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# Geometric Invariant Theory and Generalized Eigenvalue Problem

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## Abstract

Let  $G$  be a connected reductive subgroup of a complex connected reductive group  $\hat{G}$ . Fix maximal tori and Borel subgroups of  $G$  and  $\hat{G}$ . Consider the cone  $\mathcal{LR}(\hat{G}, G)$  generated by the pairs  $(\nu, \hat{\nu})$  of dominant characters such that  $V_\nu$  is a submodule of  $V_{\hat{\nu}}$  (with usual notation). Here we give a minimal set of inequalities describing  $\mathcal{LR}(\hat{G}, G)$  as a part of the dominant chamber.

In way, we obtain results about the faces of the Dolgachev-Hu's  $G$ -ample cone and variations of this cone.

## 1 Introduction

Let  $G$  be a connected reductive subgroup of a connected reductive group  $\hat{G}$  both defined over an algebraic closed field  $\mathbb{K}$  of characteristic zero. We consider the following question:

What irreducible representations of  $G$  appear in a given irreducible representation of  $\hat{G}$  ?

Let  $T \subset B$  (resp.  $\hat{T} \subset \hat{B}$ ) be a maximal torus and a Borel subgroup of  $G$  (resp.  $\hat{G}$ ). Let  $X(T)$  and  $X(\hat{T})$  denote the groups of characters of  $T$  and  $\hat{T}$ . For a dominant character  $\nu$  (resp.  $\hat{\nu}$ ) of  $T$  (resp.  $\hat{T}$ ), we denote by  $V_\nu$  (resp.  $V_{\hat{\nu}}$ ) the irreducible representation of  $G$  (resp.  $\hat{G}$ ). The question is to understand the set  $\text{LR}(\hat{G}, G)$  of pairs  $(\nu, \hat{\nu})$  of dominant characters such that  $V_\nu^*$  (the dual of  $V_\nu$ ) is a submodule of  $V_{\hat{\nu}}$ , viewed as a  $G$ -module. By a result of M. Brion and F. Knop (see [É92]),  $\text{LR}(\hat{G}, G)$  is a finitely generated submonoid of  $X(T) \times X(\hat{T})$ . Our purpose is to study the linear inequalities satisfied by this monoid. More precisely, we consider the

convex cone  $\mathcal{LR}(\hat{G}, G)$  generated by  $\text{LR}(\hat{G}, G)$  in the rational vector space  $(X(T) \times X(\hat{T})) \otimes \mathbb{Q}$ . This cone is characterized by finitely many inequalities that we want determine.

If  $\hat{G} = G^s$ , an irreducible representation of  $\hat{G}$  is simply a tensor product of  $s$  irreducible representations of  $G$ . So, if  $G$  is diagonally embedded in  $G^s$ , the concerned restriction problem is the problem of decomposition of the tensor product. In the case when  $G = \text{GL}_n$  diagonally embedded in  $\text{GL}_n \times \text{GL}_n$  the cone  $\mathcal{LR}(\text{GL}_n \times \text{GL}_n, \text{GL}_n)$  has surprisingly numerous interpretations (see [Ful00]). One of them is the Horn's problem:  $\mathcal{LR}(\text{GL}_n \times \text{GL}_n, \text{GL}_n)$  is the set of spectrum of three complex Hermitian matrices  $A, B$  and  $C$  with  $A + B + C = 0$ .

Let us introduce some notation. Let  $Y(T)$  denote the group of one parameter subgroups of  $T$ . We denote by  $\langle \cdot, \cdot \rangle$ , the natural paring between  $Y(T)$  and  $X(T)$ . We extend this paring linearly  $\langle \cdot, \cdot \rangle : Y(T) \times (X(T) \otimes \mathbb{Q}) \rightarrow \mathbb{Q}$ . Let  $W$  denote the Weyl group of  $(G, T)$ . Let  $\lambda$  be a one parameter subgroup of  $T$ . We denote by  $P(\lambda)$  the associated parabolic subgroup (see Section 3.6) and  $W_{P(\lambda)}$  its Weyl group. Consider the homogeneous space  $G/P(\lambda)$ . For  $w \in W/W_{P(\lambda)}$ , we consider the Schubert class  $[\Lambda_w] \in H^*(G/P(\lambda), \mathbb{Z})$  of  $\overline{BwP(\lambda)/P(\lambda)}$ .

Let  $\rho$  be the half sum of positive roots of  $G$ . Let  $w \in W/W_{P(\lambda)}$ . Let  $\tilde{w} \in W$  denote the longest element of the class of  $w$ . We set  $\chi_w = \rho + \tilde{w}^{-1}\rho$ .

We also consider  $\hat{W}, \hat{P}(\lambda), [\Lambda_{\hat{w}}]$  and  $\chi_{\hat{w}}$ , in the same way but with  $\hat{G}$  in place of  $G$ . Let us also assume that  $T \subset \hat{T}$  and denote by  $r_T : X(\hat{T}) \rightarrow X(T)$  the restriction map. Since  $P(\lambda) = G \cap \hat{P}(\lambda)$ ,  $G/P(\lambda)$  is embedded in  $\hat{G}/\hat{P}(\lambda)$ . Consider the morphism  $\iota^* : H^*(\hat{G}/\hat{P}(\lambda), \mathbb{Z}) \rightarrow H^*(G/P(\lambda), \mathbb{Z})$  induced in cohomology by this embedding.

Let  $\mathfrak{g}$  and  $\hat{\mathfrak{g}}$  denote the Lie algebras of  $G$  and  $\hat{G}$ . A one parameter subgroup of  $T$  which is not equal to  $n\lambda$  for an integer  $n \geq 2$  and another one parameter subgroup  $\lambda$  of  $T$  is said to be indivisible. A one parameter subgroup of  $T$  whose the image is the neutral component of the intersection of the kernels of weights of  $T$  in  $\hat{\mathfrak{g}}/\mathfrak{g}$  is said to be admissible. Let us consider the set  $\lambda_1, \dots, \lambda_n$  of indivisible dominant admissible one parameter subgroups of  $T$ .

Corollary 2 of the paper can now be stated (with slightly different notation):

**Theorem A** *We assume that no ideal of  $\mathfrak{g}$  is an ideal of  $\hat{\mathfrak{g}}$ . Then,  $\mathcal{LR}(\hat{G}, G)$  has non empty interior in  $(X(\hat{T}) \times X(T)) \otimes \mathbb{Q}$ .*

A dominant weight  $(\hat{\nu}, \nu)$  belongs to  $\mathcal{LR}(\hat{G}, G)$  if and only if

$$\langle \lambda_i, \hat{w}^{-1}\hat{\nu} + w^{-1}\nu \rangle \geq 0,$$

for all  $i = 1, \dots, n$  and for all  $(\hat{w}, w) \in \hat{W}/W_{\hat{P}(\lambda)} \times W/W_{P(\lambda)}$  such that

$$(i) \ \iota^*([\Lambda_{\hat{w}}]).[\Lambda_w] = [\Lambda_e] \in H^*(G/P(\lambda), \mathbb{Z}), \text{ and}$$

$$(ii) \ \langle \lambda_i, r_T(\hat{\chi}_{\hat{w}}) + \chi_w - \chi_e \rangle = 0.$$

Moreover, the above inequalities are pairwise distinct and no one can be omitted.

In [BS00], Berenstein and Sjamaar showed that  $(\hat{\nu}, \nu)$  belongs to  $\mathcal{LR}(\hat{G}, G)$  if and only if  $\langle \lambda_i, \hat{w}^{-1}\hat{\nu} + w^{-1}\nu \rangle \geq 0$ , for all  $i = 1, \dots, n$  and for all  $(\hat{w}, w) \in \hat{W}/W_{\hat{P}(\lambda)} \times W/W_{P(\lambda)}$  such that  $\iota^*([\Lambda_{\hat{w}}]).[\Lambda_w] = d.[\Lambda_e] \in H^*(G/P(\lambda), \mathbb{Z})$  for some positive integer  $d$ . Theorem A improves the result of Berenstein and Sjamaar by selecting exactly the essential inequalities in their list. In the case when  $G$  is diagonally embedded in  $G^s$ , Kapovich already proved that one may assume that  $d = 1$ . Again in the case when  $G$  is diagonally embedded in  $G^s$ , Belkale and Kumar obtained the same inequalities as in Theorem A in [BK06] but without proving that this list is optimal. In fact Theorem A were known in some particular cases. Indeed, Knutson, Tao and Woodward shown in [KTW04] the case when  $G = \mathrm{SL}_n$  is diagonally embedded in  $\mathrm{SL}_n \times \mathrm{SL}_n$  by combinatorial tools. Using the interpretation of the Littlewood-Richardson coefficients as structure coefficient of the cohomology ring of the Grassmann varieties, Belkale made a geometric proof of Knutson-Tao-Woodward's result (see [Bel05]). Using explicit calculation with the help of a computer, Kapovich, Kumar and Millson proves the case when  $G = \mathrm{SO}(8)$  is diagonally embedded in  $G \times G$  in [KKM06].

Theorem A can be thought as a description of the faces of codimension one of  $\mathcal{LR}(\hat{G}, G)$  which intersect the interior of the dominant chamber. In Theorem 10 below, we study the smaller faces of  $\mathcal{LR}(\hat{G}, G)$ . In the introduction, we only give our results about these faces in the case when  $G$  is diagonally embedded in  $G^s$  for an integer  $s \geq 2$ ; that is, in the case of decomposition of the tensor product.

We need some notation. If  $\alpha$  is a simple root of  $G$ ,  $\omega_{\alpha^\vee}$  denote the corresponding fundamental weight. If  $I$  is a set of simple roots,  $P(I)$  denote the standard parabolic subgroup associated to  $I$  and  $W_I$  its Weyl group. In [BK06], Belkale and Kumar defined a new product  $\odot_0$  on the cohomology groups  $H^*(G/P, \mathbb{Z})$ .

**Theorem B** We assume that  $G$  is semisimple diagonally embedded in  $\hat{G} = G^s$  for an integer  $s \geq 2$ .

- (i) Let  $I$  be a set of  $d$  simple roots and  $(w_1, \dots, w_{s+1}) \in (W/W_I)^{s+1}$  such that  $[\Lambda_{w_1}] \odot_0 \cdots \odot_0 [\Lambda_{w_{s+1}}] = [\Lambda_e] \in H^*(G/P(I), \mathbb{Z})$ . Then, the set of  $(\nu_1, \dots, \nu_{s+1}) \in \mathcal{LR}(G^s, G)$  such that

$$\forall \alpha \in I \quad \sum_i \langle \omega_{\alpha^\vee}, w_i^{-1} \nu_i \rangle = 0,$$

is a face of codimension  $d$  of  $\mathcal{LR}(G^s, G)$ .

- (ii) Any face of  $\mathcal{LR}(G^s, G)$  intersecting the interior of the dominant chamber of  $G^{s+1}$  is obtained in this way.

Let us explain the starting point of the proofs of Theorems A and B. Consider the variety  $X = \hat{G}/\hat{B} \times G/B$  endowed with the diagonal action of  $G$ . To any character  $(\hat{\nu}, \nu) \in X(\hat{T}) \times X(T)$ , one associate a  $G$ -linearized line bundle  $\mathcal{L}_{(\hat{\nu}, \nu)}$  on  $X$  such that  $H^0(X, \mathcal{L}_{(\hat{\nu}, \nu)}) = V_{\hat{\nu}} \otimes V_{\nu}$ . Hence,  $(\hat{\nu}, \nu)$  belongs to  $\mathcal{LR}(\hat{G}, G)$  if and only if a positive power of  $\mathcal{L}_{(\hat{\nu}, \nu)}$  admit a non zero  $G$ -invariant section. If  $\mathcal{L}_{(\hat{\nu}, \nu)}$  is ample this is equivalent to saying that  $X$  admit semistable points for  $\mathcal{L}_{(\hat{\nu}, \nu)}$ . So, one can use classical results of Geometric Invariant Theory as Hilbert-Mumford's Theorem and Luna's Slice Etale Theorem. This method was already used in [Kly98, BS00, BK06].

We obtain results in the following more general context. Consider a connected reductive group  $G$  acting on a projective variety  $X$ . To any  $G$ -linearized line bundle  $\mathcal{L}$  on  $X$  we associate the following open subset  $X^{\text{ss}}(\mathcal{L})$  of  $X$ :

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \sigma \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\}.$$

The points of  $X^{\text{ss}}(\mathcal{L})$  are said to be *semistable* for  $\mathcal{L}$ . Note that if  $\mathcal{L}$  is not ample, this notion of semistability is not the standard one. In particular, the quotient  $\pi_{\mathcal{L}} : X^{\text{ss}}(\mathcal{L}) \longrightarrow X^{\text{ss}}(\mathcal{L})//G$  is a good quotient only if  $\mathcal{L}$  is ample. In this context, we ask for:

What are the  $\mathcal{L}$ 's with non empty set  $X^{\text{ss}}(\mathcal{L})$  ?

Let us fix a freely finitely generated subgroup  $\Lambda$  of the group  $\text{Pic}^G(X)$  of  $G$ -linearized line bundles on  $X$ . Let  $\Lambda_{\mathbb{Q}}$  denote the  $\mathbb{Q}$ -vector space containing  $\Lambda$

as a lattice. The central objects of this article are the convex cones  $\mathcal{TC}_\Lambda^G(X)$  (resp.  $\mathcal{AC}_\Lambda^G(X)$ ) generated in  $\Lambda_\mathbb{Q}$  by the  $\mathcal{L}$ 's (resp. the ample  $\mathcal{L}$ 's) in  $\Lambda$  which have non zero  $G$ -invariant sections.

We need to introduce a definition due to D. Luna. Assume that  $X$  is smooth. Let  $\mathcal{O}$  be an orbit of  $G$  in  $X$ . For  $x \in \mathcal{O}$ , we consider the action of the isotropy  $G_x$  on the normal space  $N_x$  of  $\mathcal{O}$  in  $X$  at  $x$ . The pair  $(G_x, N_x)$  is called the *type* of the orbit  $\mathcal{O}$  and is defined up to conjugacy by  $G$ . The main part of Theorem 6 is:

**Theorem C** *We assume that  $X$  is smooth. Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_\Lambda^G(X)$ .*

*Then, the type of the closed orbit in  $\pi_{\mathcal{L}}^{-1}(\xi)$  for  $\xi \in X^{\text{ss}}(\mathcal{L})//G$  general does not depend on the choice of an ample  $G$ -linearized line bundle  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ .*

*We will call this type the type of  $\mathcal{F}$ .*

Theorem 7 needs too many notation to be stated in this introduction. It describes the local geometry of  $\mathcal{AC}_\Lambda^G(X)$  around a face in terms of its type.

Let  $\lambda$  be a one parameter subgroup of  $G$  and  $C$  be an irreducible component of its fix points. Consider  $C^+ = \{x \in X \mid \lim_{t \rightarrow 0} \lambda(t)x \in C\}$  and the natural  $G$ -equivariant map  $\eta : G \times_{P(\lambda)} C^+ \rightarrow X$ . The pair  $(C, \lambda)$  is said to be *well covering* if  $\eta$  induces an isomorphism over an open subset of  $X$  intersecting  $C$ .

For  $\mathcal{L} \in \text{Pic}^G(X)$ , we denote by  $\mu^{\mathcal{L}}(C, \lambda)$  the integer giving the action of  $\lambda$  on the restriction of  $\mathcal{L}$  on  $C$ .

**Theorem D** *Let  $T \subset B$  a maximal torus and a Borel subgroup of  $G$ .*

*Then, an ample  $G$ -linearized line bundle  $\mathcal{L} \in \Lambda$  belongs to  $\mathcal{AC}_\Lambda^G(X)$  if and only if for all well covering pair  $(C, \lambda)$  with a dominant one parameter subgroup  $\lambda$  of  $T$  we have  $\mu^{\mathcal{L}}(C, \lambda) \leq 0$ .*

Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_\Lambda^G(X)$  of codimension one in  $\mathcal{AC}_\Lambda^G(X)$ . Theorem D shows that there exists a well covering pair  $(C, \lambda)$  such that  $\mu^{\mathcal{L}}(C, \lambda) \leq 0$  for all  $\mathcal{L} \in \mathcal{AC}_\Lambda^G(X)$  and  $\mathcal{F}$  is the set of the  $\mathcal{L}$ 's in  $\mathcal{AC}_\Lambda^G(X)$  such that  $\mu^{\mathcal{L}}(C, \lambda) = 0$ . Theorem 8 below establishes such relations between all the faces of  $\mathcal{AC}_\Lambda^G(X)$  and pairs  $(C, \lambda)$ . The most useful part for the description of  $\mathcal{LR}(\hat{G}, G)$  made in Theorem 8 is

**Theorem E** *Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_\Lambda^G(X)$  of codimension  $d$  in  $\Lambda \otimes \mathbb{Q}$ . Let  $(H, L)$  be the type of  $\mathcal{F}$ . We assume that the neutral component  $H^\circ$  of  $H$  is a torus of dimension  $d$ .*

Then, there exists an irreducible component  $C$  of the set of fix points of  $H^\circ$  in  $X$ , and a one parameter subgroup  $\lambda$  of  $H^\circ$  such that

(i)  $(C, \lambda)$  is well covering, and

(ii)  $\mathcal{F}$  is the set of the  $\mathcal{L}$ 's in  $\mathcal{AC}_\Lambda^G(X)$  such that  $\mu^{\mathcal{L}}(C, \lambda) = 0$ .

We now assume that  $\Lambda \otimes \mathbb{Q} = \text{Pic}^G(X) \otimes \mathbb{Q}$ . Notice that if  $\text{Pic}(X)$  is not finitely generated it can be replaced by the Neron-Severi group (see [DH98]). We denote  $\mathcal{TC}_\Lambda^G(X)$  by  $\mathcal{TC}^G(X)$ . We are interested in a kind of converse of Theorem E. So, we fix a well covering pair  $(C, \lambda)$ . Consider the commutator  $G^\lambda$  of the image of  $\lambda$ . Since it is connected, it acts on  $C$ .

**Theorem F** *With above notation, consider the linear map  $\rho$  induced by the restriction:*

$$\rho : \text{Pic}^G(X) \otimes \mathbb{Q} \longrightarrow \text{Pic}^{G^\lambda}(C) \otimes \mathbb{Q}.$$

*Then, the subspace of  $\text{Pic}^G(X) \otimes \mathbb{Q}$  spanned by the  $\mathcal{L} \in \mathcal{TC}^G(X)$  such that  $\mu^{\mathcal{L}}(C, \lambda) = 0$  is the pullback by  $\rho$  of the subspace of  $\text{Pic}^{G^\lambda}(C) \otimes \mathbb{Q}$  spanned by  $\mathcal{TC}^{G^\lambda}(C)$ .*

A particular case is specially interesting. Indeed, assume that the variety  $X$  equals  $Y \times G/B$ , for a  $G$ -variety  $Y$ . Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $Y$ . Let  $\Lambda$  be the subgroup of  $\text{Pic}^G(X)$  generated by the pullback of  $\mathcal{L}$  and the pullbacks of the  $G$ -linearized line bundles on  $G/B$ . Then,  $\mathcal{TC}_\Lambda^G(X)$  is a cone over the moment polytope  $P(Y, \mathcal{L})$  defined in [Bri99]; so, the faces of  $\mathcal{TC}_\Lambda^G(X)$  correspond bijectively to the faces of  $P(Y, \mathcal{L})$ .

Following [Bri99], we show in Proposition 13 below, that any moment polytope  $P(Y, \mathcal{L})$  can be describe in terms of one which intersects the interior of the dominant chamber. We now assume that  $P(Y, \mathcal{L})$  intersects the interior of the dominant chamber and that  $Y$  is smooth. In Proposition 14 below, we associate to each face of  $P(Y, \mathcal{L})$  which intersects the interior of the dominant chamber a well  $B$ -covering (see Definition 9.3) pair of  $Y$  improving (with stronger assumptions) [Bri99, Theorem 1 and 2].

In Section 2, we fix notation about parabolic fiber products and prove a useful result on their linearized Picard group. Section 3 is concerned by the Hilbert-Mumford's numerical criterion of semistability. In Section 4, we recall some useful results about the Bialinicki-Birula's cells. In Section 5, we introduce the notion of well covering pair and prove their first relations with the  $G$ -cones. Section 6 recalls some useful consequences of Luna's Slice

Etale Theorem. In Section 7, our general results about the faces of the  $G$ -cones are obtained. In Section 8, we apply our results to the cone  $\mathcal{LR}(\hat{G}, G)$  essentially by making more explicit the notion of well covering pair. The last Section 9 is an application of our results to moment polytopes.

**Convention.** The ground field  $\mathbb{K}$  is assumed to be algebraic closed of characteristic zero. The notation introduced in the environments “**Notation.**” are fixed for all the sequence of the article.

## 2 Preliminaries on parabolic fiber products

**Notation.** Let  $\mathbb{K}^*$  denote the multiplicative group of  $\mathbb{K}$ . If  $G$  is an affine algebraic group,  $X(G)$  denotes the group of characters of  $G$ ; that is, of algebraic group homomorphisms from  $G$  on  $\mathbb{K}^*$ . If  $G$  acts algebraically on a variety  $X$ ,  $X$  is said to be a  $G$ -variety. As in [MFK94], we denote by  $\text{Pic}^G(X)$  the group of  $G$ -linearized line bundles on  $X$ . If  $\mathcal{L} \in \text{Pic}^G(X)$ ,  $H^0(X, \mathcal{L})$  denotes the  $G$ -module of regular sections of  $\mathcal{L}$ .

In this section we collect some useful properties of the fiber product. Let us fix a reductive group  $G$  and a parabolic subgroup  $P$  of  $G$ .

### 2.1 Construction

Let  $Y$  be a  $P$ -variety. Consider over  $G \times Y$  the action of  $G \times P$  given by the formula (with obvious notation):

$$(g, p) \cdot (g', y) = (gg'p^{-1}, py).$$

Since the quotient map  $G \rightarrow G/P$  is a Zariski-locally trivial principal  $P$ -bundle; one can easily construct a quotient  $G \times_P Y$  of  $G \times Y$  by the action of  $\{e\} \times P$ . The action of  $G \times \{e\}$  induces an action of  $G$  on  $G \times_P Y$ . Moreover, the first projection  $G \times Y \rightarrow G$  induces a  $G$ -equivariant map  $G \times_P Y \rightarrow G/P$  which is a locally trivial fibration with fiber  $Y$ .

The class of a pair  $(g, y) \in G \times Y$  in  $G \times_P Y$  is denoted by  $[g : y]$ . If  $Y$  is a  $P$ -stable locally closed subvariety of a  $G$ -variety  $X$ , it is well known that the map

$$\begin{aligned} G \times_P Y &\longrightarrow G/P \times X \\ [g : y] &\longmapsto (gP, gy) \end{aligned}$$

is an isomorphism onto the set of the  $(gP, x) \in G/P \times X$  such that  $g^{-1}x \in Y$ .



Let  $\nu$  be a character of  $P$ . If  $Y$  is the field  $\mathbb{K}$  endowed with the action of  $P$  defined by  $p \cdot \tau = \nu(p^{-1})\tau$  for all  $\tau \in \mathbb{K}$  and  $p \in P$ ,  $G \times_P Y$  is a  $G$ -linearized line bundle on  $G/P$ . We denote by  $\mathcal{L}_\nu$  this element of  $\text{Pic}^G(G/P)$ . In fact, the map  $X(P) \rightarrow \text{Pic}^G(G/P)$ ,  $\nu \mapsto \mathcal{L}_\nu$  is an isomorphism.

