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Nicolas Ressayre

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# Geometric Invariant Theory and Generalized Eigenvalue Problem

N. Ressayre

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## Abstract

Let  $G$  be a connected reductive subgroup of a complex semi-simple group  $\hat{G}$ . We are interested in the set of pairs  $(\hat{\nu}, \nu)$  of dominant characters for  $\hat{G}$  and  $G$  such that  $V_{\hat{\nu}} \otimes V_{\nu}$  contains nonzero  $G$ -invariant vectors. This set of pairs  $(\hat{\nu}, \nu)$  generates a convex cone  $\mathcal{C}$  in a finite dimensional vector space. Using methods of variation of quotient in Geometric Invariant Theory, we obtain a list of linear inequalities which characterize  $\mathcal{C}$ . This list is a generalization of the list that Belkale and Kumar obtained in the case when  $\hat{G} = G^s$ . Moreover, we prove that this list is no far to be minimal (and really minimal in the case when  $\hat{G} = G^s$ ). We also give a description of some lower faces of  $\mathcal{C}$ ; if  $\hat{G} = G^s$  these description gives an application of the Belkale-Kumar product  $\odot_0$  on the cohomology group of all the projective  $G$ -homogeneous spaces. Some of the results are more general than in the abstract and are obtained in the general context of Geometric Invariant Theory.

## 1 Introduction

Let  $G$  be a connected reductive group acting algebraically on a projective variety  $X$ , both defined over an algebraically closed field  $\mathbb{K}$  of characteristic zero. Geometric Invariant Theory (GIT) associates to any ample  $G$ -linearized line bundle  $\mathcal{L}$  on  $X$  the following open subset  $X^{\text{ss}}(\mathcal{L})$  of  $X$ :

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \sigma \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\}.$$

The points of  $X^{\text{ss}}(\mathcal{L})$  are said to be *semistable* for  $\mathcal{L}$ . The problem studied in this paper is to characterize the set of the  $\mathcal{L}$ 's such that  $X^{\text{ss}}(\mathcal{L})$  is not empty.

Let us make the question more precise. The set of  $G$ -linearized line bundles on  $X$  forms an abelian group denoted by  $\text{Pic}^G(X)$ . Set  $\text{Pic}^G(X)_{\mathbb{Q}} =$

$\text{Pic}^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Let  $\Lambda$  be a freely finitely generated subgroup of  $\text{Pic}^G(X)$ ;  $\Lambda_{\mathbb{Q}}$  be the vector subspace of  $\text{Pic}^G(X)_{\mathbb{Q}}$  spanned by  $\Lambda$ . Note that  $\Lambda$  is canonically embedded in  $\Lambda_{\mathbb{Q}}$ . The ample  $G$ -linearized elements of  $\Lambda$  generate a convex cone denoted by  $\Lambda_{\mathbb{Q}}^+$  which is open in  $\Lambda_{\mathbb{Q}}$ . Note that for all  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^+$ , there exists a positive integer  $n$  such that  $\mathcal{L}^{\otimes n}$  is an ample  $G$ -linearized line bundle on  $X$  in  $\Lambda$ . Since,  $X^{\text{ss}}(\mathcal{L}^{\otimes n}) = X^{\text{ss}}(\mathcal{L})$  for all positive integer  $n$ , one can define  $X^{\text{ss}}(\mathcal{L})$  for any  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^+$ . We set:

$$\mathcal{C}_{\Lambda}^G(X) := \{\mathcal{L} \in \Lambda_{\mathbb{Q}}^+ : X^{\text{ss}}(\mathcal{L}) \text{ is not empty}\}.$$

By [DH98] (see also [Res00]),  $\mathcal{C}_{\Lambda}^G(X)$  is a closed convex polyhedral cone in  $\Lambda_{\mathbb{Q}}^+$ ; that is, is defined as a part of  $\Lambda_{\mathbb{Q}}^+$  by finitely many large linear inequalities. Our aim is to study the geometry of this cone (called *the  $G$ -ample cone*) and these inequalities.

Let us explain our results in this general setting. Let  $\lambda$  be a one parameter subgroup of  $G$ ,  $P(\lambda)$  be the associated parabolic subgroup and  $C$  be an irreducible component of its fixed points. Consider the set  $C^+$  of the  $x \in X$  such that  $\lim_{t \rightarrow 0} \lambda(t)x$  belongs to  $C$ . The pair  $(C, \lambda)$  is said to be *well covering* if the natural morphism  $G \times_{P(\lambda)} C^+ \rightarrow X$  induces an isomorphism onto an open subset of  $X$  intersecting  $C$ . In the classical numerical criterion of Hilbert-Mumford, one associates a morphism  $\text{Pic}^G(X) \rightarrow \mathbb{Z}$ ,  $\mathcal{L} \mapsto \mu^{\mathcal{L}}(C, \lambda)$  to the pair  $(C, \lambda)$  (see Sections 4.2 and 7.2).

A direct application of a result of Kirwan (see Proposition 4) about the Hilbert-Mumford numerical criterion gives us the first description of  $\mathcal{C}_{\Lambda}^G(X)$ :

**Proposition 1.1** *We assume that  $X$  is normal.*

*Then, the cone  $\mathcal{C}_{\Lambda}^G(X)$  is the set of the  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^+$  such that for all well covering pair  $(C, \lambda)$ , we have  $\mu^{\mathcal{L}}(C, \lambda) \leq 0$ .*

Note that, since conjugated one parameter subgroups are "equivalent" one can easily improve this result (see also Proposition 8). Proposition 1.1 implies that the well covering pairs parametrize the faces of codimension one of  $\mathcal{C}_{\Lambda}^G(X)$ . A natural question occur: What pairs  $(C, \lambda)$  give really faces of codimension one of  $\mathcal{C}_{\Lambda}^G(X)$ ? This is one of the two main questions studied here.

If  $f$  is a linear form on  $E$  such that  $f|_{\mathcal{C}_{\Lambda}^G(X)} \geq 0$  then the set of  $x$ 's in  $\mathcal{C}_{\Lambda}^G(X)$  such that  $f(x) = 0$  is called a *face* of  $\mathcal{C}_{\Lambda}^G(X)$ . Using the notion of generic closed isotropy or generic closed orbit due to D. Luna, in Theorem 6,

we associate two invariants to any face of  $\mathcal{C}_\Lambda^G(X)$ .

Some particular cases are specially interesting. Indeed, assume that the variety  $X$  equals  $Y \times G/B$ , for a  $G$ -variety  $Y$ . Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $Y$ . Let  $\Lambda$  be the subgroup of  $\text{Pic}^G(X)$  generated by the pullback of  $\mathcal{L}$  and the pullbacks of the  $G$ -linearized line bundles on  $G/B$ . Then,  $\mathcal{C}_\Lambda^G(X)$  is a cone over the intersection of the open dominant chamber and the moment polytope  $P(Y, \mathcal{L})$  defined in [Bri99]; so, the faces of  $\mathcal{C}_\Lambda^G(X)$  correspond bijectively to the faces of  $P(Y, \mathcal{L})$  which intersect the open chamber.

We obtain more precise results in the case  $X = Y \times G/B$ . For example, Proposition 11 improves Proposition 1.1 by limiting the list of useful covering pairs; that is, it adds a necessary condition for the pair  $(C, \lambda)$  giving a face of codimension one.

Theorem 7 describes  $\mathcal{C}_\Lambda^G(X)$  if  $X = Y \times G/B$  with  $Y$  smooth and if  $\Lambda$  is abundant (see Section 8.1). The first assertion of this theorem shows that well covering pairs parametrizes all the faces of  $\mathcal{C}_\Lambda^G(X)$ ; and, not only those of codimension one. The second one gives an inductive way to decide if a given pair  $(C, \lambda)$  gives a face of  $\mathcal{C}_\Lambda^G(X)$  and what is its dimension.

A specially interesting case is when  $X = \hat{G}/\hat{B} \times G/B$ , where  $\hat{G}$  is a semi-simple group containing  $G$ . Now, we assume that  $\Lambda = \text{Pic}^G(X)$ . Then, the closure  $\overline{\mathcal{C}}_\Lambda^G(X)$  of the cone  $\mathcal{C}_\Lambda^G(X)$  is the cone studied by Berenstein and Sjamar in [BS00]. This cone has a very simple interpretation in terms of the problem of restricting the representations of  $\hat{G}$  to  $G$  (see Lemma 6).

Two improvements of Theorem 7 are obtained in this case. Firstly, we obtain a simple sufficient condition in Theorem 8 for a covering pair  $(C, \lambda)$  giving a face of given codimension. Moreover, the set of covering pairs is easy to describe in terms of cohomology (see Theorem 9). These results allow us to obtain Theorem 8: in the first assertion, we give a list of inequalities which determine  $\mathcal{C}_\Lambda^G(X)$ . In the second assertion, we give a list of faces of  $\overline{\mathcal{C}}_\Lambda^G(X)$  whose dimension is determined. Conversely, in the last assertion, we prove that any face of  $\mathcal{C}_\Lambda^G(X)$  (that is, face of  $\overline{\mathcal{C}}_\Lambda^G(X)$  which intersects  $\Lambda_{\mathbb{Q}}^+$ ) is in the above list.

The case when  $\hat{G} = G^s$  and  $G$  is diagonally embedded in  $\hat{G}$  is particularly interesting. Indeed, the cone  $\overline{\mathcal{C}}_\Lambda^G(X)$  has numerous interpretations in this case (see [Ful00]), and its study began with Weyl (see [Wey49]). Recently Belkale and Kumar defined in [BK06] a new product  $\odot_0$  on the cohomology

groups of the flag varieties. This product allows to characterize the well covering pairs (which is a generalization of  $L$ -movability of Belkale and Kumar) in a very beautiful manner. Moreover, in [MR04], P.L. Montagard and N.R. obtain results on the faces of  $\overline{\mathcal{C}}_\Lambda^G(X)$  which do not intersect  $\Lambda_\mathbb{Q}^+$ . These results allow us to obtain Theorem 11 in which we prove that all the equations obtained by Belkale and Kumar are necessary. These equations are parametrized by a condition expressed with the product  $\odot_0$  in  $H^*(G/P, \mathbb{Z})$  for the maximal parabolic subgroups  $P$  of  $G$ . The product  $\odot_0$  in  $H^*(G/P, \mathbb{Z})$  for lower parabolic subgroups  $P$  of  $G$  is used to describe the lower faces of  $\overline{\mathcal{C}}_\Lambda^G(X)$  and  $\mathcal{C}_\Lambda^G(X)$ .

## 2 General notation and definitions

The base field  $\mathbb{K}$  is algebraically closed and of characteristic zero. The multiplicative group of  $\mathbb{K}$  will be denoted by  $\mathbb{K}^*$ . All the varieties will be quasiprojective.

**About Groups.** All the algebraic groups will be affine. Let  $G$  be an algebraic group and  $H$  be a closed subgroup of  $G$ . We set:

- $G^\circ$  the neutral component of  $G$ ,
- $X(G)$  the group of characters that is the homomorphisms from  $G$  on  $\mathbb{K}^*$ ,
- $X_*(G)$  the set of one parameter subgroups of  $G$ ,
- $N_G(H)$  the normalizer of  $H$  in  $G$ ,
- $G^H$  the centralizer of  $H$  in  $G$ ,
- $\mathfrak{g}, \mathfrak{t} \dots$  the Lie algebra of  $G, T \dots$

**About Varieties.** Let  $X$  be a variety and  $x \in X$ . Let  $\mathcal{L}$  be a line bundle on  $X$  and  $Y$  be a locally closed subvariety of  $X$ . Let  $f : X \rightarrow Z$  be a morphism between varieties. We denote:

- $T_x X$  the Zariski tangent of  $X$  at  $x$ ,
- $T_x f : T_x X \rightarrow T_x Z$  the tangent map of  $f$  at  $x$ ,
- $\overline{Y}$  the closure of  $Y$  in  $X$ ,
- $\mathcal{L}_x$  the fiber in  $\mathcal{L}$  over  $x$ ,
- $\mathcal{L}|_Y$  the restriction of  $\mathcal{L}$  to  $Y$ ,
- $H^0(X, \mathcal{L})$  the set of regular sections of the line bundle  $\mathcal{L}$ ,

If  $X$  is smooth of dimension  $n$  and  $Z$  smooth, we set:

$\mathcal{T}(X)$  the tangent bundle of  $X$ ,  
 $\mathcal{D}et(X) = \bigwedge^n \mathcal{T}(X)$  the determinant bundle of  $X$ ,  
 $\mathcal{T}(f) : \mathcal{T}(X) \longrightarrow \mathcal{T}(Y)$  the tangent map to  $f$ ,  
it is a morphism of vector bundles,  
 $\mathcal{D}et f : \mathcal{D}et(Y) \longrightarrow \mathcal{D}et(Z)$  the determinant of  $\mathcal{T}f$ ,  
it is a morphism of line bundles.

