\chi \in X(T)} V_\chi.$$

Let $x \in X$ and $v \in V$ such that $[v] = \phi(x)$. There exist unique vectors $v_\chi \in V_\chi$ such that $v = \sum_\chi v_\chi$. Let \mathcal{Q} be the convex hull in $X(T)_\mathbb{R}$ of the χ 's such that $v_\chi \neq 0$. It is well known (see [Oda88]) that the fixed point of T in $\overline{T \cdot [v]}$ are exactly the $[v_\chi]$'s with χ vertex of \mathcal{Q} . Moreover, T acts by the character $-\chi$ on the fiber $\mathcal{O}(1)_{[v_\chi]}$ over $[v_\chi]$ in $\mathcal{O}(1)$. One deduces that $\mathcal{Q} = \mathcal{P}_T^{\mathcal{L}}(x)$. So, it is sufficient to prove the proposition with \mathcal{Q} in place of $\mathcal{P}_T^{\mathcal{L}}(x)$ and $\phi(x) = x$ (because of our non-standard definition of semistability): this is a statement of [Nes78]. \square

3.6 Properties of $M^\bullet(x)$

Notation. We will denote by $\text{Pic}^G(X)^+$ (resp. $\text{Pic}^G(X)^{++}$) the set of semiample (resp. ample) G -linearized line bundles on X .

An indivisible one parameter subgroup λ of G is said to be *adapted for x and \mathcal{L}* if and only if $\bar{\mu}^{\mathcal{L}}(x, \lambda) = M^{\mathcal{L}}(x)$. Denote by $\Lambda^{\mathcal{L}}(x)$ the set of adapted one parameter subgroups for x .

Using the fact that any one parameter subgroup is conjugated to one in a given torus, Proposition 2 implies easily

Corollary 1 (i) *The numbers $M^{\mathcal{L}}(x)$ are finite (even if \mathcal{L} is not semiample, see Proposition 1.1.6 in [DH98]).*

(ii) *If \mathcal{L} is semiample, $\Lambda^{\mathcal{L}}(x)$ is not empty.*

Now, we can reformulate the numerical criterion for stability: if \mathcal{L} is semiample, we have

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : M^{\mathcal{L}}(x) \leq 0\}, \quad X^s(\mathcal{L}) = \{x \in X : M^{\mathcal{L}}(x) < 0\}.$$

The following proposition is a result of finiteness for the set of functions $M^{\bullet}(x)$. It will be used to understand how $X^{\text{ss}}(\mathcal{L})$ depends on \mathcal{L} (see Lemma 6 below).

Proposition 3 *When x varies in X , one obtains only a finite number of functions $M^{\bullet}(x) : \text{Pic}^G(X)^+ \rightarrow \mathbb{R}$.*

Proof. Let T be a maximal torus of G . Consider the partial forgetful map $r^T : \text{Pic}^G(X) \rightarrow \text{Pic}^T(X)$. Since $M^{\bullet}(x) = \max_{g \in G} M^{r^T(\bullet)}(g.x)$, it is sufficient to prove the proposition for the torus T .

If z and z' belong to the same irreducible component C of X^T , the morphisms $\mu^{\bullet}(z, T)$ and $\mu^{\bullet}(z', T)$ are equal: we denote by $\mu^{\bullet}(C, T)$ this morphism.

By Proposition 2, $M^{\mathcal{L}}(x)$ only depends on $\mathcal{P}_T^{\mathcal{L}}(x)$, which only depends on the set of irreducible components of X^T which intersects $\overline{T.x}$. Since, X^T has finitely many irreducible components, the proposition follows. \square

Remark. Proposition 3 implies that the open subsets of X which can be realized as $X^{\text{ss}}(\mathcal{L})$ for some semiample G -linearized line bundle \mathcal{L} on X is finite. This is a result of Dolgachev and Hu (see Theorem 3.9 in [DH98]; see also [Sch03]).

3.7 Adapted one parameter subgroups

To describe $\Lambda^{\mathcal{L}}(x)$, we need some additional notation. To the one parameter subgroup λ of G , we associate the parabolic subgroup (see [MFK94]):

$$P(\lambda) = \left\{ g \in G : \lim_{t \rightarrow 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

The unipotent radical of $P(\lambda)$ is

$$U(\lambda) = \left\{ g \in G : \lim_{t \rightarrow 0} \lambda(t).g.\lambda(t)^{-1} = e \right\}.$$

Moreover, the centralizer G^λ of the image of λ in G is a Levi subgroup of $P(\lambda)$. For $p \in P(\lambda)$, we set $\bar{p} = \lim_{t \rightarrow 0} \lambda(t) \cdot p \cdot \lambda(t)^{-1}$. Then, we have the following short exact sequence:

$$1 \longrightarrow U(\lambda) \longrightarrow P(\lambda) \xrightarrow{p \mapsto \bar{p}} G^\lambda \longrightarrow 1.$$

For $g \in P(\lambda)$, we have $\mu^\mathcal{L}(x, \lambda) = \mu^\mathcal{L}(x, g \cdot \lambda \cdot g^{-1})$. The following theorem due to G. Kempf is a generalization of the last assertion of Proposition 2.

Theorem 1 (see [Kem78]) *Let x be an unstable point for a semiample G -linearized line bundle \mathcal{L} . Then:*

- (i) *All the $P(\lambda)$ for $\lambda \in \Lambda^\mathcal{L}(x)$ are equal. We denote by $P^\mathcal{L}(x)$ this subgroup.*
- (ii) *Any two elements of $\Lambda^\mathcal{L}(x)$ are conjugate by an element of $P^\mathcal{L}(x)$.*

We will also use the following theorem of L. Ness.

Theorem 2 (Theorem 9.3 in [Nes84]) *Let x and \mathcal{L} be as in the above theorem. Let λ be an adapted one parameter subgroup for x and \mathcal{L} . We consider $y = \lim_{t \rightarrow 0} \lambda(t) \cdot x$. Then, $\lambda \in \Lambda^\mathcal{L}(y)$ and $M^\mathcal{L}(x) = M^\mathcal{L}(y)$.*

3.8 Stratification of X induced from \mathcal{L}

Let \mathcal{L} be an ample G -linearized line bundle on X . If $d > 0$ and $\langle \tau \rangle$ is a conjugacy class of one parameter subgroups of G , we set:

$$S_{d, \langle \tau \rangle}^\mathcal{L} = \{x \in X : M^\mathcal{L}(x) = d \text{ and } \Lambda^\mathcal{L}(x) \cap \langle \tau \rangle \neq \emptyset\}.$$

If \mathcal{T} is the set of conjugacy classes of one parameter subgroups, the previous section gives us the following partition of X :

$$X = X^{\text{ss}}(\mathcal{L}) \cup \bigcup_{d > 0, \langle \tau \rangle \in \mathcal{T}} S_{d, \langle \tau \rangle}^\mathcal{L}. \quad (2)$$

W. Hesselink showed in [Hes79] that this union is a finite stratification by G -stable locally closed subvarieties of X . We will call it the *stratification induced from \mathcal{L}* .

To describe the geometry of these stratum, we need additional notation. For $\lambda \in \langle \tau \rangle$, we set:

$$S_{d, \lambda}^\mathcal{L} := \{x \in S_{d, \langle \tau \rangle}^\mathcal{L} : \lambda \in \Lambda^\mathcal{L}(x)\},$$

and

$$Z_{d,\lambda}^{\mathcal{L}} := \{x \in S_{d,\lambda}^{\mathcal{L}} : \lambda(\mathbb{K}^*) \text{ fixes } x\}.$$

By Theorem 2, we have the map

$$p_\lambda : S_{d,\lambda}^{\mathcal{L}} \longrightarrow Z_{d,\lambda}^{\mathcal{L}}, \quad x \longmapsto \lim_{t \rightarrow 0} \lambda(t).x.$$

The proof of the following result can be found in [Kir84, Section 1.3].

Proposition 4 *With above notation, if d is positive, we have:*

- (i) $Z_{d,\lambda}^{\mathcal{L}}$ is open in X^λ and stable by G^λ ;
- (ii) $S_{d,\lambda}^{\mathcal{L}} = \{x \in X : \lim_{t \rightarrow 0} \lambda(t).x \in Z_{d,\lambda}^{\mathcal{L}}\}$ and is stable by $P(\lambda)$;
- (iii) there is a birational morphism $G \times_{P(\lambda)} S_{d,\lambda}^{\mathcal{L}} \longrightarrow S_{d,(\lambda)}^{\mathcal{L}}$, which is an isomorphism if $S_{d,(\lambda)}^{\mathcal{L}}$ is normal.

3.9 A description of $Z_{d,\lambda}^{\mathcal{L}}$

Let λ be a one parameter subgroup of G . Let Z denote the neutral component of the center of G^λ and G^{ss} be the maximal semisimple subgroup of G^λ . The product induces an isogeny $Z \times G^{\text{ss}} \longrightarrow G^\lambda$. Let T_1 be a maximal torus of G^{ss} . Set $T = Z.T_1$. Note that T is a maximal torus of G^λ and G . Let S be the subtorus of Z such that $Y(S.T_1)_{\mathbb{R}}$ is the hyperplane of $Y(T)_{\mathbb{R}}$ orthogonal to λ . Set $H^\lambda = S.G^{\text{ss}}$. The map $\mathbb{G}_m \times H^\lambda \longrightarrow G^\lambda, (t, h) \longmapsto \lambda(t)h$ is an isogeny.

Theorem 3 (Ness-Kirwan) *Let \mathcal{L} be an ample G -linearized line bundle on X . The one parameter subgroup λ is assumed to be indivisible. Let $x \in X^\lambda$ be such that $\mu^{\mathcal{L}}(x, \lambda) > 0$.*

Then, λ is adapted to x and \mathcal{L} if and only if x is semistable for \mathcal{L} and the action of H^λ .

Theorem 3 is a version of [Nes84, Theorem 9.4]. Whereas the Ness' proof works without changing, the statement in [Nes84] is not correct. In [Kir84, Remark 12.21], F. Kirwan made the above correction.

3.10 The open stratum

Let \mathcal{L} be an ample G -linearized line bundle on X . We denote by $X^\circ(\mathcal{L})$ the open stratum of Stratification 2. We have the following characterization of $X^\circ(\mathcal{L})$:

Lemma 5 *If $X^{\text{ss}}(\mathcal{L}) \neq \emptyset$, set $d_0 = 0$; else, set $d_0 = \min_{x \in X} M^\mathcal{L}(x)$. Then, $X^\circ(\mathcal{L})$ is the set of $x \in X$ such that $M^\mathcal{L}(x) \leq d_0$.*

Proof. If $d_0 = 0$, $M^\mathcal{L}(x) \leq 0$ if and only if $x \in X^{\text{ss}}(\mathcal{L}) = X^\circ(\mathcal{L})$. We now assume that $d_0 > 0$.

Up to changing \mathcal{L} by a positive power, one may assume that there exists a G -module V such that X is contained in $\mathbb{P}(V)$ and \mathcal{L} is the restriction of the G -linearized line bundle $\mathcal{O}(1)$.

Let us fix the positive real d and a one parameter subgroup λ such that $X^\circ(\mathcal{L}) = S_{d, \langle \lambda \rangle}^\mathcal{L}$. For $i \in \mathbb{Z}$, set $V_i = \{v \in V \mid \lambda(t)v = t^i v\}$. Set $V^+ = \bigoplus_{i > d \|\lambda\|} V_i$, $C = \mathbb{P}(V_{d \|\lambda\|}) \cap X$ and $\overline{C^+} = \mathbb{P}(V_{d \|\lambda\|} \oplus V^+) \cap X$.

We first prove that $d = d_0$; that is, that $M^\mathcal{L}(x) \geq d$ for all $x \in X$. Consider the morphism $\eta : G \times_{P(\lambda)} \overline{C^+} \rightarrow X$. Since $G/P(\lambda)$ is projective, η is proper; but, the image of η contains $S_{d, \langle \lambda \rangle}^\mathcal{L}$; so, η is surjective. Let $x \in X$. There exists $g \in G$ such that $gx \in \overline{C^+}$. Then, $M^\mathcal{L}(x) \geq \overline{\mu}^\mathcal{L}(gx, \lambda) \geq d$.

Conversely, let $x \in X$ be such that $M^\mathcal{L}(x) = d_0$. Let $g \in G$ such that $gx \in \overline{C^+}$. Let $v_1 \in V_{d \|\lambda\|}$ and $v_2 \in V^+$ such that $gx = [v_1 + v_2]$. Since $\overline{\mu}^\mathcal{L}(gx, \lambda) \leq d_0$, v_1 is non zero; so, $\overline{\mu}^\mathcal{L}(gx, \lambda) = d_0 = M^\mathcal{L}(x)$. In particular, λ is adapted to x and \mathcal{L} ; that is, $x \in X^\circ(\mathcal{L})$. \square

We will need the following result of monotonicity for the function $\mathcal{L} \mapsto X^\circ(\mathcal{L})$:

Lemma 6 *With above notation, there exists an open neighborhood U of \mathcal{L} in $\Lambda_{\mathbb{Q}}^{++}$ such that for all $\mathcal{L}' \in U$, $X^\circ(\mathcal{L}') \subset X^\circ(\mathcal{L})$.*

Proof. By Proposition 3 and Lemma 5, there exists only finitely many open subsets of X which are of the form $X^\circ(\mathcal{M})$ for some ample \mathcal{M} . Let $X_1^\circ, \dots, X_s^\circ$ those which are not contained in $X^\circ(\mathcal{L})$. For each i , fix $x_i \in X_i^\circ - X^\circ(\mathcal{L})$. It remains to prove that for each i , there exists U_i such that $x_i \notin X^\circ(\mathcal{L}')$ for all $\mathcal{L}' \in U_i$. Indeed, $U = \bigcap_i U_i$ will work.

Set d_0 be as in Lemma 5. Since $x_i \in X^\circ(\mathcal{L}')$, $M^\mathcal{L}(x) > d_0$. Let e be such that $M^\mathcal{L}(x) > e > d_0$. By Proposition 3, there exists an open neighborhood U' of \mathcal{L} such that $\min_{x \in X} M^{\mathcal{L}'}(x) > e$ for all $\mathcal{L}' \in U'$. Moreover, there exists

U'' , such that $M^{\mathcal{L}'}(x_i) < e$ for all $\mathcal{L}' \in U'$. Lemma 5 implies that for all $\mathcal{L}' \in U' \cap U''$, $x_i \notin X^\circ(\mathcal{L}')$. \square

4 Bialynicki-Birula cells

4.1 Bialynicki-Birula's theorem

Notation. Let X be a G -variety, H be a subgroup of G and λ be a one parameter subgroup of G . The centralizer of H in G (that is, the set of fix point of H acting on G by conjugacy) will be denoted by G^H . The set of fix points of the image of λ will be denoted by X^λ ; the centralizer of this image will be denoted by G^λ .