Let  $B$  be a Borel subgroup of  $G$  contained in  $P$ , and  $T$  be a maximal torus contained in  $B$ . Then,  $X(P)$  identifies with a subgroup of  $X(T)$  which contains dominant weights. For  $\nu \in X(P)$ ,  $\mathcal{L}_\nu$  is generated by its sections if and only if it has non zero sections if and only if  $\nu$  is dominant. Moreover,  $H^0(G/P, \mathcal{L}_\nu)$  is the simple  $G$ -module of highest weight  $\nu$ . For  $\nu$  dominant,  $\mathcal{L}_\nu$  is ample if and only if  $\nu$  cannot be extended to a subgroup of  $G$  bigger than  $P$ .

## 2.2 Line bundles

We are now interested in the  $G$ -linearized line bundles on  $G \times_P Y$ .

**Lemma 1** *With above notation, we have:*

(i) *The map  $\mathcal{L} \mapsto G \times_P \mathcal{L}$  defines a morphism*

$$e : \text{Pic}^P(Y) \rightarrow \text{Pic}^G(G \times_P Y).$$

(ii) *The map  $\iota : Y \rightarrow G \times_P Y$ ,  $y \mapsto [e : y]$  is a  $P$ -equivariant immersion. We denote by  $\iota^* : \text{Pic}^G(G \times_P Y) \rightarrow \text{Pic}^P(Y)$  the associated restriction homomorphism.*

(iii) *The morphisms  $e$  and  $\iota^*$  are the inverse one of each other; in particular, they are isomorphisms.*

(iv) *For any  $\mathcal{L} \in \text{Pic}^G(G \times_P Y)$ , the restriction map from  $H^0(G \times_P Y, \mathcal{L})$  to  $H^0(Y, \iota^*(\mathcal{L}))$  induces a linear isomorphism*

$$H^0(G \times_P Y, \mathcal{L})^G \simeq H^0(Y, \iota^*(\mathcal{L}))^P.$$

**Proof.** Let  $\mathcal{M}$  be a  $P$ -linearized line bundle on  $Y$ . Since  $G \times \mathcal{M} \rightarrow G \times_P \mathcal{M}$  is a categorical quotient, we have the following commutative diagram:

$$\begin{array}{ccc} G \times \mathcal{M} & \longrightarrow & G \times_P \mathcal{M} \\ \downarrow & & \downarrow p \\ G \times Y & \longrightarrow & G \times_P Y. \end{array}$$

Since  $G \rightarrow G/P$  is locally trivial, the map  $p$  endows  $G \times_P \mathcal{M}$  with a structure of line bundle on  $G \times_P Y$ . Moreover, the action of  $G$  on  $G \times_P \mathcal{M}$

endows this line bundle with a  $G$ -linearization. This proves Assertion 1. The second one is obvious.

By construction, the restriction of  $G \times_P \mathcal{M}$  to  $Y$  is  $\mathcal{M}$ . So,  $\iota^* \circ e$  is the identity map. Conversely, let  $\mathcal{L} \in \text{Pic}^G(G \times_P Y)$ . Then, we have:

$$e \circ \iota^*(\mathcal{L}) \simeq \{(gP, l) \in G/P \times \mathcal{L} : g^{-1}l \in \mathcal{L}|_Y\}.$$

The second projection induces an isomorphism from  $e \circ \iota^*(\mathcal{L})$  onto  $\mathcal{L}$ . This ends the proof of Assertion 3.

The map  $\text{H}^0(G \times_P Y, \mathcal{L})^G \longrightarrow \text{H}^0(Y, \iota^*(\mathcal{L}))^P$  is clearly well defined and injective. Let us prove the surjectivity. Let  $\sigma \in \text{H}^0(Y, \iota^*(\mathcal{L}))^P$ . Consider the morphism

$$\begin{aligned} \hat{\sigma} : G \times Y &\longrightarrow G \times_P \mathcal{L} \\ (g, y) &\longmapsto [g : \sigma(y)]. \end{aligned}$$

Since  $\sigma$  is  $P$ -invariant, so is  $\hat{\sigma}$ ; and  $\hat{\sigma}$  induces a section of  $G \times_P \mathcal{L}$  over  $G \times_P Y$  which is  $G$ -invariant and extends  $\sigma$ .  $\square$

### 3 Numerical criterion of Hilbert-Mumford

We will use classical results in Geometric Invariant Theory (GIT) about the numerical criterion of Hilbert-Mumford. In this section, we present these results and give some useful complements. Let us fix a connected reductive group  $G$  acting on an irreducible projective algebraic variety  $X$ .

#### 3.1 An Ad Hoc notion of semistability

As in the introduction, for any  $G$ -linearized line bundle  $\mathcal{L}$  on  $X$ , we consider the following set of *semistable points*:

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \sigma \in \text{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\}.$$

To precise the acting group, we sometimes denote  $X^{\text{ss}}(\mathcal{L})$  by  $X^{\text{ss}}(\mathcal{L}, G)$ .

The subset  $X^{\text{ss}}(\mathcal{L})$  is open and stable by  $G$ . A point  $x$  which is not semistable is said to be *unstable*; and, we set  $X^{\text{us}}(\mathcal{L}) = X - X^{\text{ss}}(\mathcal{L})$ .

**Remark.** Note that this definition of  $X^{\text{ss}}(\mathcal{L})$  is NOT standard. Indeed, one usually imposes that the open subset defined by the non vanishing of  $\sigma$  to be affine. This property which is useful to construct a good quotient is automatic if  $\mathcal{L}$  is ample but not in general; hence, our definition coincides with the usual one if  $\mathcal{L}$  is ample.

If  $\mathcal{L}$  is ample, there exists a categorical quotient:

$$\pi : X^{\text{ss}}(\mathcal{L}) \longrightarrow X^{\text{ss}}(\mathcal{L})//G,$$

such that  $X^{\text{ss}}(\mathcal{L})//G$  is a projective variety and  $\pi$  is affine. A point  $x \in X^{\text{ss}}(\mathcal{L})$  is said to be *stable* if  $G_x$  is finite and  $G.x$  is closed in  $X^{\text{ss}}(\mathcal{L})$ . Then, for all stable point  $x$  we have  $\pi^{-1}(\pi(x)) = G.x$ ; and the set  $X^s(\mathcal{L})$  of stable points is open in  $X$ .

### 3.2 The functions $\mu^\bullet(x, \lambda)$

Let  $\mathcal{L} \in \text{Pic}^G(X)$ . Let  $x$  be a point in  $X$  and  $\lambda$  be a one parameter subgroup of  $G$ . Since  $X$  is complete,  $\lim_{t \rightarrow 0} \lambda(t)x$  exists; let  $z$  denote this limit. The image of  $\lambda$  fixes  $z$  and so the group  $\mathbb{K}^*$  acts via  $\lambda$  on the fiber  $\mathcal{L}_z$ . This action defines a character of  $\mathbb{K}^*$ , that is, an element of  $\mathbb{Z}$  denoted by  $\mu^\mathcal{L}(x, \lambda)$ . One can immediately prove that the numbers  $\mu^\mathcal{L}(x, \lambda)$  satisfy the following properties:

- (i)  $\mu^\mathcal{L}(g \cdot x, g \cdot \lambda \cdot g^{-1}) = \mu^\mathcal{L}(x, \lambda)$  for any  $g \in G$ ;
- (ii) the map  $\mathcal{L} \mapsto \mu^\mathcal{L}(x, \lambda)$  is a homomorphism from  $\text{Pic}^G(X)$  to  $\mathbb{Z}$ ;
- (iii) for any  $G$ -variety  $Y$  and for any  $G$ -equivariant morphism  $f : X \rightarrow Y$ ,  $\mu^{f^*(\mathcal{L})}(x, \lambda) = \mu^\mathcal{L}(f(x), \lambda)$ , where  $x \in X$  and  $\mathcal{L}$  is a  $G$ -linearized line bundle on  $Y$ .

A less direct property of the function  $\mu^\mathcal{L}(x, \lambda)$  is

**Proposition 1** *Let  $\mathcal{L}$ ,  $x$ ,  $\lambda$  and  $z$  be as above. Let  $\tilde{x}$  be a non zero point in the fiber in  $\mathcal{L}$  over  $x$ . We embed  $X$  in  $\mathcal{L}$  by the zero section. Then, we have*

- (i) *if  $\mu^\mathcal{L}(x, \lambda) > 0$ , then  $\lambda(t)\tilde{x}$  tends to  $z$  when  $t \rightarrow 0$ ;*
- (ii) *if  $\mu^\mathcal{L}(x, \lambda) = 0$ , then  $\lambda(t)\tilde{x}$  tends to a non zero point  $\tilde{z}$  in the fiber in  $\mathcal{L}$  over  $z$  when  $t \rightarrow 0$ ;*
- (iii) *if  $\mu^\mathcal{L}(x, \lambda) < 0$ , then  $\lambda(t)\tilde{x}$  has no limit in  $\mathcal{L}$  when  $t \rightarrow 0$ .*

**Proof.** Set  $V = \{\lambda(t).x \mid t \in \mathbb{K}^*\} \cup \{z\}$ : it is a locally closed subvariety of  $X$  stable by the action of  $\mathbb{K}^*$  via  $\lambda$ . Moreover,  $z$  is the unique closed orbit of  $\mathbb{K}^*$  in  $V$ . So, [Res00, Lemma 7] implies that  $\text{Pic}^{\mathbb{K}^*}(V)$  is isomorphic to  $X(\mathbb{K}^*)$ ; and finally that for all  $\mathcal{L} \in \text{Pic}^G(X)$  the restriction  $\mathcal{L}$  to  $V$  is the trivial line bundle endowed with the action of  $\mathbb{K}^*$  given by  $\mu^\mathcal{L}(x, \lambda)$ . The

proposition follows immediately.  $\square$

The numbers  $\mu^{\mathcal{L}}(x, \lambda)$  are used in [MFK94] to give a numerical criterion for stability with respect to an ample  $G$ -linearized line bundle  $\mathcal{L}$ :

$$\begin{aligned} x \in X^{\text{ss}}(\mathcal{L}) &\iff \mu^{\mathcal{L}}(x, \lambda) \leq 0 \text{ for all one parameter subgroup } \lambda, \\ x \in X^{\text{s}}(\mathcal{L}) &\iff \mu^{\mathcal{L}}(x, \lambda) < 0 \text{ for all non trivial } \lambda. \end{aligned}$$

A line bundle  $\mathcal{L}$  over  $X$  is said to be *semiample* if a positive power of  $\mathcal{L}$  is base point free. With our notion of semistability, Hilbert-Mumford's theorem admits the following direct generalization:

**Lemma 2** *Let  $\mathcal{L}$  be a  $G$ -linearized line bundle over  $X$  and  $x$  a point in  $X$ . Then,*

- (i) *if  $x$  is semistable for  $\mathcal{L}$ ,  $\mu^{\mathcal{L}}(x, \lambda) \leq 0$  for any one parameter subgroup  $\lambda$  of  $G$ ;*
- (ii) *for a one parameter subgroup  $\lambda$  of  $G$ , if  $x$  is semistable for  $\mathcal{L}$  and  $\mu^{\mathcal{L}}(x, \lambda) = 0$ , then  $\lim_{t \rightarrow 0} \lambda(t)x$  is semistable for  $\mathcal{L}$ ;*
- (iii) *if in addition  $\mathcal{L}$  is semiample,  $x$  is semistable for  $\mathcal{L}$  if and only if  $\mu^{\mathcal{L}}(x, \lambda) \leq 0$  for any one parameter subgroup  $\lambda$  of  $G$ .*

**Proof.** Assume that  $x$  is semistable for  $\mathcal{L}$  and consider a  $G$ -invariant section  $\sigma$  of  $\mathcal{L}^{\otimes n}$  which does not vanish at  $x$ . Since  $\lambda(t)\sigma(x) = \sigma(\lambda(t)x)$  tends to  $\sigma(z)$  when  $t \rightarrow 0$ , Proposition 1 shows that  $\mu^{\mathcal{L}}(x, \lambda) \leq 0$ . If in addition  $\mu^{\mathcal{L}}(x, \lambda) = 0$ , Proposition 1 shows that  $\sigma(z)$  is non zero; and so that  $z$  is semistable for  $\mathcal{L}$ . This proves the two first assertions.

Assume now that  $\mathcal{L}$  is semiample. Let  $n$  be a positive integer such that  $\mathcal{L}^{\otimes n}$  is base point free. Let  $V$  denote the dual of the space of global sections of  $\mathcal{L}^{\otimes n}$ :  $V$  is a finite dimensional  $G$ -module. Moreover, the usual map  $\phi : X \rightarrow \mathbb{P}(V)$  is  $G$ -equivariant. Let  $Y$  denote the image of  $\phi$  and  $\mathcal{M}$  denote the restriction of  $\mathcal{O}(1)$  to  $Y$ .

Then  $\mathcal{L}^{\otimes n}$  is the pullback of  $\mathcal{M}$  by  $\phi$ . Since  $X$  is projective,  $\phi$  induces isomorphisms from  $H^0(Y, \mathcal{M}^{\otimes k})$  onto  $H^0(X, \mathcal{M}^{\otimes nk})$  (for all  $k$ ). So,  $X^{\text{ss}}(\mathcal{L}) = \phi^{-1}(Y^{\text{ss}}(\mathcal{M}))$ . We deduce the last assertion of the lemma by applying Hilbert-Mumford's criterion to  $Y$  and  $\mathcal{M}$  and Property 3 of the functions  $\mu^{\bullet}(x, \lambda)$ .  $\square$

**Remark.**

- (i) If  $\mathcal{L}$  is ample, Assertion 2 of Lemma 2 is Lemma 3 in [Res00].

- (ii) The proof of Assertion 3 shows that a lot of properties of semistability for an ample line bundle are also available for semiample line bundles (see Propositions 2 and 3, Lemma 3 and Theorems 1 and 2 below).

### 3.3 Definition of the functions $M^\bullet(x)$

**Notation.** Let  $\Gamma$  be any affine algebraic group. The neutral component of  $\Gamma$  is denoted by  $\Gamma^\circ$ . Let  $Y(\Gamma)$  denote the set of one parameter subgroups of  $\Gamma$ ; that is, of group homomorphisms from  $\mathbb{K}^*$  in  $\Gamma$ . Note that if  $\Gamma^\circ$  is a torus,  $Y(\Gamma)$  is a group.

If  $\Lambda$  is an abelian group, we denote by  $\Lambda_{\mathbb{Q}}$  (resp.  $\Lambda_{\mathbb{R}}$ ) the tensor product of  $\Lambda$  with  $\mathbb{Q}$  (resp.  $\mathbb{R}$ ) over  $\mathbb{Z}$ .

Let  $T$  be a maximal torus of  $G$ . The Weyl group  $W$  of  $T$  acts linearly on  $Y(T)_{\mathbb{R}}$ . Since  $W$  is finite, there exists a  $W$ -invariant Euclidean norm  $\|\cdot\|$  on  $Y(T)_{\mathbb{R}}$ . On the other hand, if  $\lambda \in Y(G)$  there exists  $g \in G$  such that  $g \cdot \lambda \cdot g^{-1} \in Y(T)$ . Moreover, if two elements of  $Y(T)$  are conjugate by an element of  $G$ , then they are by an element of the normalizer of  $T$  (see [MFK94, Lemma 2.8]). This allows to define the norm of  $\lambda$  by  $\|\lambda\| = \|g \cdot \lambda \cdot g^{-1}\|$ .

Let  $\mathcal{L} \in \text{Pic}^G(X)$ . One can now introduce the following notation:

$$\bar{\mu}^{\mathcal{L}}(x, \lambda) = \frac{\mu^{\mathcal{L}}(x, \lambda)}{\|\lambda\|}, \quad M^{\mathcal{L}}(x) = \sup_{\lambda \in Y(G)} \bar{\mu}^{\mathcal{L}}(x, \lambda).$$

In fact, we will see in Corollary 1 that  $M^{\mathcal{L}}(x)$  is finite.

### 3.4 $M^\bullet(x)$ for a torus action

**Notation.** If  $Y$  is a variety, and  $Z$  is a part of  $Y$ , the closure of  $Y$  in  $Z$  will be denoted by  $\bar{Z}$ . If  $\Gamma$  is a group acting on  $Y$ ,  $Y^\Gamma$  denote the set of fix point of  $\Gamma$  in  $Y$ .

If  $V$  is a finite dimensional vector space, and  $v$  is a non zero vector in  $V$ ,  $[v]$  denote the class of  $v$  in the projective space  $\mathbb{P}(V)$ . If  $V$  is a  $\Gamma$ -module, and  $\chi$  is a character of  $\Gamma$ , we denote by  $V_\chi$  the set of  $v \in V$  such that  $g.v = \chi(g)v$  for all  $g \in \Gamma$ .

In this subsection we assume that  $G = T$  is a torus. Let  $z$  be a point of  $X$  fixed by  $T$ . The action of  $T$  on the fiber  $\mathcal{L}_z$  over the point  $z$  in the  $T$ -linearized line bundle  $\mathcal{L}$  define a character  $\chi_z^{\mathcal{L}}$  of  $T$ ; we obtain a morphism

$$\mu^\bullet(z, T) : \text{Pic}^T(X) \longrightarrow X(T).$$

For any point  $x$  in  $X$ , we denote by  $\mathcal{P}_T^{\mathcal{L}}(x)$  the convex hull in  $X(T)_{\mathbb{R}}$  of the characters  $-\mu^{\mathcal{L}}(z, T)$  for  $z \in \overline{T \cdot x}^T$ .

The following proposition is an adaptation of a result of L. Ness and gives a pleasant interpretation of the number  $M^{\mathcal{L}}(x)$ :

**Proposition 2** *Let  $\mathcal{L}$  be a semiample  $T$ -linearized line bundle on  $X$ . With the above notation, we have:*

- (i) *The point  $x$  is unstable if and only if  $0$  does not belong to  $\mathcal{P}_T^{\mathcal{L}}(x)$ . In this case,  $M^{\mathcal{L}}(x)$  is the distance from  $0$  to  $\mathcal{P}_T^{\mathcal{L}}(x)$ .*
- (ii) *If  $x$  is semistable, the opposite of  $M^{\mathcal{L}}(x)$  is the distance from  $0$  to the boundary of  $\mathcal{P}_T^{\mathcal{L}}(x)$ .*
- (iii) *There exists  $\lambda \in Y(T)$  such that  $\overline{\mu}^{\mathcal{L}}(x, \lambda) = M^{\mathcal{L}}(x)$ . If moreover  $\lambda$  is indivisible, we call it an adapted one parameter subgroup for  $x$ .*
- (iv) *If  $x$  is unstable, there exists a unique adapted one parameter subgroup for  $x$ .*

**Proof.** Since  $\mathcal{L}$  is semiample, there exist a positive integer  $n$ , a  $T$ -module  $V$ , and a  $T$ -equivariant morphism  $\phi : X \rightarrow \mathbb{P}(V)$  such that  $\mathcal{L}^{\otimes n} = \phi^*(\mathcal{O}(1))$ . Since  $\mu^{\bullet}(z, T)$  is a morphism, we have:  $\mathcal{P}_T^{\mathcal{L}^{\otimes n}}(x) = n\mathcal{P}_T^{\mathcal{L}}(x)$  for all  $x$ . Moreover,  $\overline{\mu}^{\mathcal{L}^{\otimes n}}(x, \lambda) = n\overline{\mu}^{\mathcal{L}}(x, \lambda)$ , for all  $x$  and  $\lambda$ ; so,  $M^{\mathcal{L}^{\otimes n}}(x) = nM^{\mathcal{L}}(x)$ . As a consequence, it is sufficient to prove the proposition for  $\mathcal{L}^{\otimes n}$ ; in other words, we may assume that  $n = 1$ .

Let us recall that:

$$V = \bigoplus_{\chi \in X(T)} V_{\chi}.$$

Let  $x \in X$  and  $v \in V$  such that  $[v] = \phi(x)$ . There exist unique vectors  $v_{\chi} \in V_{\chi}$  such that  $v = \sum_{\chi} v_{\chi}$ . Let  $\mathcal{Q}$  be the convex hull in  $X(T)_{\mathbb{R}}$  of the  $\chi$ 's such that  $v_{\chi} \neq 0$ . It is well known (see [Oda88]) that the fixed point of  $T$  in  $\overline{T \cdot [v]}$  are exactly the  $[v_{\chi}]$ 's with  $\chi$  vertex of  $\mathcal{Q}$ . One easily deduces that  $\mathcal{Q} = \mathcal{P}_T^{\mathcal{L}}(x)$ . Now, the proposition is a direct consequence of [Nes78].  $\square$

### 3.5 Properties of $M^{\bullet}(x)$

**Notation.** We will denote by  $\text{Pic}^G(X)^+$  (resp.  $\text{Pic}^G(X)^{++}$ ) the set of semiample (resp. ample)  $G$ -linearized line bundle on  $X$ .

The following very useful result of Ness relies the function  $M^{\bullet}(x)$  for  $G$  to similar ones for a maximal torus of  $G$ .

**Lemma 3** (Lemma 3.4 in [Nes78]) *Let  $\mathcal{L}$  be a semiample  $G$ -linearized line bundle and  $T$  be a maximal torus of  $G$ . We denote by  $r^T : \text{Pic}^G(X) \rightarrow \text{Pic}^T(X)$  the partial forgetful map.*

*Then, for all  $x \in X$ , the set of the numbers  $M^{r^T(\mathcal{L})}(g \cdot x)$  for  $g \in G$  is finite and  $M^{\mathcal{L}}(x) = \max_{g \in G} M^{r^T(\mathcal{L})}(g \cdot x)$ .*

An indivisible one parameter subgroup  $\lambda$  of  $G$  is said to be *adapted for  $x$  and  $\mathcal{L}$*  if and only if  $\bar{\mu}^{\mathcal{L}}(x, \lambda) = M^{\mathcal{L}}(x)$ . Denote by  $\Lambda^{\mathcal{L}}(x)$  the set of adapted one parameter subgroups for  $x$ .

**Corollary 1** (i) *The numbers  $M^{\mathcal{L}}(x)$  are finite (even if  $\mathcal{L}$  is not ample, see Proposition 1.1.6 in [DH98]).*

(ii) *If  $\mathcal{L}$  is semiample,  $\Lambda^{\mathcal{L}}(x)$  is not empty.*

Now, we can reformulate the numerical criterion for stability: if  $\mathcal{L}$  is semiample, we have

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : M^{\mathcal{L}}(x) \leq 0\}, \quad X^{\text{s}}(\mathcal{L}) = \{x \in X : M^{\mathcal{L}}(x) < 0\}.$$

The following proposition is a result of finiteness for the set of functions  $M^{\bullet}(x)$ . It will be used to understand how  $X^{\text{ss}}(\mathcal{L})$  depends on  $\mathcal{L}$  (see Proposition 5).

**Proposition 3** *When  $x$  varies in  $X$ , one obtains only a finite number of functions  $M^{\bullet}(x) : \text{Pic}^G(X)^+ \rightarrow \mathbb{R}$ .*

**Proof.** Let  $T$  be a maximal torus of  $G$ . Consider the partial forgetful map  $r^T : \text{Pic}^G(X) \rightarrow \text{Pic}^T(X)$ . Since  $M^{\bullet}(x) = \max_{g \in G} M^{r^T(\bullet)}(g \cdot x)$ , it is sufficient to prove the proposition for the torus  $T$ .