**About  $G$ -Varieties.** Let  $G$  be an algebraic group. A variety  $X$  endowed with an algebraic action of  $G$  will be called a  $G$ -variety. Given a  $G$ -variety  $X$  and a point  $x$  in  $X$ , we set:

$G.x$  the orbit of  $x$  by  $G$ ,  
 $G_x$  the stabilizer of  $x$  in  $G$ ,  
 $X^G$  the set of fixed point of  $X$ ,  
 $N_G(Y)$  the set of  $g$ 's in  $G$  such that  $g.Y \subseteq Y$ , where  $Y \subset X$ ,  
 $G_Y$  the subgroups of the  $g \in G$  such that  $g.y = y$  for all  $y \in Y$ ,  
 $X^{\text{ss}}(\mathcal{L})$  the set semistable points for  $\mathcal{L}$ ,  
 $\pi_{\mathcal{L}}$  the quotient morphism from  $X^{\text{ss}}(\mathcal{L})$  onto  $X^{\text{ss}}(\mathcal{L})//G$ .

**About Tori.** Let  $T$  be a torus, that is an algebraic group isomorphic to  $(\mathbb{K}^*)^r$ , for some positive integer  $r$ . For  $\lambda \in X_*(\Gamma)$  and  $\chi \in X(\Gamma)$  there exists a unique integer  $\langle \lambda, \chi \rangle$  such that  $\chi \circ \lambda(t) = t^{\langle \lambda, \chi \rangle}$  for all  $t \in \mathbb{K}^*$ . Moreover,  $X_*(\Gamma)$  and  $X(\Gamma)$  are free abelian groups of rank  $r$ ; and,  $\langle \cdot, \cdot \rangle : X_*(\Gamma) \times X(\Gamma)$  is a perfect paring.

If  $V$  is a  $T$ -module,  $\text{St}_T(V)$  denote the set of  $\chi \in X(T)$  such that there exists a non zero vector  $v \in V$  such that for all  $t \in T$ ,  $t.v = \chi(t)v$ . We also set:  $V_{\chi} = \{v \in V : t.v = \chi(t)v \ \forall \chi \in T\}$ . We have:

$$V = \bigoplus_{\chi \in \text{St}_T(V)} V_{\chi}.$$

**About reductive groups and root systems** Let  $G$  be a reductive group,  $B$  be a Borel subgroup of  $G$  and  $T$  be a maximal torus of  $B$ . Let  $I$  be a subset of simple roots. Let  $\lambda$  be a one parameter subgroup of  $T$ . We set:

$W$	the Weyl group of $(G, T)$ ,
$R$	the set of roots of $G$ for $T$ ,
$R^+$	the set of positive roots for $B$ ,
$\rho$	the half sum of the positive roots,
$\alpha^\vee$	the coroot associated to the root $\alpha$ ,
$\omega_\alpha$	the fundamental weight associated to the simple root $\alpha$ ,
$\omega_{\alpha^\vee}$	the fundamental coweight associated to the simple coroot $\alpha^\vee$ ,
$P(I)$	the standard parabolic subgroup associated to $I$ ,
$W(I)$	the Weyl group of the standard Levi of $P(I)$ ,
$R_\lambda$	the set of roots of $G^\lambda$ for $T$ ,
$R_\lambda^+$	$= R^+ \cap R_\lambda$ ,
$\rho_\lambda$	the half sum the elements of $R_\lambda^+$ ,
$W_\lambda$	the Weyl group of $G^\lambda$ ,
$W^\lambda$	the elements of $W$ which are of minimal length in their class in $W/W_\lambda$ ,

The one parameter subgroup  $\lambda$  is said to be *dominant* if  $\langle \lambda, \alpha \rangle \geq 0$ , for all simple root  $\alpha$ . Let  $\mathfrak{t}^+$  be the convex cone generated by the tangent vectors to the dominant one parameter subgroups of  $T$ . This cone is generated by the  $\omega_{\alpha^\vee}$  for the simple roots  $\alpha$ . An important property for us, is that every conjugacy class of one parameter subgroup of  $G$  contains a unique dominant one parameter subgroup of  $T$ .

**About Convex cones.** Let  $E$  be a rational finite dimensional vector space  $E$ . Let  $\mathcal{C}$  be a *convex cone* of  $E$  that is that for all  $v, w \in \mathcal{C}$  and for all non negative rational numbers  $\alpha, \beta$ ,  $\alpha v + \beta w \in \mathcal{C}$ . We set:

$\langle \mathcal{C} \rangle$	the subspace spanned by $\mathcal{C}$ ,
$\dim(\mathcal{C})$	the <i>dimension</i> of $\mathcal{C}$ that is the dimension of $\langle \mathcal{C} \rangle$ ,
$\mathcal{C}^\circ$	the <i>relative interior</i> of $\mathcal{C}$ that its interior in $\langle \mathcal{C} \rangle$ .

Let  $\mathcal{C}_1$  be a convex cone in  $E$ . A *closed convex rational and polyhedral cone*  $\mathcal{C}$  in  $\mathcal{C}_1$  is a part of  $\mathcal{C}_1$  defined as a part of  $\mathcal{C}_1$  by finitely many large linear and rational inequalities. If  $f$  is a linear form on  $E$  such that  $f|_{\mathcal{C}} \geq 0$  then the set of  $x$ 's in  $\mathcal{C}$  such that  $f(x) = 0$  is called a *face* of  $\mathcal{C}$ .

### 3 Preliminaries on parabolic fiber products

In this section we collect some useful properties of the fiber product. Let  $G$  be a reductive group and  $P$  be a parabolic subgroup of  $G$ .

### 3.1 Construction

Let  $Y$  be a  $P$ -variety. Consider over  $G \times Y$  the action of  $G \times P$  given by the formula (with obvious notation):

$$(g, p) \cdot (g', y) = (gg'p^{-1}, py).$$

Since the quotient map  $G \rightarrow G/P$  is a Zariski-locally trivial principal  $P$ -bundle; one can easily construct a quotient  $G \times_P Y$  of  $G \times Y$  by the action of  $\{e\} \times P$ . The action of  $G \times \{e\}$  induces an action of  $G$  on  $G \times_P Y$ . Moreover, the first projection  $G \times Y \rightarrow G$  induces a  $G$ -equivariant map  $G \times_P Y \rightarrow G/P$  which is a locally trivial fibration with fiber  $Y$ .

The class of a pair  $(g, y) \in G \times Y$  in  $G \times_P Y$  is denoted by  $[g : y]$ . If  $Y$  is a  $P$ -stable locally closed subvariety of a  $G$ -variety  $X$ , it is well known that the map

$$\begin{aligned} G \times_P Y &\longrightarrow G/P \times X \\ [g : y] &\longmapsto (gP, gy) \end{aligned}$$

is an isomorphism onto the set of the  $(gP, x) \in G/P \times X$  such that  $g^{-1}x \in Y$ .

Let  $\nu$  be a character of  $P$ . If  $Y$  is the field  $\mathbb{K}$  endowed with the action of  $P$  defined by  $p \cdot \tau = \nu(p^{-1})\tau$  for all  $\tau \in \mathbb{K}$  and  $p \in P$ ,  $G \times_P Y$  is a  $G$ -linearized line bundle on  $G/P$ . We denote by  $\mathcal{L}_\nu$  this element of  $\text{Pic}^G(G/P)$ . Actually, the map  $X(P) \rightarrow \text{Pic}^G(G/P)$ ,  $\nu \mapsto \mathcal{L}_\nu$  is an isomorphism.

Let  $B$  be a Borel subgroup of  $G$  contained in  $P$ , and  $T$  be a maximal torus contained in  $B$ . Then,  $X(P)$  identifies with a subgroup of  $X(T)$  which contains dominant weights. For  $\nu \in X(P)$ ,  $\mathcal{L}_\nu$  is generated by its sections if and only if it has non zero sections if and only if  $\nu$  is dominant. Moreover,  $H^0(G/P, \mathcal{L}_\nu)$  is the simple  $G$ -module of heights weight  $\nu$ . For  $\nu$  dominant,  $\mathcal{L}_\nu$  is ample if and only if  $\nu$  cannot be extended to a subgroup of  $G$  bigger than  $P$ .

### 3.2 Line bundles

We are now interested in the  $G$ -linearized line bundles on  $G \times_P Y$ .

**Proposition 1** *With above notation, we have:*

1. The map  $\mathcal{L} \mapsto G \times_P \mathcal{L}$  defines a morphism

$$e : \text{Pic}^P(Y) \longrightarrow \text{Pic}^G(G \times_P Y).$$



2. The map  $\iota : Y \longrightarrow G \times_P Y$ ,  $y \longmapsto [e : y]$  is a  $P$ -equivariant immersion. We denote by  $\iota^* : \text{Pic}^G(G \times_P Y) \longrightarrow \text{Pic}^P(Y)$  the associated restriction homomorphism.
3. The morphisms  $e$  and  $\iota^*$  are the inverse one of each other; in particular, they are isomorphisms.
4. For any  $\mathcal{L} \in \text{Pic}^G(G \times_P Y)$ , the restriction map from  $\text{H}^0(G \times_P Y, \mathcal{L})$  to  $\text{H}^0(Y, \iota^*(\mathcal{L}))$  induces a linear isomorphism

$$\text{H}^0(G \times_P Y, \mathcal{L})^G \simeq \text{H}^0(Y, \iota^*(\mathcal{L}))^P.$$

**Proof.** Let  $\mathcal{M}$  be a  $P$ -linearized line bundle on  $Y$ . Since  $G \times \mathcal{M} \longrightarrow G \times_P \mathcal{M}$  is a categorical quotient, we have the following commutative diagram:

$$\begin{array}{ccc} G \times \mathcal{M} & \longrightarrow & G \times_P \mathcal{M} \\ \downarrow & & \downarrow p \\ G \times Y & \longrightarrow & G \times_P Y. \end{array}$$

Since  $G \longrightarrow G/P$  is locally trivial, the map  $p$  endows  $G \times_P \mathcal{M}$  with a structure of line bundle on  $G \times_P Y$ . Moreover, the action of  $G$  on  $G \times_P \mathcal{M}$  endows this line bundle with a  $G$ -linearization. This proves Assertion 1. The second one is obvious.

By construction, the restriction of  $G \times_P \mathcal{M}$  to  $Y$  is  $\mathcal{M}$ . So,  $\iota^* \circ e$  is the identity map. Conversely, let  $\mathcal{L} \in \text{Pic}^G(G \times_P Y)$ . Then, we have:

$$e \circ \iota^*(\mathcal{L}) \simeq \{(gP, l) \in G/P \times \mathcal{L} : g^{-1}l \in \mathcal{L}|_Y\}.$$

The second projection induces an isomorphism from  $e \circ \iota^*(\mathcal{L})$  onto  $\mathcal{L}$ . This ends the proof of Assertion 3.

The map  $\text{H}^0(G \times_P Y, \mathcal{L})^G \longrightarrow \text{H}^0(Y, \iota^*(\mathcal{L}))^P$  is clearly well defined and injective. Let us prove the surjectivity. Let  $\sigma \in \text{H}^0(Y, \iota^*(\mathcal{L}))^P$ . Consider the morphism

$$\begin{aligned} \hat{\sigma} : G \times Y &\longrightarrow G \times_P \mathcal{L} \\ (g, y) &\longmapsto [g : \sigma(y)]. \end{aligned}$$

Since  $\sigma$  is  $P$ -invariant, so is  $\hat{\sigma}$ ; and  $\hat{\sigma}$  induces a section of  $G \times_P \mathcal{L}$  over  $G \times_P Y$  which is  $G$ -invariant and extends  $\sigma$ .  $\square$

## 4 Numerical criterion of Hilbert-Mumford

In the paper, we use classical results in Geometric Invariant Theory (GIT) about the numerical criterion of Hilbert-Mumford. In this section, we present these results and give some useful complements. Let  $G$  be a connected reductive group acting on a irreducible projective algebraic variety  $X$ .