If Y is a locally closed subvariety of X , and \mathcal{L} is a line bundle on X , $\mathcal{L}|_Y$ will denote the restriction of \mathcal{L} to Y .

Let X be a complete G -variety. Let λ be a one parameter subgroup of G . Let C be an irreducible component of X^λ . Since G^λ is connected, C is a G^λ -stable closed subvariety of X . We set:

$$C^+ := \{x \in X : \lim_{t \rightarrow 0} \lambda(t)x \in C\}.$$

Then, C^+ is a locally closed subvariety of X stable by $P(\lambda)$. Moreover, the map $p_\lambda : C^+ \rightarrow C$, $x \mapsto \lim_{t \rightarrow 0} \lambda(t)x$ is a morphism satisfying:

$$\forall (l, u) \in G^\lambda \times U(\lambda) \quad p_\lambda(lu.x) = lp_\lambda(x).$$

Let $x \in X^\lambda$. We consider the natural action of \mathbb{K}^* induced by λ on the Zariski tangent space $T_x X$ of X at x . We consider the following \mathbb{K}^* -submodules of $T_x X$:

$$\begin{aligned} T_x X_{>0} &= \{\xi \in T_x X : \lim_{t \rightarrow 0} \lambda(t)\xi = 0\}, \\ T_x X_{<0} &= \{\xi \in T_x X : \lim_{t \rightarrow 0} \lambda(t^{-1})\xi = 0\}, \\ T_x X_0 &= (T_x X)^\lambda, \quad T_x X_{\geq 0} = T_x X_{>0} \oplus T_x X_0 \quad \text{and} \quad T_x X_{\leq 0} = T_x X_{<0} \oplus T_x X_0. \end{aligned}$$

A classical result of Bialynicki-Birula (see [BB73]) is

Theorem 4 *Assuming in addition that X is smooth, we have:*

- (i) C is smooth and for all $x \in C$ we have $T_x C = T_x X_0$;
- (ii) C^+ is smooth and irreducible and for all $x \in C$ we have $T_x C^+ = T_x X_{\geq 0}$;
- (iii) the morphism $p_\lambda : C^+ \rightarrow C$ induces a structure of vector bundle on C with fibers isomorphic to $T_x X_{>0}$.

4.2 Line bundles on C^+

We will need some results about the line bundles on C^+ . Let \mathcal{L} be a $P(\lambda)$ -linearized line bundle on C^+ . Since $\lim_{t \rightarrow 0} \lambda(t).x$ belongs to C , the number $\mu^{\mathcal{L}}(x, \lambda)$ is well defined, for $x \in C^+$. Moreover, since C is irreducible, $\mu^{\mathcal{L}}(x, \lambda)$ does not depend on $x \in C^+$; we denote by $\mu^{\mathcal{L}}(C, \lambda)$ this integer.

Proposition 5 *We assume that X is smooth. Then, we have:*

(i) *The restriction map $\text{Pic}^{P(\lambda)}(C^+) \longrightarrow \text{Pic}^{G^\lambda}(C)$ is an isomorphism.*

Let $\mathcal{L} \in \text{Pic}^{P(\lambda)}(C^+)$.

(ii) *If $\mu^{\mathcal{L}}(C, \lambda) \neq 0$, $H^0(C, \mathcal{L}|_C)^\lambda = \{0\}$.*

(iii) *If $\mu^{\mathcal{L}}(C, \lambda) = 0$, the restriction map induces an isomorphism from $H^0(C^+, \mathcal{L})^{P(\lambda)}$ onto $H^0(C, \mathcal{L}|_C)^{G^\lambda}$. Moreover, for any $\sigma \in H^0(C^+, \mathcal{L})^{P(\lambda)}$, we have:*

$$\{x \in C^+ : \sigma(x) = 0\} = p_\lambda^{-1}(\{x \in C : \sigma(x) = 0\}).$$

Proof. Since p_λ is $P(\lambda)$ -equivariant, for any $\mathcal{M} \in \text{Pic}^{G^\lambda}(C)$, $p_\lambda^*(\mathcal{M})$ is $P(\lambda)$ -linearized. Since p_λ is a vector bundle, $p_\lambda^*(\mathcal{L}|_C)$ and \mathcal{L} are isomorphic as line bundles without linearization. But, $X(P(\lambda)) \simeq X(G^\lambda)$, so the $P(\lambda)$ -linearizations must coincide; and $p_\lambda^*(\mathcal{L}|_C)$ and \mathcal{L} are isomorphic as $P(\lambda)$ -linearized line bundles. Assertion (i) follows.

Assertion (ii) is a direct application of Lemma 3.

Let us fix $\mathcal{L} \in \text{Pic}^{P(\lambda)}(C^+)$ and denote by $p : \mathcal{L} \longrightarrow C^+$ the projection. We assume that $\mu^{\mathcal{L}}(C, \lambda) = 0$. Let $\sigma \in H^0(C^+, \mathcal{L})^{P(\lambda)}$. We just proved that

$$\mathcal{L} \simeq p_\lambda^*(\mathcal{L}|_C) = \{(x, l) \in C^+ \times \mathcal{L}|_C : p_\lambda(x) = p(l)\}.$$

Let p_2 denote the projection of $p_\lambda^*(\mathcal{L}|_C)$ onto $\mathcal{L}|_C$.

For all $x \in C^+$ and $t \in \mathbb{K}^*$, we have:

$$\begin{aligned} \sigma(\lambda(t).x) &= \left(\lambda(t).x, p_2(\sigma(\lambda(t).x)) \right) \\ &= \lambda(t). \left(x, p_2(\sigma(x)) \right) && \text{since } \sigma \text{ is invariant,} \\ &= \left(\lambda(t).x, p_2(\sigma(x)) \right) && \text{since } \mu^{\mathcal{L}}(C, \lambda) = 0. \end{aligned}$$

We deduce that for all $x \in C^+$, $\sigma(x) = (x, \sigma(p_\lambda(x)))$. Assertion (iii) follows.

□

5 A theorem by Luna

Notation. If H is a closed subgroup of an affine algebraic group G , $N_G(H)$ denote the normalizer of H in G .

We will use the following interpretation of a result of Luna:

Proposition 6 *Let \mathcal{L} be an ample G -linearized line bundle on an irreducible projective G -variety X . Let H be a reductive subgroup of G . Let C be an irreducible component of X^H . Then, the reductive groups $(G^H)^\circ$ and $N_G(H)^\circ$ act on C .*

Let x be a point in C . Then, the following are equivalent:

- (i) x is semistable for \mathcal{L} .*
- (ii) x is semistable for the action of $(G^H)^\circ$ on C and the restriction of \mathcal{L} .*
- (iii) x is semistable for the action of $N_G(H)^\circ$ on C and the restriction of \mathcal{L} .*

Proof. [LR79, Lemma 1.1] shows that $(G^H)^\circ$ and $N_G(H)^\circ$ are reductive. Changing \mathcal{L} by a positive power if necessary, one may assume that X is contained in $\mathbb{P}(V)$ where V is a G -module and $\mathcal{L} = \mathcal{O}(1)|_X$. Let $v \in V$ such that $[v] = x$. Let us recall that in this case $x \in X^{\text{us}}(\mathcal{L})$ if and only if $\overline{G.v}$ contains 0.

Let χ be the character of H such that $hv = \chi(h)v$ for all $h \in H$.

If χ is of infinite order, so is its restriction to the connected center Z of H . Then, $Z.v = \mathbb{K}^*v$ and $0 \in \overline{(G^H)^\circ.v}$. In this case, x belongs to no semistable set of the proposition.

Let us now assume that χ is of finite order. Changing \mathcal{L} by a positive power if necessary, one may assume that χ is trivial, that is H fixes v . In this case, [Lun75, Corollary 2 and Remark 1] shows that

$$0 \in \overline{G.v} \iff 0 \in \overline{N_G(H)^\circ.v} \iff 0 \in \overline{(G^H)^\circ.v}.$$

The proposition follows. □

6 First descriptions of the G -cones

6.1 Definitions

Let us recall from the introduction that Λ is a freely finitely generated subgroup of $\text{Pic}^G(X)$ and $\Lambda_{\mathbb{Q}}$ is the \mathbb{Q} -vector space containing Λ as a lattice.

Since $X^{\text{ss}}(\mathcal{L}) = X^{\text{ss}}(\mathcal{L}^{\otimes n})$, for any G -linearized line bundle and any positive integer n , we can define $X^{\text{ss}}(\mathcal{L})$ for any $\mathcal{L} \in \Lambda_{\mathbb{Q}}$. The central object of this article is the following *total G -cone*:

$$\mathcal{TC}_{\Lambda}^G(X) = \{\mathcal{L} \in \Lambda_{\mathbb{Q}} : X^{\text{ss}}(\mathcal{L}) \text{ is not empty}\}.$$

Since the tensor product of two non zero G -invariant sections is a non zero G -invariant section, $\mathcal{TC}_{\Lambda}^G(X)$ is a convex cone.

Consider the convex cones $\Lambda_{\mathbb{Q}}^+$ and $\Lambda_{\mathbb{Q}}^{++}$ generated respectively by the semiample and ample elements of Λ . For all $\mathcal{L} \in \Lambda_{\mathbb{Q}}^+$ (resp. $\Lambda_{\mathbb{Q}}^{++}$), there exists a positive integer n such that $\mathcal{L}^{\otimes n}$ is a semiample (resp. ample) G -linearized line bundle on X in Λ . So, any set of semistable points associated to a point in $\Lambda_{\mathbb{Q}}^+$ (resp. $\Lambda_{\mathbb{Q}}^{++}$) is in fact a set of semistable point associated to a semiample (resp. ample) G -linearized line bundle. We consider the following *semiample and ample G -cones*:

$$\mathcal{SAC}_{\Lambda}^G(X) = \mathcal{TC}_{\Lambda}^G(X) \cap \Lambda_{\mathbb{Q}}^+ \quad \text{and} \quad \mathcal{AC}_{\Lambda}^G(X) = \mathcal{TC}_{\Lambda}^G(X) \cap \Lambda_{\mathbb{Q}}^{++}.$$

By [DH98] (see also [Res00]), $\mathcal{AC}_{\Lambda}^G(X)$ is a closed convex rational polyhedral cone in $\Lambda_{\mathbb{Q}}^{++}$.

6.2 Well covering pairs

Here comes a central definition in this work:

Definition. Let λ be a one parameter subgroup of G and C be an irreducible component of X^{λ} . Set $C^+ := \{x \in X \mid \lim_{t \rightarrow 0} \lambda(t)x \in C\}$. Consider the following G -equivariant map

$$\begin{array}{ccc} \eta : G \times_{P(\lambda)} C^+ & \longrightarrow & X \\ [g : x] & \longmapsto & g.x. \end{array}$$

The pair (C, λ) is said to be *covering* (resp. *dominant*) if η is birational (resp. dominant). It is said to be *well covering* if η induces an isomorphism from $G \times_{P(\lambda)} \Omega$ onto an open subset of X for an open subset Ω of C^+ intersecting C .

Let us recall that $\mu^{\bullet}(C, \lambda)$ denote the common value of the $\mu^{\bullet}(x, \lambda)$, for $x \in C^+$. The first relation between covering pairs and the G -cones is the following

Lemma 7 *Let (C, λ) be a dominant pair and $\mathcal{L} \in \mathcal{TC}_{\Lambda}^G(X)$. Then, $\mu^{\mathcal{L}}(C, \lambda) \leq 0$.*

Proof. Since $X^{\text{ss}}(\mathcal{L})$ is a G -stable non empty open subset of X and (C, λ) is covering, there exists a point $x \in C^+$ semistable for \mathcal{L} . Since $x \in C^+$, $\mu^{\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(C, \lambda)$. But, since $x \in X^{\text{ss}}(\mathcal{L})$ Lemma 4 shows that $\mu^{\mathcal{L}}(x, \lambda) \leq 0$. \square

Proposition 4 allows us to give a first description of the cone $\mathcal{AC}_{\Lambda}^G(X)$:

Proposition 7 *We assume that X is normal. Let T be a maximal torus of G and B be a Borel subgroup containing T .*

Then, the cone $\mathcal{AC}_{\Lambda}^G(X)$ is the set of the $\mathcal{L} \in \Lambda_{\mathbb{Q}}^{++}$ such that for all well covering pair (C, λ) with a dominant one parameter subgroup λ of T we have $\mu^{\mathcal{L}}(C, \lambda) \leq 0$.

Proof. Lemma 7 shows that $\mathcal{AC}^G(X)$ is contained in the part of $\Lambda_{\mathbb{Q}}^{++}$ defined by the inequalities $\mu^{\mathcal{L}}(C, \lambda) \leq 0$ of the proposition.

Conversely, let $\mathcal{L} \in \Lambda_{\mathbb{Q}}^{++}$ such that $X^{\text{ss}}(\mathcal{L})$ is empty. Consider the open stratum $S_{d, \langle \tau \rangle}^{\mathcal{L}}$ in X . Let λ be a dominant one parameter subgroup of T in the class $\langle \tau \rangle$. Since $S_{d, \langle \tau \rangle}^{\mathcal{L}}$ is open in X , it is normal and irreducible. Now, Proposition 4 implies that $Z_{d, \lambda}^{\mathcal{L}}$ is open in an irreducible component C of X^{λ} . It shows also that $S_{d, \lambda}^{\mathcal{L}}$ is open in C^+ and intersects C . Finally, the last assertion of Proposition 4 shows that (C, λ) is well covering. Moreover, $\mu^{\mathcal{L}}(C, \lambda) = d > 0$. \square

7 Faces of the G -ample cone and well covering pairs

Definition. Let φ be a linear form on $\Lambda_{\mathbb{Q}}$ which is non negative on $\mathcal{AC}_{\Lambda}^G(X)$. The set of $\mathcal{L} \in \mathcal{AC}_{\Lambda}^G(X)$ such that $\varphi(\mathcal{L}) = 0$ is called a *face* of $\mathcal{AC}_{\Lambda}^G(X)$. Note that a face can be empty or equal to $\mathcal{AC}_{\Lambda}^G(X)$. A face different from $\mathcal{AC}_{\Lambda}^G(X)$ is said to be strict.

Lemma 7 implies that for any dominant pair (C, λ) , by intersecting $\mathcal{AC}_{\Lambda}^G(X)$ with the hyperplane with equation $\mu^{\bullet}(C, \lambda) = 0$, one obtain a face of $\mathcal{AC}_{\Lambda}^G(X)$ (eventually empty): this face is said to be *associated to* (C, λ) . The proof of Proposition 7 and Lemma 6 imply easily (see the proof of Theorem 6 below) that for any codimension one face \mathcal{F} of $\mathcal{AC}_{\Lambda}^G(X)$, there exist a well covering pair (C, λ) whose associated face is \mathcal{F} . The aim of this

section is to prove that the relations between faces and well covering pairs are much more deeper.