If  $z$  and  $z'$  belong to the same irreducible component  $C$  of  $X^T$ , the morphisms  $\mu^{\bullet}(z, T)$  and  $\mu^{\bullet}(z', T)$  are equal: we denote by  $\mu^{\mathcal{L}}(C, T)$  this morphism.

By Proposition 2,  $M^{\mathcal{L}}(x)$  only depends on  $\mathcal{P}_T^{\mathcal{L}}(x)$ , which only depends on the set of irreducible components of  $X^T$  which intersects  $\overline{T \cdot x}$ . Since,  $X^T$  has finitely many irreducible components, the proposition follows.  $\square$

**Remark.** Proposition 3 implies that the open subsets of  $X$  which can be realized as  $X^{\text{ss}}(\mathcal{L})$  for some semiample  $G$ -linearized line bundle  $\mathcal{L}$  on  $X$  is finite. This is essentially a result of Dolgachev and Hu (see Theorem 3.9 in [DH98]; see also [Sch03]).

### 3.6 Adapted one parameter subgroups

To describe  $\Lambda^{\mathcal{L}}(x)$ , we need some additional notation. To the one parameter subgroup  $\lambda$  of  $G$ , we associate the parabolic subgroup (see [MFK94]):

$$P(\lambda) = \left\{ g \in G : \lim_{t \rightarrow 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1} \text{ exists in } G \right\}.$$

The unipotent radical of  $P(\lambda)$  is

$$U(\lambda) = \left\{ g \in G : \lim_{t \rightarrow 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1} = e \right\}.$$

Moreover, the centralizer  $G^\lambda$  of the image of  $\lambda$  in  $G$  is a Levi subgroup of  $P(\lambda)$ . For  $p \in P(\lambda)$ , we set  $\bar{p} = \lim_{t \rightarrow 0} \lambda(t) \cdot p \cdot \lambda(t)^{-1}$ . Then, we have the following short exact sequence:

$$1 \longrightarrow U(\lambda) \longrightarrow P(\lambda) \xrightarrow{p \mapsto \bar{p}} G^\lambda \longrightarrow 1.$$

For  $g \in P(\lambda)$ , we have  $\mu^{\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(x, g \cdot \lambda \cdot g^{-1})$ . The following theorem due to G. Kempf is a generalization of the last assertion of Proposition 2.

**Theorem 1** (see [Kem78]) *Let  $x$  be an unstable point for a semiample  $G$ -linearized line bundle  $\mathcal{L}$ . Then:*

- (i) *All the  $P(\lambda)$  for  $\lambda \in \Lambda^{\mathcal{L}}(x)$  are equal. We denote by  $P^{\mathcal{L}}(x)$  this subgroup.*
- (ii) *Any two elements of  $\Lambda^{\mathcal{L}}(x)$  are conjugate by an element of  $P^{\mathcal{L}}(x)$ .*

We will also use the following theorem of L. Ness.

**Theorem 2** (Theorem 9.3 in [Nes84]) *Let  $x$  and  $\mathcal{L}$  be as in the above theorem. Let  $\lambda$  be an adapted one parameter subgroup for  $x$  and  $\mathcal{L}$ . We consider  $y = \lim_{t \rightarrow 0} \lambda(t) \cdot x$ . Then,  $\lambda \in \Lambda^{\mathcal{L}}(y)$  and  $M^{\mathcal{L}}(x) = M^{\mathcal{L}}(y)$ .*

### 3.7 Stratification of $X$ induced from $\mathcal{L}$

Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $X$ . If  $d > 0$  and  $\langle \tau \rangle$  is a conjugacy class of one parameter subgroups of  $G$ , we set:

$$S_{d, \langle \tau \rangle}^{\mathcal{L}} = \{ x \in X : M^{\mathcal{L}}(x) = d \text{ and } \Lambda^{\mathcal{L}}(x) \cap \langle \tau \rangle \neq \emptyset \}.$$



If  $\mathcal{T}$  is the set of conjugacy classes of one parameter subgroups, the previous section gives us the following decomposition of  $X$ :

$$X = X^{\text{ss}}(L) \cup \bigcup_{d>0, \langle \tau \rangle \in \mathcal{T}} S_{d, \langle \tau \rangle}^{\mathcal{L}}.$$

W. Hesselink showed in [Hes79] that this union is a finite stratification by  $G$ -stable locally closed subvarieties of  $X$ . We will call it the *stratification induced from  $\mathcal{L}$* .

To describe the geometry of these stratum, we need additional notation:

$$Z_{d, \langle \tau \rangle}^{\mathcal{L}} := \{x \in S_{d, \langle \tau \rangle}^{\mathcal{L}} : \lambda(\mathbb{K}^*) \text{ fixes } x \text{ for some } \lambda \in \langle \tau \rangle\}.$$

For  $\lambda \in \langle \tau \rangle$ , we set:

$$S_{d, \lambda}^{\mathcal{L}} := \{x \in S_{d, \langle \tau \rangle}^{\mathcal{L}} : \lambda \in \Lambda^{\mathcal{L}}(x)\},$$

and

$$Z_{d, \lambda}^{\mathcal{L}} := \{x \in S_{d, \lambda}^{\mathcal{L}} : \lambda(\mathbb{K}^*) \text{ fixes } x\}.$$

We have the map

$$p_{\lambda} : S_{d, \lambda}^{\mathcal{L}} \longrightarrow Z_{d, \lambda}^{\mathcal{L}}, \quad x \longmapsto \lim_{t \rightarrow 0} \lambda(t).x.$$

The proof of the following result can be found in [Kir84, Section 1.3].

**Proposition 4** *With above notation, if  $d$  is positive, we have:*

- (i)  $Z_{d, \lambda}^{\mathcal{L}}$  is open in  $X^{\lambda}$  and stable by  $G^{\lambda}$ ;
- (ii)  $S_{d, \lambda}^{\mathcal{L}} = \{x \in X : \lim_{t \rightarrow 0} \lambda(t).x \in Z_{d, \lambda}^{\mathcal{L}}\}$  and is stable by  $P(\lambda)$ ;
- (iii) there is a bijective morphism  $G \times_{P(\lambda)} S_{d, \lambda}^{\mathcal{L}} \longrightarrow S_{d, \langle \tau \rangle}^{\mathcal{L}}$ , which is an isomorphism if  $S_{d, \langle \tau \rangle}^{\mathcal{L}}$  is normal.

### 3.8 Some technical results

**Notation.** If  $x$  is a point of a  $G$ -variety, we will denote by  $G_x$  its isotropy subgroup and by  $G.x$  its orbit.

The following lemma is easy and well known:

**Lemma 4** *Let  $\mathcal{L}$  be an  $G$ -linearized line bundle on  $X$  and  $x \in X$  be a point semistable for  $\mathcal{L}$ .*

*Then, the restriction of  $\mathcal{L}$  to  $G.x$  is of finite order.*

**Proof.** Let us recall that for any  $\mathcal{L} \in \text{Pic}^G(G.x)$ , the action of  $G_x$  on the fiber over  $x$  in  $\mathcal{L}$  determines a character  $\mu^{\mathcal{L}}(x, G_x)$  of  $G_x$ . Moreover, the map  $\mathcal{L} \mapsto \mu^{\mathcal{L}}(x, G_x)$  is an injective homomorphism.

Now, let  $\mathcal{L}$  be a  $G$ -linearized line bundle on  $X$  such that the character  $\mu^{\mathcal{L}}(x, T)$  is of infinite order. It remains to prove that  $x$  is unstable for  $\mathcal{L}$ . Let  $\sigma$  be a  $G$ -invariant section of  $\mathcal{L}^{\otimes n}$  for some  $n > 0$ . Then  $\sigma(x)$  is a  $G_x$  fix point of the fiber in  $\mathcal{L}^{\otimes n}$  over  $x$ . Since,  $n \cdot \mu^{\mathcal{L}}(x, T)$  is non trivial,  $\sigma(x)$  must be zero. So,  $x$  is unstable.  $\square$

A point  $x \in X^{\text{ss}}(\mathcal{L})$  is said to be *semisimple* for the ample  $G$ -linearized line bundle  $\mathcal{L}$  if its  $G$ -orbit is closed in  $X^{\text{ss}}(\mathcal{L})$ .

**Proposition 5** *Let  $\mathcal{L}_0$  be an ample  $G$ -linearized line bundle on  $X$  in  $\Lambda$  and  $x$  be a point semisimple for  $\mathcal{L}_0$ . Consider the following vector subspace  $\mathcal{K}_x$  of  $\Lambda_{\mathbb{Q}}$ :*

$$\mathcal{K}_x := \{\mathcal{L} \in \Lambda_{\mathbb{Q}} \mid \mu^{\mathcal{L}}(x, G_x) \text{ is of finite order}\}.$$

*Then, there exists an open neighborhood  $\Omega$  of  $\mathcal{L}_0$  in  $\mathcal{K}_x$  such that  $x$  is semistable for any  $\mathcal{L} \in \Omega$ .*

**Proof.** Let  $T$  be a maximal torus of  $G$ . We have to prove that there exists an open neighborhood  $\Omega$  of  $\mathcal{L}_0$  in  $\mathcal{K}_x$  such that for all  $g \in G$ ,  $M^{r^T(\bullet)}(g.x) \leq 0$  on  $\Omega$ .

Using Proposition 3, one easily checks that there exists an open neighborhood  $\Omega'$  of  $\mathcal{L}_0$  in  $\Lambda_{\mathbb{Q}}$  such that  $M^{r^T(\bullet)}(g.x) < 0$  on  $\Omega'$  if  $M^{r^T(\mathcal{L}_0)}(g.x) < 0$ .

Let us now fix  $g \in G$  such that  $M^{r^T(\mathcal{L}_0)}(g.x) = 0$ . Let  $\mathcal{F}$  be the face of  $\mathcal{P}_T^{\mathcal{L}_0}(g.x)$  containing 0 in its relative interior. Let  $\lambda$  be a one parameter subgroup of  $T$  such that the point  $z = \lim_{t \rightarrow 0} \lambda(t).g.x$  satisfies  $\mathcal{P}_T^{\mathcal{L}_0}(z) = \mathcal{F}$ . By Assertion 2 of Lemma 2,  $z$  is semistable for  $\mathcal{L}_0$ ; and, since  $x$  is semisimple  $z \in G.x$ .

Let  $\mathcal{L} \in \mathcal{K}_x$ . The group  $T_z^{\circ}$  acts trivially on the fiber  $\mathcal{L}_z$ , and so on all the fibers over  $T.z$ , and so on all the fibers over  $\overline{T.z}$ . Therefore,  $\mathcal{P}_T^{\mathcal{L}}(z)$  is contained in the linear subspace  $F$  of  $X(T)_{\mathbb{R}}$  spanned by the characters  $\chi \in X(T)$  trivial in restriction to  $T_z^{\circ}$ .

On the other hand, since 0 belongs to  $\mathcal{F}$ ,  $F$  is the linear subspace spanned by  $\mathcal{F}$ . Since 0 belongs to the relative interior of  $\mathcal{F}$ , there exists an open

neighborhood  $\Omega_z$  of  $\mathcal{L}_0$  in  $\mathcal{K}_x$  such that

$$\forall \mathcal{L} \in \Omega_z \quad M^{r^T(\mathcal{L})}(z) = 0 = M^{r^T(\mathcal{L})}(x).$$

We conclude by using again Proposition 3.  $\square$

## 4 Bialynicki-Birula cells

### 4.1 Bialynicki-Birula's theorem

**Notation.** Let  $X$  be a  $G$ -variety,  $H$  be a subgroup of  $G$  and  $\lambda$  be a one parameter subgroup of  $G$ . The commutator of  $H$  in  $G$  (that is, the set of fix point of  $H$  acting on  $G$  by conjugacy) will be denoted by  $G^H$ . The set of fix points of the image of lambda will be denoted by  $X^\lambda$ ; the commutator of this image will be denoted by  $G^\lambda$ .

If  $Y$  is a locally closed subvariety of  $X$ , and  $\mathcal{L}$  is a line bundle on  $X$ ,  $\mathcal{L}|_Y$  will denote the restriction of  $\mathcal{L}$  to  $Y$ .

Let  $X$  be a complete  $G$ -variety. Let  $\lambda$  be a one parameter subgroup of  $G$ . Let  $C$  be an irreducible component of  $X^\lambda$ . Since  $G^\lambda$  is connected,  $C$  is a  $G^\lambda$ -stable closed subvariety of  $X$ . We set:

$$C^+ := \{x \in X : \lim_{t \rightarrow 0} \lambda(t)x \in C\}.$$

Then,  $C^+$  is a locally closed subvariety of  $X$  stable by  $P(\lambda)$ . Moreover, the map  $p_\lambda : C^+ \rightarrow C$ ,  $x \mapsto \lim_{t \rightarrow 0} \lambda(t)x$  is a morphism satisfying:

$$\forall (l, u) \in G^\lambda \times U(\lambda) \quad p_\lambda(lu.x) = lp_\lambda(x).$$

Let  $x \in X^\lambda$ . We consider the natural action of  $\mathbb{K}^*$  induced by  $\lambda$  on the Zariski tangent space  $T_x X$  of  $X$  at  $x$ . We consider the following  $\mathbb{K}^*$ -submodules of  $T_x X$ :

$$\begin{aligned} T_x X_{>0} &= \{\xi \in T_x X : \lim_{t \rightarrow 0} \lambda(t)\xi = 0\}, \\ T_x X_{<0} &= \{\xi \in T_x X : \lim_{t \rightarrow 0} \lambda(t^{-1})\xi = 0\}, \\ T_x X_0 &= (T_x X)^\lambda, \quad T_x X_{\geq 0} = T_x X_{>0} \oplus T_x X_0 \text{ and } T_x X_{\leq 0} = T_x X_{<0} \oplus T_x X_0. \end{aligned}$$

A classical result of Bialynicki-Birula (see [BB73]) is

**Theorem 3** *Assuming in addition that  $X$  is smooth, we have:*

- (i)  $C$  is smooth and for all  $x \in C$  we have  $T_x C = T_x X_0$ ;

- (ii)  $C^+$  is smooth and irreducible and for all  $x \in C$  we have  $T_x C^+ = T_x X_{\geq 0}$ ;
- (iii) the morphism  $p_\lambda : C^+ \rightarrow C$  induces a structure of vector bundle on  $C$  with fibers isomorphic to  $T_x X_{> 0}$ .

## 4.2 Line bundles on $C^+$

We will need some results about the line bundles on  $C^+$ . Let  $\mathcal{L}$  be a  $P(\lambda)$ -linearized line bundle on  $C^+$ . Since  $\lim_{t \rightarrow 0} \lambda(t).x$  belongs to  $C$ , the number  $\mu^{\mathcal{L}}(x, \lambda)$  is well defined. Moreover, since  $C$  is irreducible,  $\mu^{\mathcal{L}}(x, \lambda)$  does not depend on  $x \in C^+$ ; we denote by  $\mu^{\mathcal{L}}(C, \lambda)$  this integer.

**Proposition 6** *We assume that  $X$  is smooth. Then, we have:*

- (i) The restriction map  $\text{Pic}^{P(\lambda)}(C^+) \rightarrow \text{Pic}^{G^\lambda}(C)$  is an isomorphism. Let  $\mathcal{L} \in \text{Pic}^{P(\lambda)}(C^+)$ .
- (ii) If  $\mu^{\mathcal{L}}(C, \lambda) \neq 0$ ,  $H^0(C, \mathcal{L}|_C)^\lambda = \{0\}$ .
- (iii) If  $\mu^{\mathcal{L}}(C, \lambda) = 0$ , the restriction map induces an isomorphism from  $H^0(C^+, \mathcal{L})^{P(\lambda)}$  onto  $H^0(C, \mathcal{L}|_C)^{G^\lambda}$ . Moreover, for any  $\sigma \in H^0(C^+, \mathcal{L})^{P(\lambda)}$ , we have:

$$\{x \in C^+ : \sigma(x) = 0\} = p_\lambda^{-1}(\{x \in C : \sigma(x) = 0\}).$$

**Proof.** Since  $p_\lambda$  is  $P(\lambda)$ -equivariant, for any  $\mathcal{M} \in \text{Pic}^{G^\lambda}(C)$ ,  $p_\lambda^*(\mathcal{M})$  is  $P(\lambda)$ -linearized. Since  $p_\lambda$  is a vector bundle,  $p_\lambda^*(\mathcal{L}|_C)$  and  $\mathcal{L}$  are isomorphic as line bundles without linearization. But,  $X(P(\lambda)) \simeq X(G^\lambda)$ , so the  $P(\lambda)$ -linearizations must coincide; and  $p_\lambda^*(\mathcal{L}|_C)$  and  $\mathcal{L}$  are isomorphic as  $P(\lambda)$ -linearized line bundles. Assertion 1 follows.

Assertion 2 is a direct application of Lemma 4.

Let us fix  $\mathcal{L} \in \text{Pic}^{P(\lambda)}(C^+)$  and denote by  $p : \mathcal{L} \rightarrow C^+$  the projection. We assume that  $\mu^{\mathcal{L}}(C, \lambda) = 0$ . Let  $\sigma \in H^0(C^+, \mathcal{L})^{P(\lambda)}$ . We just proved that

$$\mathcal{L} \simeq p_\lambda^*(\mathcal{L}|_C) = \{(x, l) \in C^+ \times \mathcal{L}|_C : p_\lambda(x) = p(l)\}.$$

Let  $p_2$  denote the projection of  $p_\lambda^*(\mathcal{L}|_C)$  onto  $\mathcal{L}|_C$ .

For all  $x \in C^+$  and  $t \in \mathbb{K}^*$ , we have:

$$\begin{aligned} \sigma(\lambda(t).x) &= \left( \lambda(t).x, p_2(\sigma(\lambda(t).x)) \right) \\ &= \lambda(t). \left( x, p_2(\sigma(x)) \right) && \text{since } \sigma \text{ is invariant,} \\ &= \left( \lambda(t).x, p_2(\sigma(x)) \right) && \text{since } \mu^{\mathcal{L}}(C, \lambda) = 0. \end{aligned}$$

We deduce that for all  $x \in C^+$ ,  $\sigma(x) = (x, \sigma(p_\lambda(x)))$ . Assertion 3 follows.  $\square$

## 5 First descriptions of the $G$ -cones

### 5.1 Definitions

Let us recall from the introduction that  $\Lambda$  is a freely finitely generated subgroup of  $\text{Pic}^G(X)$  and  $\Lambda_{\mathbb{Q}}$  is the  $\mathbb{Q}$ -vector space containing  $\Lambda$  as a lattice. Since  $X^{\text{ss}}(\mathcal{L}) = X^{\text{ss}}(\mathcal{L}^{\otimes n})$ , for any  $G$ -linearized line bundle and any positive integer  $n$ , we can define  $X^{\text{ss}}(\mathcal{L})$  for any  $\mathcal{L} \in \Lambda_{\mathbb{Q}}$ . The central object of this article is the following *total  $G$ -cone*:

$$\mathcal{TC}_{\Lambda}^G(X) = \{\mathcal{L} \in \Lambda_{\mathbb{Q}} : X^{\text{ss}}(\mathcal{L}) \text{ is not empty}\}.$$

Since the tensor product of two non zero  $G$ -invariant sections is a non zero  $G$ -invariant section,  $\mathcal{TC}_{\Lambda}^G(X)$  is a convex cone.

Consider the convex cones  $\Lambda_{\mathbb{Q}}^+$  and  $\Lambda_{\mathbb{Q}}^{++}$  generated respectively by the semiample and ample elements of  $\Lambda$ . For all  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^+$  (resp.  $\Lambda_{\mathbb{Q}}^{++}$ ), there exists a positive integer  $n$  such that  $\mathcal{L}^{\otimes n}$  is a semiample (resp. ample)  $G$ -linearized line bundle on  $X$  in  $\Lambda$ . So, any set of semistable points associated to a point in  $\Lambda_{\mathbb{Q}}^+$  (resp.  $\Lambda_{\mathbb{Q}}^{++}$ ) is in fact a set of semistable point associated to a semiample (resp. ample)  $G$ -linearized line bundle. We consider the following *semiample and ample  $G$ -cones*:

$$\mathcal{SAC}_{\Lambda}^G(X) = \mathcal{TC}_{\Lambda}^G(X) \cap \Lambda_{\mathbb{Q}}^+ \quad \text{and} \quad \mathcal{AC}_{\Lambda}^G(X) = \mathcal{TC}_{\Lambda}^G(X) \cap \Lambda_{\mathbb{Q}}^{++}.$$

By [DH98] (see also [Res00]),  $\mathcal{AC}_{\Lambda}^G(X)$  is a closed convex rational polyhedral cone in  $\Lambda_{\mathbb{Q}}^{++}$ .

Two points  $\mathcal{L}$  and  $\mathcal{L}'$  in  $\mathcal{AC}_{\Lambda}^G(X)$  are said to be *GIT-equivalent* if  $X^{\text{ss}}(\mathcal{L}) = X^{\text{ss}}(\mathcal{L}')$ . An equivalence class is simply called a *GIT-class*.

For  $x \in X$ , the *stability set of  $x$*  is the set of  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^{++}$  such that  $X^{\text{ss}}(\mathcal{L})$  contains  $x$ ; it is denoted by  $\Omega_{\Lambda}(x)$  or  $\Omega_{\Lambda}(G.x)$ . In [Res00], we have studied

the geometry of the GIT-classes and the stability sets with lightly different assumptions (no  $\Lambda$  for example). However all the results and proofs of [Res00] remain valuable here. In particular, there are only finitely many GIT-classes (see also the remark in Section 3.5); and each GIT-class is the relative interior of a closed convex polyhedral cone of  $\Lambda_{\mathbb{Q}}^{++}$ .

## 5.2 Well covering pairs

Here comes a central definition in this work:

**Definition.** Let  $\lambda$  be a one parameter subgroup of  $G$  and  $C$  be an irreducible component of  $X^\lambda$ . Set  $C^+ := \{x \in X \mid \lim_{t \rightarrow 0} \lambda(t)x \in C\}$ . Consider the following  $G$ -equivariant map

$$\begin{aligned} \eta : G \times_{P(\lambda)} C^+ &\longrightarrow X \\ [g : x] &\longmapsto g.x. \end{aligned}$$

The pair  $(C, \lambda)$  is said to be *covering* if  $\eta$  is birational. It is said to be *well covering* if  $\eta$  induces an isomorphism from  $G \times_{P(\lambda)} \Omega$  onto an open subset of  $X$  for an open subset  $\Omega$  of  $C^+$  intersecting  $C$ .