### 4.1 The quotient map

As in the introduction, for any ample  $G$ -linearized line bundle  $\mathcal{L}$  on  $X$ , we set

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \sigma \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\}.$$

Then,  $X^{\text{ss}}(\mathcal{L})$  is open in  $X$ , and there exists a categorical quotient:

$$\pi : X^{\text{ss}}(\mathcal{L}) \longrightarrow X^{\text{ss}}(\mathcal{L})//G,$$

such that  $X^{\text{ss}}(\mathcal{L})//G$  is a projective variety and  $\pi$  is affine. A point  $x \in X^{\text{ss}}(\mathcal{L})$  is said to be *stable* if  $G_x$  is finite and  $G.x$  is closed in  $X^{\text{ss}}(\mathcal{L})$ . Then, for all stable point  $\pi^{-1}(\pi(x)) = G.x$ ; and the set  $X^{\text{s}}(\mathcal{L})$  of stable points is open in  $X$ . A point  $x$  which is not semistable is said to be *unstable*; and, we set  $X^{\text{us}}(\mathcal{L}) = X - X^{\text{ss}}(\mathcal{L})$ .

### 4.2 The functions $\mu^\bullet(x, \lambda)$

As in [MFK94], we denote by  $\text{Pic}^G(X)$  the group of  $G$ -linearized line bundles on  $X$ . Let  $\mathcal{L} \in \text{Pic}^G(X)$ . Let  $x$  be a point in  $X$  and  $\lambda$  be a one parameter subgroup of  $G$ . Since  $X$  is complete,  $\lim_{t \rightarrow 0} \lambda(t)x$  exists; let  $x_0$  denote this limit. The image of  $\lambda$  fixes  $x_0$  and so the group  $k^\times$  acts via  $\lambda$  on the fiber  $\mathcal{L}_{x_0}$ . This action defines a character of  $k^\times$ , that is, an element of  $\mathbb{Z}$  denoted by  $\mu^\mathcal{L}(x, \lambda)$ . One can immediately prove that the numbers  $\mu^\mathcal{L}(x, \lambda)$  satisfy the following properties:

1.  $\mu^\mathcal{L}(g \cdot x, g \cdot \lambda \cdot g^{-1}) = \mu^\mathcal{L}(x, \lambda)$  for any  $g \in G$ ;
2. for fixed  $x$  and  $\lambda$ , the map  $\mathcal{L} \mapsto \mu^\mathcal{L}(x, \lambda)$  is a homomorphism from  $\text{Pic}^G(X)$  to  $\mathbb{Z}$ ;
3. for any  $G$ -variety  $Y$  and for any  $G$ -equivariant morphism  $f : Y \longrightarrow X$ ,  $\mu^{f^*(\mathcal{L})}(y, \lambda) = \mu^\mathcal{L}(f(y), \lambda)$ , where  $y \in Y$ .

The numbers  $\mu^{\mathcal{L}}(x, \lambda)$  are used in [MFK94] to give a numerical criterion for stability with respect to an ample  $G$ -linearized line bundle  $\mathcal{L}$ :

$$\begin{aligned} x \in X^{\text{ss}}(\mathcal{L}) &\iff \mu^{\mathcal{L}}(x, \lambda) \leq 0 \text{ for all one parameter subgroups } \lambda, \\ x \in X^{\text{s}}(\mathcal{L}) &\iff \mu^{\mathcal{L}}(x, \lambda) < 0 \text{ for all non trivial } \lambda. \end{aligned}$$

### 4.3 Definition of the functions $M^{\bullet}(x)$

Let  $T$  be a maximal torus of  $G$ . We denote the real vector space  $X_*(T) \otimes \mathbb{R}$  by  $X_*(T)_{\mathbb{R}}$ . The Weyl group  $W$  of  $T$  acts linearly on  $X_*(T)_{\mathbb{R}}$ . Since  $W$  is finite, there exists a  $W$ -invariant Euclidean norm  $\|\cdot\|$  on  $X_*(T)_{\mathbb{R}}$ . On the other hand, if  $\lambda \in X_*(G)$  there exists  $g \in G$  such that  $g \cdot \lambda \cdot g^{-1} \in X_*(T)$ . Moreover, if two elements of  $X_*(T)$  are conjugate by an element of  $G$ , then they are by an element of the normalizer of  $T$  (see Lemma 2.8 in [MFK94]). This allows us to define the norm of  $\lambda$  by  $\|\lambda\| = \|g \cdot \lambda \cdot g^{-1}\|$ .

Let  $\mathcal{L} \in \text{Pic}^G(X)$ . We can now introduce the following notation:

$$\bar{\mu}^{\mathcal{L}}(x, \lambda) = \frac{\mu^{\mathcal{L}}(x, \lambda)}{\|\lambda\|}, \quad M^{\mathcal{L}}(x) = \sup_{\lambda \in X_*(G)} \bar{\mu}^{\mathcal{L}}(x, \lambda).$$

Actually, we will see in Corollary 1 that  $M^{\mathcal{L}}(x)$  is finite.

### 4.4 $M^{\bullet}(x)$ for a torus action

In this subsection we assume that  $G = T$  is a torus. Let  $z$  be a point of  $X$  fixed by  $T$ . The action of  $T$  on the fiber  $\mathcal{L}_z$  over the point  $z$  in the  $T$ -linearized line bundle  $\mathcal{L}$  define a character  $\chi_z^{\mathcal{L}}$  of  $T$ ; we obtain a morphism

$$\chi_z^{\bullet} : \text{Pic}^T(X) \longrightarrow X(T).$$

For any point  $x$  in  $X$ , we denote by  $\mathcal{P}_T^{\mathcal{L}}$  the convex hull in  $X(T)_{\mathbb{R}}$  of the characters  $-\chi_z^{\mathcal{L}}$  for  $z \in \overline{T \cdot x}^T$ .

The following proposition is an adaptation of a result of L. Ness and gives a pleasant interpretation of the number  $M^{\mathcal{L}}(x)$ :

**Proposition 2** *Let  $\mathcal{L}$  be an ample  $T$ -linearized line bundle on  $X$ . With the above notation, we have:*

1. *The point  $x$  is unstable if and only if  $0$  does not belong to  $\mathcal{P}_T^{\mathcal{L}}(x)$ . In this case,  $M^{\mathcal{L}}(x)$  is the distance from  $0$  to  $\mathcal{P}_T^{\mathcal{L}}(x)$ .*
2. *If  $x$  is semistable, the opposite of  $M^{\mathcal{L}}(x)$  is the distance from  $0$  to the boundary of  $\mathcal{P}_T^{\mathcal{L}}(x)$ .*

3. There exists  $\lambda \in X_*(T)$  such that  $\overline{\mu}^{\mathcal{L}}(x, \lambda) = M^{\mathcal{L}}(x)$ . If moreover  $\lambda$  is indivisible, we call it an adapted one parameter subgroup for  $x$ .
4. If  $x$  is unstable, there exists a unique adapted one parameter subgroup for  $x$ .

**Proof.** Since  $\mathcal{L}$  is ample, there exist a positive integer  $n$  and a  $T$ -module  $V$  such that  $X$  can be equivariantly embedded in  $\mathbb{P}(V)$  in such a way  $\mathcal{L}^{\otimes n}$  is the restriction of  $\mathcal{O}(1)$  to  $X$ . Since  $\chi_z^\bullet$  is a morphism, we have:  $\mathcal{P}_T^{\mathcal{L}^{\otimes n}}(x) = n\mathcal{P}_T^{\mathcal{L}}(x)$ . Moreover,  $\overline{\mu}^{\mathcal{L}^{\otimes n}}(x, \lambda) = n\overline{\mu}^{\mathcal{L}}(x, \lambda)$ , for all  $x$  and  $\lambda$ ; so,  $M^{\mathcal{L}^{\otimes n}}(x) = nM^{\mathcal{L}}(x)$ . As a consequence, it is sufficient to prove the proposition for  $\mathcal{L}^{\otimes n}$ ; in other words, we may assume that  $n = 1$ .

Let us recall that:

$$V = \bigoplus_{\chi \in \text{St}_T(V)} V_\chi.$$

Let  $x \in X$  and  $v \in V$  such that  $[v] = x$ . There exist unique vectors  $v_\chi \in V_\chi$  such that  $v = \sum_\chi v_\chi$ . Let  $\mathcal{Q}$  be the convex hull in  $X(T)_\mathbb{R}$  of the  $\chi$ 's such that  $v_\chi \neq 0$ . It is well known (see [Oda88]) that the fixed point of  $T$  in  $\overline{T \cdot x}$  are exactly the  $[v_\chi]$ 's with  $\chi$  vertex of  $\mathcal{Q}$ . One easily deduces that  $\mathcal{Q} = \mathcal{P}_T^{\mathcal{L}}(x)$ .

Now, the proposition is a direct consequence of [Nes78].  $\square$

#### 4.5 Properties of $M^\bullet(x)$

The following very useful result of Ness relies the function  $M^\bullet(x)$  for  $G$  to similar ones for a maximal torus of  $G$ .

**Lemma 1** (Lemma 3.4 in [Nes78]) *Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle and  $T$  be a maximal torus of  $G$ . We denote by  $r_T : \text{Pic}^G(X) \rightarrow \text{Pic}^T(X)$  the partial forgetful map.*

*Then, for all  $x \in X$ , the set of the numbers  $M^{r_T(\mathcal{L})}(g \cdot x)$  for  $g \in G$  is finite and  $M^{\mathcal{L}}(x) = \max_{g \in G} M^{r_T(\mathcal{L})}(g \cdot x)$ .*

An indivisible one parameter subgroup  $\lambda$  of  $G$  is said to be *adapted for  $x$  and  $\mathcal{L}$*  if and only if  $\overline{\mu}^{\mathcal{L}}(x, \lambda) = M^{\mathcal{L}}(x)$ . Denote by  $\Lambda^{\mathcal{L}}(x)$  the set of adapted one parameter subgroups for  $x$ .

**Corollary 1** 1. *The numbers  $M^{\mathcal{L}}(x)$  are finite (even if  $\mathcal{L}$  is not ample, see Proposition 1.1.6 in [DH98]).*

2. *If  $\mathcal{L}$  is ample,  $\Lambda^{\mathcal{L}}(x)$  is not empty.*

Now, we can reformulate the numerical criterion for stability: if  $\mathcal{L}$  is ample, we have

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : M^{\mathcal{L}}(x) \leq 0\}, \quad X^s(\mathcal{L}) = \{x \in X : M^{\mathcal{L}}(x) < 0\}.$$

The following proposition is a result of finiteness for the set of functions  $M^\bullet(x)$ . It will be used to understand how  $X^{\text{ss}}(\mathcal{L})$  depends on  $\mathcal{L}$  (see Proposition 5).

**Proposition 3** *When  $x$  varies in  $X$ , one obtains only a finite number of functions  $M^\bullet(x) : \text{Pic}^G(X) \rightarrow \mathbb{R}$ .*

**Proof.** Let  $T$  be a maximal torus of  $G$ . Consider the partial forgetful map  $r_T : \text{Pic}^G(X) \rightarrow \text{Pic}^T(X)$ . Since  $M^\bullet(x) = \max_{g \in G} M^{r_T(\bullet)}(g.x)$ , it is sufficient to prove the proposition for the torus  $T$ .

If  $z$  and  $z'$  belong to the same irreducible component  $C$  of  $X^T$ , the morphisms  $\chi_z^\bullet$  and  $\chi_{z'}^\bullet$  are equal: we denote by  $\chi_C^\mathcal{L}$  this morphism.

By Proposition 2,  $M^\mathcal{L}(x)$  only depends on  $\mathcal{P}_T^\mathcal{L}(x)$ , which only depends on the set of irreducible components of  $X^T$  which intersects  $\overline{T.x}$ . Since,  $X^T$  has finitely many irreducible components, the proposition follows.  $\square$

**Remark.** Proposition 3 implies that the set possible open subsets of  $X$  which can be realized as  $X^{\text{ss}}(\mathcal{L})$  for some ample  $G$ -linearized line bundle  $\mathcal{L}$  on  $X$  is finite. This is a result of Dolgachev and Hu (see Theorem 3.9 in [DH98]; see also [Sch03]).