7.1 Dominant pairs and quotient varieties

The following theorem is a description of the quotient variety associated to a point in $\mathcal{AC}_\Lambda^G(X)$ which belongs to a face associated to a dominant pair.

Theorem 5 *Let (C, λ) be a dominant pair and \mathcal{L} be an ample G -linearized line bundle on X such that $\mu^\mathcal{L}(C, \lambda) = 0$. Consider the action of G^λ on C and the associated set of semistable points $C^{\text{ss}}(\mathcal{L}, G^\lambda)$. Then, $C^{\text{ss}}(\mathcal{L}, G^\lambda) = X^{\text{ss}}(\mathcal{L}, G) \cap G$. Consider the morphism θ which makes the following diagram commutative:*

$$\begin{array}{ccc} C^{\text{ss}}(\mathcal{L}|_C, G^\lambda) & \hookrightarrow & X^{\text{ss}}(\mathcal{L}) \\ \pi_C \downarrow & & \downarrow \pi \\ C^{\text{ss}}(\mathcal{L}|_C, G^\lambda) // G^\lambda & \xrightarrow{\theta} & X^{\text{ss}}(\mathcal{L}) // G. \end{array}$$

Then, θ is finite and surjective.

Proof. The first assertion is a direct consequence of Proposition 6. Since $C^{\text{ss}}(\mathcal{L}, G^\lambda) // G^\lambda$ is projective, to prove the second one, it is sufficient to prove that θ is dominant and its fibers are finite.

Since (C, λ) is dominant, C^+ must intersect $X^{\text{ss}}(\mathcal{L})$. Let $x \in C^+ \cap X^{\text{ss}}(\mathcal{L})$. Set $z = \lim_{t \rightarrow 0} \lambda(t)x$. By Assertion (ii) of Lemma 4, z is semistable for \mathcal{L} . So, C intersects $X^{\text{ss}}(\mathcal{L})$. Moreover, we just proved that $\pi(C^+ \cap X^{\text{ss}}(\mathcal{L})) = \pi(C^{\text{ss}}(\mathcal{L}, G^\lambda))$. It follows that θ is dominant.

Let $\xi \in X^{\text{ss}}(\mathcal{L}) // G$. Let \mathcal{O}_ξ be the unique closed G -orbit in $\pi^{-1}(\xi)$. The points in the fiber $\theta^{-1}(\xi)$ correspond bijectively to the closed G^λ -orbits in $\pi^{-1}(\xi) \cap C$. But, [Lun75, Corollary 2 and Remark 1] implies that these orbits are contained in \mathcal{O}_ξ . We conclude that $\theta^{-1}(\xi)$ is finite, by using a result of [Ric82] which implies that \mathcal{O}_ξ^λ contains only finitely many G^λ -orbits. \square

7.2 From faces to well covering pairs

Notation. If λ is a one parameter subgroup of G , $\text{Im}\lambda$ will denote its image. If \mathcal{F} is a convex cone in a vector space, $\langle \mathcal{F} \rangle$ will denote the subspace spanned by \mathcal{F} . The dimension of $\langle \mathcal{F} \rangle$ will be called the dimension of \mathcal{F} .

Let \mathcal{L} be an ample line bundle in Λ without semistable point. Let d be the positive real number and λ be a one parameter subgroup of G such that $X^\circ(\mathcal{L}) = S_{d, \langle \lambda \rangle}^{\mathcal{L}}$.

Lemma 8 *With above notation, $Z_{d, \lambda}^{\mathcal{L}}$ (see Section 3.8) is irreducible; and the closure C of $Z_{d, \lambda}^{\mathcal{L}}$ is an irreducible component of X^λ .*

The pair (C, λ) is covering. It is well covering, if, in addition, X is normal. Moreover, the conjugacy class of the pair (C, λ) only depends on \mathcal{L} .

Proof. Since $S_{d, \langle \lambda \rangle}^{\mathcal{L}}$ is open in X , it is irreducible. But, $P(\lambda)$ is connected, so the last assertion of Proposition 4 implies that $S_{d, \lambda}^{\mathcal{L}}$ is irreducible. It follows that $Z_{d, \lambda}^{\mathcal{L}}$ is irreducible. Since $Z_{d, \lambda}^{\mathcal{L}}$ is open in X^λ its closure is an irreducible component of X^λ .

The fact that (C, λ) is covering (resp. well covering, if X is normal) is a direct consequence of Proposition 4.

The last assertion is obvious since λ is unique up to conjugacy. \square

Lemma 8 implies that the linear form $\mu^\bullet(C, \lambda)$ on $\Lambda_{\mathbb{Q}}$ only depends on \mathcal{L} . We set:

$$\mathcal{H}(\mathcal{L}) = \{\mathcal{L} \in \Lambda_{\mathbb{Q}} : \mu^{\mathcal{L}}(C, \lambda) = 0\} \quad \text{and} \quad \mathcal{H}(\mathcal{L})^{>0} = \{\mathcal{L} \in \Lambda_{\mathbb{Q}} : \mu^{\mathcal{L}}(C, \lambda) > 0\}.$$

By Proposition 4, the pair (C, λ) is dominant. So, Lemma 7 implies that $\mathcal{H} \cap \mathcal{AC}_{\Lambda}^G(X)$ is a face of $\mathcal{AC}_{\Lambda}^G(X)$: this face is denoted by $\mathcal{F}(\mathcal{L})$ and called the *face viewed from \mathcal{L}* .

The aim of this section is to prove

Theorem 6 *Any strict face \mathcal{F} of $\mathcal{AC}_{\Lambda}^G(X)$ is viewed from some ample point in Λ with no semistable point.*

We start by proving Lemmas 9, 10 and 11 about the function $\mathcal{L} \mapsto \mathcal{F}(\mathcal{L})$. Consider the subgroup H^λ of G^λ defined in Section 3.9. Consider the morphism of restriction $p : \Lambda \longrightarrow \text{Pic}^{H^\lambda}(C)$.

Lemma 9 *With above notation, $\mathcal{F}(\mathcal{L})$ equals the intersection $\Lambda_{\mathbb{Q}}^{++} \cap \mathcal{H}(\lambda) \cap p^{-1}(\mathcal{AC}^{H^\lambda}(C))$.*

Proof. Let \mathcal{M} belong to the intersection of the lemma. There exists $x \in C$ which is semistable for the action of H^λ and \mathcal{L} . Since $\mathcal{M} \in \mathcal{H}(\mathcal{L})$, λ acts trivially on $\mathcal{M}|_C$; so x is semistable for G^λ . Proposition 6 shows that x is semistable for G and \mathcal{M} . So, $\mathcal{M} \in \mathcal{AC}_{\Lambda}^G(X)$.

Conversely, let $\mathcal{M} \in \mathcal{F}(\mathcal{L})$. Since (C, λ) is covering, $X^{\text{ss}}(\mathcal{M})$ intersects C^+ . But $\mu^{\mathcal{M}}(C, \lambda) = 0$; so, Assertion (ii) of Lemma 4 shows that C intersects $X^{\text{ss}}(\mathcal{L})$. So, $p(\mathcal{M}) \in \mathcal{AC}_{p(\Lambda)}^{H^\lambda}(C)$. \square

Lemma 10 *Let $\mathcal{L}' \in \mathcal{H}(\lambda)^{>0}$. We assume that $p(\mathcal{L}') \in \mathcal{AC}^{H^\lambda}(C)$. Then, the face of $\mathcal{AC}_\Lambda^G(X)$ viewed from \mathcal{L}' equals those viewed from \mathcal{L} . More precisely, the pair (C, λ) satisfies Lemma 8 for \mathcal{L}' .*

Proof. Let $x \in C^+$ such that $\lim_{t \rightarrow 0} \lambda(t)x \in C^{\text{ss}}(p(\mathcal{L}'), H^\lambda)$. Note that the set of such x 's is open in C^+ . By Theorem 3, λ is adapted to x and \mathcal{L}' . Since (C, λ) is covering, $S_{\overline{\mu}^{\mathcal{L}'}(x, \lambda)}^{\mathcal{L}'(x, \lambda)} = X^\circ(\mathcal{L}')$. The lemma follows. \square

Lemma 11 *There exists an open neighborhood U of \mathcal{L} such that for any \mathcal{L}' in U , \mathcal{L}' has no semistable point and the face of $\mathcal{AC}_\Lambda^G(X)$ viewed from \mathcal{L}' is contained in those viewed from \mathcal{L} .*

If in addition $p(\mathcal{L}')$ has no semistable point, we have $\mathcal{F}(\mathcal{L}')$ is the set of $\mathcal{M} \in \mathcal{H}(\lambda)$ such that $p(\mathcal{M})$ belongs to the face of $\mathcal{AC}_{p(\Lambda)}^{H^\lambda}(C)$ viewed from $p(\mathcal{L}')$.

Proof. Let T be a maximal torus of G containing the image of λ . By Lemma 2, there exists an open neighborhood U of \mathcal{L} such that for any \mathcal{L}' in U we have:

For any $x \in C^+ \cap X^\circ(\mathcal{L})$, the face of $\mathcal{P}_T^{\mathcal{L}'}(x)$ viewed from 0 is contained in the hyperplane $\langle \lambda, \cdot \rangle = \mu^{\mathcal{L}'}(x, \lambda)$.

Let us fix $\mathcal{L}' \in U$ and $x \in C^+ \cap X^\circ(\mathcal{L})$ such that $z = \lim_{t \rightarrow 0} \lambda(t).x$ belongs to $C^\circ(\mathcal{L}', H^\lambda)$. By [Kir84, Lemma 12.19], there exists a one parameter subgroup adapted to z and \mathcal{L}' which commutes with λ . So, there exists $h_0 \in H^\lambda$ and a one parameter subgroup ζ of T which is adapted to h_0z and \mathcal{L}' . Since $h_0x \in C^+ \cap X^\circ(\mathcal{L})$, the orthogonal projection of 0 on $\mathcal{P}_T^{\mathcal{L}'}(h_0x)$ belongs to $\mathcal{P}_T^{\mathcal{L}'}(h_0z)$. In particular, $\lim_{t \rightarrow 0} \zeta(t)h_0x = \lim_{t \rightarrow 0} \zeta(t)h_0z =: z'$. Let H^ζ the subgroup of G^ζ defined in Section 3.9. Since ζ is adapted to h_0z and \mathcal{L}' , Theorem 3 implies that z' is semistable for the action H^ζ and \mathcal{L}' . So, by Theorem 3 again, ζ is adapted for h_0x and \mathcal{L}' .

One easily checks that $C^\circ(\mathcal{L}', H^\lambda) = C^\circ(\mathcal{L}', G^\lambda)$. Let d be such that $C^\circ(\mathcal{L}', G^\lambda) = S_{d', \langle \zeta \rangle}^{\mathcal{L}'}$. So, for any $z' \in C^\circ(\mathcal{L}', H^\lambda)$, there exists $h' \in H^\lambda$ such that ζ is adapted to $h'z'$ and \mathcal{L}' . So, we just proved that for any x in a nonempty open subset of C^+ , ζ is adapted to $h.x$ for some $h \in H$. Since $G.C^+$ contains an open subset of X , we can deduce that $X^\circ(\mathcal{L}') = S_{d', \langle \zeta \rangle}^{\mathcal{L}'}$.

Note that the equality $\lim_{t \rightarrow 0} \zeta(t)h_0x = \lim_{t \rightarrow 0} \zeta(t)h_0z$ implies that $Z_{d', \zeta}^{\mathcal{L}'}$ is contained in C . Let C_ζ denote the closure of $Z_{d', \zeta}^{\mathcal{L}'}$.

Since $\mathcal{P}_T^{\mathcal{L}'}(h_0z)$ is contained in the hyperplane $\langle \lambda, \cdot \rangle = \mu^{\mathcal{L}'}(z, \lambda) > 0$, there exists a one parameter subgroup λ_1 of $T \cap H^\lambda$ such that $\zeta = \overline{\lambda + \lambda_1}$. Since z is semistable for the action of H^λ and \mathcal{L} , 0 belongs to $\mathcal{P}_{T \cap H^\lambda}^{\mathcal{L}}(h_0z)$. We deduce that λ_1 tends to 0 when \mathcal{L}' tends to \mathcal{L} .

Since $C_\zeta \subset C$, there exists a positive constant c such that:

$$\mu^\bullet(C_\zeta, \zeta) = c \cdot \mu^\bullet(C_\zeta, \lambda_1 + \lambda) = c \cdot (\mu^\bullet(C_\zeta, \lambda_1) + \mu^\bullet(C, \lambda)).$$

In particular, $\mu^\bullet(C_\zeta, \zeta)$ tends to $\mu^\bullet(C, \lambda)$ when \mathcal{L}' tends to \mathcal{L} . Up to changing U , we may assume that $\mathcal{F}(\mathcal{L}') \subset \mathcal{F}(\mathcal{L})$ for any $\mathcal{L}' \in U$.

Now, Lemma 9 implies that for any $\mathcal{L}' \in U$, $\mathcal{F}(\mathcal{L}')$ is the set of $\mathcal{M} \in \Lambda^+$ such that $\mathcal{M} \in \mathcal{H}(\lambda)$ and $p(\mathcal{M}) \in \mathcal{F}(p(\mathcal{L}'))$. The lemma is proved. \square

We are now ready to prove the theorem.

Proof.[of Theorem 6] We will prove the following assertion, by induction on the codimension $\text{codim}_\Lambda(\mathcal{F})$ of \mathcal{F} in Λ :

Let X be a not necessarily normal variety. Let G , Λ and \mathcal{F} as in the theorem. Let U be an open subset of $\Lambda_\mathbb{Q}$ intersecting \mathcal{F} . Then, there exists $\mathcal{L} \in U$ such that $\mathcal{F}(\mathcal{L}) = \mathcal{F}$.

Let \mathcal{M} be a point in the relative interior of $\mathcal{F} \cap U$. By Lemma 6, there exists $\mathcal{L} \in U$ without semistable point and such that $X^\circ(\mathcal{L}) \subset X^{\text{ss}}(\mathcal{M})$. Let (C, λ) associated to \mathcal{L} as in Lemma 8. Since C intersects $X^\circ(\mathcal{L})$ it intersect $X^{\text{ss}}(\mathcal{M})$; so, $\mu^{\mathcal{M}}(C, \lambda) = 0$ and $\mathcal{M} \in \mathcal{F}(\mathcal{L})$. Since \mathcal{M} belongs to the relative interior of \mathcal{F} , we deduce that $\mathcal{F} \subset \mathcal{F}(\mathcal{L})$.