Let us recall that  $\mu^\bullet(C, \lambda)$  denote the common value of the  $\mu^\bullet(x, \lambda)$ , for  $x \in C^+$ . The first relation between covering pairs and the  $G$ -cones is the following

**Lemma 5** *Let  $(C, \lambda)$  be a covering pair and  $\mathcal{L} \in \mathcal{TC}_\Lambda^G(X)$ . Then,  $\mu^\mathcal{L}(C, \lambda) \leq 0$ .*

**Proof.** Since  $X^{\text{ss}}(\mathcal{L})$  is a  $G$ -stable non empty open subset of  $X$  and  $(C, \lambda)$  is covering, there exists a point  $x \in C^+$  semistable for  $\mathcal{L}$ . Since  $x \in C^+$ ,  $\mu^\mathcal{L}(x, \lambda) = \mu^\mathcal{L}(C, \lambda)$ . But, since  $x \in X^{\text{ss}}(\mathcal{L})$  Lemma 2 shows that  $\mu^\mathcal{L}(x, \lambda) \leq 0$ .  $\square$

Proposition 4 allows us to give a first description of the cone  $\mathcal{AC}_\Lambda^G(X)$ :

**Proposition 7** *We assume that  $X$  is normal. Let  $T$  be a maximal torus of  $G$  and  $B$  be a Borel subgroup containing  $T$ .*

*Then, the cone  $\mathcal{AC}_\Lambda^G(X)$  is the set of the  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^{++}$  such that for all well covering pair  $(C, \lambda)$  with a dominant one parameter subgroup  $\lambda$  of  $T$  we have  $\mu^\mathcal{L}(C, \lambda) \leq 0$ .*

**Proof.** Lemma 5 shows that  $\mathcal{AC}^G(X)$  is contained in the part of  $\Lambda_{\mathbb{Q}}^{++}$  defined by the inequalities  $\mu^{\mathcal{L}}(C, \lambda) \leq 0$  of the proposition.

Conversely, let  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^{++}$  such that  $X^{\text{ss}}(\mathcal{L})$  is empty. Consider the open stratum  $S_{d, \langle \tau \rangle}^{\mathcal{L}}$  in  $X$ . Let  $\lambda$  be a dominant one parameter subgroup of  $T$  in the class  $\langle \tau \rangle$ . Since  $S_{d, \langle \tau \rangle}^{\mathcal{L}}$  is open in  $X$ , it is normal and irreducible. Now, Proposition 4 implies that  $Z_{d, \lambda}^{\mathcal{L}}$  is open in an irreducible component  $C$  of  $X^{\lambda}$ . It shows also that  $S_{d, \lambda}^{\mathcal{L}}$  is open in  $C^+$  and intersects  $C$ . Finally, the last assertion of Proposition 4 shows that  $(C, \lambda)$  is well covering. Moreover,  $\mu^{\mathcal{L}}(C, \lambda) = d > 0$ .  $\square$

### 5.3 An example of $G$ -ample cone

**Notation.** If  $\Gamma$  is an affine algebraic group,  $[\Gamma, \Gamma]$  will denote its derived subgroup.

For later use, we consider here a  $G$ -ample cone for the action of  $G$  over an affine variety. More precisely, let  $V$  be an irreducible affine  $G$ -variety containing a fix point  $O$  as unique closed orbit. The action of  $G$  over the fiber gives a morphism  $\mu^{\bullet}(O, G) : \text{Pic}^G(V) \rightarrow X(G)$ . By [Res00, Lemma 7],  $\mu^{\bullet}(O, G)$  is an isomorphism. We denote by  $V^{\text{ss}}(\chi)$  the set of semistable points for the  $G$ -linearized line bundle  $\mathcal{L}_{\chi}$  associated to  $\chi \in X(G)$ ; that is, the trivial line bundle on  $V$  linearized by  $\chi$ . As in the projective case, we consider the  $G$ -ample cone  $\mathcal{AC}^G(V)$  in  $X(G) \otimes \mathbb{Q}$ .

For any  $\chi \in X(G)$ , we have:

$$\Gamma(V, \mathcal{L}_{\chi})^G = \{f \in \mathbb{K}[V] : \forall x \in V \ (g.f)(x) = \chi(g)f(x)\} = \mathbb{K}[V]_{\chi}.$$

Note the  $\Gamma(V, \mathcal{L}_{\chi})^G$  is contained in  $\mathbb{K}[V]^{[G, G]}$ . Set

$$S = \{\chi \in X(G) : \Gamma(V, \mathcal{L}_{\chi}) \text{ is non trivial}\}.$$

It is the set of weights of  $G/[G, G]$  in  $\mathbb{K}[V]^{[G, G]}$ . We have:

**Lemma 6** *With above notation,  $S$  is a finitely generated semigroup in  $X(G)$ . Moreover,  $\mathcal{AC}^G(V)$  is the convex cone generated by  $S$ ; it is strictly convex.*

**Proof.** Since  $\mathbb{K}[V]^{[G, G]}$  is a finitely generated algebra,  $S$  is a finitely generated semigroup. The fact that  $\mathcal{AC}^G(V)$  is generated by  $S$  is obvious. Finally,  $\mathcal{AC}^G(V)$  is strictly convex since  $\Gamma(V, \mathcal{L}_0)^G = \mathbb{K}$ .  $\square$

## 6 Slice Etale Theorem

In this section, we recall some very useful results of D. Luna. We fix an ample  $G$ -linearized line bundle  $\mathcal{L}$  on the irreducible projective  $G$ -variety  $X$ .

### 6.1 Semistability for normalizer and the centralizer of a stabilizer

**Notation.** If  $H$  is a closed subgroup of an affine algebraic group  $G$ ,  $N_G(H)$  denote the normalizer of  $H$  in  $G$ .

We will use the following interpretation of a result of Luna:

**Proposition 8** *Let  $H$  be a reductive subgroup of  $G$ . Let  $C$  be an irreducible component of  $X^H$ . Then, the reductive groups  $(G^H)^\circ$  and  $N_G(H)^\circ$  act on  $C$ .*

*Let  $x$  be a point in  $C$ . Then, the following are equivalent:*

- (i)  $x$  is semistable for  $\mathcal{L}$ .
- (ii)  $x$  is semistable for the action of  $(G^H)^\circ$  on  $C$  and the restriction of  $\mathcal{L}$ .
- (iii)  $x$  is semistable for the action of  $N_G(H)^\circ$  on  $C$  and the restriction of  $\mathcal{L}$ .

**Proof.** Lemma 1.1. of [LR79] shows that  $(G^H)^\circ$  and  $N_G(H)^\circ$  are reductive. Changing  $\mathcal{L}$  by a positive power if necessary, one may assume that  $X$  is contained in  $\mathbb{P}(V)$  where  $V$  is a  $G$ -module and  $\mathcal{L} = \mathcal{O}(1)|_X$ . Let  $v \in V$  such that  $[v] = x$ . Let us recall that in this case  $x \in X^{\text{us}}(\mathcal{L})$  if and only if  $\overline{G.v}$  contains 0.

Let  $\chi$  be the character of  $H$  such that  $hv = \chi(h)v$  for all  $h \in H$ .

If  $\chi$  is of infinite order, so is its restriction to the connected center  $Z$  of  $H$ . Then,  $Z.v = \mathbb{K}^*v$  and  $0 \in \overline{(G^H)^\circ.v}$ . In this case,  $x$  belongs to no semistable set of the proposition.

Let us now assume that  $\chi$  is of finite order. Changing  $\mathcal{L}$  by a positive power if necessary, one may assume that  $\chi$  is trivial, that is  $H$  fixes  $v$ . In this case, Corollary 2 and Remark 1 of [Lun75] show that

$$0 \in \overline{G.v} \iff 0 \in \overline{N_G(H)^\circ.v} \iff 0 \in \overline{(G^H)^\circ.v}.$$

The proposition follows. □



## 6.2 Closed orbits in general position

**Notation.** Consider the quotient  $\pi : X^{\text{ss}}(\mathcal{L}) \longrightarrow X^{\text{ss}}(\mathcal{L})//G$ . For all  $\xi \in X^{\text{ss}}(\mathcal{L})//G$ , we denote by  $T(\xi)$  the unique closed orbit of  $G$  in  $\pi^{-1}(\xi)$ . We denote by  $(X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$  the set of  $\xi$  such that there exists an open neighborhood  $\Omega$  of  $\xi$  in  $X^{\text{ss}}(\mathcal{L})//G$  such that the orbits  $T(\xi')$  are isomorphic to  $T(\xi)$ , for all  $\xi' \in \Omega$ .

Since  $\pi$  is a gluing of affine quotients, some results on the actions of  $G$  on affine variety remains true for  $X^{\text{ss}}(\mathcal{L})$ . For example, the following theorem is a result of Luna and Richardson (see Section 3 of [LR79] and Corollary 4 of [Lun75] or Section 7 of [PV91]):

**Theorem 4** *With above notation, if  $X$  is normal, we have:*

- (i) *The set  $(X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$  is a non empty open subset of  $X^{\text{ss}}(\mathcal{L})//G$ . Let  $H$  be the isotropy of a point in  $T(\xi)$  with  $\xi \in (X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$ . The group  $H$  has fixed points in  $T(\xi)$  for any  $\xi \in X^{\text{ss}}(\mathcal{L})//G$ .*
- (ii) *Let  $Y$  be the closure of  $\pi^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})^H$  in  $X$ . It is the union of some components of  $X^H$ . Then,  $H$  acts trivially on some positive power  $\mathcal{L}_{|Y}^{\otimes n}$  of  $\mathcal{L}_{|Y}$ . Moreover, the natural map*

$$Y^{\text{ss}}(\mathcal{L}_{|Y}^{\otimes n})//((N_G(H)/H)) \longrightarrow X^{\text{ss}}(\mathcal{L})//G$$

*is an isomorphism.*

A subgroup  $H$  as in Theorem 4 will be called a *generic closed isotropy* of  $X^{\text{ss}}(\mathcal{L})$ . The conjugacy class of  $H$  which is obviously unique is called the *generic closed isotropy* of  $X^{\text{ss}}(\mathcal{L})$ .

## 6.3 The principal Luna stratum

When  $X$  is smooth, the open subset  $(X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$  is called the *principal Luna stratum* and has very useful properties (see [Lun73] or [PV91]):

**Theorem 5 (Luna)** *We assume that  $X$  is smooth. Let  $H$  be a generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$ .*

*Then, there exists a  $H$ -module  $L$  such that for any  $\xi \in (X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$ , there exist points  $x$  in  $T(\xi)$  satisfying:*

- (i)  $G_x = H$ ;

- (ii) the  $H$ -module  $T_x X/T_x(G.x)$  is isomorphic to the sum of  $L$  and its fix points, in particular, it is independent of  $\xi$  and  $x$ ;
- (iii) for any  $v \in L$ ,  $0$  belongs to the closure of  $H.v$ ;
- (iv) the fiber  $\pi^{-1}(\xi)$  is isomorphic to  $G \times_H L$ .

## 7 About faces of the $G$ -ample cone

### 7.1 Isotropy subgroups associated to faces of $\mathcal{AC}_\Lambda^G(X)$

Let  $\varphi$  be a linear form on  $\Lambda_{\mathbb{Q}}$  which is non negative on  $\mathcal{AC}_\Lambda^G(X)$ . If the set of  $\mathcal{L} \in \mathcal{AC}_\Lambda^G(X)$  such that  $\varphi(\mathcal{L}) = 0$  is non empty it will be called a *face* of  $\mathcal{AC}_\Lambda^G(X)$ . Now, we associate two invariants to a face  $\mathcal{F}$  of  $\mathcal{AC}_\Lambda^G(X)$ .

**Theorem 6** *Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_\Lambda^G(X)$ . Then, we have:*

- (i) *The generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$  does not depends on the point  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ . We call this isotropy the generic closed isotropy of  $\mathcal{F}$ .*

*Let us fix a generic closed isotropy  $H$  of  $\mathcal{F}$ .*

- (ii) *For any  $\mathcal{L} \in \mathcal{F}$ ,  $H$  fixes points in any closed orbit of  $G$  in  $X^{\text{ss}}(\mathcal{L})$ .*
- (iii) *The closure  $Y$  of  $\left(\pi_{\mathcal{L}}^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})\right)^H$  in  $X$  does not depends on a choice of  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ . Let  $Y_{\mathcal{F}}$  denote this subvariety of  $X^H$ ; it is the union of some components of  $X^H$ .*
- (iv) *Let  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ . Then,  $H$  acts trivially on some positive power  $\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n}$  of  $\mathcal{L}_{|Y_{\mathcal{F}}}$ . Moreover, the natural map*

$$Y_{\mathcal{F}}^{\text{ss}}(\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n})//((N_G(H)/H)) \longrightarrow X^{\text{ss}}(\mathcal{L})//G$$

*is an isomorphism.*

- (v) *Set  $Y_{\mathcal{F}}^+ := \{x \in X : \overline{H.x} \cap Y_{\mathcal{F}} \neq \emptyset\}$ . Then  $G.Y_{\mathcal{F}}^+$  contains an open subset of  $X$ .*
- (vi) *For all  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ ,  $Y_{\mathcal{F}}$  contains stable points for the action of  $N_G(H)/H$  and the line bundle  $\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n}$ .*

**Proof.** Let  $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{AC}_\Lambda^G(X)$ . By an easy argument of convexity, to prove Assertion 1 it is sufficient to prove that the generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$  does not depend on  $\mathcal{L}$  in the open segment  $] \mathcal{L}_1, \mathcal{L}_2 [$ . Let us fix  $\mathcal{L}, \mathcal{L}' \in ] \mathcal{L}_1, \mathcal{L}_2 [$ . Let  $x \in X$  which maps in  $(X^{\text{ss}}(\mathcal{M})//G)_{\text{pr}}$ , for  $\mathcal{M} = \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}$  and  $\mathcal{L}'$  by the quotients maps.

Recall that  $\Omega_\Lambda(x)$  is a polyhedral convex cone. Since  $\mathcal{L}_1$  and  $\mathcal{L}_2$  belong to  $\Omega_\Lambda(x)$ ,  $\mathcal{L}$  and  $\mathcal{L}'$  belong to the relative interior of the same face of  $\Omega_\Lambda(x)$ . By [Res00, Proposition 6, Assertion (iii)], there exists  $x' \in \overline{G.x}$  such that this face is  $\Omega_\Lambda(x')$ . But, [Res00, Proposition 6, Assertion (i)] shows that the closed orbits in  $X^{\text{ss}}(\mathcal{L}) \cap \overline{G.x'}$  and  $X^{\text{ss}}(\mathcal{L}') \cap \overline{G.x'}$  are equal. Now, our choice of point  $x$  implies that the generic closed isotropies of  $X^{\text{ss}}(\mathcal{L})$  and  $X^{\text{ss}}(\mathcal{L}')$  are equal.

Let  $H$  be a generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$ . Let  $Y$  be the closure of  $X^H \cap \pi_{\mathcal{L}}^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})$ . By Theorem 4,  $N_G(H)$  acts transitively on the set of irreducible components of  $Y$ . Let  $Y_1$  be such a component of  $X^H$ . Again by Theorem 4,  $\pi_{\mathcal{L}}(Y_1 \cap X^{\text{ss}}(\mathcal{L})) = X^{\text{ss}}(\mathcal{L})//G$ ; that is, any closed  $G$ -orbit in  $X^{\text{ss}}(\mathcal{L})$  intersects  $Y_1$ . Finally,  $Y$  is the union of irreducible components of  $X^H$  which intersect a general closed  $G$ -orbit in  $X^{\text{ss}}(\mathcal{L})$ . But, the above proof of Assertion 1 shows that a general closed orbit in  $X^{\text{ss}}(\mathcal{L})$  is also a closed orbit in  $X^{\text{ss}}(\mathcal{L}')$ . In particular,  $Y$  is the closure of  $X^H \cap \pi_{\mathcal{L}'}^{-1}((X^{\text{ss}}(\mathcal{L}')//G)_{\text{pr}})$ . Assertion 3 follows.

Let us now fix a generic closed isotropy  $H$  of  $\mathcal{F}$ . Let  $\mathcal{M}_1 \in \mathcal{F}$ . By Assertion 1 of Theorem 4, to prove the second assertion, it is sufficient to prove that the generic closed isotropy of  $X^{\text{ss}}(\mathcal{M}_1)$  contains  $H$ . By [Res00, Theorem 4], there exists a point  $\mathcal{M}_2$  in the relative interior of  $\mathcal{F}$  such that  $X^{\text{ss}}(\mathcal{M}_1)$  contains  $X^{\text{ss}}(\mathcal{M}_2)$ . The inclusion  $X^{\text{ss}}(\mathcal{M}_2) \subset X^{\text{ss}}(\mathcal{M}_1)$  induces a surjective morphism  $\eta : X^{\text{ss}}(\mathcal{M}_2)//G \rightarrow X^{\text{ss}}(\mathcal{M}_1)//G$ . Let  $\xi' \in (X^{\text{ss}}(\mathcal{M}_2)//G)_{\text{pr}}$  such that  $\xi = \eta(\xi') \in (X^{\text{ss}}(\mathcal{M}_1)//G)_{\text{pr}}$ . Let  $x$  be a point in the closed  $G$ -orbit in  $X^{\text{ss}}(\mathcal{M}_1)$  over  $\xi$ . The fiber in  $X^{\text{ss}}(\mathcal{M}_1)$  over  $\xi$  is fibered over  $G.x$ ; hence, for any  $y$  in this fiber,  $G_y$  is conjugated to a subgroup of  $G_x$ . Since this fiber contains the fiber in  $X^{\text{ss}}(\mathcal{M}_2)$  over  $\xi'$ ,  $H$  is conjugated to a subgroup of  $G_x$ . The second assertion is proved.

Now, let  $\mathcal{M}$  be any point in the relative interior of  $\mathcal{F}$ . Let  $Y$  be the subvariety of  $X^H$  of Assertion 3 for  $\mathcal{M}$ . By Theorem 4,  $Y$  satisfies Assertion 4. Moreover,  $G.Y^+$  contains  $\pi_{\mathcal{M}}^{-1}((X^{\text{ss}}(\mathcal{M})//G)_{\text{pr}})$ ; and, Assertion 5 is proved.

Since  $\pi_1$  is affine Corollary 1 of [Lun75] shows that the  $N_G(H)$ -orbit of any element of  $\left(\pi_{\mathcal{M}}^{-1}((X^{\text{ss}}(\mathcal{M})//G)_{\text{pr}})\right)^H$  is closed in  $X^{\text{ss}}(\mathcal{M})$ . By Proposition 8, we can deduce Assertion 6.

□

## 7.2 Local structure of the $G$ -ample cone around a face

**Notation.** Let  $E$  be a prime Cartier divisor on a variety  $X$  endowed with a line bundle  $\mathcal{L}$  and  $\sigma$  be a rational section for  $\mathcal{L}$ . We will denote by  $\nu_E(\sigma) \in \mathbb{Z}$  the cancellation order of  $\sigma$  along  $E$ .

Let  $\mathcal{P}$  be a polyhedron in a rational vector space  $V$  and  $\mathcal{F}$  be a face of  $\mathcal{P}$ . Let  $x$  in the relative interior of  $\mathcal{F}$ . The cone of  $V$  generated by the vectors  $y - x$  for  $y \in \mathcal{P}$  does not depend on the choice of  $x$  in the relative interior of  $\mathcal{F}$  and will be called *the cone of  $\mathcal{P}$  viewed from  $\mathcal{F}$* . This cone carries the geometry of  $\mathcal{P}$  in a neighborhood of  $x$ .

Let  $H$  be a reductive subgroup of  $G$  and  $C$  be an irreducible component of  $X^H$ . For any  $x$  in  $C$ , the action of  $H$  on the fiber over  $x$  defines a morphism  $\Lambda \rightarrow X(H)$ . Since  $X(H)$  is discrete, this morphism does not depend on  $x \in C$ ; we will denote it by  $\mu^\bullet(C, H)$ . The subspace of  $\Lambda_{\mathbb{Q}}$  spanned by the kernel of  $\mu^\bullet(C, H)$  will be denoted by  $\mathcal{K}_C^H$ .

**Theorem 7** *We assume  $X$  smooth. Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_\Lambda^G(X)$ . Let  $H$  be a generic closed isotropy of  $\mathcal{F}$  and  $C$  be an irreducible component of  $Y_{\mathcal{F}}$ .*

*Let  $x$  in  $C$  such that  $G_x = H$ . Let  $N_x = T_x X / T_x(G \cdot x)$  and  $L$  be the  $H$ -stable supplementary of  $N_x^H$  in  $N_x$ .*

*Then, the cone of  $\mathcal{AC}_\Lambda^G(X)$  viewed from  $\mathcal{F}$  is the pullback of  $\mathcal{AC}^H(L)$  by  $\mu^\bullet(C, H)$ .*

**Proof.** Let  $\mathcal{L}_0$  and  $\mathcal{L}$  be two different ample  $G$ -linearized line bundles in  $\Lambda$ . We assume that  $\mathcal{L}_0$  is the only point in the segment  $[\mathcal{L}; \mathcal{L}_0]$  which belongs to  $\mathcal{AC}_\Lambda^G(X)$ . For convenience, we set  $U = X^{\text{ss}}(\mathcal{L}_0)$ . By assumption, there is no  $G$ -invariant rational section of  $\mathcal{L}$  which is regular on  $X$ ; we claim that there is no such section which is regular on  $U$ .

Let us prove the claim. Let us fix a non zero regular  $G$ -invariant section  $\sigma_0$  of  $\mathcal{L}_0^{\otimes m}$  for a positive integer  $m$ . Let  $\sigma$  be a  $G$ -invariant rational section of  $\mathcal{L}$  which is regular on  $U$ . For any positive integer  $k$ ,  $\sigma \otimes \sigma_0^{\otimes k}$  is a rational  $G$ -invariant section of  $\mathcal{L} \otimes \mathcal{L}_0^{\otimes mk}$  which is regular on  $U$ . Let  $E$  be an irreducible component of codimension one of  $X - U$ . By definition of  $U$ ,  $\sigma_0$  is

zero along  $E$ ; and,  $\nu_E(\sigma_0) > 0$ . Then,  $\nu_E(\sigma \otimes \sigma_0^{\otimes k}) = \nu_E(\sigma) + k \cdot \nu_E(\sigma_0)$  is positive for  $k$  big enough. We deduce that  $\sigma \otimes \sigma_0^{\otimes k}$  is regular on  $X$  for  $k$  big enough. Since by assumption  $\mathcal{L} \otimes \mathcal{L}_0^{\otimes mk}$  does not belong to  $\mathcal{AC}_\Lambda^G(X)$ , this implies that  $\sigma \otimes \sigma_0^{\otimes k}$  and finally  $\sigma$  are zero. The claim is proved.

We now fix a point  $\mathcal{L}_0$  in the relative interior of  $\mathcal{F}$ . By an elementary argument of convexity, there exists an open neighborhood  $\Omega$  of  $\mathcal{L}_0$  in  $\Lambda_{\mathbb{Q}}$  such that

- (i) for any  $\mathcal{L} \in \Omega$ , if  $\mathcal{L}$  does not belong to  $\mathcal{AC}_\Lambda^G(X)$  then  $\mathcal{L}_0$  is the only point in  $[\mathcal{L}, \mathcal{L}_0] \cap \mathcal{AC}_\Lambda^G(X)$ .

It remains to prove that for any  $\mathcal{L} \in \Omega$ ,  $\mathcal{L} \in \mathcal{AC}_\Lambda^G(X)$  if and only if  $\mu^{\mathcal{L}}(C, H) \in \mathcal{AC}^H(\mathcal{L})$ .