## 4.6 Adapted one parameter subgroups

To describe  $\Lambda^\mathcal{L}(x)$ , we need some additional notation. To the one parameter subgroup  $\lambda$  of  $G$ , we associate the parabolic subgroup (see [MFK94]):

$$P(\lambda) = \left\{ g \in G \text{ such that } \lim_{t \rightarrow 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

The unipotent radical of  $P(\lambda)$  is

$$U(\lambda) = \left\{ g \in G \text{ such that } \lim_{t \rightarrow 0} \lambda(t).g.\lambda(t)^{-1} = e \right\}.$$

Moreover, the centralizer  $G^\lambda$  of the image of  $\lambda$  in  $G$  is a Levi subgroup of  $P(\lambda)$ . For  $p \in P(\lambda)$ , we set  $\bar{p} = \lim_{t \rightarrow 0} \lambda(t).p.\lambda(t)^{-1}$ . Then, we have the following short exact sequence:

$$1 \longrightarrow U(\lambda) \longrightarrow P(\lambda) \xrightarrow{p \mapsto \bar{p}} G^\lambda \longrightarrow 1.$$

For  $g \in P(\lambda)$ , we have  $\mu^{\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(x, g \cdot \lambda \cdot g^{-1})$ . The following theorem due to G. Kempf is a generalization of the last assertion of Proposition 2.

**Theorem 1** (see [Kem78]) *Let  $x$  be an unstable point for an ample  $G$ -linearized line bundle  $\mathcal{L}$ . Then:*

1. *All the  $P(\lambda)$  for  $\lambda \in \Lambda^{\mathcal{L}}(x)$  are equal. We denote by  $P^{\mathcal{L}}(x)$  this subgroup.*
2. *Any two elements of  $\Lambda^{\mathcal{L}}(x)$  are conjugate by an element of  $P^{\mathcal{L}}(x)$ .*

We will also use the following theorem of L. Ness.

**Theorem 2** (Theorem 9.3 in [Nes84]) *Let  $x$  and  $\mathcal{L}$  be as in the above theorem. Let  $\lambda$  be an adapted one parameter subgroup for  $x$  and  $\mathcal{L}$ . We consider  $y = \lim_{t \rightarrow 0} \lambda(t) \cdot x$ . Then,  $\lambda \in \Lambda^{\mathcal{L}}(y)$  and  $M^{\mathcal{L}}(x) = M^{\mathcal{L}}(y)$ .*

#### 4.7 Stratification of $X$ induced from $\mathcal{L}$

If  $d > 0$  and  $\langle \tau \rangle$  is a conjugacy class of one parameter subgroups of  $G$ , we set:

$$S_{d, \langle \tau \rangle}^{\mathcal{L}} = \{x \in X \text{ such that } M^{\mathcal{L}}(x) = d \text{ and } \Lambda^{\mathcal{L}}(x) \cap \langle \tau \rangle \neq \emptyset\}.$$

If  $\mathcal{T}$  is the set of conjugacy classes of one parameter subgroups, the previous section gives us the following decomposition of  $X$ :

$$X = X^{\text{ss}}(L) \cup \bigcup_{d > 0, \langle \tau \rangle \in \mathcal{T}} S_{d, \langle \tau \rangle}^{\mathcal{L}}.$$

W. Hesselink showed in [Hes79] that this union is a finite stratification by  $G$ -stable locally closed subvarieties of  $X$ . We will call it the *stratification induced from  $\mathcal{L}$* .

To describe the geometry of these stratum, we need additional notation:

$$Z_{d, \langle \tau \rangle}^{\mathcal{L}} := \{x \in S_{d, \langle \tau \rangle}^{\mathcal{L}} : \lambda(\mathbb{K}^*) \text{ fixes } x \text{ for some } \lambda \in \langle \tau \rangle\}.$$

For  $\lambda \in \langle \tau \rangle$ , we set:

$$S_{d, \lambda}^{\mathcal{L}} := \{x \in S_{d, \langle \tau \rangle}^{\mathcal{L}} : \lambda \in \Lambda^{\mathcal{L}}(x)\},$$

and

$$Z_{d, \lambda}^{\mathcal{L}} := \{x \in S_{d, \lambda}^{\mathcal{L}} : \lambda(k^*) \text{ fixes } x\}.$$

We have the map

$$p_\lambda : S_{d,\lambda}^\mathcal{L} \longrightarrow Z_{d,\lambda}^\mathcal{L}, \quad x \longmapsto \lim_{t \rightarrow 0} \lambda(t).x.$$

The proof of the following result can be found in [Kir84], Section 1.3.

**Proposition 4**    1.  $S_{d,\lambda}^\mathcal{L} = \{x \in X : \lim_{t \rightarrow 0} \lambda(t).x \in Z_{d,\lambda}^\mathcal{L}\}$ .

2. for each connected component  $Z_{d,\lambda,i}^\mathcal{L}$  of  $Z_{d,\lambda}^\mathcal{L}$  the restriction of the map  $p_\lambda$  over  $Z_{d,\lambda,i}^\mathcal{L}$  is a vector bundle with zero section equal to  $Z_{d,\lambda,i}^\mathcal{L}$ , assuming in addition that  $X$  is smooth;
3.  $S_{d,\lambda}^\mathcal{L}$  is  $P(\lambda)$ -invariant,  $Z_{d,\lambda,i}^\mathcal{L}$  is  $G^\lambda$ -invariant; moreover, for  $p \in P(\lambda)$  and  $x \in S_{d,\lambda}^\mathcal{L}$  we have  $p_\lambda(p.x) = \bar{p}.p_\lambda(x)$ ;
4. there is a surjective finite morphism  $G \times_{P(\lambda)} S_{d,\lambda}^\mathcal{L} \longrightarrow S_{d,<\tau>}^\mathcal{L}$ . It is bijective if  $d > 0$  and an isomorphism if  $S_{d,<\tau>}^\mathcal{L}$  is normal.

## 4.8 Some technical results

Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $X$ . A point  $x \in X^{\text{ss}}(\mathcal{L})$  is said to be *semisimple* for  $\mathcal{L}$  if its  $G$ -orbit is closed in  $X^{\text{ss}}(\mathcal{L})$ . The following lemma is easy and well known:

**Lemma 2** *Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $X$  and  $x \in X$  be a point semisimple for  $\mathcal{L}$ .*

*Then, the restriction of  $\mathcal{L}$  to  $G.x$  is of finite order.*

**Proof.** Let us recall that for any  $\mathcal{L} \in \text{Pic}^G(G.x)$ , the action of  $G_x$  on the fiber over  $x$  in  $\mathcal{L}$  determines a character  $\chi_x^\mathcal{L}$  of  $G_x$ . Moreover, the map  $\mathcal{L} \mapsto \chi_x^\mathcal{L}$  is an injective homomorphism.

Now, let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $X$  such that the character  $\chi_x^\mathcal{L}$  is of infinite order. To obtain the lemma, it is sufficient to prove that  $x$  is unstable for  $\mathcal{L}$ . Let  $\sigma$  be a  $G$ -invariant section of  $\mathcal{L}^{\otimes n}$  for some  $n > 0$ . Then  $\sigma(x)$  is a  $G_x$  fix point of the fiber in  $\mathcal{L}^{\otimes n}$  over  $x$ . Since,  $n.\chi_x^\mathcal{L}$  is non trivial,  $\sigma(x)$  must be zero. So,  $x$  is unstable.  $\square$

The following result is Lemma 3.1 in [Res00]. We recall its statement because it will be often used.

**Lemma 3** *Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $X$  and  $x \in X$  be a point semistable for  $\mathcal{L}$ . Let  $\lambda$  be a one parameter subgroup of  $G$ . Set  $z = \lim_{t \rightarrow 0} \lambda(t).x$ .*

*If  $\mu^{\mathcal{L}}(x, \lambda) = 0$  then  $z$  is semistable for  $\mathcal{L}$ .*

The first using of Lemma 3 is the

**Lemma 4** *Let  $\mathcal{L}$  be an ample  $G$ -linearized line bundle on  $X$ . Let  $T$  be a maximal torus of  $G$  and  $x$  be a semistable point for  $\mathcal{L}$ .*

*If  $0$  belongs to a face of  $\mathcal{P}_T^{\mathcal{L}}(x)$  of codimension  $c$  in  $X(T)_{\mathbb{R}}$  then there exists a point  $z$  in  $\overline{T}.x \cap X^{\text{ss}}(\mathcal{L})$  such that the dimension of  $T_z$  is  $c$ .*

**Proof.** Let  $\mathcal{F}$  be a face of  $\mathcal{P}_T^{\mathcal{L}}(x)$  of codimension  $c$  and containing  $0$ . There exists a one parameter subgroup  $\lambda$  of  $T$  such that  $z = \lim_{t \rightarrow 0} \lambda(t).x$  satisfies  $\mathcal{P}_T^{\mathcal{L}}(z) = \mathcal{F}$ . Since  $\mathcal{F}$  has codimension  $c$  the stabilizer  $T_z$  has dimension  $c$ .

Since  $\mathcal{F}$  contains  $0$ ,  $\mu^{\mathcal{L}}(z, \lambda) = 0$ . So, Lemma 3 shows that  $z$  is semistable for  $\mathcal{L}$ .  $\square$

**Proposition 5** *Let  $\mathcal{L}_0$  be an ample  $G$ -linearized line bundle on  $X$  in  $\Lambda$  and  $x$  be a point semisimple for  $\mathcal{L}_0$ . Set*

$$\mathcal{K}_x := \{\mathcal{L} \in \Lambda_{\mathbb{Q}} \mid \mathcal{L}|_{G.x} \text{ is of finite order}\}.$$

*Then, there exists an open neighborhood  $\Omega$  of  $\mathcal{L}_0$  in  $\mathcal{K}_x$  such that  $x$  is semistable for any  $\mathcal{L} \in \Omega$ .*

**Proof.** Let  $T$  be a maximal torus of  $G$ . We have to prove that there exists an open neighborhood  $\Omega$  of  $\mathcal{L}_0$  in  $\mathcal{K}_x$  such that for all  $g \in G$ ,  $M^{rT(\bullet)}(g.x) \leq 0$  on  $\Omega$ .

Using Proposition 3, one easily checks that there exists an open neighborhood  $\Omega'$  of  $\mathcal{L}_0$  in  $\Lambda_{\mathbb{Q}}$  such that  $M^{rT(\bullet)}(g.x) < 0$  on  $\Omega'$  if  $M^{rT(\mathcal{L}_0)}(g.x) < 0$ .

Let us now fix  $g \in G$  such that  $M^{rT(\mathcal{L}_0)}(g.x) = 0$ . Let  $\mathcal{F}$  be the face of  $\mathcal{P}_T^{\mathcal{L}_0}(g.x)$  containing  $0$  in its relative interior. Let  $\lambda$  be a one parameter subgroup of  $T$  such that the point  $z = \lim_{t \rightarrow 0} \lambda(t).g.x$  satisfies  $\mathcal{P}_T^{\mathcal{L}_0}(z) = \mathcal{F}$ . By Lemma 3,  $z$  is semistable for  $\mathcal{L}_0$ ; and, since  $x$  is semisimple  $z \in G.x$ .

Let  $\mathcal{L} \in \mathcal{K}_x$ . The group  $T_z^{\circ}$  acts trivially on the fiber  $\mathcal{L}_z$ , and so on all the fibers over  $T.z$ , and so on all the fibers over  $\overline{T}.z$ . Therefore,  $\mathcal{P}_T^{\mathcal{L}}(z)$  is contained in the linear subspace  $F$  of  $X(T)_{\mathbb{R}}$  spanned by the characters  $\chi \in X(T)$  trivial in restriction to  $T_z^{\circ}$ .