If $\text{codim}_\Lambda(\mathcal{F}) = 1$ then $\mathcal{F} = \mathcal{F}(\mathcal{L})$, and the theorem is proved. Let us now assume that $\text{codim}_\Lambda(\mathcal{F}) \geq 2$.

By Lemma 9, $\mathcal{F} \cap U$ is a face of $U \cap \mathcal{H}(\mathcal{L}) \cap p^{-1}(\mathcal{AC}_{p(\Lambda)}^{H^\lambda}(C))$. So, there exists a face $\tilde{\mathcal{F}}_1$ of $p^{-1}(\mathcal{AC}_{p(\Lambda)}^{H^\lambda}(C))$ such that $\mathcal{F} = \Lambda_\mathbb{Q}^{++} \cap \mathcal{H}(\mathcal{L}) \cap \tilde{\mathcal{F}}_1$. Let us assume that $\tilde{\mathcal{F}}_1$ is of maximal dimension among such faces. Since $\mathcal{H}(\mathcal{L})^{>0} \cap p^{-1}(\mathcal{AC}_{p(\Lambda)}^{H^\lambda}(C))$ is non empty (it contains \mathcal{L} !), we may assume that $\tilde{\mathcal{F}}_1$ intersects $\mathcal{H}(\mathcal{L})^{>0}$. Let \mathcal{F}_1 denote the unique face of $\mathcal{AC}_{p(\Lambda)}^{H^\lambda}(C)$ such that $\tilde{\mathcal{F}}_1 = p^{-1}(\mathcal{F}_1)$.

Since $\tilde{\mathcal{F}}_1$ intersects $\mathcal{H}(\mathcal{L})^{>0}$, Lemma 10 allows to move \mathcal{L} on $\tilde{\mathcal{F}}_1$ without changing $\mathcal{F}(\mathcal{L})$ nor (C, λ) . From now on, we assume that $\mathcal{L} \in \tilde{\mathcal{F}}_1$.

Up to restricting U if necessary, we may assume it satisfies Lemma 11 and it is contained in $\mathcal{H}(\mathcal{L})^{>0}$. Since $\mathcal{F} \subset \tilde{\mathcal{F}}_1 \cap \mathcal{H}(\mathcal{L})$ and $\tilde{\mathcal{F}}_1$ intersects $\mathcal{H}(\mathcal{L})^{>0}$, we have: $\text{codim}_\Lambda(\tilde{\mathcal{F}}_1) < \text{codim}_\Lambda \mathcal{F}$. But, $\text{codim}_\Lambda(\tilde{\mathcal{F}}_1) = \text{codim}_{p(\Lambda)}(\mathcal{F}_1)$. Moreover, p is linear and so open. We apply the induction to the action of H^λ on C , the face \mathcal{F}_1 and the open subset $p(U)$ of $p(\Lambda)_\mathbb{Q}$: there exists \mathcal{L}' in U such that $\mathcal{F}(p(\mathcal{L}')) = \mathcal{F}_1$. Lemma 11 implies now that $\mathcal{F}(\mathcal{L}') = \mathcal{F}$. \square

Theorem 6 can be restated in terms of well covering pairs:

Corollary 2 *Let \mathcal{F} be a face of $\mathcal{AC}_\Lambda^G(X)$ of codimension r in Λ . Then, there exists a r -dimensional torus S in G , a one parameter subgroup λ of S and an irreducible component C of X^S such that:*

- (i) C is an irreducible component of X^λ ;
- (ii) the pair (C, λ) is covering (respectively, well covering if X is normal);
- (iii) \mathcal{F} is the face of $\mathcal{AC}_\Lambda^G(X)$ associated to (C, λ) .

Proof. Let \mathcal{L} be an ample line bundle on X without semistable point and such that \mathcal{F} is viewed from \mathcal{L} . Let d be the positive number and λ be a one parameter subgroup of G such that $S_{d, \langle \lambda \rangle}^\mathcal{L}$ is open in X . Let C be the closure of $Z_{d, \lambda}^\mathcal{L}$.

By Lemma 8, it remains to prove that there exists a torus S of dimension r containing the image of λ and acting trivially on C . The proof of the existence of such a S can easily be integrated in the induction of the proof of Theorem 6. \square

7.3 From well covering pairs to faces of total G -cone

In Corollary 2, starting with a face of $\mathcal{AC}^G(X)$ we have constructed a well covering pair. Conversely, in the following theorem, we start with a well covering pair and study the associated face of $\mathcal{TC}^G(X)$.

Theorem 7 *Let X be a projective G -variety. We assume the rank of $\text{Pic}^G(X)$ is finite and consider $\mathcal{TC}^G(X)$. Let (C, λ) be a well covering pair. Consider the linear map ρ induced by the restriction:*

$$\rho : \text{Pic}^G(X)_\mathbb{Q} \longrightarrow \text{Pic}^{G^\lambda}(C)_\mathbb{Q}.$$

Then, the subspace of $\text{Pic}^G(X)_\mathbb{Q}$ spanned by the $\mathcal{L} \in \mathcal{TC}^G(X)$ such that $\mu^\mathcal{L}(C, \lambda) = 0$ is the pullback by ρ of the intersection of the image of ρ and $\mathcal{TC}^{G^\lambda}(C)$.

Proof. Let $\mathcal{L} \in \mathcal{TC}^G(X)$ such that $\mu^{\mathcal{L}}(C, \lambda) = 0$. Since $X^{\text{ss}}(\mathcal{L})$ is open and G -stable, and (C, λ) is covering, there exists $x \in C^+$ semistable for \mathcal{L} . Set $z = \lim_{t \rightarrow 0} \lambda(t)x$. Since $\mu^{\mathcal{L}}(x, \lambda) = 0$, Lemma 4 shows that z is semistable for \mathcal{L} . This implies that the restriction of \mathcal{L} belongs to $\mathcal{TC}^{G^\lambda}(C)$. This proves that the first subspace of the theorem is contained in the second one.

We denote by F the pullback by ρ of the subspace spanned by $\mathcal{TC}^{G^\lambda}(C)$. Let $\mathcal{L}_1, \dots, \mathcal{L}_n \in \text{Pic}^G(X)$ which span F and whose the restrictions to C belong to $\mathcal{TC}^{G^\lambda}(C)$. Denote by \mathcal{M}_i the restriction of \mathcal{L}_i to C . For each i , let us fix a non zero regular G^λ -invariant section σ_i of \mathcal{M}_i .

Consider the G -linearized line bundle $G \times_{P(\lambda)} p_\lambda^*(\mathcal{M}_i)$ on $G \times_{P(\lambda)} C^+$ with notation of Section 4. Since $\eta^*(\mathcal{L}_i)$ and $G \times_{P(\lambda)} p_\lambda^*(\mathcal{M}_i)$ have the same restriction to C , Lemma 1 and Proposition 5 show that $\eta^*(\mathcal{L}_i) = G \times_{P(\lambda)} p_\lambda^*(\mathcal{M}_i)$. Moreover, since $\mu^{\mathcal{M}_i}(C, \lambda) = 0$, Proposition 5 shows that σ_i admits a unique $P(\lambda)$ -invariant extension to a section σ'_i of $p_\lambda^*(\mathcal{M}_i)$. On the other hand, Lemma 1 shows that σ'_i admits a unique G -invariant extension $\tilde{\sigma}_i$ from C^+ to $G \times_{P(\lambda)} C^+$. So, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{L}_i & \longleftarrow & \eta^*(\mathcal{L}_i) = G \times_{P(\lambda)} p_\lambda^*(\mathcal{M}_i) & \longleftarrow & p_\lambda^*(\mathcal{M}_i) & \longrightarrow & \mathcal{M}_i \\
\downarrow & & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
X & \xleftarrow{\eta} & G \times_{P(\lambda)} C^+ & \xleftarrow{\tilde{\sigma}_i} & C^+ & \xrightarrow{p_\lambda} & C \\
& & & & \uparrow & & \uparrow \\
& & & & \sigma'_i & & \sigma_i
\end{array}$$

Since η is birational, $\tilde{\sigma}_i$ descends to a rational G -invariant section τ_i of \mathcal{L}_i . Let X° be a G -stable open subset of X such that η induces an isomorphism from $\eta^{-1}(X^\circ)$ onto X° . Since (C, λ) is well covering, we may (and shall) assume that X° intersects C . Let E_j be the irreducible components of codimension one of $X - X^\circ$. For any j we denote by a_j the maximum of 0 and the $-\nu_{E_j}(\tau_i)$'s with $i = 1, \dots, n$. Consider the line bundle $\mathcal{L}_0 = \mathcal{O}(\sum a_j E_j)$ on X . Since the E_j 's are stable by the action of G , \mathcal{L}_0 is canonically G -linearized. By construction, the τ_i 's induce G -invariant regular sections τ'_i of $\mathcal{L}'_i := \mathcal{L}_i \otimes \mathcal{L}_0$. Moreover, since no E_j contains C , the restriction of τ'_i to C is non zero. In particular, the \mathcal{L}'_i 's belong to $\mathcal{TC}^G(X)$ and their restrictions belong to $\mathcal{TC}^{G^\lambda}(C)$. Moreover, replacing \mathcal{L}_0 by $\mathcal{L}_0^{\otimes 2}$ if necessary, we may (and shall) assume that the \mathcal{L}'_i 's span F . This ends the proof of the theorem. \square

7.4 If $X = G/B \times Y \dots$

In this section, we will explain how Theorems 5 and 6 can be improved when $X = G/B \times Y$ for a G -variety Y .

Complement to Theorem 5.

Theorem 8 *We use notation of Theorem 5, assuming in addition that $X = G/B \times Y$ with a normal projective G -variety Y . We also assume that $X^{\text{ss}}(\mathcal{L})$ is not empty.*

Then, θ is an isomorphism and (C, λ) is a well covering pair.

Proof. Since X is normal, so is $X^{\text{ss}}(\mathcal{L})//G$. But, Theorem 5 shows that θ is finite; it is sufficient to prove that it is birational. Since the base field has characteristic zero, it is sufficient to prove that θ is bijective to obtain the first assertion.

Let $\xi \in X^{\text{ss}}(\mathcal{L})//G$ and \mathcal{O} denote the only closed G -orbit in $\pi^{-1}(\xi)$. As noticed during the proof of Theorem 5, $\theta^{-1}(\xi)$ correspond bijectively with the set of G^λ -orbits in $\mathcal{O} \cap C$.

Consider the first projection $p_1 : X \rightarrow G/B$ and fix $x \in \mathcal{O} \cap C$. Let B denote the stabilizer in G of $p_1(x)$. Since \mathcal{O} is closed in $\pi^{-1}(\xi)$, it is affine and so, G_x is reductive. But, G_x is contained in B , so G_x is diagonalisable. Let T be a maximal torus of G such that $G_x \subset T \subset B$. Consider the G -equivariant morphisms $q : \mathcal{O} \rightarrow G/T$, $x \mapsto T/T$ and $\hat{q} : G/T \rightarrow G/B$, $T/T \mapsto B/B$ unduced by these inclusions; we have, $p_1 = \hat{q} \circ q$.

We claim that $\hat{q}^{-1}(G^\lambda B/B)^\lambda = G^\lambda T/T$. Since G^λ is connected, each irreducible component of $\hat{q}^{-1}(G^\lambda B/B)^\lambda$ is G^λ -stable, and so, it maps onto $G^\lambda T/T$. In particular, it intersects $\hat{q}^{-1}(B/B) = B/T$. But, B/T is isomorphic to the Lie algebra of the unipotent radical of B as a T -variety; in particular, $(B/T)^\lambda$ is irreducible and so is $\hat{q}^{-1}(G^\lambda B/B)^\lambda$. The claim now follows from [Ric82].

Note that C is the product of one irreducible component of $(G/B)^\lambda$ and one of Y^λ ; this implies that $p_1(C) = G^\lambda B/B$. So, $\mathcal{O} \cap C$ is contained in $p_1^{-1}(G^\lambda B/B)^\lambda$ and so in $\mathcal{O} \cap q^{-1}(G^\lambda T/T)$. But, since T is contained in G^λ , $q^{-1}(G^\lambda T/T) = G^\lambda x$. It follows that $\mathcal{O} \cap C = G^\lambda x$; and so, that $\theta^{-1}(\xi)$ is reduced to one point. This ends the proof of the first assertion.

Consider $\eta : G \times_{P(\lambda)} C^+ \rightarrow X$. We claim that for any $x \in C^+ \cap X^{\text{ss}}(\mathcal{L})$, $\eta^{-1}(x)$ is only one point. Since η is dominant and the ground field has characteristic zero, the claim implies that η is birational. Since $X^{\text{ss}}(\mathcal{L}) \cap C$ is non empty, the claim implies that η is bijective over on open subset of

X intersecting C . Since X is normal, Zariski's main Theorem (see [GD66, §8.12]) implies that (C, λ) is well covering.

Let us prove the claim. Let $g \in G$ such that $g^{-1}x \in C^+$. We have to prove that $g \in P(\lambda)$. Set $x' = g^{-1}x$, $z = \lim_{t \rightarrow 0} \lambda(t)x$ and $z' = \lim_{t \rightarrow 0} \lambda(t)x'$. Obviously $x' \in X^{\text{ss}}(\mathcal{L})$, and by Assertion (ii) of Lemma 4, $z, z' \in X^{\text{ss}}(\mathcal{L})$ as well. It is also clear that $\pi(x) = \pi(x') = \pi(z) = \pi(z') =: \xi$.

Let $x_0 \in C \cap \pi^{-1}(\xi)$ whose orbit is closed in $X^{\text{ss}}(\mathcal{L})$. Set $H = G_{x_0}$. Consider Σ (resp. Σ_C) the set of y in X (resp. C) such that x_0 is contained in the closure of $H.y$ (resp. $H^\lambda.y$). By [Lun73] (see also [PV91]), $\pi^{-1}(\xi) \simeq G \times_H \Sigma$ and $\pi_C^{-1}(\theta^{-1}(\xi)) \simeq G^\lambda \times_{H^\lambda} \Sigma_C$. Consider the natural G -equivariant morphism $\gamma : G \times_H \Sigma \rightarrow G/H$. Since $\pi^{-1}(\xi) \cap C = \pi_C^{-1}(\theta^{-1}(\xi))$, it equals $G^\lambda \cdot \Sigma_C$. So, $\gamma(C \cap \pi^{-1}(\xi)) = G^\lambda H/H$. Since γ is "continuous" and G -equivariant, we deduce that $\lim_{t \rightarrow 0} \lambda(t)\gamma(x)$ and $\lim_{t \rightarrow 0} \lambda(t)\gamma(x')$ belong to $G^\lambda H/H$. Lemma 12 below proves that there exists p and p' in $P(\lambda)$ such that $\gamma(x) = pH/H$ and $\gamma(x') = p'H/H$. Since $gx' = x$, we have $g \in pHp'^{-1}$. But, H is a reductive subgroup (since $G.x_0$ is closed in the affine variety $\pi^{-1}(\xi)$) contained in a Borel subgroup of G (since $X = G/B \times Y$) and containing the image of λ : it follows that $H \subset G^\lambda$. Finally, $g \in P(\lambda)$. \square

It remains to prove the following lemma. In fact, it is an adaptation of the main result of [Ric82]:

Lemma 12 *Let \mathcal{O} be any G -homogeneous space. Let λ be a one parameter subgroup of G . Let C be an irreducible component of fixed points of λ in \mathcal{O} . Set $C^+ := \{x \in \mathcal{O} : \lim_{t \rightarrow 0} \lambda(t)x \text{ exists and belongs to } C\}$.*

Then, C^+ is a $P(\lambda)$ -orbit.