Let  $\mathcal{L} \in \Omega$  which does not belong to  $\mathcal{AC}_\Lambda^G(X)$ . Let  $H$  be a generic closed isotropy of  $U$  and  $L$  be the  $H$ -module of the statement. Let  $\pi_{\mathcal{L}_0}^{-1}(\xi)$  denote a fiber of the quotient map  $\pi : X^{\text{ss}}(\mathcal{L}_0) \rightarrow X^{\text{ss}}(\mathcal{L}_0)//G$ . By the beginning of the proof, for any positive  $n$ ,  $\mathrm{H}^0(U, \mathcal{L}^{\otimes n})^G = \{0\}$ . Since  $\pi_{\mathcal{L}_0}^{-1}(\xi)$  is closed in  $U$ , this implies that  $\mathrm{H}^0(\pi_{\mathcal{L}_0}^{-1}(\xi), \mathcal{L}^{\otimes n})^G = \{0\}$  for all positive  $n$ . By Theorem 5, we may assume that  $\pi_{\mathcal{L}_0}^{-1}(\xi)$  is isomorphic to  $G \times_H L$ . So, for all positive  $n$ ,  $\mathrm{H}^0(L, \mathcal{L}|_L^{\otimes n})^H = \{0\}$ ; and, so  $\mu^{\mathcal{L}}(C, H)$  does not belong to  $\mathcal{AC}^H(L)$ .

Let now  $\mathcal{L} \in \Omega \cap \mathcal{AC}_\Lambda^G(X)$ . Since  $X^{\text{ss}}(\mathcal{L}) \cap X^{\text{ss}}(\mathcal{L}_0)$  is open and non empty, there exists  $y \in X^{\text{ss}}(\mathcal{L}) \cap X^{\text{ss}}(\mathcal{L}_0)$  such that  $\pi_{\mathcal{L}_0}(y)$  belongs to the principal Luna stratum of  $X^{\text{ss}}(\mathcal{L}_0)//G$ . Then, by Theorem 5,  $L$  can be embedded in  $X^{\text{ss}}(\mathcal{L}_0)$  as a closed  $H$ -stable subvariety intersecting  $X^{\text{ss}}(\mathcal{L})$ . It follows that  $L^{\text{ss}}(\mu^{\mathcal{L}}(C, H), H)$  is non empty; that is,  $\mu^{\mathcal{L}}(C, H)$  belongs to  $\mathcal{AC}^H(L)$ .  $\square$

Theorem 7 shows that the vector subspace of  $\Lambda_{\mathbb{Q}}$  spanned by  $\mathcal{F}$  is the kernel of  $\mu^\bullet(C, H)$ . With notation of Proposition 5, this means that for all  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$  and for  $x$  a point in the closed orbit of the fiber over a generic point in  $X^{\text{ss}}(\mathcal{L})//G$ ,  $\mathcal{K}_x$  is the subspace spanned by  $\mathcal{F}$ . The following proposition gives a more precise result about the vector subspace spanned by a face of  $\mathcal{AC}_\Lambda^G(X)$ .

**Proposition 9** *Let  $F$  be a GIT-class of  $\mathcal{AC}_\Lambda^G(X)$  of non empty interior in a face  $\mathcal{F}$  of  $\mathcal{AC}_\Lambda^G(X)$ . Let  $x$  be a semisimple point of  $X^{\text{ss}}(F)$ .*

*Then,  $\mathcal{K}_x$  is the subspace spanned by  $\mathcal{F}$ .*

**Proof.** Let  $x$  be a semisimple point for  $F$ . Since  $\pi$  is affine and  $x$  is semisimple,  $G.x$  is affine. By Matsushima's theorem (see [Mat60] or [Lun73]),  $G_x$  is reductive. By Lemma 4,  $F$  and so  $\mathcal{F}$  is contained in  $\mathcal{K}_x$ .

By Proposition 5,  $\mathcal{K}_x \cap \mathcal{AC}_\Lambda^G(X)$  is of non empty interior in  $\mathcal{K}_x$  and  $F$  is contained in this interior. So,  $\mathcal{F}$  is a face of  $\mathcal{K}_x \cap \mathcal{AC}_\Lambda^G(X)$  intersecting its interior. It follows that  $\mathcal{F} = \mathcal{K}_x \cap \mathcal{AC}_\Lambda^G(X)$  and  $\mathcal{F}$  spans  $\mathcal{K}_x$ .  $\square$

### 7.3 From faces of the $G$ -ample cone to covering pairs

**Notation.** If  $\lambda$  is a one parameter subgroup of  $G$ ,  $\text{Im}\lambda$  will denote its image.

Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_\Lambda^G(X)$  of codimension one in  $\mathcal{AC}_\Lambda^G(X)$ . Proposition 7 shows that there exists a well covering pair  $(C, \lambda)$  such that  $\mu^\mathcal{L}(C, \lambda) \leq 0$  for all  $\mathcal{L} \in \mathcal{AC}_\Lambda^G(X)$  and  $\mathcal{F}$  is the set of the  $\mathcal{L}$ 's in  $\mathcal{AC}_\Lambda^G(X)$  such that  $\mu^\mathcal{L}(C, \lambda) = 0$ . The following theorem establishes such relations between all the faces of  $\mathcal{AC}_\Lambda^G(X)$  and covering pairs.

**Theorem 8** *We assume  $X$  smooth. Let  $\mathcal{F}$  be a face of  $\mathcal{AC}_\Lambda^G(X)$  of codimension  $d$  in  $\Lambda$ .*

(i) *Then, there exists a subtorus  $S$  of  $G$  of dimension  $d$ , an irreducible component  $C$  of  $X^S$  and a one parameter subgroup  $\lambda$  of  $S$  such that*

- (a)  *$C$  is an irreducible component of  $X^\lambda$ ;*
- (b)  *$(C, \lambda)$  is a covering pair;*
- (c)  *$\mathcal{F}$  is the set of the  $\mathcal{L}$ 's in  $\mathcal{AC}_\Lambda^G(X)$  such that  $\mu^\mathcal{L}(C, \lambda) = 0$ .*

(ii) *We assume that the neutral component of the generic closed isotropy of  $\mathcal{F}$  is a torus.*

*Then, any pair  $(C, \lambda)$  satisfying Assertion 1 is well covering.*

(iii) *Now, we assume that the dimension of the generic closed isotropy  $H$  of  $\mathcal{F}$  equals  $d$ . Then,  $H^\circ$  is a torus and there exists a well covering pair  $(C, \lambda)$  satisfying Assertion 2 with  $S = H^\circ$ . Moreover,  $G^\lambda = G^{H^\circ}$ .*

In the proof of the theorem, we will use the following

**Lemma 7** *Let  $(C, \lambda)$  be a covering pair and  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $X$  such that  $\mu^\mathcal{L}(C, \lambda) = 0$ .*

*Then,  $C$  intersects all the closed orbits of  $G$  in  $X^{\text{ss}}(\mathcal{L})$ .*

**Proof.** Since  $(C, \lambda)$  is covering,  $C^+$  must intersect  $X^{\text{ss}}(\mathcal{L})$ . Let  $x \in C^+ \cap X^{\text{ss}}(\mathcal{L})$ . Set  $z = \lim_{t \rightarrow 0} \lambda(t)x$ . By Assertion 2 of Lemma 2,  $z$  is semistable for  $\mathcal{L}$ . So,  $C$  intersects  $X^{\text{ss}}(\mathcal{L})$ .

By Proposition 8,  $C^{\text{ss}}(\mathcal{L}|_C, G^\lambda) = X^{\text{ss}}(\mathcal{L}) \cap C$ . Consider the morphism  $\theta$  which makes the following diagram commutative:

$$\begin{array}{ccc} C^{\text{ss}}(\mathcal{L}|_C, G^\lambda) & \hookrightarrow & X^{\text{ss}}(\mathcal{L}) \\ \downarrow & & \downarrow \pi \\ C^{\text{ss}}(\mathcal{L}|_C, G^\lambda)//G^\lambda & \xrightarrow{\theta} & X^{\text{ss}}(\mathcal{L})//G. \end{array}$$

Since  $(C, \lambda)$  is covering,  $\pi(C^+ \cap X^{\text{ss}}(\mathcal{L}))$  is dense in  $X^{\text{ss}}(\mathcal{L})//G$ . But, the same argument as in the beginning of the proof shows that  $\pi(C^+ \cap X^{\text{ss}}(\mathcal{L})) = \pi(C^{\text{ss}}(\mathcal{L}|_C, G^\lambda))$ . So,  $\theta$  is dominant. Since,  $C^{\text{ss}}(\mathcal{L}|_C, G^\lambda)//G^\lambda$  is projective,  $\theta$  is surjective; finally,  $\pi(C^{\text{ss}}(\mathcal{L}|_C, G^\lambda)) = X^{\text{ss}}(\mathcal{L})//G$ .

Let  $\mathcal{O}$  be a closed  $G$ -orbit in  $X^{\text{ss}}(\mathcal{L})$ . Set  $\xi = \pi(\mathcal{O})$ . We just proved that there exists  $z \in \pi^{-1}(\xi) \cap C$ . Let  $z_0$  in the unique closed  $G^\lambda$ -orbit in  $\overline{G^\lambda z} \cap C^{\text{ss}}(\mathcal{L}|_C, G^\lambda)$ . Applying [Lun75, Corollary 1] in the affine  $G$ -variety  $\pi^{-1}(\xi)$ , we obtain that  $G \cdot z_0$  is closed in  $\pi^{-1}(\xi)$  and so in  $X^{\text{ss}}(\mathcal{L})$ . The lemma follows.  $\square$

**Proof.**[of Theorem 8] Let us prove Assertion 1 by induction on the codimension  $d$  of  $\mathcal{F}$  in  $\Lambda$ . Notice that the induction assumption is assumed to hold for all reductive group acting on a smooth projective variety. If  $d = 1$ , the assertion is a direct consequence of Proposition 7. Let us assume that  $d \geq 2$ .

By Proposition 7 again, there exists a well covering pair  $(C, \lambda)$  such that  $\Lambda_0 = \{\mathcal{L} \in \Lambda \mid \mu^{\mathcal{L}}(C, \lambda) = 0\}$  spans an hyperplane of  $\Lambda_{\mathbb{Q}}$  containing  $\mathcal{F}$ . Consider the morphism of restriction

$$\rho : \Lambda_0 \longrightarrow \text{Pic}^{G^\lambda}(C).$$

Let us denote by  $\Lambda_C$  the image of  $\rho$ . Notice that  $\lambda$  acts trivially of the bundles in  $\Lambda_C$ . We claim that

$$\mathcal{AC}_{\Lambda_0}^G(X) = \Lambda_0^+_{\mathbb{Q}} \cap \rho_{\mathbb{Q}}^{-1}(\mathcal{AC}_{\Lambda_C}^{G^\lambda}(C)). \quad (1)$$

Indeed, if  $\mathcal{L}$  belongs to the intersection, there exists  $x \in C$  which is semistable for the action of  $G^\lambda$  and  $\mathcal{L}$ . But since  $x$  is fixed by  $\lambda$ , Proposition 8 shows that  $x$  is semistable for  $G$  and  $\mathcal{L}$ . So,  $\mathcal{L} \in \mathcal{AC}_{\Lambda_0}^G(X)$ . Conversely, let  $\mathcal{L} \in \mathcal{AC}_{\Lambda_0}^G(X)$ . Since  $(C, \lambda)$  is covering,  $X^{\text{ss}}(\mathcal{L})$  intersects  $C^+$ .

But  $\mu^{\mathcal{L}}(C, \lambda) = 0$ ; so, Assertion 2 of Lemma 2 shows that  $C$  intersects  $X^{\text{ss}}(\mathcal{L})$ . So,  $\rho(\mathcal{L}) \in \mathcal{AC}_{\Lambda_C}^{G^\lambda}(C)$ .

The claim implies that  $\rho(\mathcal{F})$  is a face of  $\mathcal{AC}_{\Lambda_C}^{G^\lambda}(C)$  of codimension  $d - 1$  in  $\Lambda_C$ . On the other hand, since  $X$  is smooth, so is  $C$ . By induction, there exists a one parameter subgroup  $\lambda_1$  of  $G^\lambda$  and an irreducible component  $C_1$  of  $C^{\lambda_1}$  satisfying Assertion 1 for the  $G^\lambda$ -variety  $C$  and  $\Lambda_C$ .

For positive integers  $p, q$  we set  $\lambda_{(p,q)} = q\lambda + p\lambda_1$ . Let us now prove that for  $\varepsilon = \frac{p}{q}$  smaller enough, the pair  $(C_1, \lambda_{(p,q)})$  is a covering pair.

Let  $P^\lambda(\lambda_1)$  denote the parabolic subgroup of  $G^\lambda$  associated to  $\lambda_1$ . For  $\varepsilon$  smaller enough, we have  $P(\lambda_{(p,q)}) = P^u(\lambda).P^\lambda(\lambda_1) \subset P(\lambda)$ . Let  $x$  be a point in  $C_1$ . We will make use notation of Section 4 for one parameter subgroups  $\lambda, \lambda_1$  and  $\lambda_{(p,q)}$ . To avoid any confusion, we put the concerned one parameter subgroup in subscribe. One easily checks that for  $\varepsilon$  smaller enough,  $(T_x X_{\lambda_{(p,q)}})_0$  is contained in  $(T_x X_\lambda)_0 = T_x C$ ; so,  $C_1$  is an irreducible component of  $X^{\lambda_{(p,q)}}$ . Consider the natural map  $f : G^\lambda \times C_{1\lambda_{(p,q)}}^+ \longrightarrow C_\lambda^+$ . Let  $x$  be a general point in  $C_1$ . Since  $(C_1, \lambda_1)$  is covering in  $C$  and  $T_x C_\lambda^+ = T_x C + (T_x X_\lambda)_{>0}$ ,  $T_x f$  is surjective. Since  $G^\lambda \times C_{1\lambda_{(p,q)}}^+$  and  $C^+$  are smooth, this implies that  $f$  is dominant.

We are now ready to prove that for  $\varepsilon$  smaller enough, the map  $\eta_1 : G \times_{P(\lambda_{(p,q)})} C_{1\lambda_{(p,q)}}^+ \longrightarrow X$  is birational. Let us recall that  $\eta : G \times_{P(\lambda)} C_\lambda^+ \longrightarrow X$  is birational. Let  $x$  be a general point in  $C_{1\lambda_{(p,q)}}^+$  and  $g \in G$  such that  $g.x \in C_{1\lambda_{(p,q)}}^+$ . To prove that  $\eta_1^{-1}(\eta_1(x)) = \{x\}$ , we have to prove that  $g \in P(\lambda_{(p,q)})$ . Since  $f$  is dominant and  $G^\lambda$ -equivariant, we may assume that  $\eta^{-1}(\eta(x)) = \{x\}$ . Since  $C_{1\lambda_{(p,q)}}^+$  is contained in  $C_\lambda^+$ , this implies that  $g \in P(\lambda)$ . Let us write  $g = ul$  with  $u \in P^u(\lambda)$  and  $l \in G^\lambda$ . Since  $P^u(\lambda)$  is contained in  $P(\lambda_{(p,q)})$ ,  $u^{-1}gx = lx \in C_{1\lambda_{(p,q)}}^+$  and it is sufficient to prove that  $l \in P(\lambda_{(p,q)})$ .

Let us consider the point  $z = \lim_{t \rightarrow 0} \lambda(t)x$ . Then  $z$  and  $lz = \lim_{t \rightarrow 0} \lambda(t)lx$  belong to  $C$  and  $C_{1\lambda_{(p,q)}}^+$ . Since the map  $C_\lambda^+ \longrightarrow C, y \longmapsto \lim_{t \rightarrow 0} \lambda(t)y$  is surjective, for  $x$  general enough,  $z$  belongs to the open subset of  $C \cap C_{1\lambda_{(p,q)}}^+$  where  $G^\lambda \times_{P^\lambda(\lambda_1)} (C \cap C_{1\lambda_{(p,q)}}^+) \longrightarrow C$  is an isomorphism. In this case,  $z, lz \in C \cap C_{1\lambda_{(p,q)}}^+$  implies that  $l \in P^\lambda(\lambda_1) \subset P(\lambda_{(p,q)})$ . This proves that  $(C_1, \lambda_{(p,q)})$  is covering for  $\varepsilon$  smaller enough.



To ends the proof of Assertion 1, it is now sufficient to prove that  $\mathcal{F} = \{\mathcal{L} \in \Lambda_{\mathbb{Q}} \mid \mu^{\mathcal{L}}(C_1, \lambda_{(p,q)}) = 0\} \cap \mathcal{AC}_{\Lambda}^G(X)$  for  $\varepsilon$  smaller enough. By Lemma 5,  $\{\mathcal{L} \in \Lambda_{\mathbb{Q}} \mid \mu^{\mathcal{L}}(C_1, \lambda_{(p,q)}) = 0\} \cap \mathcal{AC}_{\Lambda}^G(X)$  is a face of  $\mathcal{AC}_{\Lambda}^G(X)$ . Then, by general convexity arguments, for  $\varepsilon$  smaller enough, this face is contained in  $\mathcal{F}$ . Since  $\mu^{\bullet}(C_1, \lambda_{(p,q)}) = q\mu^{\bullet}(C, \lambda) + p\mu^{\bullet}(C_1, \lambda_1)$ , this implies that  $\{\mathcal{L} \in \Lambda_{\mathbb{Q}} \mid \mu^{\mathcal{L}}(C_1, \lambda_{(p,q)}) = 0\} \cap \mathcal{AC}_{\Lambda}^G(X) = \{\mathcal{L} \in \Lambda_{\mathbb{Q}} \mid \mu^{\mathcal{L}}(C_1, \lambda_1) = 0\} \cap \mathcal{F}$ . We can conclude using Equality (1).

Let us prove Assertion 2. Let  $(C, \lambda)$  be a covering pair such that  $\mathcal{F} = \{\mathcal{L} \in \Lambda_{\mathbb{Q}} \mid \mu^{\mathcal{L}}(C, \lambda) = 0\} \cap \mathcal{AC}_{\Lambda}^G(X)$ . Let  $\mathcal{L}$  be a point in the relative interior of  $\mathcal{F}$  and  $\xi \in (X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$ . By Lemma 7, there exists a point  $x \in C$  such that  $G \cdot x$  is the closed  $G$ -orbit of  $\pi_{\mathcal{L}}^{-1}(\xi)$ . Let  $Y$  be the irreducible component of  $X^{G_x}$  containing  $x$ . We have:  $Y \subset C$  and  $\text{Im}(\lambda) \subset G_x$ . Consider the map  $\eta : G \times_{P(\lambda)} C^+ \longrightarrow X$ . Since  $G \times_{P(\lambda)} C^+$  and  $X$  are smooth, to prove Assertion 2, it is sufficient to prove that  $T_{[e:x]}\eta$  is an isomorphism, or even that it is surjective.

Let  $N_x$  be a  $G_x$ -stable supplementary of  $T_x(G \cdot x)$  in  $T_x X$  and  $L$  be the  $G_x$ -stable supplementary of  $N_x^{G_x}$  in  $N_x$ . Since  $\eta$  is  $G$ -equivariant, the image of  $T_{[e:x]}\eta$  contains  $T_x(G \cdot x)$ . Since  $N_x^{G_x} \subset T_x X^{G_x} \subset T_x C$ , the image of  $T_{[e:x]}\eta$  contains  $N_x^{G_x}$ . Consider the map  $\mu^{\bullet}(x, G_x) : \Lambda_{\mathbb{Q}} \longrightarrow X(G_x)_{\mathbb{Q}}$  induced by the action of  $G_x$  on the fiber over  $x$  in the line bundles. Since  $G_x$  contains the image of  $\lambda$ , there exists  $\tilde{\mu} : X(G_x)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$  such that  $\mu^{\bullet}(C, \lambda) = \tilde{\mu} \circ \mu^{\bullet}(x, G_x)$ . By Theorem 7,  $\mathcal{AC}^{G_x}(L)$  is contained in the set of the  $\chi$ 's such that  $\tilde{\mu}(\chi) \leq 0$  and 0 is the only point  $\chi$  in  $\mathcal{AC}^{G_x}(L)$  such that  $\tilde{\mu}(\chi) = 0$ . By Lemma 6, this implies that  $\langle \lambda, \chi \rangle > 0$  for all weight  $\chi$  of  $G_x^{\circ}$  in  $L$ . In other words,  $L \subset (T_x X_{\lambda})_{>0}$ . In particular,  $L$  is contained in the image of  $T_{[e:x]}\eta$ . Finally,  $T_{[e:x]}\eta$  is surjective.

Let us prove the last assertion. Let  $\mathcal{L}$  be a point in the relative interior of  $\mathcal{F}$  and  $\xi \in (X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$ . Let us fix a point  $x \in C$  which belongs to the closed orbit of  $G$  in  $\pi_{\mathcal{L}}^{-1}(\xi)$ . By assumption  $G_x$  has dimension  $d$  and contains  $S$ ; hence,  $G_x^{\circ} = S$ . Note that the above construction of  $(C, \lambda)$  prove that the set of  $\lambda \in Y(S)$  satisfying Assertion 1 contains the entire points of an open cone in  $Y(S)_{\mathbb{Q}}$ . So, there exists one satisfying  $G^{\lambda} = G^S$ .  $\square$

## 7.4 From well covering pairs to faces of total $G$ -cone

In Theorem 8, starting with a face of  $\mathcal{AC}^G(X)$  we have constructed covering and well covering pairs. Conversely, in the following theorem, we start with a well covering pair and study the associated face of  $\mathcal{TC}^G(X)$ .

**Theorem 9** *Let  $X$  be a projective  $G$ -variety. We assume the rank of  $\text{Pic}^G(X)$  is finite and consider  $\mathcal{TC}^G(X)$ . Let  $(C, \lambda)$  be a well covering pair. Consider the linear map  $\rho$  induced by the restriction:*

$$\rho : \text{Pic}^G(X)_{\mathbb{Q}} \longrightarrow \text{Pic}^{G^\lambda}(C)_{\mathbb{Q}}.$$

*Then, the subspace of  $\text{Pic}^G(X)_{\mathbb{Q}}$  spanned by the  $\mathcal{L} \in \mathcal{TC}^G(X)$  such that  $\mu^{\mathcal{L}}(C, \lambda) = 0$  is the pullback by  $\rho$  of the subspace of  $\text{Pic}^{G^\lambda}(C)_{\mathbb{Q}}$  spanned by  $\mathcal{TC}^{G^\lambda}(C)$ .*

**Proof.** Let  $\mathcal{L} \in \mathcal{TC}^G(X)$  such that  $\mu^{\mathcal{L}}(C, \lambda) = 0$ . Since  $X^{\text{ss}}(\mathcal{L})$  is open and  $G$ -stable, and  $(C, \lambda)$  is covering, there exists  $x \in C^+$  semistable for  $\mathcal{L}$ . Set  $z = \lim_{t \rightarrow 0} \lambda(t)x$ . Since  $\mu^{\mathcal{L}}(x, \lambda) = 0$ , Lemma 2 shows that  $z$  is semistable for  $\mathcal{L}$ . This implies that the restriction of  $\mathcal{L}$  belongs to  $\mathcal{TC}^{G^\lambda}(C)$ . This proves that the first subspace of the theorem is contained in the second one.

We denote by  $F$  the pullback by  $\rho$  of the subspace spanned by  $\mathcal{TC}^{G^\lambda}(C)$ . Let  $\mathcal{L}_1, \dots, \mathcal{L}_n \in \text{Pic}^G(X)$  which span  $F$  and whose the restrictions to  $C$  belong to  $\mathcal{TC}^{G^\lambda}(C)$ . Denote by  $\mathcal{M}_i$  the restriction of  $\mathcal{L}_i$  to  $C$ . For each  $i$ , let us fix a non zero regular  $G^\lambda$ -invariant section  $\sigma_i$  of  $\mathcal{M}_i$ .