On the other hand, since  $0$  belongs to  $\mathcal{F}$ ,  $F$  is the linear subspace spanned by  $\mathcal{F}$ . Since  $0$  belongs to the relative interior of  $\mathcal{F}$ , there exists an open neighborhood  $\Omega_z$  of  $\mathcal{L}_0$  in  $\mathcal{K}_x$  such that

$$\forall \mathcal{L} \in \Omega_z \quad M^{r_T(\mathcal{L})}(z) = 0 = M^{r_T(\mathcal{L})}(x).$$

We conclude by using again Proposition 3. □

## 5 Bialynicki-Birula cells

### 5.1 Bialynicki-Birula's theorem

Let  $X$  be a complete  $G$ -variety. Let  $\lambda$  be a one parameter subgroup of  $G$ . Let  $C$  be an irreducible component of  $X^\lambda$ . Since  $G^\lambda$  is connected,  $C$  is a  $G^\lambda$  closed subvariety of  $X$ . We set:

$$C^+ := \{x \in X : \lim_{t \rightarrow 0} \lambda(t)x \in C\}.$$

Then,  $C^+$  is a locally closed subvariety of  $X$  stable by  $P(\lambda)$ . Moreover, the map  $p_\lambda : C^+ \rightarrow C$ ,  $x \mapsto \lim_{t \rightarrow 0} \lambda(t)x$  is a morphism satisfying:

$$\forall (l, u) \in G^\lambda \times U(\lambda) \quad p_\lambda(lu.x) = lp_\lambda(x).$$

Let  $x \in X^\lambda$ . We consider the natural action of  $\mathbb{K}^*$  induced by  $\lambda$  on the Zariski tangent space  $T_x X$  of  $X$  at  $x$ . We consider the following  $\mathbb{K}^*$ -submodules of  $T_x X$ :

$$\begin{aligned} T_x X_{>0} &= \{\xi \in T_x X : \lim_{t \rightarrow 0} \lambda(t)\xi = 0\}, \\ T_x X_{<0} &= \{\xi \in T_x X : \lim_{t \rightarrow 0} \lambda(t^{-1})\xi = 0\}, \\ T_x X_0 &= (T_x X)^\lambda, \quad T_x X_{\geq 0} = T_x X_{>0} \oplus T_x X_0 \quad \text{and} \quad T_x X_{\leq 0} = T_x X_{<0} \oplus T_x X_0. \end{aligned}$$

A classical result of Bialynicki-Birula (see [BB73]) is

**Theorem 3** *Assuming in addition that  $X$  is smooth, we have:*

1.  $C$  is smooth and for all  $x \in C$  we have  $T_x C = T_x X_0$ ;
2.  $C^+$  is smooth and irreducible and for all  $x \in C$  we have  $T_x C^+ = T_x X_{\geq 0}$ ;
3. the morphism  $p_\lambda : C^+ \rightarrow C$  induces a structure of vector bundle on  $C$  with fibers isomorphic to  $T_x X_{>0}$ .

## 5.2 Line bundles on $C^+$

We will need some results about the line bundles on  $C^+$ . Let  $\mathcal{L}$  be a  $P(\lambda)$ -linearized line bundle on  $C^+$ . Since  $C$  is irreducible, the number  $\mu^{\mathcal{L}}(x, \lambda)$  does not depend on  $x \in C^+$ ; we denote by  $\mu^{\mathcal{L}}(C, \lambda)$  this integer.

**Proposition 6** *We assume that  $X$  is smooth. Then, we have:*

1. *The restriction map  $\text{Pic}^{P(\lambda)}(C^+) \longrightarrow \text{Pic}^{G^\lambda}(C)$  is an isomorphism.  
Let  $\mathcal{L} \in \text{Pic}^{P(\lambda)}(C^+)$ .*
2. *If  $\mu^{\mathcal{L}}(C, \lambda) \neq 0$ ,  $H^0(C, \mathcal{L}|_C)^\lambda = \{0\}$ .*
3. *If  $\mu^{\mathcal{L}}(C, \lambda) = 0$ , the restriction map induces an isomorphism from  $H^0(C^+, \mathcal{L})^{P(\lambda)}$  onto  $H^0(C, \mathcal{L}|_C)^{G^\lambda}$ . Moreover, for any  $\sigma \in H^0(C^+, \mathcal{L})^{P(\lambda)}$ , we have:*

$$\{x \in C^+ : \sigma(x) = 0\} = p_\lambda^{-1}(\{x \in C : \sigma(x) = 0\}).$$

**Proof.** Since  $p_\lambda$  is  $P(\lambda)$ -equivariant, for any  $\mathcal{M} \in \text{Pic}^{G^\lambda}(C)$ ,  $p_\lambda^*(\mathcal{M})$  is  $P(\lambda)$ -linearized. Since  $p_\lambda$  is a vector bundle,  $p_\lambda^*(\mathcal{L}|_C)$  and  $\mathcal{L}$  are isomorphic as line bundles without linearization. But,  $X(P(\lambda)) \simeq X(G^\lambda)$ , so the  $P(\lambda)$ -linearizations must coincide; and  $p_\lambda^*(\mathcal{L}|_C)$  and  $\mathcal{L}$  are isomorphic as  $P(\lambda)$ -linearized line bundles. Assertion 1 follows.

Assertion 2 is a direct application of Lemma 2.

Let us fix  $\mathcal{L} \in \text{Pic}^{P(\lambda)}(C^+)$  and denote by  $p : \mathcal{L} \longrightarrow C^+$  the projection. We assume that  $\mu^{\mathcal{L}}(C, \lambda) = 0$ . Let  $\sigma \in H^0(C^+, \mathcal{L})^{P(\lambda)}$ . We just proved that

$$\mathcal{L} \simeq p_\lambda^*(\mathcal{L}|_C) = \{(x, l) \in C^+ \times \mathcal{L}|_C : p_\lambda(x) = p(l)\}.$$

Let  $p_2$  denote the projection of  $p_\lambda^*(\mathcal{L}|_C)$  onto  $\mathcal{L}|_C$ .

For all  $x \in C^+$  and  $t \in \mathbb{K}^*$ , we have:

$$\begin{aligned} \sigma(\lambda(t).x) &= \left( \lambda(t).x, p_2(\sigma(\lambda(t).x)) \right) \\ &= \lambda(t). \left( x, p_2(\sigma(x)) \right) && \text{since } \sigma \text{ is invariant,} \\ &= \left( \lambda(t).x, p_2(\sigma(x)) \right) && \text{since } \mu^{\mathcal{L}}(C, \lambda) = 0. \end{aligned}$$

We deduce that for all  $x \in C^+$ ,  $\sigma(x) = (x, \sigma(p_\lambda(x)))$ . Assertion 3 follows.

□

## 6 Slice Etale Theorem

In this section, we fix an ample  $G$ -linearized line bundle  $\mathcal{L}$  on the irreducible projective  $G$ -variety  $X$ .

### 6.1 Semistability for normalizer and the centralizer of a subgroup

We will use the following interpretation of a result of Luna:

**Proposition 7** *Let  $H$  be a reductive subgroup of  $G$ . Let  $C$  be an irreducible component of  $X^H$ . Then, the reductive groups  $(G^H)^\circ$  and  $N_G(H)^\circ$  act on  $C$ .*

*Let  $x$  be a point in  $C$ . Then, the following are equivalent:*

1.  $x$  is semistable for  $\mathcal{L}$ .
2.  $x$  is semistable for the action of  $(G^H)^\circ$  on  $C$  and the restriction of  $\mathcal{L}$ .
3.  $x$  is semistable for the action of  $N_G(H)^\circ$  on  $C$  and the restriction of  $\mathcal{L}$ .

**Proof.** Lemma 1.1. of [LR79] shows that  $(G^H)^\circ$  and  $N_G(H)^\circ$  are reductive. Changing  $\mathcal{L}$  by a positive power if necessary, one may assume that  $X$  is contained in  $\mathbb{P}(V)$  where  $V$  is a  $G$ -module and  $\mathcal{L} = \mathcal{O}(1)|_X$ . Let  $v \in V$  such that  $[v] = x$ . Let us recall that in this case  $x \in X^{\text{us}}(\mathcal{L})$  if and only if  $\overline{G.v}$  contains 0.

Let  $\chi$  be the character of  $H$  such that  $hv = \chi(h)v$  for all  $h \in H$ .

If  $\chi$  is of infinite order, so is its restriction to the connected center  $Z$  of  $H$ . Then,  $Z.v = \mathbb{K}^*v$  and  $0 \in \overline{(G^H)^\circ.v}$ . In this case,  $x$  belongs to no semistable set of the statement of the proposition.

Let us now assume that  $\chi$  is of finite order. Changing  $\mathcal{L}$  by a positive power if necessary, one may assume that  $\chi$  is trivial, that is  $H$  fixes  $v$ . In this case, Corollary 2 and Remark 1 of [Lun75] show that

$$0 \in \overline{G.v} \iff 0 \in \overline{N_G(H)^\circ.v} \iff 0 \in \overline{G^H.v}.$$

The proposition follows. □

## 6.2 Closed orbits in General position

Consider the quotient  $\pi : X^{\text{ss}}(\mathcal{L}) \longrightarrow X^{\text{ss}}(\mathcal{L})//G$ . For all  $\xi \in X^{\text{ss}}(\mathcal{L})//G$ , we denote by  $T(\xi)$  the unique closed orbit of  $G$  in  $\pi^{-1}(\xi)$ . We denote by  $(X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$  the set of  $\xi$  such that there exists an open neighborhood  $V$  of  $\xi$  in  $X^{\text{ss}}(\mathcal{L})//G$  such that the orbits  $T(\xi')$  are isomorphic to  $T(\xi)$ , for all  $\xi' \in V$ .

Since  $\pi$  is a gluing of affine quotients, some results on the actions of  $G$  on affine variety remains true for  $X^{\text{ss}}(\mathcal{L})$ . For example, the following theorem is a result of Luna and Richardson (see Section 3 of [LR79] and Corollary 4 of [Lun75] or Section 7 of [PV91]):

**Theorem 4** *With above notation, if  $X$  is normal, we have:*

1. *The set  $(X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$  is a non empty open subset of  $X^{\text{ss}}(\mathcal{L})//G$ . Let  $H$  be the isotropy of a point in  $T(\xi)$  with  $\xi \in (X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$ . The group  $H$  has fixed points in  $T(\xi)$  for any  $\xi \in X^{\text{ss}}(\mathcal{L})//G$ .*
2. *Let  $Y$  be the closure of  $\pi^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})^H$  in  $X$ . It is the union of some components of  $X^H$ . Then,  $H$  acts trivially on some positive power  $\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n}$  of  $\mathcal{L}_{|Y_{\mathcal{F}}}$ . Moreover, the natural map*

$$Y_{\mathcal{F}}^{\text{ss}}(\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n})//N_G(H)/H \longrightarrow X^{\text{ss}}(\mathcal{L})//G$$

*is an isomorphism.*

A subgroup  $H$  as in Theorem 4 will be called a *generic closed isotropy* of  $X^{\text{ss}}(\mathcal{L})$ . The conjugacy class of  $H$  which is obviously unique is called the *generic closed isotropy* of  $X^{\text{ss}}(\mathcal{L})$ .

## 6.3 The principal Luna stratum

When  $X$  is smooth, the open subset  $(X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$  is called the *principal Luna stratum* and has very useful properties (see [Lun73] or [PV91]):

**Theorem 5 (Luna)** *We assume that  $X$  is smooth. Let  $H$  be a generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$ .*

*Then, there exists a  $H$ -module  $L$  such that for any  $\xi \in (X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}}$ , there exists point  $x$  in  $T(\xi)$  satisfying:*

1.  $G_x = H$ ;
2. *the  $H$ -module  $T_x X/T_x(G.x)$  is isomorphic to the sum of  $L$  and its fix points, in particular, it is independent of  $\xi$  and  $x$ ;*

3. for any  $v \in L$ ,  $0$  belongs to the closure of  $H.v$ ;
4. the fiber  $\pi^{-1}(\xi)$  is isomorphic to  $G \times_H L$ .

## 7 First properties of the $G$ -ample cone

### 7.1 Definitions

Let us recall from the introduction that  $\Lambda$  is a freely finitely generated subgroup of  $\text{Pic}^G(X)$  and  $\Lambda_{\mathbb{Q}}$  is the  $\mathbb{Q}$ -vector space containing  $\Lambda$  as a lattice. Moreover, the convex cone  $\Lambda_{\mathbb{Q}}^+$  generated by the ample elements of  $\Lambda$  is open in  $\Lambda_{\mathbb{Q}}$  and for all  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^+$ , there exists a positive integer  $n$  such that  $\mathcal{L}^{\otimes n}$  is an ample  $G$ -linearized line bundle on  $X$  in  $\Lambda$ . Using this property, we defined  $X^{\text{ss}}(\mathcal{L})$  for any  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^+$ . The central object of this article is the following  $G$ -ample cone:

$$\mathcal{C}_{\Lambda}^G(X) := \{\mathcal{L} \in \Lambda_{\mathbb{Q}}^+ : X^{\text{ss}}(\mathcal{L}) \text{ is not empty}\}.$$

By [DH98] (see also [Res00]),  $\mathcal{C}_{\Lambda}^G(X)$  is a closed convex rational polyhedral cone in  $\Lambda_{\mathbb{Q}}^+$ .