Proof. Let $x \in C$. The differential of the map $g \mapsto g.x$ induces a surjective linear map $\phi : \mathfrak{g} \rightarrow T_x\mathcal{O}$. Since x is fixed by λ , λ acts on $T_x\mathcal{O}$; it also acts by the adjoint action on \mathfrak{g} , and ϕ is equivariant. In particular, the restriction of ϕ , $\bar{\phi} : \mathfrak{g}_{\geq 0} \rightarrow (T_x\mathcal{O})_{\geq 0}$ is also surjective. But, on one hand $(T_x\mathcal{O})_{\geq 0} = T_x C^+$ by Theorem 4 and on the other hand $\mathfrak{g}_{\geq 0}$ is the Lie algebra of $P(\lambda)$. One can conclude that $T_x C^+ = T_x P(\lambda)x$. Since $P(\lambda)x$ is smooth, this implies that $P(\lambda)x$ is open in C^+ . Since C^+ is irreducible, it contains a unique open $P(\lambda)$ -orbit \mathcal{O}_0 which contains C .

Note that $C^+ - \mathcal{O}_0$ is $P(\lambda)$ -stable and closed in C^+ . Since for any $y \in C^+$ $\lim_{t \rightarrow 0} \lambda(t)y \in C \subset \mathcal{O}_0$, this implies that $C^+ - \mathcal{O}_0$ is empty. \square

Remark. Here, is an example which proves that the assumption on X is useful. Let V be a vector space of dimension 2. Make the group $G =$

$\mathbb{K}^* \times \mathrm{SL}(V)$ acting on $X = \mathbb{P}(\mathbb{K} \oplus V \oplus V)$ by:

$$(t, g) \cdot [\tau : v_1 : v_2] = [\tau, gv_1, t^2gv_2]. \quad (3)$$

Consider (with obvious notation), the following one parameter subgroup $\lambda(t) = (t^{-1}, \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix})$. Note that $C = \{[1 : 0 : 0]\}$ is an irreducible component of X^λ . Formula 3 gives also a linearization \mathcal{L} of the bundle $\mathcal{O}(1)$. Then, $\mu^\mathcal{L}(C, \lambda) = 0$, the map η associated to (C, λ) is birational, but the fiber over the only point of C is \mathbb{P}^1 : so (C, λ) is covering but not well covering.

Our assumption on Λ . We call a subgroup Γ' of an abelian group Γ *cofinite* if Γ/Γ' is finite. The following definition is an adaptation of those of Dolgachev and Hu (see [DH98]).

Definition. The subgroup Λ is said to be *abundant* if for any x in X such that G_x is reductive, the image of the restriction $\Lambda \longrightarrow \mathrm{Pic}^G(G.x)$ is cofinite.

The main example of abundant subgroups comes from the case when $X = G/B \times Y$.

Proposition 8 *Let $X = G/B \times Y$ for a G -variety Y . Let $\pi : X \longrightarrow G/B$ denote the projection map and $\pi^* : \mathrm{Pic}^G(G/B) \longrightarrow \mathrm{Pic}^G(X)$ the associated homomorphism.*

Then, any subgroup Λ of $\mathrm{Pic}^G(X)$ containing the image of π^ is abundant.*

Proof. Let $x = (y, gB/B) \in G/B \times Y$. Changing x by $g^{-1}x$, we assume that $g = e$. Let $\chi \in X(G_x)$. Note that $G_x = B_y$. Since the restriction map $X(B) \longrightarrow X(B_y)$ is surjective, there exists $\nu \in X(B)$ such that $\nu|_{B_y} = \chi$. The restriction of $\pi^*(\mathcal{L}_\nu)$ to $G.x$ equals \mathcal{L}_χ ; the proposition follows. \square

The next statement is a precision of Corollary 2.

Corollary 3 *Let $X = G/B \times Y$ for a normal G -variety Y . We assume that $\Lambda \subset \mathrm{Pic}^G(X)$ is abundant. Let \mathcal{F} be a face of $\mathcal{AC}_\Lambda^G(X)$ of codimension r in Λ . Then, there exists a r -dimensional torus S in G , a one parameter subgroup λ of S and an irreducible component C of X^S such that:*

(i) for generic $x \in C$, we have $G_x^\circ = S$;

- (ii) C is an irreducible component of X^λ ;
- (iii) the pair (C, λ) is covering (respectively, well covering if X is normal);
- (iv) \mathcal{F} is the face of $\mathcal{AC}_\Lambda^G(X)$ associated to (C, λ) .

Proof. We just have to prove that if (C, λ) and S satisfy Corollary 2, they also satisfy the first assertion.

Let \mathcal{L} be an ample G -linearized line bundle on X and x be a semistable point for \mathcal{L} . We claim that G_x is diagonalizable. Consider $\pi : X^{\text{ss}}(\mathcal{L}) \rightarrow X^{\text{ss}}(\mathcal{L})//G$. Let y be a point in the closure of $G.x$ such that $G.y$ is closed in $X^{\text{ss}}(\mathcal{L})$. Since $G.y$ is closed in $\pi^{-1}(\pi(y))$, it is affine; so, G_y is reductive. But G_y is contained in a Borel subgroup of G ; so, G_y is diagonalizable. By Luna's Slice Theorem, $\pi^{-1}(\pi(y)) \simeq G \times_{G_y} \Sigma$ for an affine G_y variety Σ . Since $x \in \pi^{-1}(\pi(y))$, we deduce that G_x is conjugated to a subgroup of G_y . In particular, G_x is diagonalizable.

Let us fix a point x in the (finite) intersection of the set $X^{\text{ss}}(\mathcal{L}) \cap C$ for $\mathcal{L} \in \mathcal{F}$. Consider $\rho : \Lambda \rightarrow X(G_x)$. Since Λ is abundant and G_x diagonalizable, the rank of ρ equal the dimension of G_x . Since \mathcal{F} is contained in the kernel of ρ , the dimension of G_x is less or equal to r . Since $S \subset G_x$, it follows that $G_x^\circ = S$. \square

8 If $X = G/Q \times \hat{G}/\hat{Q} \dots$

8.1 Interpretations of the G -cones

From now on, we assume that G is a connected reductive subgroup of a connected reductive group \hat{G} . Let us fix maximal tori T (resp. \hat{T}) and Borel subgroups B (resp. \hat{B}) of G (resp. \hat{G}) such that $T \subset B \subset \hat{B} \supset \hat{T} \supset T$. Let Q (resp. \hat{Q}) be a parabolic subgroup of G (resp. \hat{G}) containing B (resp. \hat{B}); let L (resp. \hat{L}) denote the Levi subgroup of Q (resp. \hat{Q}) containing T (resp. \hat{T}).

In this section, X denote the variety $G/Q \times \hat{G}/\hat{Q}$ endowed with the diagonal action of G . We will apply the results of Section 7 to X with $\Lambda = \text{Pic}^G(X)$. The cones $\mathcal{TC}^G(X)$, $\mathcal{SAC}^G(X)$ and $\mathcal{AC}^G(X)$ will be denoted without the Λ in subscribe.

Let us describe $\text{Pic}^G(X)_{\mathbb{Q}}$. Consider the natural action of $G \times \hat{G}$ on X . By applying the construction of Section 2.1 to the $G \times \hat{G}$ -homogeneous space X , one obtains the following isomorphism $X(Q) \times X(\hat{Q}) \rightarrow \text{Pic}^{G \times \hat{G}}(X)$, $(\nu, \hat{\nu}) \mapsto \mathcal{L}_{(\nu, \hat{\nu})}$.

Lemma 13 *The following short sequence is exact*

$$0 \longrightarrow X(\hat{G})_{\mathbb{Q}} \xrightarrow{\hat{\nu} \mapsto \mathcal{L}_{(-\hat{\nu}|_Q, \hat{\nu}|_{\hat{Q}})}} \text{Pic}^{G \times \hat{G}}(X)_{\mathbb{Q}} \xrightarrow{r_{\mathbb{Q}}^{\Delta G}} \text{Pic}^G(X)_{\mathbb{Q}} \longrightarrow 0$$

where the second linear map is induced by the restriction $r^{\Delta G}$ of the action of $G \times \hat{G}$ to G diagonally embedded in $G \times \hat{G}$.

Proof. Let χ and $\hat{\chi}$ be characters of G and \hat{G} respectively. The trivial bundle on X linearized by $(\chi, \hat{\chi})$ belongs to $\text{Pic}^{G \times \hat{G}}(X)$. The image of this line bundle in $\text{Pic}^G(X)$ is the trivial line bundle linearized by the character $\chi + \hat{\chi}|_G$ of G . In particular, any G -linearization of the trivial bundle belongs to the image of $r^{\Delta G}$.

Let $\mathcal{L} \in \text{Pic}^G(X)$. Let $\mathcal{L}' \in \text{Pic}(X)$ obtained from \mathcal{L} by forgetting the action of G . By [FHT84], there exists a $G \times \hat{G}$ -linearization \mathcal{M} of $\mathcal{L}'^{\otimes n}$ for a positive integer n . Then $\mathcal{M}^* \otimes \mathcal{L}^{\otimes n}$ is the trivial line bundle over X ; so, it belongs to the image $r^{\Delta G}$. Finally, $\mathcal{L}^{\otimes n}$ belongs to the image of the restriction. This ends the proof of the surjectivity of $r_{\mathbb{Q}}^{\Delta G}$.

Let \mathcal{L} in the kernel of $r^{\Delta G}$. Since \mathcal{L} is trivial as a line bundle, there exists characters χ and $\hat{\chi}$ of G and \hat{G} such that $\mathcal{L} = \mathcal{L}_{(\chi, \hat{\chi})}$. The G -linearization of this last character is trivial if and only if $\chi + \hat{\chi}|_G$ is trivial. This ends the proof of the lemma. \square

Proposition 9 (i) $\mathcal{TC}^G(X) = \mathcal{SAC}^G(X)$ is a closed convex polyhedral cone.

(ii) Let $(\nu, \hat{\nu}) \in X(Q)_{\mathbb{Q}} \times X(\hat{Q})_{\mathbb{Q}}$. Then $r_{\mathbb{Q}}^{\Delta G}(\mathcal{L}_{(\nu, \hat{\nu})}) \in \mathcal{TC}^G(X)$ if and only if $\hat{\nu}$ and ν are dominant and for n big enough $V_{n\nu} \otimes V_{n\hat{\nu}}$ contains non zero G -invariant vectors.

(iii) If $\mathcal{AC}^G(X)$ is non empty, its closure in $\text{Pic}^G(X)_{\mathbb{Q}}$ is $\mathcal{SAC}^G(X)$.

(iv) If Q and \hat{Q} are Borel subgroups of G and \hat{G} then $\mathcal{AC}^G(X)$ is non empty.

Proof. Since X is homogeneous under the action of $G \times \hat{G}$ and since any line bundle has a linearizable power, every bundle with a non zero section is semiample. Hence, $\mathcal{TC}^G(X) = \mathcal{SAC}^G(X)$. Lemma 4 and Proposition 3 implies that only finitely many non empty open subsets of X are of the form $X^{\text{ss}}(\mathcal{L})$ with $\mathcal{L} \in \text{Pic}^G(X)$. Let x be a point in the intersection of these

subsets. Lemma 4 shows that $\mathcal{SAC}^G(X)$ is the set of points $\mathcal{L} \in \text{Pic}^G(X)_{\mathbb{Q}}$ semiample such that $\mu^{\mathcal{L}}(x, \lambda) \leq 0$ for all one parameter subgroup λ of G . In particular, $\mathcal{SAC}^G(X)$ is a closed convex cone.

Let now ν and $\hat{\nu}$ be characters of Q and \hat{Q} . If ν or $\hat{\nu}$ is not dominant then $\mathcal{L}_{(\nu, \hat{\nu})}$ has no regular section. Else, Borel-Weil's theorem shows that $H^0(X, \mathcal{L}_{(\nu, \hat{\nu})})$ is isomorphic as a $G \times \hat{G}$ -module to $V_{\nu}^* \otimes V_{\hat{\nu}}^*$. The second assertion of the proposition follows.

The third assertion is satisfied since the ample cone in $\text{Pic}^G(X)$ is the interior of the semiample cone.

Let w_0 be the longest element of W . Assume that $Q = B$ and $\hat{Q} = \hat{B}$ are Borel subgroups of G and \hat{G} . Let $\hat{\nu}_0$ be any character of \hat{B} such that $\mathcal{L}_{\hat{\nu}_0}$ is ample over \hat{G}/\hat{B} . Let ν be any dominant weight of the G -module $V_{\hat{\nu}_0}^*$. Then, $r^{\Delta G}(\mathcal{L}_{(-w_0\nu, \hat{\nu}_0)})$ belongs to $\mathcal{SAC}^G(X)$.

Let ν_0 be any character of B such that \mathcal{L}_{ν_0} is ample over G/B . Since the restriction $\mathbb{K}[\hat{G}] \rightarrow \mathbb{K}[G]$ is surjective, Frobenius's theorem implies that $V_{\nu_0}^*$ is contained in an irreducible \hat{G} -module $V_{\hat{\nu}}^*$. Then, $r^{\Delta G}(\mathcal{L}_{(-w_0\nu_0, \hat{\nu})})$ belongs to $\mathcal{SAC}^G(X)$.

Since $\mathcal{SAC}^G(X)$ is convex, it contains $r^{\Delta G}(\mathcal{L}_{(-w_0(\nu_0+\nu), \hat{\nu}+\hat{\nu}_0)})$. But the line bundle $\mathcal{L}_{(-w_0(\nu_0+\nu), \hat{\nu}+\hat{\nu}_0)}$ is ample. The last assertion is proved. \square

8.2 Covering and well covering pairs

Notation. Let W and \hat{W} denote the Weyl groups of G and \hat{G} . If P is a parabolic subgroup of G containing T , W_P denote the Weyl group of the Levi subgroup of P containing T . This group W_P is canonically embedded in W .