Consider the  $G$ -linearized line bundle  $G \times_{P(\lambda)} p_\lambda^*(\mathcal{M}_i)$  on  $G \times_{P(\lambda)} C^+$  with notation of Section 4. Since  $\eta^*(\mathcal{L}_i)$  and  $G \times_{P(\lambda)} p_\lambda^*(\mathcal{M}_i)$  have the same restriction to  $C$ , Lemma 1 and Proposition 6 show that  $\eta^*(\mathcal{L}_i) = G \times_{P(\lambda)} p_\lambda^*(\mathcal{M}_i)$ . Moreover, since  $\mu^{\mathcal{M}_i}(C, \lambda) = 0$ , Proposition 6 shows that  $\sigma_i$  admits a unique  $P(\lambda)$ -invariant extension to a section  $\sigma'_i$  of  $p_\lambda^*(\mathcal{M}_i)$ . On the other hand, Lemma 1 shows that  $\sigma'_i$  admits a unique  $G$ -invariant extension  $\tilde{\sigma}_i$  from  $C^+$  to  $G \times_{P(\lambda)} C^+$ . So, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{L}_i & \longleftarrow & \eta^*(\mathcal{L}_i) = G \times_{P(\lambda)} p_\lambda^*(\mathcal{M}_i) & \longleftarrow & p_\lambda^*(\mathcal{M}_i) & \longrightarrow & \mathcal{M}_i \\
\downarrow & & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
& & & \tilde{\sigma}_i & & \sigma'_i & \sigma_i \\
X & \xleftarrow{\eta} & G \times_{P(\lambda)} C^+ & \xleftarrow{\quad} & C^+ & \xrightarrow{p_\lambda} & C
\end{array}$$

Since  $\eta$  is birational,  $\tilde{\sigma}_i$  descends to a rational  $G$ -invariant section  $\tau_i$  of  $\mathcal{L}_i$ . Let  $X^\circ$  be a  $G$ -stable open subset of  $X$  such that  $\eta$  induces an isomorphism from  $\eta^{-1}(X^\circ)$  onto  $X^\circ$ . Since  $(C, \lambda)$  is well covering, we may (and shall) assume that  $X^\circ$  intersects  $C$ . Let  $E_j$  be the irreducible components of

codimension one of  $X - X^\circ$ . For any  $j$  we denote by  $a_j$  the maximum of 0 and the  $-\nu_{E_j}(\tau_i)$ 's with  $i = 1, \dots, n$ . Consider the line bundle  $\mathcal{L}_0 = \mathcal{O}(\sum a_j E_j)$  on  $X$ . Since the  $E_j$ 's are stable by the action of  $G$ ,  $\mathcal{L}_0$  is canonically  $G$ -linearized. By construction, the  $\tau_i$ 's induce  $G$ -invariant regular sections  $\tau'_i$  of  $\mathcal{L}'_i := \mathcal{L}_i \otimes \mathcal{L}_0$ . Moreover, since no  $E_j$  contains  $C$ , the restriction of  $\tau'_i$  to  $C$  is non zero. In particular, the  $\mathcal{L}'_i$ 's belong to  $\mathcal{TC}^G(X)$  and their restrictions belong to  $\mathcal{TC}^{G^\lambda}(C)$ . Moreover, replacing  $\mathcal{L}_0$  by  $\mathcal{L}_0^{\otimes 2}$  if necessary, we may (and shall) assume that the  $\mathcal{L}'_i$ 's span  $F$ . This ends the proof of the theorem.  $\square$

## 8 The case $X = \hat{G}/\hat{Q} \times G/Q$

### 8.1 Interpretations of the $G$ -cones

From now on, we assume that  $G$  is a connected reductive subgroup of a connected reductive group  $\hat{G}$ . Let us fix maximal tori  $T$  (resp.  $\hat{T}$ ) and Borel subgroups  $B$  (resp.  $\hat{B}$ ) of  $G$  (resp.  $\hat{G}$ ) such that  $T \subset B \subset \hat{B} \supset \hat{T} \supset T$ . Let  $Q$  (resp.  $\hat{Q}$ ) be a parabolic subgroup of  $G$  (resp.  $\hat{G}$ ) containing  $B$  (resp.  $\hat{B}$ ); let  $L$  (resp.  $\hat{L}$ ) denote the Levi subgroup of  $Q$  (resp.  $\hat{Q}$ ) containing  $T$  (resp.  $\hat{T}$ ).

In this section,  $X$  denote the variety  $\hat{G}/\hat{Q} \times G/Q$  endowed with the diagonal action of  $G$ . We will apply the results of Section 7 to  $X$  with  $\Lambda = \text{Pic}^G(X)$ . The cones  $\mathcal{TC}^G(X)$ ,  $\mathcal{SAC}^G(X)$  and  $\mathcal{AC}^G(X)$  will be denoted without the  $\Lambda$  in subscribe.

Let us describe  $\text{Pic}^G(X)_\mathbb{Q}$ . Consider the natural action of  $\hat{G} \times G$  on  $X$ . By applying the construction of Section 2.1 to the  $\hat{G} \times G$ -homogeneous space  $X$ , one obtains the following isomorphism  $X(\hat{Q}) \times X(Q) \longrightarrow \text{Pic}^{\hat{G} \times G}(X)$ ,  $(\hat{\nu}, \nu) \longmapsto \mathcal{L}_{(\hat{\nu}, \nu)}$ .

**Lemma 8** *The following short sequence is exact*

$$0 \longrightarrow X(\hat{G})_\mathbb{Q} \xrightarrow{\hat{\nu} \mapsto \mathcal{L}_{(\hat{\nu}|_{\hat{Q}}, -\hat{\nu}|_Q)}} \text{Pic}^{\hat{G} \times G}(X)_\mathbb{Q} \xrightarrow{r_\mathbb{Q}^{\Delta G}} \text{Pic}^G(X)_\mathbb{Q} \longrightarrow 0$$

where the second linear map is induced by the restriction  $r^{\Delta G}$  of the action of  $\hat{G} \times G$  to  $G$  diagonally embedded in  $\hat{G} \times G$ .

**Proof.** Let  $\hat{\chi}$  and  $\chi$  be characters of  $\hat{G}$  and  $G$  respectively. The trivial bundle on  $X$  linearized by  $(\hat{\chi}, \chi)$  belongs to  $\text{Pic}^{\hat{G} \times G}(X)$ . The image of this line bundle in  $\text{Pic}^G(X)$  is the trivial line bundle linearized by the character

$\hat{\chi}|_G + \chi$  of  $G$ . In particular, any  $G$ -linearization of the trivial bundle belongs to the image of  $r^{\Delta G}$ .

Let  $\mathcal{L} \in \text{Pic}^G(X)$ . Let  $\mathcal{L}' \in \text{Pic}(X)$  obtained from  $\mathcal{L}$  by forgetting the action of  $G$ . By [FHT84], there exists a  $\hat{G} \times G$ -linearization  $\mathcal{M}$  of  $\mathcal{L}'^{\otimes n}$  for a positive integer  $n$ . Then  $\mathcal{M}^* \otimes \mathcal{L}^{\otimes n}$  is the trivial line bundle over  $X$ ; so, it belongs to the image  $r^{\Delta G}$ . Finally,  $\mathcal{L}^{\otimes n}$  belongs to the image of the restriction. This ends the proof of the surjectivity of  $r^{\Delta G}_{\mathbb{Q}}$ .

Let  $\mathcal{L}$  in the kernel of  $r^{\Delta G}$ . Since  $\mathcal{L}$  is trivial as a line bundle, there exists characters  $\hat{\chi}$  and  $\chi$  of  $\hat{G}$  and  $G$  such that  $\mathcal{L} = \mathcal{L}_{(\hat{\chi}, \chi)}$ . The  $G$ -linearization of this last character is trivial if and only if  $\hat{\chi}|_G + \chi$  is trivial. This ends the proof of the lemma.  $\square$

**Proposition 10** (i)  $\mathcal{TC}^G(X) = \mathcal{SAC}^G(X)$  is a closed convex polyhedral cone.

(ii) Let  $(\hat{\nu}, \nu) \in X(\hat{Q})_{\mathbb{Q}} \times X(Q)_{\mathbb{Q}}$ . Then  $r^{\Delta G}_{\mathbb{Q}}(\mathcal{L}_{(\hat{\nu}, \nu)}) \in \mathcal{TC}^G(X)$  if and only if  $\hat{\nu}$  and  $\nu$  are dominant and for  $n$  big enough  $V_{n\hat{\nu}} \otimes V_{n\nu}$  contains non zero  $G$ -invariant vectors.

(iii) If  $\mathcal{AC}^G(X)$  is non empty, its closure in  $\text{Pic}^G(X)_{\mathbb{Q}}$  is  $\mathcal{SAC}^G(X)$ .

(iv) If  $Q$  and  $\hat{Q}$  are Borel subgroups of  $G$  and  $\hat{G}$  then  $\mathcal{AC}^G(X)$  is non empty.

**Proof.** Since  $X$  is homogeneous under the action of  $\hat{G} \times G$  and since any line bundle has a linearizable power, every bundle with a non zero section is semiample. Hence,  $\mathcal{TC}^G(X) = \mathcal{SAC}^G(X)$ . Lemma 2 and Proposition 3 implies that only finitely many non empty open subsets of  $X$  are of the form  $X^{\text{ss}}(\mathcal{L})$  with  $\mathcal{L} \in \text{Pic}^G(X)$ . Let  $x$  be a point in the intersection of these subsets. Lemma 2 shows that  $\mathcal{SAC}^G(X)$  is the set of points  $\mathcal{L} \in \text{Pic}^G(X)_{\mathbb{Q}}$  semiample such that  $\mu^{\mathcal{L}}(x, \lambda) \leq 0$  for all one parameter subgroup  $\lambda$  of  $G$ . In particular,  $\mathcal{SAC}^G(X)$  is a closed convex cone.

Let now  $\hat{\nu}$  and  $\nu$  be characters of  $\hat{Q}$  and  $Q$ . If  $\hat{\nu}$  or  $\nu$  is not dominant then  $\mathcal{L}_{(\hat{\nu}, \nu)}$  has no regular section. Else, Borel-Weyl's theorem shows that  $H^0(X, \mathcal{L}_{(\hat{\nu}, \nu)})$  is isomorphic as a  $\hat{G} \times G$ -module to  $V_{\hat{\nu}} \otimes V_{\nu}$ . The second assertion of the proposition follows.

The third assertion is satisfied since the ample cone in  $\text{Pic}^G(X)$  is the interior of the semiample cone.

Let  $w_0$  be the longest element of  $W$ . Assume that  $Q = B$  and  $\hat{Q} = \hat{B}$  are Borel subgroups of  $G$  and  $\hat{G}$ . Let  $\hat{\nu}_0$  be any character of  $\hat{B}$  such that  $\mathcal{L}_{\hat{\nu}_0}$  is ample over  $\hat{G}/\hat{B}$ . Let  $\nu$  be any dominant weight of the  $G$ -module  $V_{\nu_0}$ . Then,  $r^{\Delta G}(\mathcal{L}_{(\hat{\nu}_0, -w_0\nu)})$  belongs to  $\mathcal{SAC}^G(X)$ .

Let  $\nu_0$  be any character of  $B$  such that  $\mathcal{L}_{\nu_0}$  is ample over  $G/B$ . Since the restriction  $\mathbb{K}[\hat{G}] \rightarrow \mathbb{K}[G]$  is surjective, Frobenius's theorem implies that  $V_{\nu_0}$  is contained in an irreducible  $\hat{G}$ -module  $V_{\hat{\nu}}$ . Then,  $r^{\Delta G}(\mathcal{L}_{(\hat{\nu}, -w_0\nu_0)})$  belongs to  $\mathcal{SAC}^G(X)$ .

Since  $\mathcal{SAC}^G(X)$  is convex, it contains  $r^{\Delta G}(\mathcal{L}_{(\hat{\nu}+\hat{\nu}_0, -w_0(\nu_0+\nu))})$ . But the line bundle  $\mathcal{L}_{(\hat{\nu}+\hat{\nu}_0, -w_0(\nu_0+\nu))}$  is ample. The last assertion is proved.  $\square$

## 8.2 Covering and well covering pairs

**Notation.** Let  $W$  and  $\hat{W}$  denote the Weyl groups of  $G$  and  $\hat{G}$ . If  $P$  is a parabolic subgroup of  $G$  containing  $T$ ,  $W_P$  denote the Weyl group of the Levi subgroup of  $P$  containing  $T$ . This group  $W_P$  is canonically embedded in  $W$ .

In this subsection, we explain how to find the well covering pairs in the case when  $X = \hat{G}/\hat{Q} \times G/Q$ .

**8.2.1** — Let  $\lambda$  be a one parameter subgroup of  $T$  and so of  $\hat{T}$ . We can describe the fix point set  $X^\lambda$ :

$$X^\lambda = \bigcup_{\substack{\hat{w} \in W_{\hat{P}(\lambda)} \backslash \hat{W}/W_{\hat{Q}} \\ w \in W_{P(\lambda)} \backslash W/W_Q}} \hat{G}^\lambda \hat{w} \hat{Q}/\hat{Q} \times G^\lambda w Q/Q.$$

We will denote by  $C(\hat{w}, w)$  the component  $\hat{G}^\lambda \hat{w} \hat{Q}/\hat{Q} \times G^\lambda w Q/Q$  of  $X^\lambda$ . Note that

$$C^+(\hat{w}, w) = \hat{P}(\lambda) \hat{w} \hat{Q}/\hat{Q} \times P(\lambda) w Q/Q.$$

**8.2.2** — We consider the cohomology ring  $H^*(G/P(\lambda), \mathbb{Z})$  of  $G/P(\lambda)$ . Here, we use simplicial cohomology with integers coefficients. If  $V$  is a closed subvariety of  $G/P(\lambda)$ , we denote by  $[V]$  its class in  $G/P(\lambda)$ . If  $w \in W$  (or  $W/W_{P(\lambda)}$ ), we set  $\Lambda_w = BwP(\lambda)/P(\lambda)$  and denote by  $[\Lambda_w]$  the class in

$H^*(G/P(\lambda), \mathbb{Z})$  of the closure in  $G/P(\lambda)$  of  $\Lambda_w$ . Let us recall that

$$H^*(G/P(\lambda), \mathbb{Z}) = \bigoplus_{w \in W/W_{P(\lambda)}} \mathbb{Z}[\Lambda_w].$$

We use similar notation for  $\hat{G}/\hat{P}(\lambda)$ . Since  $P(\lambda) = G \cap \hat{P}(\lambda)$ ,  $G/P(\lambda)$  identifies with the orbit by  $G$  of  $\hat{P}(\lambda)/\hat{P}(\lambda)$  in  $\hat{G}/\hat{P}(\lambda)$ ; let  $\iota : G/P(\lambda) \rightarrow \hat{G}/\hat{P}(\lambda)$  denote this closed immersion. The map  $\iota$  induces a map  $\iota^*$  in cohomology:

$$\iota^* : H^*(\hat{G}/\hat{P}(\lambda), \mathbb{Z}) \rightarrow H^*(G/P(\lambda), \mathbb{Z}).$$

**Lemma 9** *Let  $\hat{w} \in \hat{W}$  and  $w \in W$ . The pair  $(C(\hat{w}, w), \lambda)$  is covering if and only if  $\iota^*([\hat{Q}\hat{w}^{-1}\hat{P}(\lambda)/\hat{P}(\lambda)].[\overline{Qw^{-1}P(\lambda)/P(\lambda)}]) = [\Lambda_e]$ .*

**Proof.** Set  $P = P(\lambda)$  and  $\hat{P} = \hat{P}(\lambda)$ . Consider the map:

$$\eta : G \times_P C^+(\hat{w}, w) \rightarrow X.$$

Since the characteristic of  $\mathbb{K}$  is zero,  $\eta$  is birational if and only if for  $x$  in an open subset of  $X$ ,  $\eta^{-1}(x)$  is reduced to a point. Consider the projection  $\pi : G \times_P C^+(\hat{w}, w) \rightarrow G/P$ . For any  $x$  in  $X$ ,  $\pi$  induces an isomorphism from  $\eta^{-1}(x)$  onto the following locally closed subvariety of  $G/P$ :  $F_x := \{gP \in G/P : g^{-1}x \in C^+(\hat{w}, w)\}$ .

Let  $(\hat{g}, g) \in \hat{G} \times G$  and set  $x = (\hat{g}\hat{Q}/\hat{Q}, gQ/Q) \in X$ . We have:

$$\begin{aligned} F_x &= \{hP/P \in G/P : h^{-1}gQ/Q \in PwQ/Q \text{ and } h^{-1}\hat{g}\hat{Q}/\hat{Q} \in \hat{P}\hat{w}\hat{Q}/\hat{Q}\} \\ &= \{hP/P \in G/P : h^{-1} \in (PwQg^{-1}) \cap (\hat{P}\hat{w}\hat{Q}\hat{g}^{-1})\} \\ &= (gQw^{-1}P/P) \cap (\hat{g}\hat{Q}\hat{w}^{-1}\hat{P}/\hat{P}). \end{aligned}$$

Let us fix  $g$  arbitrarily. By Kleiman's Theorem (see [Kle74]), there exists an open subset of  $\hat{g}$ 's in  $\hat{G}$  such that the intersection  $\overline{gQw^{-1}P/P} \cap \hat{g}\hat{Q}\hat{w}^{-1}\hat{P}/\hat{P}$  is transverse. Moreover (see for example [BK06]), one may assume that  $(gQw^{-1}P/P) \cap (\hat{g}\hat{Q}\hat{w}^{-1}\hat{P}/\hat{P})$  is dense in  $\overline{gQw^{-1}P/P} \cap \hat{g}\hat{Q}\hat{w}^{-1}\hat{P}/\hat{P}$ . We deduce that the following are equivalent:

- (i) for generic  $\hat{g}$ ,  $F_x$  is reduced to a point,
- (ii)  $\iota^*([\hat{Q}\hat{w}^{-1}\hat{P}/\hat{P}]).[\overline{Qw^{-1}P/P}] = [\Lambda_e]$ .

Since  $\eta$  is  $G$ -equivariant, the above Condition 1 is clearly equivalent to the fact that  $\eta$  is birational.  $\square$

**8.2.3 — Notation.** From now on,  $\mathfrak{g}$  and  $\mathfrak{b}$  will denote the Lie algebras of  $G$  and  $B$ ,  $R$  will denote the set of roots of  $\mathfrak{g}$  and  $R^+$  those of positive ones. We denote by  $\rho$  the half sum of the positive roots of  $\mathfrak{g}$ . We will also use the following similar notation for  $\hat{G}$ :  $\hat{\mathfrak{g}}, \hat{\mathfrak{b}}, \hat{R}, \hat{R}^+, \hat{\rho}$ .

Let  $w \in W$ . Consider  $\tilde{w}$  the longest element in the class of  $w$  in  $W_P \backslash W$ . We define  $\theta_w^P$  to be the sum of the weights of  $T$  in the normal space at  $P/P$  of  $\tilde{w}B\tilde{w}^{-1}P/P$  in  $G/P$ . Obviously,  $\theta_w^P$  only depends on the class of  $w$  in  $W_P \backslash W$ . If  $P = P(\lambda)$ , we set  $\theta_w^\lambda = \theta_w^{P(\lambda)}$ . Similarly, we define  $\hat{\theta}$ .

**Lemma 10** *If  $w$  is of maximal length in its class in  $W_P \backslash W$ , then*

$$\theta_w^P = -(\rho + w\rho).$$

**Proof.** By considering the natural map  $G \rightarrow G/P$ , one sees that  $\theta_w^P$  is the sum of the weights of  $T$  in the normal space at  $e$  of  $wBw^{-1}P$  in  $G$ . Our assumption on  $w$  implies that  $wBw^{-1}B$  is open in  $wBw^{-1}P$ ; hence,  $\theta_w^P$  is the sum of the weights of  $T$  in  $\mathfrak{g}/(w\mathfrak{b}w^{-1} + \mathfrak{b})$ . Since the sum of all the roots is zero, we obtain that  $-\theta_w^P$  is the sum of the elements of  $R^+ \cup wR^+$ . So, we have:

$$\begin{aligned} -\theta_w^P &= \sum_{\alpha \in R^+ \cup wR^+} \alpha \\ &= \sum_{\alpha \in R^+} \alpha + \sum_{\alpha \in wR^+} \alpha - \sum_{\alpha \in R^+ \cap wR^+} \alpha \\ &= 2\rho + 2w\rho - \left[ \sum_{\alpha \in R^+ \cup wR^+} \alpha - \sum_{\alpha \in R^+ \setminus wR^+} \alpha - \sum_{\alpha \in wR^+ \setminus R^+} \alpha \right] \\ &= 2\rho + 2w\rho + \theta_w^P, \end{aligned}$$

where the last equality holds since  $R^+ \setminus wR^+ = -(wR^+ \setminus R^+)$ . The lemma follows immediately.  $\square$

**Remark.** In [BK06], Belkale and Kumar defined characters  $\chi_{w^{-1}}$  for  $w$  of minimal length its class in  $W/W_P$ . If  $\tilde{w}$  is the element of the class of  $w$  in  $W/W_P$ , we have  $\theta_{\tilde{w}} = -\chi_{w^{-1}}$ .

**8.2.4 —** Let us fix again a dominant one parameter subgroup  $\lambda$  of  $T$ ,  $\hat{w} \in \hat{W}$  and  $w \in W$ .

To simplify notation, we set  $P = P(\lambda)$ ,  $C = C(\hat{w}, w)$  and  $C^+ = C^+(\hat{w}, w)$ . Consider

$$\eta : G \times_P C^+ \rightarrow X = \hat{G}/\hat{Q} \times G/Q.$$

**Notation.** If  $Y$  is a smooth variety of dimension  $n$ ,  $TY$  denotes its tangent bundle. The line bundle  $\bigwedge^n TY$  over  $Y$  will be called the *determinant*

bundle and denoted by  $\mathcal{D}etY$ . If  $\varphi : Y \rightarrow Y'$  is a morphism between smooth variety, we denote by  $T\varphi : \mathcal{T}Y \rightarrow \mathcal{T}Y'$  its tangent map, and by  $\mathcal{D}et\varphi : \mathcal{D}etY \rightarrow \mathcal{D}etY'$  its determinant.

Consider the restriction of  $T\eta$  to  $C^+$ :

$$T\eta|_{C^+} : \mathcal{T}(G \times_P C^+)|_{C^+} \rightarrow \mathcal{T}(X)|_{C^+},$$

and the restriction of  $\mathcal{D}et\eta$  to  $C^+$ :

$$\mathcal{D}et\eta|_{C^+} : \mathcal{D}et(G \times_P C^+)|_{C^+} \rightarrow \mathcal{D}et(X)|_{C^+}.$$

Since  $\eta$  is  $G$ -equivariant, the morphism  $\mathcal{D}et\eta|_{C^+}$  is  $P$ -equivariant; it can be thought as a  $P$ -invariant section of the line bundle  $\mathcal{D}et(G \times_P C^+)|_{C^+}^* \otimes \mathcal{D}et(X)|_{C^+}$  over  $C^+$ . We denote by  $\mathcal{L}_{P, \hat{w}, w}$  this last  $P$ -linearized line bundle on  $C^+$ .