Two points  $\mathcal{L}$  and  $\mathcal{L}'$  in  $\mathcal{C}_{\Lambda}^G(X)$  are said to be *GIT-equivalent* if  $X^{\text{ss}}(\mathcal{L}) = X^{\text{ss}}(\mathcal{L}')$ . An equivalence class is simply called a *GIT-class*.

For  $x \in X$ , the *stability set of  $x$*  is the set of  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^+$  such that  $X^{\text{ss}}(\mathcal{L})$  contains  $x$ ; it is denoted by  $\Omega_{\Lambda}(x)$ . In [Res00], we have studied the geometry of the GIT-classes and the stability sets with lightly different assumptions (no  $\Lambda$  for example). However all the results and proofs of [Res00] remain valuable here. In particular, there are only finitely many GIT-classes (see also the remark in Section 4.5); and each GIT-class is the relative interior of a closed convex polyhedral cone of  $\Lambda_{\mathbb{Q}}^+$ .

### 7.2 A first description of the $G$ -ample cone

Here comes a central definition in this work:

**Definition.** Let  $\lambda$  be a one parameter subgroup of  $G$  and  $C$  be an irreducible component of its fix points. Set  $C^+ := \{x \in X \mid \lim_{t \rightarrow 0} \lambda(t)x \in C\}$ .

The pair  $(C, \lambda)$  is said to be *well covering* if the natural  $G$ -equivariant map  $\eta : G \times_{P(\lambda)} C^+ \rightarrow X$  induces an isomorphism from  $G \times_{P(\lambda)} \Omega$  onto an open subset of  $X$  for an open subset  $\Omega$  of  $C^+$  intersecting  $C$ .

Let us recall that  $\mu^{\bullet}(C, \lambda)$  denote the common value of the  $\mu^{\bullet}(x, \lambda)$ , for  $x \in C$ . Proposition 4 allows us to give a first description of the cone  $\mathcal{C}_{\Lambda}^G(X)$ :

**Proposition 8** *We assume that  $X$  is normal. Let  $T$  be a maximal torus of  $G$  and  $B$  be a Borel subgroup containing  $T$ .*

*Then, the cone  $\mathcal{C}_\Lambda^G(X)$  is the set of the  $\mathcal{L} \in \Lambda_\mathbb{Q}^+$  such that for all well covering pair  $(C, \lambda)$  with a dominant one parameter subgroup  $\lambda$  of  $T$  we have  $\mu^\mathcal{L}(C, \lambda) \leq 0$ .*

**Proof.** Let  $\lambda$  be a one parameter subgroup of  $G$  and  $C$  be an irreducible component of its fix points such that the morphism  $\eta : G \times_{P(\lambda)} C^+ \rightarrow X$  is dominant. Let  $\mathcal{L} \in \mathcal{C}_\Lambda^G(X)$ . Since  $X^{\text{ss}}(\mathcal{L})$  is a  $G$ -stable open subset of  $X$  and  $\eta$  is dominant,  $X^{\text{ss}}(\mathcal{L})$  intersects  $C^+$ . Let  $x$  be a point in this intersection and  $y = \lim_{t \rightarrow 0} \lambda(t).x$ . By the Mumford numerical criterion,  $\mu^\mathcal{L}(x, \lambda) \leq 0$ . But,  $\mu^\mathcal{L}(x, \lambda) = \mu^\mathcal{L}(y, \lambda) = \mu^\mathcal{L}(C, \lambda)$ . We deduce that  $\mathcal{C}_\Lambda^G(X)$  is contained in the part of  $\Lambda_\mathbb{Q}^+$  defined by the inequalities  $\mu^\mathcal{L}(C, \lambda) \leq 0$  of the proposition.

Conversely, let  $\mathcal{L} \in \Lambda_\mathbb{Q}^+$  such that  $X^{\text{ss}}(\mathcal{L})$  is empty.

When  $\lambda$  runs over the dominant one parameter subgroups of  $T$ , we obtain only finitely many locally closed subvarieties of  $X$  of the form  $C^+$ . Moreover, we obtain finitely many parabolic subgroups  $P(\lambda)$ . In particular, we can chose a point  $x$  in  $X$  contained in no image of a non dominant morphism  $G \times_{P(\lambda)} C^+ \rightarrow X$ .

Moreover, we may assume that  $x$  belongs to the open stratum  $S_{d, <\tau>}^\mathcal{L}$  of  $X^{\text{us}}(\mathcal{L}) = X$ . Let  $\lambda \in \Lambda^\mathcal{L}(x)$ . Eventually, changing  $x$  by another point of  $G.x$ , we may assume that  $\lambda$  is a dominant one parameter subgroup of  $T$ . Let  $C$  denote the irreducible component of  $X^\lambda$  containing  $x$ . Since  $X$  is normal, so is  $S_{d, <\tau>}^\mathcal{L}$ ; thus Assertion 4 of Proposition 4 shows that the pair  $(C, \lambda)$  is well covering. But,  $\mu^\mathcal{L}(C, \lambda) = \mu^\mathcal{L}(x, \lambda) = d > 0$ .  $\square$

**Remark.** Proposition 8 asserts that any facet of  $\mathcal{C}_\Lambda^G(X)$  is obtained by intersecting  $\mathcal{C}_\Lambda^G(X)$  with an hyperplane  $\mu^\bullet(C, \lambda) = 0$  for a well covering pair  $(C, \lambda)$ . In the sequence of this article, we will precise (with more assumptions) this result in two directions. Firstly, we will give a description of the smaller faces of  $\mathcal{C}_\Lambda^G(X)$ . Secondly, we are interested in kind of converse: given  $(C, \lambda)$ , what is the dimension of the intersection of the hyperplane  $\mu^\bullet(C, \lambda) = 0$  and  $\mathcal{C}_\Lambda^G(X)$ .

### 7.3 Faces of $\mathcal{C}_\Lambda^G(X)$

In this section we associate two invariants to a face  $\mathcal{F}$  of  $\mathcal{C}_\Lambda^G(X)$ .

**Theorem 6** *Let  $\mathcal{F}$  be a face of  $\mathcal{C}_\Lambda^G(X)$ . Then, we have:*

1. *The generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$  does not depend on the point  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ . We call this isotropy the generic closed isotropy of  $\mathcal{F}$ .*

*Let us fix a generic closed isotropy  $H$  of  $\mathcal{F}$ .*

2. *For any  $\mathcal{L} \in \mathcal{F}$ ,  $H$  fixes points in any closed orbit of  $G$  in  $X^{\text{ss}}(\mathcal{L})$ .*
3. *The closure  $Y$  of  $\left(\pi_{\mathcal{L}}^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})\right)^H$  in  $X$  does not depend on a choice of a point  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ . Let  $Y_{\mathcal{F}}$  denote this subvariety of  $X^H$ ; it is the union of some components of  $X^H$ .*
4. *Let  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ . Then,  $H$  acts trivially on some positive power  $\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n}$  of  $\mathcal{L}_{|Y_{\mathcal{F}}}$ . Moreover, the natural map*

$$Y_{\mathcal{F}}^{\text{ss}}(\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n})//((N_G(H)/H)) \longrightarrow X^{\text{ss}}(\mathcal{L})//G$$

*is an isomorphism.*

5. *Set  $Y_{\mathcal{F}}^{\pm} := \{x \in X : \overline{H.x} \cap Y_{\mathcal{F}} \neq \emptyset\}$ . Then  $G.Y_{\mathcal{F}}^{\pm}$  contains an open subset of  $X$ .*
6. *For all  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ ,  $Y_{\mathcal{F}}$  contains stable points for the action of  $N_G(H)/H$  and the line bundle  $\mathcal{L}_{|Y_{\mathcal{F}}}^{\otimes n}$ .*

**Proof.** Let  $F$  be a GIT-class in  $\mathcal{F}$  which has the same dimension as  $\mathcal{F}$ . Let  $\mathcal{L} \in \mathcal{F}$ . Let us fix a point  $x$  in  $\pi_F^{-1}((X^{\text{ss}}(F)//G)_{\text{pr}}) \cap \pi_{\mathcal{L}}^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})$ . Let  $\mathcal{O}_1$  (resp.  $\mathcal{O}_2$ ) be the unique closed orbit of  $G$  in  $\overline{G.x} \cap X^{\text{ss}}(F)$  (resp.  $\overline{G.x} \cap X^{\text{ss}}(\mathcal{L})$ ). By Proposition 3.6 of [Res00] and since  $\mathcal{O}_1$  is closed in  $X^{\text{ss}}(F)$ ,  $\Omega_{\Lambda}(\mathcal{O}_1)$  is the face of  $\Omega_{\Lambda}(x)$  which contains  $F$  in its relative interior. So, our assumption on  $F$  implies that  $\Omega_{\Lambda}(\mathcal{O}_1)$  contains  $\mathcal{F} \cap \Omega_{\Lambda}(x)$ . In particular,  $\mathcal{O}_1 \subset X^{\text{ss}}(\mathcal{L})$ , and  $\mathcal{O}_2 \subset \overline{\mathcal{O}_1}$ . It follows that the generic closed isotropy of  $X^{\text{ss}}(F)$  is contained in those of  $X^{\text{ss}}(\mathcal{L})$ .

Let  $\mathcal{M}$  be a point in the relative interior of  $\mathcal{F}$ . Then, there exists  $F$  and  $\mathcal{L}$  as above such that  $\mathcal{M}$  belongs to the convex hull of  $F \cup \mathcal{L}$ . With the notation of the last paragraph,  $\mathcal{M}$  belongs to the relative interior of  $\Omega_{\Lambda}(\mathcal{O}_1)$ . Now, Proposition 3.6 of [Res00] shows that  $\mathcal{O}_1$  is closed in  $X^{\text{ss}}(\mathcal{L})$ . Remembering that  $x$  is generic this proves that the generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$  and  $X^{\text{ss}}(F)$  coincide. Assertion 1 is proved.

Moreover, we had that for any  $\mathcal{L} \in \mathcal{F}$ , the generic closed isotropy of  $\mathcal{L}$  contains  $H$ . Now, Assertion 2 follows from Theorem 4.

Let  $Y$  be the subvariety of  $X^H$  of Assertion 3 for  $\mathcal{M}$ . By Theorem 4,  $Y$  satisfies Assertion 4. Moreover,  $G.Y^+$  contains  $\pi_{\mathcal{M}}^{-1}((X^{\text{ss}}(\mathcal{M})//G)_{\text{pr}})$ ; and, Assertion 5 is proved for  $Y$ .

Since  $\pi_1$  is affine Corollary 1 of [Lun75] shows that the  $N_G(H)$ -orbit of any element of  $\left(\pi_{\mathcal{M}}^{-1}((X^{\text{ss}}(\mathcal{M})//G)_{\text{pr}})\right)^H$  is closed in  $X^{\text{ss}}(\mathcal{M})$ . By Proposition 7, we can deduce Assertion 6 for  $Y$  and  $\mathcal{M}$ .

Let  $\Omega_{\mathcal{F}}(Y)$  denote the set of  $\mathcal{L}$  in  $\mathcal{F}$  such that  $Y^{\text{ss}}(\mathcal{L})$  is not empty. By the numerical criterion of semistability  $\Omega_{\mathcal{F}}(Y)$  is closed in  $\mathcal{F}$ . Moreover, by Assertion 5, for all  $\mathcal{L} \in \Omega_{\mathcal{F}}(Y)$ ,  $\pi_{\mathcal{L}}(Y^{\text{ss}}(\mathcal{L}))$  is dense in  $X^{\text{ss}}(\mathcal{L})//G$ . In particular,  $Y$  intersects  $\pi_{\mathcal{L}}^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})$ . This ends the proof of Assertion 3 and of the proposition.  $\square$

## 8 Abundance

### 8.1 Definition and examples

We call a subgroup  $\Gamma'$  of an abelian group  $\Gamma$  *cofinite* if  $\Gamma/\Gamma'$  is finite. The following definition is an adaptation of those of Dolgachev and Hu (see [DH98]).