In this subsection, we fix two parabolic subgroups Q and \hat{Q} of G and \hat{G} containing respectively B and \hat{B} . We will explain how to find the well covering pairs in the case when $X = G/Q \times \hat{G}/\hat{Q}$.

8.2.1 — Let λ be a one parameter subgroup of T and so of \hat{T} . We can describe the fix point set X^{λ} :

$$X^{\lambda} = \bigcup_{\substack{w \in W_{P(\lambda)} \backslash W/W_Q \\ \hat{w} \in \hat{W}_{\hat{P}(\lambda)} \backslash \hat{W}/\hat{W}_{\hat{Q}}}} G^{\lambda} w Q / Q \times \hat{G}^{\lambda} \hat{w} \hat{Q} / \hat{Q}.$$

For $(w, \hat{w}) \in W_Q \backslash W/W_{P(\lambda)} \times \hat{W}_{\hat{Q}} \backslash \hat{W}/\hat{W}_{\hat{P}(\lambda)}$, we set

$$C(w, \hat{w}) = G^\lambda w^{-1} Q/Q \times \hat{G}^\lambda \hat{w}^{-1} \hat{Q}/\hat{Q}.$$

Note that, for later use, we have introduced $^{-1}$ in this definition. Note also that

$$C^+(w, \hat{w}) = P(\lambda)w^{-1} Q/Q \times \hat{P}(\lambda)\hat{w}^{-1} \hat{Q}/\hat{Q}.$$

8.2.2 — Let P be a parabolic subgroup of G . We consider the cohomology ring $H^*(G/P, \mathbb{Z})$ of G/P . Here, we use singular cohomology with integers coefficients. If V is a closed subvariety of G/P , we denote by $[V]$ the Poincaré dual class in $H^*(G/P, \mathbb{Z})$.

Notation. The elements of W/W_P will be denoted w as elements of W : in other word, the notation do not distinguish a class and a representative. Each times we use this abuse the reader has to check that the considered quantities does not depend on the representative.

If $w \in W/W_P$, we denote the corresponding Schubert variety $\overline{BwP/P}$ by Λ_w^P . We denote by $[\Lambda_w^P]$ the Poincaré dual class in $H^*(G/P, \mathbb{Z})$ of Λ_w . We also denote by $[\text{pt}]$ the Poincaré dual class of the point; note that, $[\text{pt}] = [\Lambda_e]$. Let us recall that

$$H^*(G/P, \mathbb{Z}) = \bigoplus_{w \in W/W_{P(\lambda)}} \mathbb{Z}[\Lambda_w^P].$$

We use similar notation for \hat{G}/\hat{P} .

We now consider the case when $P = P(\lambda)$ and $\hat{P} = \hat{P}(\lambda)$. Since $P = G \cap \hat{P}(\lambda)$, $G/P(\lambda)$ identifies with the orbit by G of $\hat{P}(\lambda)/\hat{P}(\lambda)$ in $\hat{G}/\hat{P}(\lambda)$; let $\iota : G/P(\lambda) \rightarrow \hat{G}/\hat{P}(\lambda)$ denote this closed immersion. The map ι induces a map ι^* in cohomology:

$$\iota^* : H^*(\hat{G}/\hat{P}(\lambda), \mathbb{Z}) \rightarrow H^*(G/P(\lambda), \mathbb{Z}).$$

To simplify notation, we set $P = P(\lambda)$ and $\hat{P} = \hat{P}(\lambda)$.

Lemma 14 For $(w, \hat{w}) \in W_Q \backslash W/W_P \times \hat{W}_{\hat{Q}} \backslash \hat{W}/\hat{W}_{\hat{P}}$.

Then the following are equivalent:

(i) the pair $(C(w, \hat{w}), \lambda)$ is covering,

(ii) $\iota^*([\overline{\hat{Q}\hat{w}\hat{P}/\hat{P}}]) \cdot [\overline{QwP/P}] = [\text{pt}]$.

Proof. Consider the map:

$$\eta : G \times_P C^+(w, \hat{w}) \longrightarrow X.$$

Since the characteristic of \mathbb{K} is zero, η is birational if and only if for x in an open subset of X , $\eta^{-1}(x)$ is reduced to a point. Consider the projection $\pi : G \times_P C^+(w, \hat{w}) \longrightarrow G/P$. For any x in X , π induces an isomorphism from $\eta^{-1}(x)$ onto the following locally closed subvariety of G/P : $F_x := \{hP \in G/P : h^{-1}x \in C^+(w, \hat{w})\}$.

Let $(g, \hat{g}) \in G \times \hat{G}$ and set $x = (gQ/Q, \hat{g}\hat{Q}/\hat{Q}) \in X$. We have:

$$\begin{aligned} F_x &= \{hP/P \in G/P : h^{-1}gQ/Q \in Pw^{-1}Q/Q \text{ and } h^{-1}\hat{g}\hat{Q}/\hat{Q} \in \hat{P}\hat{w}^{-1}\hat{Q}/\hat{Q}\} \\ &= \{hP/P \in G/P : h^{-1} \in (Pw^{-1}Qg^{-1}) \cap (\hat{P}\hat{w}^{-1}\hat{Q}\hat{g}^{-1})\} \\ &= \iota(gQwP/P) \cap (\hat{g}\hat{Q}\hat{w}\hat{P}/\hat{P}). \end{aligned}$$

Let us fix g arbitrarily. By Kleiman's Theorem (see [Kle74]), there exists an open subset of \hat{g} 's in \hat{G} such that the intersection $\overline{gQwP/P} \cap \overline{\hat{g}\hat{Q}\hat{w}\hat{P}/\hat{P}}$ is transverse. Moreover (see for example [BK06]), one may assume that $(gQwP/P) \cap (\hat{g}\hat{Q}\hat{w}\hat{P}/\hat{P})$ is dense in $\overline{gQwP/P} \cap \overline{\hat{g}\hat{Q}\hat{w}\hat{P}/\hat{P}}$. We deduce that the following are equivalent:

- (i) for generic \hat{g} , F_x is reduced to a point,
- (ii) $\iota^*([\hat{Q}\hat{w}\hat{P}/\hat{P}]) \cdot [\overline{QwP/P}] = [\text{pt}]$.

Since η is G -equivariant, the above Condition (i) is clearly equivalent to the fact that η is birational. \square

8.2.3 — Notation. From now on, \mathfrak{g} and \mathfrak{b} will denote the Lie algebras of G and B , R will denote the set of roots of \mathfrak{g} and R^+ those of positive ones. We denote by ρ the half sum of the positive roots of \mathfrak{g} . We will also use the following similar notation for \hat{G} : $\hat{\mathfrak{g}}, \hat{\mathfrak{b}}, \hat{R}, \hat{R}^+, \hat{\rho}$.

Let $w \in W/W_P$ and consider the associated B -orbit in G/P . We define γ_w^P to be the sum of the weights of T in the normal space at wP/P of BwP/P in G/P . Similarly, we define $\hat{\gamma}_{\hat{w}}^{\hat{P}}$.

Lemma 15 *We assume that P is standard (that is, contain B , that is, λ dominant). We keep above notation. Then, γ_w^P is the sum of the weights of T in $\mathfrak{g}/(\mathfrak{b} + w \cdot \mathfrak{p})$.*

Let \tilde{w} be the longest element in the class $w \in W/W_P$. We have:

$$\gamma_w^P = -(\rho + \tilde{w}\rho).$$

Proof. Consider the map $G \longrightarrow G/P$, $g \mapsto gwP/P$ and its tangent map $\varphi : \mathfrak{g} \longrightarrow T_{wP/P}G/P$. Since φ is T -equivariant, surjective and $\mathfrak{b} + w.\mathfrak{p} = \varphi^{-1}(T_{wP/P}BwP/P)$, γ_w^P is the sum of the weights of T in $\mathfrak{g}/(\mathfrak{b} + w.\mathfrak{p})$.

Consider the G -equivariant surjective map $\pi : G/B \longrightarrow G/P$. The definition of \tilde{w} implies that $\pi(\tilde{w}B/B) = wP/P$ and $B\tilde{w}B/B$ is open in $\pi^{-1}(\overline{BwP/P})$. It follows that γ_w^P is the sum of the weights of T in the normal space at $\tilde{w}B/B$ of $B\tilde{w}B/B$ in G/B ; that is, the sum of the weights of T in $\mathfrak{g}/(\mathfrak{b} + \tilde{w}.\mathfrak{b})$. Since the sum of all the roots is zero, we obtain that

$$\begin{aligned} -\gamma_w^P &= \sum_{\alpha \in R^+ \cup \tilde{w}R^+} \alpha \\ &= \frac{1}{2} \left(\sum_{\alpha \in R^+} \alpha + \sum_{\alpha \in \tilde{w}R^+} \alpha + \sum_{\alpha \in R^+ \setminus \tilde{w}R^+} \alpha + \sum_{\alpha \in \tilde{w}R^+ \setminus R^+} \alpha \right) \\ &= \frac{1}{2} \left(2\rho + 2\tilde{w}\rho + \sum_{\alpha \in R^+ \setminus \tilde{w}R^+} \alpha + \sum_{\alpha \in \tilde{w}R^+ \setminus R^+} \alpha \right) \\ &= \rho + \tilde{w}\rho \end{aligned}$$

where the last equality holds since $R^+ \setminus \tilde{w}R^+ = -(\tilde{w}R^+ \setminus R^+)$. \square

Remark. In [BK06], Belkale and Kumar defined characters $\chi_{w^{-1}}$ for w of minimal length its coset in $W_P \setminus W$. We have $\gamma_{w^{-1}}^{B,P} = -\tilde{w}^{-1}w^P(\chi_{w^{-1}})$, where w^P denote the longest element of W_P .

8.2.4 — Let us fix again a dominant one parameter subgroup λ of T , $w \in W$ and $\hat{w} \in \hat{W}$.

To simplify notation, we set $P = P(\lambda)$, $C = C(w, \hat{w})$ and $C^+ = C^+(w, \hat{w})$. Consider

$$\eta : G \times_P C^+ \longrightarrow X = G/Q \times \hat{G}/\hat{Q}.$$

Notation. If Y is a smooth variety of dimension n , TY denotes its tangent bundle. The line bundle $\bigwedge^n TY$ over Y will be called the *determinant bundle* and denoted by $\mathcal{D}etY$. If $\varphi : Y \longrightarrow Y'$ is a morphism between smooth variety, we denote by $T\varphi : TY \longrightarrow TY'$ its tangent map, and by $\mathcal{D}et\varphi : \mathcal{D}etY \longrightarrow \mathcal{D}etY'$ its determinant.

Consider the restriction of $T\eta$ to C^+ :

$$T\eta|_{C^+} : \mathcal{T}(G \times_P C^+)|_{C^+} \longrightarrow \mathcal{T}(X)|_{C^+},$$

and the restriction of $\mathcal{D}et\eta$ to C^+ :

$$\mathcal{D}et\eta|_{C^+} : \mathcal{D}et(G \times_P C^+)|_{C^+} \longrightarrow \mathcal{D}et(X)|_{C^+}.$$

Since η is G -equivariant, the morphism $\text{Det}\eta|_{C^+}$ is P -equivariant; it can be thought as a P -invariant section of the line bundle $\text{Det}(G \times_P C^+)^*_{|C^+} \otimes \text{Det}(X)|_{C^+}$ over C^+ . We denote by $\mathcal{L}_{P,w,\hat{w}}$ this last P -linearized line bundle on C^+ .

Lemma 16 *We assume that P, Q, \hat{P} and \hat{Q} are standard. Let $(w, \hat{w}) \in W/W_P \times \hat{W}/\hat{W}_{\hat{P}}$ be such that BwP/P and $\hat{B}\hat{W}\hat{P}/\hat{P}$ are open in QwP/P and $\hat{Q}\hat{W}\hat{P}/\hat{P}$ respectively. Let \tilde{w} (resp. $\tilde{\hat{w}}$) be the longest element in the class of w (resp. \hat{w}) in W/W_P (resp. $\hat{W}/\hat{W}_{\hat{P}}$).*

Then, the torus T acts on the fiber over the point $(\tilde{w}^{-1}Q/Q, \tilde{\hat{w}}^{-1}\hat{Q}/\hat{Q})$ in $\mathcal{L}_{P,w,\hat{w}}$ by the character

$$r_T(\tilde{\hat{w}}^{-1}\gamma_{\tilde{\hat{w}}}^{\hat{P}}) + \tilde{w}^{-1}\gamma_w^P - \gamma_e^P.$$

Proof. If Z is a locally closed subvariety of a variety Y and z is a point of Z , we denote by $N_z^Y(Z)$ the quotient T_zY/T_zZ of the tangent spaces at z of Y and Z . If V is a T -module $\text{St}_T(V)$ denote the multiset of the weights of T in V . Let χ denote the character of the action of T on the fiber over the point $x = (\tilde{w}^{-1}Q/Q, \tilde{\hat{w}}^{-1}\hat{Q}/\hat{Q})$ in $\mathcal{L}_{P,w,\hat{w}}$. Let \mathfrak{p} denote the Lie algebra of P .

Since η induces the identity on C^+ (canonically embedded in $G \times_P C^+$), we have:

$$\chi = - \sum_{\alpha \in \text{St}_T(N_x^{G \times_P C^+}(C^+))} \alpha + \sum_{\alpha \in \text{St}_T(N_x^X(C^+))} \alpha.$$

Moreover, we have the following T -equivariant isomorphisms:

$$\begin{aligned} N_x^{G \times_P C^+}(C^+) &\simeq \mathfrak{g}/\mathfrak{p} \simeq N_e^G(P) \simeq \mathfrak{g}/\mathfrak{p}, \\ N_x^X(C^+) &\simeq N_{\tilde{\hat{w}}^{-1}\hat{Q}/\hat{Q}}^{\hat{G}/\hat{Q}}(\hat{P}\tilde{\hat{w}}^{-1}\hat{Q}/\hat{Q}) \oplus N_{\tilde{w}^{-1}Q/Q}^{G/Q}(Pw^{-1}Q/Q) \\ &\simeq N_{\tilde{\hat{w}}^{-1}\hat{B}/\hat{B}}^{\hat{G}/\hat{B}}(\hat{P}\tilde{\hat{w}}^{-1}\hat{B}/\hat{B}) \oplus N_{\tilde{w}^{-1}B/B}^{G/B}(Pw^{-1}B/B) \\ &\simeq \hat{\mathfrak{g}}/(\hat{\mathfrak{p}} + \tilde{\hat{w}}^{-1}\hat{\mathfrak{b}}) \oplus \mathfrak{g}/(\mathfrak{p} + \tilde{w}^{-1}\mathfrak{b}) \end{aligned}$$

Now, the lemma is direct consequence of Lemma 15. \square

Notation. With notation of Lemma 16, we set $\theta_w^P = \tilde{w}^{-1}\gamma_w^P$ and $\theta_{\hat{w}}^{\hat{P}} = \tilde{\hat{w}}^{-1}\gamma_{\tilde{\hat{w}}}^{\hat{P}}$.