**Lemma 11** *We assume that  $w$  (resp.  $\hat{w}$ ) is of maximal length in its class in  $W_P \setminus W/W_Q$  (resp.  $W_{\hat{P}} \setminus \hat{W}/W_{\hat{Q}}$ ). Then, the torus  $T$  acts on the fiber over the point  $(\hat{w}\hat{Q}/\hat{Q}, wQ/Q)$  in  $\mathcal{L}_{P, \hat{w}, w}$  by the character*

$$r_T(\hat{\theta}_{\hat{w}}^P) + \theta_w^P - \theta_e^P,$$

where  $w^P$  is the longest element in  $W_P$ .

**Proof.** If  $Z$  is a locally closed subvariety of a variety  $Y$  and  $z$  is a point of  $Z$ , we denote by  $N_z^Y(Z)$  the quotient  $T_z Y / T_z Z$  of the tangent spaces at  $z$  of  $Y$  and  $Z$ . If  $V$  is a  $T$ -module  $\text{St}_T(V)$  denote the multiset of the weights of  $T$  in  $V$ . Let  $\chi$  denote the character of the action of  $T$  on the fiber over the point  $x = (\hat{w}\hat{Q}/\hat{Q}, wQ/Q)$  in  $\mathcal{L}_{P, \hat{w}, w}$ . Let  $\mathfrak{p}$  denote the Lie algebra of  $P$ .

Since  $\eta$  induces the identity on  $C^+$  (canonically embedded in  $G \times_P C^+$ ), we have:

$$\chi = - \sum_{\alpha \in \text{St}_T(N_x^{G \times_P C^+}(C^+))} \alpha + \sum_{\alpha \in \text{St}_T(N_x^X(C^+))} \alpha.$$

Moreover, we have the following  $T$ -equivariant isomorphisms:

$$\begin{aligned} N_x^{G \times_P C^+}(C^+) &\simeq \mathfrak{g}/\mathfrak{p} \simeq N_e^G(P), \\ N_x^X(C^+) &\simeq N_{\hat{w}\hat{Q}/\hat{Q}}^{\hat{G}/\hat{Q}}(\hat{P}\hat{w}\hat{Q}/\hat{Q}) \oplus N_{wQ/Q}^{G/Q}(PwQ/Q) \\ &\simeq \hat{w}N_e^{\hat{G}}(\hat{w}^{-1}\hat{P}\hat{w}\hat{Q}) \oplus wN_e^G(w^{-1}PwQ). \end{aligned}$$



Moreover, our assumption on  $w$  (resp.  $\hat{w}$ ) implies that  $w^{-1}PwB$  (resp.  $\hat{w}^{-1}\hat{P}\hat{w}\hat{B}$ ) is open in  $w^{-1}PwQ$  (resp.  $\hat{w}^{-1}\hat{P}\hat{w}\hat{Q}$ ). So, we obtain that:

$$\chi = -\theta_e^P + r_T(\hat{w}\hat{\theta}_{\hat{w}^{-1}}^{\hat{P}}) + w\theta_{w^{-1}}^P.$$

Since  $w\theta_{w^{-1}}^P = \theta_w^P$ , the lemma is proved.  $\square$

One can now describe the well covering pairs of  $X$ :

**Proposition 11** *Let  $\lambda$  be a dominant one parameter subgroup of  $T$ . Let  $(\hat{w}, w) \in \hat{W} \times W$  of maximal length in  $W_{\hat{P}} \backslash \hat{W}/W_{\hat{Q}}$  and  $W_P \backslash W/W_Q$  respectively.*

*The following are equivalent:*

(i) *The pair  $(C(\hat{w}, w), \lambda)$  is well covering.*

(ii)  *$\iota^*([\Lambda_{\hat{w}^{-1}}]).[\Lambda_{w^{-1}}] - [\Lambda_e] = 0$ , and  $\langle r_T(\hat{\theta}_{\hat{w}}^{\hat{P}}) + \theta_w^P - \theta_e^P, \lambda \rangle = 0$ .*

**Proof.** By Lemma 9, we may (and shall) assume that  $(C(\hat{w}, w), \lambda)$  is covering. Then, it remains to prove that  $(C(\hat{w}, w), \lambda)$  is well covering if and only if the restriction of  $r_T(\hat{\theta}_{\hat{w}}^{\hat{P}}) + \theta_w^P - \theta_e^P$  to  $\mathfrak{t}_\lambda$  is trivial.

Assume that  $(C(\hat{w}, w), \lambda)$  is well covering. Then  $\mathcal{D}et\eta|_C$  is non identically zero. Since  $\mathcal{D}et\eta|_C$  is a  $G^\lambda$ -invariant section of  $\mathcal{L}_{P, \hat{w}, w|_C}$ ,  $\lambda$  which fixes point wise  $C$  has to act trivially on  $\mathcal{L}_{P, \hat{w}, w|_C}$  (see for example Lemma 4). By Lemma 11 this implies that the restriction of  $r_T(\hat{\theta}_{\hat{w}}^{\hat{P}}) + \theta_w^P - \theta_e^P$  to  $\text{Im}(\lambda)$ .

Conversely, assume that  $\langle r_T(\hat{\theta}_{\hat{w}}^{\hat{P}}) + \theta_w^P - \theta_e^P, \lambda \rangle = 0$ . By Lemma 11, this implies that  $\mu^{\mathcal{L}_{P, \hat{w}, w|_C}}(C, \lambda) = 0$ . But, since  $\eta$  is birational,  $\mathcal{D}et\eta$  is  $G$ -invariant and non zero; hence,  $\mathcal{D}et\eta|_{C^+}$  is  $P$ -invariant and non zero. So, Proposition 6 show that the restriction of  $\mathcal{D}et\eta|_C$  is non identically zero. Since  $\eta$  is birational, this implies that  $\eta$  is an isomorphism over an open subset intersecting  $C$ .  $\square$

### 8.3 The case $X = \hat{G}/\hat{B} \times G/B$

**8.3.1** — We denote by  $\mathcal{LR}(\hat{G}, G)$  the cone of the pairs  $(\hat{\nu}, \nu) \in X(\hat{T})_{\mathbb{Q}} \times X(T)_{\mathbb{Q}}$  such that for a positive integer  $n$ ,  $n\hat{\nu}$  and  $n\nu$  are dominant weights such that  $V_{n\hat{\nu}} \otimes V_{n\nu}$  contains non zero  $G$ -invariant vectors.

From now on,  $X = \hat{G}/\hat{B} \times G/B$ . By Proposition 10, a point  $(\hat{\nu}, \nu)$  belongs to  $\mathcal{LR}(\hat{G}, G)$  if and only if  $r^{\Delta G}(\mathcal{L}_{(\hat{\nu}, \nu)})$  belongs to  $\mathcal{TC}^G(X) = \mathcal{SAC}^G(X)$ .

**8.3.2** — Consider the  $G$ -module  $\hat{\mathfrak{g}}/\mathfrak{g}$ . Let  $\chi_1, \dots, \chi_n$  be the set of the non trivial weights of  $T$  on  $\hat{\mathfrak{g}}/\mathfrak{g}$ . For  $I \subset \{1, \dots, n\}$ , we will denote by  $T_I$  the neutral component of the intersection of the kernels of the  $\chi_i$ 's with  $i \in I$ . A subtorus of the form  $T_I$  is said to be *admissible*. The subtorus  $T_I$  is said to be *dominant* if  $Y(T_I)_{\mathbb{Q}}$  is spanned by its intersection with the dominant chamber of  $Y(T)_{\mathbb{Q}}$ . Notice that the set of the  $\chi_i$ 's being stable by the action of  $W$ , any admissible subtorus is conjugated by an element of  $W$  to a dominant admissible subtorus. A one parameter subgroup of  $T$  is said to be *admissible* if its image is.

To each  $\chi_i$ , we associate the hyperplane  $\mathcal{H}_i$  in  $Y(T)_{\mathbb{Q}}$  spanned by the  $\lambda \in Y(T)$  such that  $\chi \circ \lambda$  is trivial. The  $\mathcal{H}_i$ 's form an  $W$ -invariant arrangement of hyperplane in  $Y(T)_{\mathbb{Q}}$ . Moreover,  $Y(T_I)_{\mathbb{Q}}$  is the intersection of the  $\mathcal{H}_i$ 's with  $i \in I$ .

To simplify, in the following statement we assume that  $\mathcal{AC}^G(X)$  has a non empty interior in  $\text{Pic}^G(X)_{\mathbb{Q}}$ . In fact, this assumption is equivalent to say that no ideal of  $\mathfrak{g}$  is an ideal of  $\hat{\mathfrak{g}}$ .

**Theorem 10** *We assume that no ideal of  $\mathfrak{g}$  is an ideal of  $\hat{\mathfrak{g}}$ .*

- (i) *The interior of  $\mathcal{LR}(\hat{G}, G)$  in  $X(\hat{T})_{\mathbb{Q}} \times X(T)_{\mathbb{Q}}$  is not empty.*
- (ii) *Let  $\mathcal{F}$  be a face of  $\mathcal{LR}(\hat{G}, G)$  of codimension  $d$ . Then there exists a dominant admissible subtorus  $T_I$  (with  $I \subset \{1, \dots, n\}$ ) of  $T$  of dimension  $d$ , a dominant one parameter subgroup  $\lambda$  of  $T_I$ , and an irreducible component  $C(\hat{w}, w)$  of  $X^\lambda$  such that:*
  - (a)  $\iota^*([\Lambda_{\hat{w}-1}]) \cdot [\Lambda_{w-1}] = [\Lambda_e] \in H^*(G/P(\lambda), \mathbb{Z})$ ,
  - (b)  $r_T(\hat{\theta}_{\hat{w}}^\lambda) + \theta_w^\lambda - \theta_e^\lambda$  is trivial on  $T_I$  (we assume that  $w$  and  $\hat{w}$  have maximal length in their class in  $W_{P(\lambda)} \setminus W$  and  $W_{\hat{P}(\lambda)} \setminus \hat{W}$ ), and
  - (c)  $\mathcal{F}$  is the  $(\hat{\nu}, \nu) \in \mathcal{LR}(\hat{G}, G)$  such that  $\langle \lambda, \hat{w}\hat{\nu} + w\nu \rangle = 0$ .

Moreover, we have:

- (a)  $C$  is an irreducible component of  $X^{T_I}$ ,
  - (b)  $T_I$  is the neutral component of a generic closed isotropy of  $\mathcal{F}$ , and
  - (c) there exists  $x \in C(\hat{w}, w)$  such that  $(G_x^\lambda)^\circ = T_I$ .
- (iii) *Conversely, let  $\lambda$  be a dominant one parameter subgroup of  $T$  and  $C = C(\hat{w}, w)$  be an irreducible component  $X^\lambda$ . Set  $I = \{i = 1, \dots, n \mid \chi \circ \lambda \text{ is trivial}\}$  and denote by  $d$  the dimension of  $T_I$ . If*

- (a)  $\iota^*([\Lambda_{\hat{w}^{-1}}]).[\Lambda_{w^{-1}}] = [\Lambda_e] \in H^*(G/P(\lambda), \mathbb{Z})$  and  
(b)  $r_T(\hat{\theta}_{\hat{w}}^\lambda) + \theta_w^\lambda - \theta_e^\lambda$  is trivial on  $T_I$  (we assume that  $w$  and  $\hat{w}$  have maximal length in their class in  $W_{P(\lambda)} \setminus W$  and  $W_{\hat{P}(\lambda)} \setminus \hat{W}$ ),

then the set of  $(\hat{\nu}, \nu) \in \mathcal{LR}(\hat{G}, G)$  such that  $\langle \lambda, \hat{w}\hat{\nu} + w\nu \rangle = 0$  is a face of  $\mathcal{LR}(\hat{G}, G)$  of codimension  $d$ .

**Proof.** By [MR08, Corollaire 1], the codimension of  $\mathcal{AC}^G(X)$  in  $\text{Pic}^G(X)_{\mathbb{Q}}$  is the dimension of the generic isotropy of  $T$  acting on  $\hat{\mathfrak{g}}/\mathfrak{g}$ . This generic isotropy is also the kernel of the action of  $T$  on  $\hat{G}/G$ . So, it is contained in  $\bigcap_{\hat{g} \in \hat{G}} \hat{g}G\hat{g}^{-1}$ . Since this group is distinguish in  $\hat{G}$  and  $G$ , it is finite by assumption. Hence, the interior of  $\mathcal{AC}^G(X)$  in  $\text{Pic}^G(X)_{\mathbb{Q}}$  is non empty.

Let  $\mathcal{F}$  be a face of  $\mathcal{AC}^G(X)$  of codimension  $d$ . Let  $H$  be a generic closed isotropy of  $\mathcal{F}$  and  $C_{\mathcal{F}}$  be an irreducible component of  $Y_{\mathcal{F}}$  (see Theorem 6). We firstly prove the following claims.

Claim 1. The subgroup  $H$  is diagonalizable and has dimension  $d$ .

By conjugating by an element of  $G$ , one may assume that  $C$  intersects  $\hat{G}/B \times \{B/B\}$ . Consider the second projection  $p_{G/B} : X \rightarrow G/B$ . Consider also the map  $\mu^\bullet(C, H)$  used in Theorem 7. Notice that  $H$  is contained in  $B$  and for any  $\chi \in X(B)$ ,  $\mu^\bullet(C, H) \circ p_{G/B}^*(\mathcal{L}_\chi) = \chi|_H$ . In particular,  $\mu^\bullet(C, H)$  is surjective. But, Theorem 6 implies that the subspace spanned by  $\mathcal{F}$  is the kernel of  $\mu^\bullet(C, H)$ . We deduce that the rank of  $X(H)$  equal  $d$ . Since  $H$  is reductive subgroup of  $B$ , Claim 1 follows.

Claim 2. The neutral component of a generic closed isotropy of  $\mathcal{F}$  is a dominant admissible subtorus of  $T$ .

By Claim 1, one may assume that  $H \subset T$ . Notice that  $C$  is isomorphic to the variety of the complete flags  $\hat{G}^H \times G^H$  and  $H$  is the isotropy of a general point in  $C$ . Hence,  $H$  is the isotropy in  $B^H$  of a general point in  $\hat{G}^H/\hat{B}^H$ , that by Bruhat's Theorem of a general point in  $\hat{B}^H/\hat{T}^H$ . Since  $U^H$  acts freely on  $\hat{B}^H/\hat{T}^H$ ,  $H$  is conjugated to the isotropy in  $T$  of a general point in  $\hat{U}^H/U^H$  which is isomorphic to  $\hat{\mathfrak{u}}^H/\mathfrak{u}^H$ . Finally, one may assume that  $H$  is the isotropy of a general point in  $\hat{\mathfrak{g}}^H/\mathfrak{g}^H = (\hat{\mathfrak{g}}/\mathfrak{g})^H$ ; that is, since  $H$  is diagonalizable, that  $H$  is the kernel of the action of  $T$  on  $(\hat{\mathfrak{g}}/\mathfrak{g})^H$ . Then,  $H$  is admissible. By conjugating by an element of  $W$  one may assume that  $H$  is dominant.

Let  $I \subset \{1, \dots, n\}$  such that  $H^\circ = T_I$ . By Theorem 8 there exists a dominant one parameter subgroup  $\lambda$  of  $H$  and an irreducible component

$C = C(\hat{w}, w)$  of  $X^H$  such that  $C$  is an irreducible component of  $X^\lambda$ ,  $(C, \lambda)$  is well covering and  $\mathcal{F}$  is obtained by vanishing  $\mu^\bullet(C, \lambda)$ . By Proposition 11,  $w, \hat{w}$  and  $\lambda$  satisfy Conditions 2a and 2b of the theorem. Assertion 2 follows.

Conversely, let  $w, \hat{w}, \lambda$  and  $I$  as in Assertion 3 of the theorem. By Proposition 11,  $(C(\hat{w}, w), \lambda)$  is a well covering pair. By Lemma 5, the set of  $\mathcal{L} \in \mathcal{TC}^G(X)$  such that  $\mu^\mathcal{L}(C, \lambda) = 0$  is a face  $\mathcal{F}'$  of  $\mathcal{TC}^G(X)$  eventually empty. It remains to prove that the codimension of  $\mathcal{F}'$  equal the dimension  $d$  of  $T_I$ .

Consider the linear map  $\rho$  induced by the restriction:  $\rho : \text{Pic}^G(X)_\mathbb{Q} \longrightarrow \text{Pic}^{G^\lambda}(C(\hat{w}, w))_\mathbb{Q}$ . We firstly prove the

**Claim 3.**  $\rho$  is surjective.

Let  $x$  be a point in  $C(\hat{w}, w)$ . Let  $(\hat{B}_1, B_1)$  denote the isotropy of  $x$  in  $\hat{G} \times G$ . Let  $\mathcal{L} \in \text{Pic}^{G^\lambda}(C(\hat{w}, w))$ . By Lemma 8 applied to  $C$ , there exists characters  $\hat{\nu}$  and  $\nu$  of  $\hat{B}_1^\lambda$  and  $B_1^\lambda$  and a positive integer  $n$  such that  $\mathcal{L}^\otimes$  is the restriction of the  $\hat{G}^\lambda \times G^\lambda$ -linearized line bundle  $\mathcal{L}_{(\hat{\nu}, \nu)}$ . But,  $\hat{\nu}$  and  $\nu$  can be extended to characters  $\hat{\nu}'$  and  $\nu'$  of  $\hat{B}_1$  and  $B_1$ . Then,  $\mathcal{L}^{\otimes n}$  is the image by  $\rho$  of  $r^{\Delta G}(\mathcal{L}_{\hat{\nu}', \nu'})$ .

By Claim 3 and Theorem 9, it remains to prove that  $\mathcal{TC}^{G^\lambda}(C(\hat{w}, w))$  has codimension  $d$  in  $\text{Pic}^{G^\lambda}(C(\hat{w}, w))_\mathbb{Q}$ . By [MR08, Corollaire 1], this codimension is the dimension of the kernel of the action of  $T$  over  $\hat{\mathfrak{g}}^\lambda/\mathfrak{g}^\lambda = (\hat{\mathfrak{g}}/\mathfrak{g})^\lambda$ . By definition  $T_I$  is the neutral component of this kernel.  $\square$

**Corollary 2** *We assume that no ideal of  $\mathfrak{g}$  is an ideal of  $\hat{\mathfrak{g}}$ . Any dominant weight  $(\hat{\nu}, \nu)$  belongs to  $\mathcal{LR}(\hat{G}, G)$  if and only if*

$$\langle \lambda, \hat{w}\hat{\nu} + w\nu \rangle \geq 0,$$

for all indivisible dominant admissible one parameter subgroup  $\lambda$  of  $T$  and for all  $(\hat{w}, w) \in W_{\hat{P}(\lambda)} \setminus \hat{W} \times W_{P(\lambda)} \setminus W$  such that

$$(i) \quad \iota^*([\Lambda_{\hat{w}^{-1}}], [\Lambda_{w^{-1}}]) = [\Lambda_e] \in H^*(G/P(\lambda), \mathbb{Z}), \text{ and}$$

$$(ii) \quad \langle \lambda, r_T(\hat{\theta}_{\hat{w}}^\lambda) + \theta_w^\lambda - \theta_e^\lambda \rangle = 0 \text{ (we assume that } w \text{ and } \hat{w} \text{ have maximal length in their class in } W_{P(\lambda)} \setminus W \text{ and } W_{\hat{P}(\lambda)} \setminus \hat{W}).$$

Moreover, the above inequalities are pairwise distinct and no one can be omitted.

**Proof.** Since  $\mathcal{SAC}^G(X)$  has non empty interior, it is defined by the inequalities corresponding to faces of codimension one. By the theorem, all these inequalities are in the corollary. The first part of the corollary is proved.

Consider an inequality  $\langle \lambda, \hat{w}\hat{\nu} + w\nu \rangle \geq 0$  as in the statement of the corollary. By Theorem 10, the set  $(\hat{\nu}, \nu)$  such that  $\langle \lambda, \hat{w}\hat{\nu} + w\nu \rangle = 0$  is a face  $\mathcal{F}$  of codimension one of  $\mathcal{LR}(\hat{G}, G)$ . This inequality cannot be omitted unless  $\mathcal{F}$  is a face of the dominant chamber in  $X(\hat{T})_{\mathbb{Q}} \times X(T)_{\mathbb{Q}}$ . It is easy to check that this is not possible.

It remains to prove that these inequalities are pairwise distinct. Let  $(C_1, \lambda_1)$  and  $(C_2, \lambda_2)$  be two well covering pairs with indivisible dominant admissible one parameter subgroups of  $\lambda_1$  and  $\lambda_2$ . Assume that these two pairs correspond to the same face  $\mathcal{F}$  of codimension one of  $\mathcal{SAC}^G(X)$ . It remains to prove that  $(C_1, \lambda_1) = (C_2, \lambda_2)$ . Let us fix a point  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ . Let  $\mathcal{O}$  be a closed orbit of  $G$  in  $X^{\text{ss}}(\mathcal{L})$  which maps in  $(X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$ . By Lemma 7, there exists  $x \in C_1 \cap \mathcal{O}$ , and,  $g \in G$  such that  $g.x \in C_2$ . Recall that the generic closed isotropy of  $\mathcal{F}$  is a diagonalizable subgroup of  $G$  of dimension one. Then,  $\text{Im}(\lambda_1)$  (resp.  $\text{Im}(\lambda_2)$ ) is the neutral component of  $G_x$  (resp.  $G_{g.x}$ ). It follows that  $g\lambda_1g^{-1} = \pm\lambda_2$ . So,  $gC_1$  is an irreducible component of  $X^{\lambda_2}$  containing  $g.x$ :  $gC_1 = C_2$ .

By absurd, assume that  $g\lambda_1g^{-1} = -\lambda_2$ . Then,  $(C_2, -\lambda_2) = (gC_1, g\lambda_1g^{-1})$  is a well covering pair. Since, by assumption,  $(C_2, \lambda_2)$  is also well covering, Lemma 5 shows that  $\mathcal{TC}^G(X)$  is contained in the hyperplane of equation  $\mu^{\bullet}(C_2, \lambda_2)$ . This contradicts Assertion 1 of Theorem 10.

So,  $g\lambda_1g^{-1} = \lambda_2$ . Since  $\lambda_1$  and  $\lambda_2$  are indivisible and dominant this implies that  $\lambda_1 = \lambda_2$  and  $g \in G^{\lambda_1}$ . Hence  $C_2 = gC_1 = C_1$ .  $\square$

## 8.4 Application to the tensor product

**8.4.1** — In this section,  $G$  is assumed to be semisimple. We also fix an integer  $s \geq 2$  and set  $\hat{G} = G^s$ ,  $\hat{T} = T^s$  and  $\hat{B} = B^s$ . We embed  $G$  diagonally in  $\hat{G}$ . Then  $\mathcal{SAC}^G(X) \cap X(T)^{s+1}$  identifies with the  $(s+1)$ -uple  $(\nu_1, \dots, \nu_{s+1}) \in X(T)^{s+1}$  such that the for  $n$  big enough  $n\nu_i$ 's are dominant weights and  $V_{n\nu_1} \otimes \dots \otimes V_{n\nu_{s+1}}$  contains a non zero  $G$ -invariant vector.