**Definition.** The subgroup  $\Lambda$  is said to be *abundant* if for any  $x$  in  $X$  such that  $G_x$  is reductive, the image of the restriction  $\Lambda \rightarrow \text{Pic}^G(G.x)$  is cofinite.

The main example of abundant subgroups comes from the case when  $X = Y \times G/B$ .

**Proposition 9** *Let  $X = Y \times G/B$  for a  $G$ -variety  $Y$ . Let  $\pi : X \rightarrow G/B$  denote the projection map and  $\pi^* : \text{Pic}^G(G/B) \rightarrow \text{Pic}^G(X)$  the associated homomorphism.*

*Then, any subgroup  $\Lambda$  of  $\text{Pic}^G(X)$  containing the image of  $\pi^*$  is abundant.*

**Proof.** Let  $x = (y, gB/B) \in Y \times G/B$ . Changing  $x$  by  $g^{-1}x$ , we assume that  $g = e$ . Let  $\chi \in X(G_x)$ . Note that  $G_x = B_y$ . Since the restriction map  $X(B) \rightarrow X(B_y)$  is surjective, there exists  $\nu \in X(B)$  such that  $\nu|_{B_y} = \chi$ . The restriction of  $\pi^*(\mathcal{L}_\nu)$  to  $G.x$  equals  $\mathcal{L}_\chi$ ; the proposition follows.  $\square$



## 8.2 Rank of isotropies and dimension of GIT-classes

If  $\Lambda$  is abundant, then we can control the dimension of the closed isotropies of  $X^{\text{ss}}(\mathcal{L})$  for  $\mathcal{L} \in \mathcal{F}$  by the dimension of  $\mathcal{F}$ . More precisely, we have:

**Proposition 10** *We assume that  $\Lambda$  is abundant. Let  $F$  be a GIT-class of  $\mathcal{C}_\Lambda^G(X)$ .*

*Then, we have:*

1. *For any point  $x$  semisimple for  $F$ , the rank of the character group of  $G_x$  is less than  $\text{rk}(\Lambda) - \dim(F)$ . Moreover,  $G_x$  is reductive.*
2. *There exists a point  $x$  semisimple for  $F$  such that the rank of the character group of  $G_x$  equals  $\dim(\mathcal{C}_\Lambda^G(X)) - \dim F$ .*
3. *We assume that  $\dim(\mathcal{C}_\Lambda^G(X)) = \dim(\Lambda_{\mathbb{Q}})$ . If  $F$  is contained in a face  $\mathcal{F}$  of  $\mathcal{C}_\Lambda^G(X)$  of the same dimension than  $F$  then for any point  $x$  semisimple for  $F$  the rank of the character group of  $G_x$  equals  $\dim(\Lambda_{\mathbb{Q}}) - \dim F$ .*

**Proof.** Let  $x$  be a semisimple point for  $F$ . Since  $\pi$  is affine and  $x$  is semisimple,  $G_x$  is affine. By the theorem of Matsushima (see [Mat60] or [Lun73]),  $G_x$  is reductive. Since  $\text{Pic}(G_x) \simeq X(G_x)$  and  $\Lambda$  is abundant, the restriction map induces a surjective linear map  $\rho_{\mathbb{Q}} : \Lambda_{\mathbb{Q}} \rightarrow X(G_x)_{\mathbb{Q}}$ . By Lemma 2,  $F$  is contained in the kernel of  $\rho_{\mathbb{Q}}$  whose the dimension is equal to  $\dim(\Lambda_{\mathbb{Q}}) - \text{rk}(X(G_x))$ . The first assertion of the proposition follows.

For any  $x \in X$ , we consider the stability set  $\Omega_\Lambda(x) = \{\mathcal{L} \in \Lambda_{\mathbb{Q}}^+ : x \text{ is semistable for } \mathcal{L}\}$ . By Lemma 4.2 of [Res00], the closure of  $F$  in  $\Lambda_{\mathbb{Q}}$  is the intersection of the  $\Omega_\Lambda(x)$  over all the points  $x$  semisimple for  $F$ . By Corollary 3.3 and Proposition 3.2 of [Res00], this intersection is a finite intersection of convex cones. We deduce that there exists a point  $x$  semisimple for  $F$  such that  $\dim(\Omega(x)) = \dim(F)$ . But, Proposition 6.5 of [Res00] shows that the rank of  $X(G_x)$  is more than  $\dim(\mathcal{C}_\Lambda^G(X)) - \dim(\Omega(x))$ . Assertion 2 of the proposition is proved.

Let  $x$  be a semisimple point for  $F$ . Let  $\mathcal{L}_0$  be a point in  $F$ . By Proposition 5, there exists an open neighborhood  $\Omega$  of  $\mathcal{L}_0$  in the kernel of  $\rho_{\mathbb{Q}}$  such that  $x$  is semisimple for all  $\mathcal{L}$  in  $\Omega$ . In particular,  $\Omega$  is contained in  $\mathcal{C}_\Lambda^G(X)$ . Since  $F$  is contained in a face of  $\mathcal{C}_\Lambda^G(X)$  of the same dimension as  $F$ , this implies that the interior of  $F$  is not empty in the kernel of  $\rho_{\mathbb{Q}}$ . The last assertion of the proposition follows.  $\square$

Let  $\mathcal{F}$  be a face of  $\mathcal{C}_\Lambda^G(X)$ . Let  $H_{\mathcal{F}}$  be a generic closed isotropy of  $\mathcal{F}$  and  $C_{\mathcal{F}}$  be an irreducible component of  $Y_{\mathcal{F}}$  with the notation of Theorem 6.

There exists a well defined morphism:

$$r_{\Lambda}^{H_{\mathcal{F}}, C_{\mathcal{F}}} : \Lambda \longrightarrow X(H)$$

such that for all  $\mathcal{L} \in \Lambda$ , for all  $v \in \mathcal{L}$  over a point  $x$  of  $C_{\mathcal{F}}$ , and for all  $h \in H$  we have:  $h.v = r_{\Lambda}^{H_{\mathcal{F}}, C_{\mathcal{F}}}(\mathcal{L})(h)v$ .

**Corollary 2** *We use above notation and assume that  $\Lambda$  is abundant.*

*The subspace spanned by a face  $\mathcal{F}$  of  $\mathcal{C}_{\Lambda}^G(X)$  and by the kernel of  $r_{\Lambda}^{H_{\mathcal{F}}, C_{\mathcal{F}}}$  coincide.*

**Proof.** By Lemma 2,  $\mathcal{F}$  is contained in the subspace spanned by the kernel of  $r_{\Lambda}^{H_{\mathcal{F}}, C_{\mathcal{F}}}$ . Now, the equality of the corollary is a consequence of the last assertion of Proposition 10.  $\square$

## 9 The case $X = Y \times G/B$

Let  $B$  be a Borel subgroup of  $G$ . From now on, we assume that  $X = Y \times G/B$  (except in Assertion 4 of Proposition 12), with a normal  $G$ -variety  $Y$ . As explained in the introduction, this case is particularly interesting. Moreover, any reductive isotropy of a point of  $x$  is a diagonalizable subgroup of  $G$ ; this remark will simplify a lot the description of the  $G$ -ample cone.

### 9.1 Equations defining $\mathcal{C}_{\Lambda}^G(Y \times G/B)$

In the situation of this section, one can make Proposition 8 more precise:

**Proposition 11** *Let  $T$  be a maximal torus of  $G$  and  $B$  be a Borel subgroup containing  $T$ . Let  $X = Y \times G/B$  for a normal  $G$ -variety  $Y$ . Let  $\Lambda$  be an abundant subgroup of  $\text{Pic}^G(X)$ .*

*Then, a point  $\mathcal{L}$  of  $\Lambda_{\mathbb{Q}}^+$  belongs to  $\mathcal{C}_{\Lambda}^G(X)$  if and only if for any dominant one parameter subgroup  $\lambda$  of  $T$  and for any irreducible component  $C$  of  $X^{\lambda}$  such that  $(C, \lambda)$  is well covering and  $G_C^{\circ}$  equals the image of  $\lambda$ , we have  $\mu^{\mathcal{L}}(C, \lambda) \leq 0$ .*

**Proof.** By Proposition 8, it is sufficient to prove that any face  $\mathcal{F}$  of codimension one of  $\mathcal{C}_{\Lambda}^G(X)$  is contained in the set of  $\mathcal{L}$ 's such that  $\mu^{\mathcal{L}}(C, \lambda) = 0$  for a dominant one parameter subgroup  $\lambda$  of  $T$  and for an irreducible component  $C$  of  $X^{\lambda}$  such that  $(C, \lambda)$  is well covering and  $G_C^{\circ} = \text{Im}(\lambda)$  is one.

Let  $F$  be a GIT-class contained in  $\mathcal{F}$  and of the same dimension as  $\mathcal{F}$ . By Proposition 8, there exists a well covering pair  $(C, \lambda)$  such that the points  $\mathcal{L}$  of  $F$  satisfy  $\mu^{\mathcal{L}}(C, \lambda) = 0$ . It is sufficient to show that  $G_C^{\circ} = \text{Im}(\lambda)$ .

On one hand, since  $X = Y \times G/B$ ,  $G_C$  is contained in the center of a Levi subgroup of  $G$ ; in particular, it is diagonalizable. So, for a generic point  $z$  in  $C$ , we have  $G_z = G_C$ . On the other hand, since  $X^{\text{ss}}(F)$  is open in  $X$  and  $G$ -stable and  $(C, \lambda)$  is well covering,  $X^{\text{ss}}(F)$  intersects  $C^+$ . Finally, one can choose a point  $x \in C^+ \cap X^{\text{ss}}(F)$  such that the point  $z = \lim_{t \rightarrow 0} \lambda(t).x$  satisfies  $G_z = G_C$ .

By Lemma 3,  $z$  is semistable for  $F$ . Since  $\Lambda$  is abundant, Proposition 10 shows that the rank of  $X(G_z)$  is less than one. Since  $G_z$  is diagonalizable and contains the image of  $\lambda$ , it follows that  $\text{Im}(\lambda) = G_z^\circ = G_C^\lambda$ .  $\square$

## 9.2 The principal Luna stratum

Let us fix an ample  $G$ -linearized line bundle  $\mathcal{L}$  on  $X$ . Let  $H$  be a generic closed isotropy of  $X^{\text{ss}}(\mathcal{L})$ . Note that  $H^\circ$  is a torus. Let  $L$  be the  $H$ -module satisfying Theorem 5. We assume that  $X$  is smooth.

Set  $\mathfrak{h}_\mathbb{R} := X_*(H^\circ) \otimes \mathbb{R}$ . Consider the set  $\mathfrak{h}_X^+$  of the  $\xi \in \mathfrak{h}_\mathbb{R}$  such that:

1. for all  $x \in X^{H^\circ}$  and for all non trivial character  $\chi$  of  $H^\circ$  in  $\text{St}_{H^\circ}(\text{T}_x X)$ , we have  $\langle \xi, \chi \rangle \neq 0$ ;
2. for all non trivial  $\alpha \in \text{St}_{H^\circ}(\mathfrak{g})$ , we have  $\langle \xi, \alpha \rangle \neq 0$ ; and,
3. for all  $\chi \in \text{St}_{H^\circ}(L)$ , we have  $\langle \xi, \chi \rangle > 0$ .

This set  $\mathfrak{h}_X^+$  is an open convex cone of  $\mathfrak{h}_\mathbb{R}$ ; in particular, their images generate  $H^\circ$ . Moreover, for all  $\lambda \in X_*(H^\circ) \cap \mathfrak{h}_X^+$ , we have:

1.  $X^\lambda = X^{H^\circ}$ ,
2.  $G^\lambda = G^{H^\circ}$ ,
3. for all  $v \in L$ ,  $\lim_{t \rightarrow 0} \lambda(t).v = 0$ .