One can now describe the well covering pairs of X :

Proposition 10 *Let λ be a dominant one parameter subgroup of T . Let $(w, \hat{w}) \in W/W_P \times \hat{W}/\hat{W}_{\hat{P}}$ be such that BwP/P and $\hat{B}\hat{w}\hat{P}/\hat{P}$ are open in QwP/P and $\hat{Q}\hat{w}\hat{P}/\hat{P}$ respectively.*

The following are equivalent:

- (i) *The pair $(C(w, \hat{w}), \lambda)$ is well covering.*
- (ii) $\iota^*([\Lambda_{\hat{w}}^{\hat{P}}]) \cdot [\Lambda_w^P] - [\Lambda_e^P] = 0$, and $\langle \hat{w}\lambda, \gamma_{\hat{w}}^{\hat{P}} \rangle + \langle w\lambda, \gamma_w^P \rangle - \langle \lambda, \gamma_e^P \rangle = 0$.

Proof. By Lemma 14, we may (and shall) assume that $(C(w, \hat{w}), \lambda)$ is covering. Note that $w\lambda$ is well defined, since W_P is precisely the stabilizer of λ in W . We chose \tilde{w} and $\tilde{\hat{w}}$ as in Lemma 16. Note that

$$\langle \lambda, r_T(\tilde{w}^{-1} \gamma_{\tilde{\hat{w}}}^{\hat{P}}) + \tilde{w}^{-1} \gamma_w^P - \gamma_e^P \rangle = \langle \hat{w}\lambda, \gamma_{\hat{w}}^{\hat{P}} \rangle + \langle w\lambda, \gamma_w^P \rangle - \langle \lambda, \gamma_e^P \rangle.$$

In particular, by Lemma 16, $\langle \hat{w}\lambda, \gamma_{\hat{w}}^{\hat{P}} \rangle + \langle w\lambda, \gamma_w^P \rangle - \langle \lambda, \gamma_e^P \rangle = 0$ if and only if λ acts trivially on the restriction of $\mathcal{L}_{P,w,\hat{w}}$ to $C(w, \hat{w})$.

Assume that $(C(w, \hat{w}), \lambda)$ is well covering. Then $\mathcal{D}et\eta|_C$ is non identically zero. Since $\mathcal{D}et\eta|_C$ is a G^λ -invariant section of $\mathcal{L}_{P,w,\hat{w}|_C}$, λ which fixes point wise C has to act trivially on $\mathcal{L}_{P,w,\hat{w}|_C}$ (see for example Lemma 3).

Conversely, assume Condition (ii) satisfied. By Lemma 16, this implies that $\mu^{\mathcal{L}_{P,w,\hat{w}|_C}}(C, \lambda) = 0$. But, since η is birational, $\mathcal{D}et\eta$ is G -invariant and non zero; hence, $\mathcal{D}et\eta|_{C^+}$ is P -invariant and non zero. So, Proposition 5 show that the restriction of $\mathcal{D}et\eta|_C$ is non identically zero. Since η is birational, this implies that η is an isomorphism over an open subset intersecting C . \square

8.3 The case $X = G/B \times \hat{G}/\hat{B}$

8.3.1 — We denote by $\mathcal{LR}(G, \hat{G})$ the cone of the pairs $(\nu, \hat{\nu}) \in X(T)_{\mathbb{Q}} \times X(\hat{T})_{\mathbb{Q}}$ such that for a positive integer n , $n\hat{\nu}$ and $n\nu$ are dominant weights such that $V_{n\nu} \otimes V_{n\hat{\nu}}$ contains non zero G -invariant vectors.

From now on, $X = G/B \times \hat{G}/\hat{B}$. By Proposition 9, a point $(\nu, \hat{\nu})$ belongs to $\mathcal{LR}(G, \hat{G})$ if and only if $r^{\Delta G}(\mathcal{L}_{(\nu, \hat{\nu})})$ belongs to $\mathcal{TC}^G(X) = \mathcal{SAC}^G(X)$.

8.3.2 — Consider the G -module $\hat{\mathfrak{g}}/\mathfrak{g}$. Let χ_1, \dots, χ_n be the set of the non trivial weights of T on $\hat{\mathfrak{g}}/\mathfrak{g}$. For $I \subset \{1, \dots, n\}$, we will denote by T_I the neutral component of the intersection of the kernels of the χ_i 's with $i \in I$. A subtorus of the form T_I is said to be *admissible*. The subtorus

T_I is said to be *dominant* if $Y(T_I)_{\mathbb{Q}}$ is spanned by its intersection with the dominant chamber of $Y(T)_{\mathbb{Q}}$. Notice that the set of the χ_i 's being stable by the action of W , any admissible subtorus is conjugated by an element of W to a dominant admissible subtorus. A one parameter subgroup of T is said to be *admissible* if its image is.

To each χ_i , we associate the hyperplane \mathcal{H}_i in $Y(T)_{\mathbb{Q}}$ spanned by the $\lambda \in Y(T)$ such that $\chi \circ \lambda$ is trivial. The \mathcal{H}_i 's form an W -invariant arrangement of hyperplane in $Y(T)_{\mathbb{Q}}$. Moreover, $Y(T_I)_{\mathbb{Q}}$ is the intersection of the \mathcal{H}_i 's with $i \in I$.

To simplify, in the following statement we assume that $\mathcal{AC}^G(X)$ has a non empty interior in $\text{Pic}^G(X)_{\mathbb{Q}}$. In fact, this assumption is equivalent to say that no ideal of \mathfrak{g} is an ideal of $\hat{\mathfrak{g}}$.

Theorem 9 *We assume that no ideal of \mathfrak{g} is an ideal of $\hat{\mathfrak{g}}$.*

- (i) *The interior of $\mathcal{LR}(G, \hat{G})$ in $X(T)_{\mathbb{Q}} \times X(\hat{T})_{\mathbb{Q}}$ is not empty.*
- (ii) *Let \mathcal{F} be a face of $\mathcal{LR}(G, \hat{G})$ of codimension r which intersects the interior of the Weyl chamber. Then there exists a dominant admissible subtorus T_I (with $I \subset \{1, \dots, n\}$) of T of dimension r , a dominant indivisible one parameter subgroup λ of T_I , and an irreducible component $C(w, \hat{w})$ of X^λ (and X^{T_I}) such that:*

- (a) $\iota^*([\Lambda_{\hat{w}}^{\hat{P}(\lambda)}]).[\Lambda_w^{P(\lambda)}] = [\Lambda_e^{P(\lambda)}] \in H^*(G/P(\lambda), \mathbb{Z})$,
- (b) $r_T(\hat{\theta}_{\hat{w}}^{\hat{P}(\lambda)}) + \theta_w^{P(\lambda)} - \theta_e^{P(\lambda)}$ is trivial on T_I , and
- (c) \mathcal{F} is the set of $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$ such that $\langle w\lambda, \nu \rangle + \langle \hat{w}\lambda, \hat{\nu} \rangle = 0$.

- (iii) *Conversely, let λ be a dominant one parameter subgroup of T and $C = C(w, \hat{w})$ be an irreducible component X^λ . Set $I = \{i = 1, \dots, n \mid \chi \circ \lambda \text{ is trivial}\}$ and denote by r the dimension of T_I . If*

- (a) $\iota^*([\Lambda_{\hat{w}}^{\hat{P}(\lambda)}]).[\Lambda_w^{P(\lambda)}] = [\Lambda_e^{P(\lambda)}] \in H^*(G/P(\lambda), \mathbb{Z})$ and
- (b) $r_T(\hat{\theta}_{\hat{w}}^{\hat{P}(\lambda)}) + \theta_w^{P(\lambda)} - \theta_e^{P(\lambda)}$ is trivial on T_I ,

then the set of $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$ such that $\langle w\lambda, \nu \rangle + \langle \hat{w}\lambda, \hat{\nu} \rangle = 0$ is a face of $\mathcal{LR}(G, \hat{G})$ of codimension r .

Proof. By [MR08, Corollaire 1], the codimension of $\mathcal{AC}^G(X)$ in $\text{Pic}^G(X)_{\mathbb{Q}}$ is the dimension of the generic isotropy of T acting on $\hat{\mathfrak{g}}/\mathfrak{g}$. This generic

isotropy is also the kernel of the action of T on \hat{G}/G . So, it is contained in $\bigcap_{\hat{g} \in \hat{G}} \hat{g}G\hat{g}^{-1}$. Since this group is distinguish in \hat{G} and G , it is finite by assumption. Hence, the interior of $\mathcal{AC}^G(X)$ in $\text{Pic}^G(X)_{\mathbb{Q}}$ is non empty.

Let \mathcal{F} be a face of $\mathcal{AC}^G(X)$ of codimension r . Let S be a torus in G , λ be an indivisible one parameter subgroup of S and C be an irreducible component of X^S satisfying Corollary 3. Up to conjugacy, we may assume that S is contained in T and λ is dominant.

Claim 1. The subtorus S is admissible.

Notice that C is isomorphic to the variety of the complete flags $G^S \times \hat{G}^S$ and S is the isotropy of a general point in C . Hence, S is the isotropy in B^S of a general point in \hat{G}^S/\hat{B}^S , that is, by Bruhat's Theorem, of a general point in \hat{B}^S/\hat{T}^S . Since U^S acts freely on \hat{B}^S/\hat{T}^S , S is conjugated to the isotropy in T of a general point in \hat{U}^S/U^S which is isomorphic to \hat{u}^S/u^S . Finally, one may assume that S is the isotropy of a general point in $\hat{\mathfrak{g}}^S/\mathfrak{g}^S = (\hat{\mathfrak{g}}/\mathfrak{g})^S$; that is, since S is diagonalizable, that S is the kernel of the action of T on $(\hat{\mathfrak{g}}/\mathfrak{g})^S$. Then, S is admissible.

Let $I \subset \{1, \dots, n\}$ such that $S = T_I$. Since (C, λ) is well covering, Proposition 10 shows that w, \hat{w} and λ satisfy Conditions (ii)a and (ii)b of the theorem. Assertion (ii) follows.

Conversely, let w, \hat{w}, λ and I as in Assertion (iii) of the theorem. By Proposition 10, $(C(w, \hat{w}), \lambda)$ is a well covering pair. By Lemma 7, the set of $\mathcal{L} \in \mathcal{TC}^G(X)$ such that $\mu^{\mathcal{L}}(C, \lambda) = 0$ is a face \mathcal{F}' of $\mathcal{TC}^G(X)$ eventually empty. It remains to prove that the codimension of \mathcal{F}' equal the dimension r of T_I .

Consider the linear map ρ induced by the restriction: $\rho : \text{Pic}^G(X)_{\mathbb{Q}} \longrightarrow \text{Pic}^{G^\lambda}(C(w, \hat{w}))_{\mathbb{Q}}$. We firstly prove the

Claim 3. ρ is surjective.

Let x be a point in $C(w, \hat{w})$. Let (\hat{B}_1, B_1) denote the isotropy of x in $G \times \hat{G}$. Let $\mathcal{L} \in \text{Pic}^{G^\lambda}(C(w, \hat{w}))$. By Lemma 13 applied to C , there exists characters $\hat{\nu}$ and ν of \hat{B}_1^λ and B_1^λ and a positive integer n such that $\mathcal{L}^{\otimes n}$ is the restriction of the $G^\lambda \times \hat{G}^\lambda$ -linearized line bundle $\mathcal{L}_{(\nu, \hat{\nu})}$. But, ν and $\hat{\nu}$ can be extended to characters ν' and $\hat{\nu}'$ of B_1 and \hat{B}_1 . Then, $\mathcal{L}^{\otimes n}$ is the image by ρ of $r^{\Delta G}(\mathcal{L}_{\nu', \hat{\nu}'})$.

By Claim 3 and Theorem 7, it remains to prove that $\mathcal{TC}^{G^\lambda}(C(w, \hat{w}))$ has

codimension r in $\text{Pic}^{G^\lambda}(C(w, \hat{w}))_{\mathbb{Q}}$. By [MR08, Corollaire 1], this codimension is the dimension of the kernel of the action of T over $\hat{\mathfrak{g}}^\lambda/\mathfrak{g}^\lambda = (\hat{\mathfrak{g}}/\mathfrak{g})^\lambda$. By definition T_I is the neutral component of this kernel. \square

Corollary 4 *We assume that no ideal of \mathfrak{g} is an ideal of $\hat{\mathfrak{g}}$. Any dominant weight $(\nu, \hat{\nu})$ belongs to $\mathcal{LR}(G, \hat{G})$ if and only if*

$$\langle w\lambda, \nu \rangle + \langle \hat{w}\lambda, \hat{\nu} \rangle \geq 0,$$

for all indivisible dominant admissible one parameter subgroup λ of T and for all $(w, \hat{w}) \in W/W_{P(\lambda)} \times \hat{W}/\hat{W}_{\hat{P}(\lambda)}$ such that

$$(i) \iota^*([\Lambda_{\hat{w}}^{\hat{P}(\lambda)}]).[\Lambda_w^{P(\lambda)}] = [\Lambda_e^{P(\lambda)}] \in H^*(G/P(\lambda), \mathbb{Z}), \text{ and}$$

$$(ii) \langle \hat{w}\lambda, \gamma_{\hat{w}}^{\hat{P}(\lambda)} \rangle + \langle w\lambda, \gamma_w^{P(\lambda)} \rangle = \langle \lambda, \gamma_e^{P(\lambda)} \rangle.$$

Moreover, the above inequalities are pairwise distinct and no one can be omitted.

Proof. By Proposition 7, the cone $\mathcal{SAC}^G(X)$ is characterized as a part of the dominant chamber by the inequalities $\mu^{\mathcal{L}}(C, \lambda) \leq 0$, for all well covering pair (C, λ) with a dominant one parameter subgroup λ of T . Since $\mathcal{SAC}^G(X)$ has non empty interior, it is sufficient to keep only the inequalities corresponding to faces of codimension one. By the theorem, all these inequalities are in the corollary. The first part of the corollary is proved.

Consider an inequality $\langle w\lambda, \nu \rangle + \langle \hat{w}\lambda, \hat{\nu} \rangle \geq 0$ as in the statement of the corollary. By Theorem 9, the set $(\nu, \hat{\nu})$ such that $\langle w\lambda, \nu \rangle + \langle \hat{w}\lambda, \hat{\nu} \rangle = 0$ is a face \mathcal{F} of codimension one of $\mathcal{LR}(G, \hat{G})$. This inequality cannot be omitted unless \mathcal{F} is a face of the dominant chamber in $X(T)_{\mathbb{Q}} \times X(\hat{T})_{\mathbb{Q}}$. It is easy to check that this is not possible.