The set of weights of  $T$  in  $\hat{\mathfrak{g}}/\mathfrak{g}$  is simply the root system  $\Phi$  of  $G$ . Let  $\Delta$  be the set of simple roots of  $G$  for  $T \subset B$ . For a part  $I$  of  $\Delta$ , the torus  $T_I$  is the neutral component of the center of the Levi subgroup associated to  $I$ ;  $T_I$  is dominant. Note that the dimension of  $T_I$  is the cardinality of  $I$ . Moreover, all dominant admissible subtorus of  $T$  is obtained in such a way. We will also denote by  $P(I)$  the standard parabolic subgroup associated to  $I$ . We denote by  $W_I$  the Weyl group  $P(I)$ .

Let  $\lambda$  be a dominant one parameter subgroup of  $T$ . For  $(\hat{w}, w_{s+1}) = (w_1, \dots, w_{s+1}) \in \hat{W} \times W = W^{s+1}$ , and  $(\hat{\nu}, \nu) = (\nu_1, \dots, \nu_{s+1}) \in \text{Pic}^G(\mathbf{X})_{\mathbb{Q}} = \mathbf{X}(T)_{\mathbb{Q}}^{s+1}$  we have:

- $r_T(\hat{w}\hat{\nu}) = \sum_{i=1}^s w_i \nu_i$ , and  $r_T(\hat{\theta}_{\hat{w}}^{\lambda}) = \sum_{i=1}^s \theta_{w_i}^{\lambda}$ ,
- $\iota^*([\Lambda_{\hat{w}^{-1}}]) = [\Lambda_{w_1^{-1}}] \cdot \dots \cdot [\Lambda_{w_s^{-1}}]$ ,

In [BK06], Belkale and Kumar defined a new product denoted  $\odot_0$  on the cohomology groups  $H^*(G/P, \mathbb{Z})$  for any parabolic subgroup  $P$  of  $G$ . By Proposition 17 of [BK06], this product  $\odot_0$  has the following very interesting property.

For  $w_i \in W$ , the following are equivalent:

- (i)  $[\Lambda_{w_1^{-1}}] \cdot \dots \cdot [\Lambda_{w_{s+1}^{-1}}] - [\Lambda_e] = 0$  and the restriction of  $\theta_{w_1}^{P(I)} + \dots + \theta_{w_{s+1}}^{P(I)} - \theta_e^{P(I)}$  to  $T_I$  is trivial (we assume that  $w$  and  $\hat{w}$  have maximal length in their class in  $_{W_{P(\lambda)}}W$  and  $_{W_{\hat{P}(\lambda)}}\hat{W}$ );
- (ii)  $[\Lambda_{w_1^{-1}}] \odot_0 \dots \odot_0 [\Lambda_{w_{s+1}^{-1}}] = [\Lambda_e]$ .

Using this result of Belkale and Kumar our Theorem 10 gives the following corollary. If  $\alpha$  is a root of  $G$ ,  $\alpha^{\vee}$  denote the corresponding coroot. If  $\alpha$  is a simple root,  $\omega_{\alpha^{\vee}}$  denote the corresponding fundamental weight.

**Corollary 3** (i) A point  $(\nu_1, \dots, \nu_{s+1}) \in X(T)_{\mathbb{Q}}^{s+1}$  belongs to the cone  $\mathcal{LR}(G^s, G)$  if and only if

- (a) each  $\nu_i$  is dominant; that is  $\langle \alpha^{\vee}, \nu_i \rangle \geq 0$  for all simple root  $\alpha$ .
- (b) for all simple root  $\alpha$ ; for all  $(w_1, \dots, w_{s+1}) \in (_{W_{P(\alpha)}}W)^{s+1}$  such that  $[\Lambda_{w_1^{-1}}] \odot_0 \dots \odot_0 [\Lambda_{w_{s+1}^{-1}}] = [\Lambda_e] \in H^*(G/P(\alpha), \mathbb{Z})$ , we have:

$$\sum_i \langle \omega_{\alpha^{\vee}}, w_i \nu_i \rangle \geq 0.$$

(ii) In the above description of  $\mathcal{LR}(G^s, G)$ , the inequalities are pairwise distinct and no one can be omitted (neither in 1a nor 1b).

(iii) Let  $\mathcal{F}$  be a face of  $\mathcal{LR}(G^s, G)$  of codimension  $d$  which intersects the interior of the dominant chamber. There exist a subset  $I$  of  $d$  simple roots and  $(w_1, \dots, w_{s+1}) \in (W_I \setminus W)^{s+1}$  such that:

- (a)  $[\Lambda_{w_1^{-1}}] \odot_0 \dots \odot_0 [\Lambda_{w_{s+1}^{-1}}] = [\Lambda_e] \in H^*(G/P(I), \mathbb{Z})$ ,

(b) the subspace spanned by  $\mathcal{F}$  is the set  $(\nu_1, \dots, \nu_{s+1}) \in X(T)_{\mathbb{Q}}^{s+1}$  such that:

$$\forall \alpha \in I \quad \sum_i \langle \omega_{\alpha^\vee}, w_i \nu_i \rangle = 0.$$

(iv) Conversely, let  $I$  be a subset of  $d$  simple roots and  $(w_1, \dots, w_{s+1}) \in (W_I \setminus W)^{s+1}$  such that  $[\Lambda_{w_1^{-1}}] \odot_0 \cdots \odot_0 [\Lambda_{w_{s+1}^{-1}}] = [\Lambda_e] \in H^*(G/P(I), \mathbb{Z})$ . Then, the set of  $(\nu_1, \dots, \nu_{s+1}) \in \mathcal{LR}(G^s, G)$  such that

$$\forall \alpha \in I \quad \sum_i \langle \omega_{\alpha^\vee}, w_i \nu_i \rangle = 0,$$

is a face of codimension  $d$  of  $\mathcal{LR}(G^s, G)$ .

**Proof.** Equations 1a are all different and are not repeated in Equations 1b. Moreover, by [MR08, Proposition 7] they define codimension one faces of  $\mathcal{LR}(G^s, G)$ .

The rest of the corollary is a simple rephrasing of Theorem 10 and Corollary 2.  $\square$

**Remark.** The description of the smaller faces of  $\mathcal{C}^G((G/B)^{s+1})$  gives an application of the Belkale-Kumar product  $\odot_0$  for all the complete homogeneous spaces.

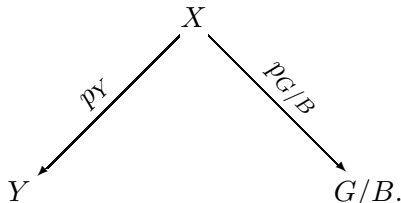
## 9 The case $X = Y \times G/B$

In this section, we are interested in the case when  $X = Y \times G/B$ , with a normal  $G$ -variety  $Y$ . Let us firstly explain the relation mentioned in the introduction between the moment polytopes of  $Y$  and some total  $G$ -cones of  $X$ .

### 9.1 Relation between semiample $G$ -cones and moment polytopes

Let us fix a maximal torus  $T$  of  $G$  and a Borel subgroup  $B$  containing  $T$ . Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $Y$ . We consider the set  $P(Y, \mathcal{L})$  of the points  $p \in X(T)_{\mathbb{Q}}$  such that for a positive integer  $n$ ,  $np$  is a dominant character of  $T$  and the dual  $V_{np}^*$  of  $V_{np}$  is a submodule of  $H^0(Y, \mathcal{L}^{\otimes n})$ . In fact,  $P(Y, \mathcal{L})$  is a polytope, called *moment polytope*. Notice that “the dual” is not usual in the definition; but it will be practical in our context.

Consider the two projections:



We denote by  $\Lambda$  the subgroup of  $\text{Pic}^G(X)$  generated by  $p_{G/B}^*(\text{Pic}^G(G/B))$  and  $p_Y^*(\mathcal{L})$ .

**Proposition 12** *With above notation, we have:*

- (i)  $\mathcal{TC}_\Lambda^G(X) = \mathcal{SAC}_\Lambda^G(X)$ ;
- (ii)  $\mathcal{SAC}_\Lambda^G(X)$  is a cone over  $P(Y, \mathcal{L})$ ; more precisely, for all positive rational number  $m$  and  $p \in X(T)_\mathbb{Q}$ , we have:

$$mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_p) \in \mathcal{SAC}_\Lambda^G(X) \iff \frac{p}{m} \in P(Y, \mathcal{L}).$$

**Proof.** Let  $n$  be a non negative integer and  $\nu$  be a character of  $T$ . As a  $G$ -module,  $H^0(X, p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu))$  is isomorphic to  $H^0(Y, \mathcal{L}^{\otimes n}) \otimes H^0(G/B, \mathcal{L}_\nu)$ . In particular, if  $p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$  has non zero global sections then  $n \geq 0$  and  $\nu$  is dominant; in this case, it is semiample. The first assertion follows.

Assume now that  $\nu$  is dominant. Then,  $H^0(G/B, \mathcal{L}_\nu) = V_\nu$ . Hence,  $p_Y^*(\mathcal{L}^{\otimes n}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$  has non zero  $G$ -invariant if and only if  $H^0(Y, \mathcal{L}^{\otimes n}) \otimes V_\nu$  contains non zero  $G$ -invariant vectors; that is, if and only if  $V_\nu^*$  is a submodule of  $H^0(Y, \mathcal{L}^{\otimes n})$ . The second assertion follows.  $\square$

## 9.2 A reduction

It is possible that  $\mathcal{AC}_\Lambda^G(X)$  is empty. In this case, our results about the faces of  $\mathcal{TC}_\Lambda^G$  cannot be applied directly.

Let  $mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu)$  in the relative interior of  $\mathcal{SAC}^G(X)$  such that  $H^0(X, mp_Y^*(\mathcal{L}) \otimes p_{G/B}^*(\mathcal{L}_\nu))^G$  is non zero, that is such that  $V_\nu^*$  can be equivariantly embedded in  $H^0(Y, \mathcal{L}^{\otimes m})$ . Let  $P$  be the standard parabolic subgroup of  $G$  associated to the face of the dominant chamber containing  $\nu$ . Let  $L$  denote the Levi subgroup of  $P$  containing  $T$  and  $D$  denote its derived subgroup.



**Proposition 13** *With above notation, there exists an irreducible component  $C_Y$  of  $Y^D$  such that a point  $\mathcal{L} \in \Lambda_{\mathbb{Q}}$  belongs to  $\mathcal{SAC}^G(X)$  if and only if  $\mathcal{L}|_C$  belongs to  $\mathcal{SAC}^{G^D}(C_Y \times G^D.B/B)$ .*

*Moreover,  $G^D.B/B$  is isomorphic to the variety of complete flags of  $G^D$  and  $\mathcal{AC}^G(C_Y \times G^D.B/B)$  is non empty.*

**Proof.** The inclusion  $V_{\nu}^* \subset \mathbb{H}^0(Y, \mathcal{L}^{\otimes m})$  gives a  $G$ -equivariant rational map  $\phi : Y \rightarrow \mathbb{P}(V_{\nu})$ . Let  $v_{\nu}$  be a vector of highest weight in  $V_{\nu}$ ;  $P$  is the stabilizer in  $G$  of  $[v_{\nu}] \in \mathbb{P}(V_{\nu})$ . Let  $\sigma \in V_{\nu}^*$  be an eigenvector of the Borel subgroup  $B^-$  opposite to  $B$  and containing  $T$ . Let  $Q$  be the the stabilizer in  $G$  of  $[\sigma] \in \mathbb{P}(V_{\nu}^*)$  and  $Q^u$  be its unipotent radical.

Let  $Y_{\sigma}$  denote the set of  $y \in Y$  such that  $\sigma(y) \neq 0$ . Let  $W$  be a  $L$ -stable supplementary subspace of  $\mathbb{K}.v_{\nu}$  in  $V_{\nu}$ . By  $w \rightarrow [v_{\nu} + w]$  we identify  $W$  with an open subspace of  $\mathbb{P}(V_{\nu})$ . Then  $\phi$  induces by restriction  $\tilde{\phi} : Y_{\sigma} \rightarrow W$ .

Let  $S$  be a  $L$ -stable supplementary to  $T_{[v_{\nu}]}G.[v_{\nu}]$  in  $W$  (actually,  $W$  canonically identify with  $T_{[v_{\nu}]} \mathbb{P}(V_{\nu})$ ). Set  $Z = \tilde{\phi}^{-1}(S)$ . By [Bri99, Remark in Section 5],  $Z$  is point wise fixed by  $D$  and the action of  $Q^u$  induces an isomorphism  $Q^u \times Z \simeq Y_{\sigma}$ .

Consider  $X' = Y \times G/P$ . Let  $\Lambda'$  be the subgroup of  $\text{Pic}^G(X')$  generated by  $p_Y^*(\mathcal{L})$  and  $p_{G/P}^*(\text{Pic}^G(G/P))$  (with obvious notation). It is clear that  $\mathcal{TC}_{\Lambda'}^G(X)$  identifies with  $\mathcal{TC}_{\Lambda'}^G(X')$ . Moreover,  $\mathcal{AC}_{\Lambda'}^G(X')$  is not empty. Consider a generic closed isotropy  $H$  of  $\mathcal{TC}_{\Lambda'}^G(X')$  viewed as a face  $\mathcal{F}$  of itself. Since  $Q^u \times Z \simeq Y_{\sigma} \subset X^{\text{ss}}(mp_Y^*(\mathcal{L}) \otimes p_{G/P}^*(\mathcal{L}_{\nu}))$ , up to conjugacy, one may assume that  $D \subset H \subset L$ . Since  $Y_{\sigma} \times \{P/P\} \subset X^{\text{ss}}(mp_Y^*(\mathcal{L}) \otimes p_{G/P}^*(\mathcal{L}_{\nu}))$ ,  $X'_{\mathcal{F}}$  intersects  $Z \times \{P/P\}$ .

Consider the irreducible component  $C_Y$  of  $Y^D$  which contains  $Z$ . By Theorem 6, for any ample  $\mathcal{L} \in \Lambda'$ ,  $X'^{\text{ss}}(\mathcal{L})$  intersects  $C_Y \times \{P/P\}$  if it is non empty. By continuity, this is also true if  $\mathcal{L}$  is only semiample. The proposition follows easily.  $\square$

### 9.3 Faces of $\mathcal{SAC}^G(X)$ if $\mathcal{AC}^G(Y)$ is non empty

From now on, we assume that  $Y$  is smooth. We will first adapt the notion of covering and well covering pairs for the situation.

Recall that  $T \subset B$  are fixed. Let  $\lambda$  be a one parameter subgroup of  $T$ . Set  $B(\lambda) = B \cap P(\lambda)$ . Let  $C$  be an irreducible component of  $Y^{\lambda}$  and  $C^+$  the associated Bialinicki-Birula cell.

**Definition.** The pair  $(C, \lambda)$  is said to be *B-covering* if the natural map  $\eta : B \times_{B(\lambda)} C^+ \rightarrow Y$  is birational. It is said to be *well B-covering* if  $\eta$

induces an isomorphism over an open subset of  $Y$  intersecting  $C$ .

The proof of the following lemma is obvious.

**Lemma 12** *With above notation, the pair  $(C, \lambda)$  is  $B$ -covering (resp. well  $B$ -covering) if and only if  $(C \times G^\lambda B/B)$  is covering (resp. well covering).*

Let us recall that the subtori of  $T$  correspond bijectively to the linear subspaces of  $X(T)_\mathbb{Q}$ . If  $V$  is a linear subspace of  $X(T)_\mathbb{Q}$ , the associated torus is the neutral component of the intersection of kernels of elements in  $X(T) \cap V$ . If  $F$  is a convex part of  $X(T)_\mathbb{Q}$ , the direction  $\text{dir}(F)$  of  $F$  is the linear subspace spanned by the differences of two elements of  $F$ .

**Proposition 14** *We keep the above notation and assume that  $Y$  is smooth and  $\mathcal{P}^G(Y, \mathcal{L})$  intersects the interior of the dominant chamber. Let  $\mathcal{F}$  be a face of codimension  $d$  of  $\mathcal{P}^G(Y, \mathcal{L})$  which intersect the interior of the dominant chamber. Let  $\tilde{\mathcal{F}}$  be the corresponding face of  $\mathcal{TC}^G(X)$ . Let  $S$  the subtorus of  $T$  associated to  $\text{dir}(\mathcal{F})$ .*

*There exists a unique irreducible component  $C$  of  $Y^S$  and a one parameter subgroup  $\lambda$  of  $S$  such that  $G^\lambda = G^S$  and  $(C, \lambda)$  is a well  $B$ -covering pair such that  $\mathcal{F}$  is the set of  $\nu \in \mathcal{P}(Y, \mathcal{L})$  such that  $\langle \lambda, \nu \rangle \geq -\mu^\mathcal{L}(C, \lambda)$ .*

**Proof.** Let  $H$  be the neutral component of a generic closed isotropy of  $\tilde{\mathcal{F}}$ . The argue used in Claim 1 of the proof of Theorem 10 shows  $H$  is a torus of dimension  $d$ . By Theorem 8 there exists a well covering pair  $(C_X, \lambda)$  with  $\lambda$  a one parameter subgroup of  $H$ , and  $C$  an irreducible component of  $X^H$  such that  $\tilde{\mathcal{F}}$  is the set of  $\mathcal{L} \in \mathcal{TC}^G(X)$  such that  $\mu^\mathcal{L}(C_X, \lambda) = 0$ .

Replacing  $C$ ,  $H$  and  $\lambda$  by  $gC$ ,  $gHg^{-1}$  and  $g\lambda g^{-1}$  for some  $g \in G$  if necessary, one may assume that  $C_X$  intersects  $Y \times \{B/B\}$  and  $H$  is a subtorus of  $T$ . Then,  $C_X = C \times G^\lambda B/B$  for some irreducible component  $C$  of  $Y^H$ . By Lemma 12,  $(C, \lambda)$  is well  $B$ -covering. Note that  $\mu^{p_Y^*(\mathcal{L}) \otimes p_{G/B}(\mathcal{L}_\nu)}(C_X, \lambda) = \mu^\mathcal{L}(C, \lambda) + \langle \lambda, \nu \rangle$ . The fact that  $H$  is the subtorus associated to  $\text{dir}(\mathcal{F})$  follows from Proposition 9. The existence of the proposition is proved.

Let  $(C', \lambda')$  be another pair satisfying the proposition. It remains to prove that  $C = C'$ . Let  $\mathcal{M}$  be an ample  $G$ -linearized line bundle on  $X$  belonging to the relative interior of  $\tilde{\mathcal{F}}$ . Let  $\mathcal{O}$  be a closed  $G$ -orbit in  $X^{\text{ss}}(\mathcal{M})$  over  $(X^{\text{ss}}(\mathcal{M})//G)_{\text{pr}}$ . By Lemma 7, there exists  $y \in C$  and  $y' \in C'$  such that  $x = (y, B/B)$  and  $x' = (y', B/B)$  belong to  $\mathcal{O}$ . Since the isotropies of  $x$  and  $x'$  contain  $S$  and are generic closed isotropy of  $\mathcal{M}$ ,  $S$  is the neutral component of

$G_x$  and  $G_{x'}$ . Moreover by [Ric82], we have:  $(G.x)^S = N_G(S).x$ ; in particular, there exists  $g \in N_G(S)$  such that  $x' = g.x$ . But, the action on the right hand of  $X$  proves that  $g \in B$ . Hence,  $g \in N_B(S) = S$ ; and,  $y = y'$ . Since  $C$  and  $C'$  are connected components of  $Y^S$ , it follows that  $C = C'$ .  $\square$

## References

- [BB73] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. **98** (1973), 480–497.
- [Bel05] Prakash Belkale, *Geometric Proof of a Conjecture of Fulton*, 2005, <http://www.citebase.org/abstract?id=oai:arXiv.org:math/0511664>.
- [BK06] Prakash Belkale and Shrawan Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. **166** (2006), no. 1, 185–228.
- [Bri99] Michel Brion, *On the general faces of the moment polytope*, Internat. Math. Res. Notices (1999), no. 4, 185–201.
- [BS00] Arkady Berenstein and Reyer Sjamaar, *Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion*, J. Amer. Math. Soc. **13** (2000), no. 2, 433–466 (electronic).
- [DH98] Igor V. Dolgachev and Yi Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. **87** (1998), 5–56, With an appendix by Nicolas Ressayre.
- [É92] A. G. Élashvili, *Invariant algebras*, Lie groups, their discrete subgroups, and invariant theory, Adv. Soviet Math., vol. 8, Amer. Math. Soc., Providence, RI, 1992, pp. 57–64.
- [FHT84] Knop F., Kraft H., and Vust T., *The Picard group of a G-variety*, Algebraic transformation group and invariant theory, Birkhäuser, 1984, pp. 77–87.
- [Ful00] William Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 3, 209–249 (electronic).
- [Hes79] W. Hesselink, *Desingularization of varieties of null forms*, Inven. Math. **55** (1979), 141–163.

- [Kem78] G. Kempf, *Instability in invariant theory*, Ann. of Math. **108** (1978), 2607–2617.
- [Kir84] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Princeton University Press, Princeton, N.J., 1984.
- [KKM06] Misha Kapovich, Shrawan Kumar, and John Millson, *Saturation and irredundancy for  $spin(8)$* , Preprint <http://www.math.ucdavis.edu/~kapovich/eprints.html> (2006), 1–28.
- [Kle74] Steven L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297.
- [Kly98] Alexander A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Math. (N.S.) **4** (1998), no. 3, 419–445.
- [KTW04] Allen Knutson, Terence Tao, and Christopher Woodward, *The honeycomb model of  $GL_n(\mathbb{C})$  tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), no. 1, 19–48 (electronic).
- [LR79] D. Luna and R. W. Richardson, *A generalization of the Chevalley restriction theorem*, Duke Math. J. **46** (1979), no. 3, 487–496.
- [Lun73] Domingo Luna, *Slices étales*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33.
- [Lun75] D. Luna, *Adhérences d'orbite et invariants*, Invent. Math. **29** (1975), no. 3, 231–238.
- [Mat60] Yozô Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. J **16** (1960), 205–218.
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3d ed., Springer Verlag, New York, 1994.
- [MR08] Pierre-Louis Montagard and Nicolas Ressayre, *Sur des faces du  $lr$ -cône généralisé*, Bull. SMF (paratre) (2008), no. arXiv:math/0406272, 1–19.

- [Nes78] L. Ness, *Mumford's numerical function and stable projective hypersurfaces*, Algebraic geometry (Copenhagen), Lecture Notes in Math., 1978.
- [Nes84] ———, *A stratification of the null cone via the moment map*, Amer. Jour. of Math. **106** (1984), 1281–1325.
- [Oda88] Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988, An introduction to the theory of toric varieties, Translated from the Japanese.
- [PV91] V. L. Popov and È. B. Vinberg, *Algebraic Geometry IV*, Encyclopedia of Mathematical Sciences, vol. 55, ch. Invariant Theory, pp. 123–284, Springer-Verlag, 1991.
- [Res00] N. Ressayre, *The GIT-equivalence for G-line bundles*, Geom. Dedicata **81** (2000), no. 1-3, 295–324.
- [Ric82] R. W. Richardson, *On orbits of algebraic groups and Lie groups*, Bull. Austral. Math. Soc. **25** (1982), no. 1, 1–28.
- [Sch03] Alexander Schmitt, *A simple proof for the finiteness of GIT-quotients*, Proc. Amer. Math. Soc. **131** (2003), no. 2, 359–362 (electronic).

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