It remains to prove that these inequalities are pairwise distinct. But the stabilizer in W (resp. \hat{W}) of $\lambda \in Y(T)$ (resp. $\lambda \in Y(\hat{T})$) is precisely $W_{P(\lambda)}$ (resp. $\hat{W}_{\hat{P}(\lambda)}$). It follows that the inequalities of the corollary are pairwise distinct. \square

8.3.3 — Corollary 4 gives a minimal list of inequalities which determine $\mathcal{LR}(G, \hat{G})$ as a part of the dominant chamber. In [BS00], Berenstein-Sjamaar gave another list containing redundant inequalities. An hyperplane given by one redundant inequality cannot intersect $\mathcal{LR}(G, \hat{G})$ along a face of codimension one containing pairs of strictly dominant weights. We now want to understand better this intersection.

Let us fix an indivisible dominant admissible one parameter subgroup λ . Let $(\Lambda_w, \Lambda_{\hat{w}})$ be a pair of Schubert varieties in $G/P(\lambda)$ and $\hat{G}/\hat{P}(\lambda)$ such that $\iota^*([\Lambda_{\hat{w}}]) \cdot [\Lambda_w] = d \cdot [\text{pt}] \in H^*(G/P(\lambda), \mathbb{Z})$ for some positive integer d . By [BS00], for all $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$, we have $\langle w\lambda, \nu \rangle + \langle \hat{w}\lambda, \hat{\nu} \rangle \geq 0$. In particular, the set \mathcal{F} of $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$ such that $\langle w\lambda, \nu \rangle + \langle \hat{w}\lambda, \hat{\nu} \rangle = 0$ is a face of $\mathcal{LR}(G, \hat{G})$.

Theorem 10 *With above notation, we assume that $d \neq 1$ or that $\langle \hat{w}\lambda, \gamma_{\hat{w}}^{\hat{P}(\lambda)} \rangle + \langle w\lambda, \gamma_w^{P(\lambda)} \rangle \neq \langle \lambda, \gamma_e^{P(\lambda)} \rangle$.*

Then, the face \mathcal{F} does not contain any weight $(\nu, \hat{\nu})$ with ν strictly dominant.

Proof. Set $x = (wB/B, \hat{w}\hat{B}/\hat{B})$. Let C denote the irreducible component of X^λ containing x . By absurd, we assume that \mathcal{F} contains a weight $(\nu, \hat{\nu})$ with ν strictly dominant. According to Proposition 10, it remains to prove that (C, λ) is well covering to obtain a contradiction.

Consider the parabolic subgroup of \hat{G} containing \hat{B} such that $\mathcal{L}_{(\nu, \hat{\nu})}$ is an ample line bundle on $G/B \times \hat{G}/\hat{Q}$ (denoted \overline{X} from now on). Consider the natural $G \times \hat{G}$ -equivariant morphism $p : X \rightarrow \overline{X}$. Set $\overline{x} = p(x)$. Let \overline{C} denote the irreducible component of \overline{X}^λ containing \overline{x} and \overline{C}^+ the corresponding Bialinicki-Birula cell.

Since \overline{C} (resp. \overline{C}^+) is an orbit of $G^\lambda \times \hat{G}^\lambda$ (resp. $P(\lambda) \times \hat{P}(\lambda)$), we have $p(C) = \overline{C}$ and $p(C^+) = \overline{C}^+$. Since $\iota^*([\Lambda_{\hat{w}}]) \cdot [\Lambda_w] = d \cdot [\text{pt}]$, the proof of Lemma 14 shows that (C, λ) is dominant. We deduce that (\overline{C}, λ) is dominant. Now, Theorem 8 implies that (\overline{C}, λ) is well covering.

Let $y \in C$ general and $g \in G$ such that $g^{-1}y \in C^+$. Since $p(C) = \overline{C}$, $p(y)$ is general in \overline{C} . But, (\overline{C}, λ) is well covering and $g^{-1}p(y) \in \overline{C}^+$; so, $g \in P(\lambda)$. This proves that (C, λ) is well covering. \square

8.4 Application to the tensor product

8.4.1 — In this section, G is assumed to be semisimple. We also fix an integer $s \geq 2$ and set $\hat{G} = G^s$, $\hat{T} = T^s$ and $\hat{B} = B^s$. We embed G diagonally in \hat{G} . Then $\mathcal{SAC}^G(X) \cap X(T)^{s+1}$ identifies with the $(s+1)$ -uple $(\nu_1, \dots, \nu_{s+1}) \in X(T)^{s+1}$ such that the for n big enough $n\nu_i$'s are dominant weights and $V_{n\nu_1} \otimes \dots \otimes V_{n\nu_{s+1}}$ contains a non zero G -invariant vector.

The set of weights of T in $\hat{\mathfrak{g}}/\mathfrak{g}$ is simply the root system Φ of G . Let Δ be the set of simple roots of G for $T \subset B$. Let I be a part I of Δ . Let $L(I)$ denote the Levi subgroup of G containing T and having $\Delta - I$ as its simple

roots. Let T_I denote the neutral component of the center of $L(I)$; T_I is dominant. Note that the dimension of T_I is the cardinality of I . Moreover, all dominant admissible subtorus of T is obtained in such a way. We will also denote by $P(I)$ the standard parabolic subgroup with Levi subgroup $L(I)$. We denote by W_I the Weyl group $L(I)$.

Let λ be a dominant one parameter subgroup of T . For $(w_1, \hat{w}) = (w_1, \dots, w_{s+1}) \in W \times \hat{W} = W^{s+1}$, and $(\nu, \hat{\nu}) = (\nu_1, \dots, \nu_{s+1}) \in \text{Pic}^G(\mathbf{X})_{\mathbb{Q}} = X(T)_{\mathbb{Q}}^{s+1}$ we have:

- $r_T(\hat{w}\hat{\nu}) = \sum_{i=2}^{s+1} w_i \nu_i$, and $r_T(\hat{\theta}_{\hat{w}}^{\hat{P}(\lambda)}) = \sum_{i=2}^{s+1} \theta_{w_i}^{P(\lambda)}$,
- $\iota^*([\Lambda_{\hat{w}}^{\hat{P}(\lambda)}]) = [\Lambda_{w_2}^{P(\lambda)}] \cdot \dots \cdot [\Lambda_{w_{s+1}}^{P(\lambda)}]$,

In [BK06], Belkale and Kumar defined a new product denoted \odot_0 on the cohomology groups $H^*(G/P, \mathbb{Z})$ for any parabolic subgroup P of G . By Proposition 17 of [BK06], this product \odot_0 has the following very interesting property.

For $w_i \in W/W_{P(I)}$, the following are equivalent:

- (i) $[\Lambda_{w_1}^{P(I)}] \cdot \dots \cdot [\Lambda_{w_{s+1}}^{P(I)}] = [\Lambda_e^{P(I)}]$ and, the restriction of $\theta_{w_1}^{P(I)} + \dots + \theta_{w_{s+1}}^{P(I)} - \theta_e^{P(I)}$ to T_I is trivial;
- (ii) $[\Lambda_{w_1}^{P(I)}] \odot_0 \dots \odot_0 [\Lambda_{w_{s+1}}^{P(I)}] = [\Lambda_e^{P(I)}]$.

Using this result of Belkale and Kumar our Theorem 9 gives the following corollary. If α is a root of G , α^\vee denote the corresponding coroot. If α is a simple root, ω_{α^\vee} denote the corresponding fundamental weight.

Corollary 5 (i) A point $(\nu_1, \dots, \nu_{s+1}) \in X(T)_{\mathbb{Q}}^{s+1}$ belongs to the cone $\mathcal{LR}(G, G^s)$ if and only if

- (a) each ν_i is dominant; that is $\langle \alpha^\vee, \nu_i \rangle \geq 0$ for all simple root α .
- (b) for all simple root α ; for all $(w_1, \dots, w_{s+1}) \in (W/W_{P(\alpha)})^{s+1}$ such that $[\Lambda_{w_1}^{P(\alpha)}] \odot_0 \dots \odot_0 [\Lambda_{w_{s+1}}^{P(\alpha)}] = [\Lambda_e^{P(\alpha)}] \in H^*(G/P(\alpha), \mathbb{Z})$, we have:

$$\sum_i \langle w_i \omega_{\alpha^\vee}, \nu_i \rangle \geq 0.$$

- (ii) We assume either that $s \geq 3$ or \mathfrak{g} does not contain any factor of rank one. In the above description of $\mathcal{LR}(G, G^s)$, the inequalities are pairwise distinct and no one can be omitted (neither in (i)a nor (i)b).

(iii) Let \mathcal{F} be a face of $\mathcal{LR}(G, G^s)$ of codimension d which intersects the interior of the dominant chamber. There exist a subset I of d simple roots and $(w_1, \dots, w_{s+1}) \in (W/W_I)^{s+1}$ such that:

(a) $[\Lambda_{w_1}^{P(I)}] \odot_0 \cdots \odot_0 [\Lambda_{w_{s+1}}^{P(I)}] = [\Lambda_e^{P(I)}] \in H^*(G/P(I), \mathbb{Z}),$

(b) the subspace spanned by \mathcal{F} is the set $(\nu_1, \dots, \nu_{s+1}) \in X(T)_{\mathbb{Q}}^{s+1}$ such that:

$$\forall \alpha \in I \quad \sum_i \langle w_i \omega_{\alpha^\vee}, \nu_i \rangle = 0.$$

(iv) Conversely, let I be a subset of d simple roots and $(w_1, \dots, w_{s+1}) \in (W/W_I)^{s+1}$ such that $[\Lambda_{w_1}^{P(I)}] \odot_0 \cdots \odot_0 [\Lambda_{w_{s+1}}^{P(I)}] = [\Lambda_e^{P(I)}] \in H^*(G/P(I), \mathbb{Z})$. Then, the set of $(\nu_1, \dots, \nu_{s+1}) \in \mathcal{LR}(G, G^s)$ such that

$$\forall \alpha \in I \quad \sum_i \langle w_i \omega_{\alpha^\vee}, \nu_i \rangle = 0,$$

is a face of codimension d of $\mathcal{LR}(G, G^s)$.

Proof. Equations (i)a are all different and are not repeated in Equations (i)b. Moreover, by [MR08, Proposition 7] they define codimension one faces of $\mathcal{LR}(G, G^s)$.

The rest of the corollary is a simple rephrasing of Theorem 9 and Corollary 4. \square

Remark. The description of the smaller faces of $\mathcal{C}^G((G/B)^{s+1})$ gives an application of the Belkale-Kumar product \odot_0 for all the complete homogeneous spaces.

References

- [BB73] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. **98** (1973), 480–497.
- [Bel07] Prakash Belkale, *Geometric proof of a conjecture of Fulton*, Adv. Math. **216** (2007), no. 1, 346–357.
- [BK06] Prakash Belkale and Shrawan Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. **166** (2006), no. 1, 185–228.

- [Bri99] Michel Brion, *On the general faces of the moment polytope*, Internat. Math. Res. Notices (1999), no. 4, 185–201.
- [BS00] Arkady Berenstein and Reyer Sjamaar, *Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion*, J. Amer. Math. Soc. **13** (2000), no. 2, 433–466 (electronic).
- [DH98] Igor V. Dolgachev and Yi Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 5–56, With an appendix by Nicolas Ressayre.
- [É92] A. G. Élashvili, *Invariant algebras*, Lie groups, their discrete subgroups, and invariant theory, Adv. Soviet Math., vol. 8, Amer. Math. Soc., Providence, RI, 1992, pp. 57–64.
- [FHT84] Knop F., Kraft H., and Vust T., *The Picard group of a G -variety*, Algebraic transformation group and invariant theory, Birkhäuser, 1984, pp. 77–87.
- [GD66] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schéma, troisième partie*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 28, 5–255.
- [Hes79] W. Hesselink, *Desingularization of varieties of null forms*, Inven. Math. **55** (1979), 141–163.
- [Kem78] G. Kempf, *Instability in invariant theory*, Ann. of Math. **108** (1978), 2607–2617.
- [Kir84] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Princeton University Press, Princeton, N.J., 1984.
- [KKM06] Misha Kapovich, Shrawan Kumar, and John Millson, *Saturation and irredundancy for $spin(8)$* , Preprint <http://www.math.ucdavis.edu/~kapovich/eprints.html> (2006), 1–28.
- [Kle74] Steven L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297.
- [Kly98] Alexander A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Math. (N.S.) **4** (1998), no. 3, 419–445.

- [KTW04] Allen Knutson, Terence Tao, and Christopher Woodward, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), no. 1, 19–48 (electronic).
- [LR79] D. Luna and R. W. Richardson, *A generalization of the Chevalley restriction theorem*, Duke Math. J. **46** (1979), no. 3, 487–496.
- [Lun73] Domingo Luna, *Slices étales*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33.
- [Lun75] D. Luna, *Adhérences d’orbite et invariants*, Invent. Math. **29** (1975), no. 3, 231–238.
- [Man97] Laurent Manivel, *Applications de Gauss et pléthysme*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 3, 715–773.
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3d ed., Springer Verlag, New York, 1994.
- [MR08] Pierre-Louis Montagard and Nicolas Ressayre, *Sur des faces du lr -cône généralisé*, Bull. SMF (à paraître) (2008), no. arXiv:math/0406272, 1–19.
- [Nes78] L. Ness, *Mumford’s numerical function and stable projective hypersurfaces*, Algebraic geometry (Copenhagen), Lecture Notes in Math., 1978.
- [Nes84] ———, *A stratification of the null cone via the moment map*, Amer. Jour. of Math. **106** (1984), 1281–1325.
- [Oda88] Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988, An introduction to the theory of toric varieties, Translated from the Japanese.
- [PV91] V. L. Popov and È. B. Vinberg, *Algebraic Geometry IV*, Encyclopedia of Mathematical Sciences, vol. 55, ch. Invariant Theory, pp. 123–284, Springer-Verlag, 1991.
- [Res00] N. Ressayre, *The GIT-equivalence for G -line bundles*, Geom. Dedicata **81** (2000), no. 1-3, 295–324.

- [Ric82] R. W. Richardson, *On orbits of algebraic groups and Lie groups*, Bull. Austral. Math. Soc. **25** (1982), no. 1, 1–28.
- [Sch03] Alexander Schmitt, *A simple proof for the finiteness of GIT-quotients*, Proc. Amer. Math. Soc. **131** (2003), no. 2, 359–362 (electronic).
- [Ver96] Michèle Vergne, *Convex polytopes and quantization of symplectic manifolds*, Proc. Nat. Acad. Sci. U.S.A. **93** (1996), no. 25, 14238–14242, National Academy of Sciences Colloquium on Symmetries Throughout the Sciences (Irvine, CA, 1996).

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