Let us fix such a one parameter subgroup  $\lambda$  and  $C$  an irreducible component of  $X^\lambda$  which intersects  $\pi^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})$ . We consider the associated Bialynicki-Birula cell  $C^+$ . Since  $C^+$  is stable by the action of  $P(\lambda)$ , we can consider the fiber product  $G \times_{P(\lambda)} C^+$  and the natural  $G$ -equivariant map

$$\eta : G \times_{P(\lambda)} C^+ \longrightarrow X.$$

The following properties of  $\eta$  will play a central role in the sequence:

**Proposition 12** *The  $G$ -equivariant map  $\eta$  satisfies:*

1.  $\eta$  is dominant
2.  $T_x\eta$  is an isomorphism for any  $x$  in a  $G$ -invariant open subset of  $G \times_{P(\lambda)} C^+$  which intersects  $C$ .
3. If  $X = Y \times G/B$ , then  $\eta$  is birational.
4. If  $H^\circ \simeq \mathbb{K}^*$  and there exists points in  $X$  with finite isotropy then  $\eta$  is birational, even without the assumption  $X = Y \times G/B$ .

**Proof.** To prove the two first assertions, it is sufficient to prove that for all  $x \in C \cap X^{\text{ss}}(\mathcal{L})$  such that  $G_x = H$  the linear map

$$T_{[e:x]}\eta : T_{[e:x]}(G \times_{P(\lambda)} C^+) \longrightarrow T_x X$$

is an isomorphism.

Let  $N_x$  be a  $G_x$ -stable supplementary subspace to  $T_x(G.x)$  in  $T_x X$ . Let  $L_x$  be a  $G_x$ -stable supplementary subspace to  $N_x^{G_x}$  in  $N_x$ . We have

$$T_x(G.x) \simeq \mathfrak{g}/\mathfrak{h} \simeq \text{Lie}(U(\lambda^{-1})) \oplus \mathfrak{g}^\lambda/\mathfrak{h} \oplus \text{Lie}(U(\lambda)),$$

and

$$T_x X = T_x(G.x) \oplus N_x^H \oplus L_x.$$

But, with the notation of Section 5 for the action of  $\lambda$ ,  $T_x C^+ = T_x X_{\geq 0}$ , so

$$T_x C^+ \simeq \mathfrak{g}^\lambda/\mathfrak{h} \oplus N_x^{G_x} \oplus \text{Lie}(U(\lambda)) \oplus L_x.$$

On the other hand, we have the exact sequence

$$0 \longrightarrow T_x C^+ \longrightarrow T_{[e:x]}(G \times_{P(\lambda)} C^+) \longrightarrow T_e(G/P(\lambda)) \simeq \text{Lie}(U(\lambda^{-1})) \longrightarrow 0.$$

In particular,  $T_x X$  and  $T_{[e:x]}(G \times_{P(\lambda)} C^+)$  have the same dimension. Moreover, the image of  $T_{[e:x]}\eta$  contains  $T_x C^+$  and so,  $N_x^{G_x} \oplus L_x$ . Since the image of  $T_{[e:x]}\eta$  contains  $T_x(G.x)$ , we deduce that  $T_{[e:x]}\eta$  is surjective. Finally,  $T_{[e:x]}\eta$  is a linear isomorphism.

It remains to prove that with one of the additional assumption of Assertions 3 and 4,  $\eta$  is generically injective. Let  $z \in C \cap \pi^{-1}((X^{\text{ss}}(\mathcal{L})//G)_{\text{pr}})$ . Let  $y \in C^+$  such that  $\lim_{t \rightarrow 0} \lambda(t)y = z$ . Since  $\eta$  is  $G$ -equivariant, it is sufficient to prove that  $\eta^{-1}(\eta(y)) = \{y\}$ . Let  $g \in G$  such that  $g.y \in C^+$ ; we have to prove that  $g \in P(\lambda)$ .

Let  $T$  be a maximal torus of  $G$  contained in the intersection  $P(\lambda) \cap P(g^{-1}.\lambda.g)$ . Since maximal tori of  $P(\lambda)$  are  $P(\lambda)$ -conjugated there exists  $u_1 \in P(\lambda)$  such that  $\lambda_1 := u_1 \lambda u_1^{-1}$  is one parameter subgroup of  $T$ . Moreover,  $P(\lambda) = U(\lambda)G^\lambda$ , and we may assume that  $u_1$  belongs to  $U(\lambda)$ . In the same way, there exists  $u_2 \in U(g^{-1}.\lambda.g)$  such that  $\lambda_2 := u_2 g^{-1} \lambda g u_2^{-1}$  is a one parameter subgroup of  $T$ .

We have:

$$\lambda_1(t).y = u_1(\lambda(t)u_1^{-1}\lambda(t^{-1}))\lambda(t).x \xrightarrow{t \rightarrow 0} u_1 z =: z_1, \text{ and}$$

$$\lambda_2(t).y = u_2 \left[ (g^{-1}\lambda(t)g)u_2^{-1}(g^{-1}\lambda(t^{-1})g) \right] g^{-1}\lambda(t)gy \xrightarrow{t \rightarrow 0} u_2 g^{-1} z' =: z_2,$$

where  $z' = \lim_{t \rightarrow 0} \lambda(t)gy \in C$ .

We claim that  $z_1 = z_2$ . Before proving the claim we prove the following properties of the orbits and the isotropies of  $z_1$  and  $z_2$ :  $G.z_1 = G.z_2$ ,  $G_{z_1}^\circ$  and  $G_{z_2}^\circ$  are contained in  $T$ .

Since  $\lim_{t \rightarrow 0} \lambda(t)y = z$ ,  $z$  belongs to the closure of  $G.y$ . But  $X^{\text{us}}(\mathcal{L})$  is closed in  $X$  and  $z \notin X^{\text{ss}}(\mathcal{L})$ ; so,  $y$  is semistable for  $\mathcal{L}$ . Since  $gy \in C^+$ ,  $\mu^{\mathcal{L}}(gy, \lambda) = 0$ ; so, Lemma 3 shows that  $z'$  is semistable for  $\mathcal{L}$ . Since  $z' \in C$  the dimension of  $G.z'$  is less than the dimension of  $G.z$ . But,  $G.z$  is the only closed orbit in  $X^{\text{ss}}(\mathcal{L}) \cap \overline{G.y}$ ; we deduce that  $G.z = G.z'$  and  $G.z_1 = G.z_2$ .

By assumption,  $G_z^\circ = H^\circ$  is contained in the center of  $G^\lambda$ . Making  $u_1$  acting, we obtain that  $G_{z_1}^\circ$  is contained in the center of  $G^{\lambda_1}$  and in particular in  $T$ .

Since  $\lambda_1$  and  $\lambda_2$  are one parameter subgroups of  $T$  conjugated in  $G$ , Lemma 2.8 in [MFK94] shows that there exists  $w$  in the normalizer of  $T$  in  $G$  such that  $\lambda_2 = w\lambda_1 w^{-1}$ . Since  $X^\lambda = X^{H^\circ}$ ,  $X^{\lambda_1} = X^{G_{z_1}^\circ}$  and  $X^{\lambda_2} = X^{wG_{z_1}^\circ w^{-1}}$ . In particular,  $z_2$  is fixed by  $wG_{z_1}^\circ w^{-1}$  which is a subtorus of  $T$ . Because of dimension,  $G_{z_2}^\circ = wG_{z_1}^\circ w^{-1}$ ; and so is contained in  $T$ .

Now, we will prove that  $\mathcal{P}_T^\mathcal{L}(z_1) = \mathcal{P}_T^\mathcal{L}(z_2)$ . Since,  $z_1$  is semisimple for  $\mathcal{L}$  and  $G_{z_1}^\circ$  is contained in  $T$ , Lemma 4 shows that 0 belongs to the relative interior of  $\mathcal{P}_T^\mathcal{L}(z_1)$ .

On the other hand,  $\mathcal{P}_T^\mathcal{L}(z_2)$  is a face of  $\mathcal{P}_T^\mathcal{L}(\text{St}_T(y))$  containing 0 since  $z_2$  is semistable. Since  $G_{z_2}^\circ$  is a subtorus of  $T$  conjugated to  $G_{z_1}^\circ$ ,  $\mathcal{P}_T^\mathcal{L}(z_1)$  and  $\mathcal{P}_T^\mathcal{L}(z_2)$  have the same dimension.

We can deduce that  $\mathcal{P}_T^{\mathcal{L}}(z_1) = \mathcal{P}_T^{\mathcal{L}}(z_2)$ .

Replacing  $\mathcal{L}$  by  $\mathcal{L}^{\otimes n}$  if necessary, we assume that there exists a  $T$ -module  $V$  such that  $X \subset \mathbb{P}(V)$  and  $\mathcal{L} = \mathcal{O}(1)|_X$ . Then,  $\text{Conv}(\text{St}_T(\bullet)) = \mathcal{P}_T^{\mathcal{L}}(\bullet)$  by the proof of Proposition 2. Recall that  $\lambda_1$  and  $\lambda_2$  are two one parameter subgroups of  $T$  such that  $z_i = \lim_{t \rightarrow 0} \lambda_i(t)y$ , for  $i = 1, 2$ . A classical fact about toric varieties (see [Oda88]) applied to  $\overline{T.y}$  shows that  $\mathcal{P}_T^{\mathcal{L}}(z_1) = \text{Conv}(\text{St}_T(z_1)) = \text{Conv}(\text{St}_T(z_2)) = \mathcal{P}_T^{\mathcal{L}}(z_2)$  implies that  $z_1 = z_2$ .

Now, we claim that  $\lambda_1 = \lambda_2$  if  $G_y$  is finite and  $H^\circ$  is a one dimensional torus. Since  $G_y$  is finite, the interior of  $\mathcal{P}_T^{\mathcal{L}}(y)$  is non empty in  $X(T)_{\mathbb{R}}$ . Since  $H^\circ$  has dimension one, so has  $G_{z_1}$  and so  $\mathcal{P}_T^{\mathcal{L}}(z_1)$  is a face of codimension one of  $\mathcal{P}_T^{\mathcal{L}}(y)$ . It follows that  $\lambda_1$  and  $\lambda_2$  are orthogonal to  $\mathcal{P}_T^{\mathcal{L}}(z_1)$  and exiting from  $\mathcal{P}_T^{\mathcal{L}}(y)$ . So, there exists positives integers  $n_1$  and  $n_2$  such that  $n_1\lambda_1 = n_2\lambda_2$ . But,  $\lambda_1$  and  $\lambda_2$  are conjugated in  $G$ , so  $n_1 = n_2$ .

We now claim that  $\lambda_1 = \lambda_2$  if  $X = Y \times G/B$ .

We have:

$$\begin{aligned} \lambda_2 &= w\lambda_1w^{-1} \\ &= u_2g^{-1}\lambda gu_2^{-1} = u_2g^{-1}u_1^{-1}\lambda_1u_1gu_2^{-1}. \end{aligned}$$

So,  $w^{-1}u_2g^{-1}u_1^{-1} \in G^{\lambda_1}$ .

Set  $C_1 = u_1.C$ ; it is an irreducible component of  $X^{\lambda_1}$ . Recall that  $z' = gu_2^{-1}.z_2 = gu_2^{-1}.z_1 \in C$ ; and so,  $u_1gu_2^{-1}.z_1 \in C_1$  which is stable by  $G^{\lambda_1}$ . We deduce that  $w^{-1}u_2g^{-1}u_1^{-1}.ugu_2^{-1}.z_1 = w^{-1}.z_1 \in C_1$ .

Write  $C_1 = C_Y \times C'$  with  $C_Y$  (resp.  $C'$ ) an irreducible component of  $Y^{\lambda_1}$  (resp.  $(G/B)^{\lambda_1}$ ). Since  $w^{-1}.z_1$  and  $z_1$  belong to  $C_1$ ,  $C' \cap w^{-1}.C'$  is not empty. But this intersection is closed and  $T$ -stable; and so contains a fix point  $w_0B/B$  of  $T$  (with  $w_0 \in N_G(T)$ ). So,  $ww_0B/B$  and  $w_0B/B$  are fix points of  $T$  in  $C'$ . It follows that there exists  $w' \in N_{G_1}^{\lambda_1}(T)$  such that  $ww_0 = w'w_0$  modulo  $T$ ; and so,  $w = w'$  modulo  $T$ . Since  $\lambda_1$  is a one parameter subgroup of  $T$ , it follows that  $\lambda_2 = w\lambda_1w^{-1} = w'\lambda_1w'^{-1} = \lambda_1$ .

Now, we can prove that  $g \in P(\lambda)$  in the two last assertions of the theorem:

$$\begin{aligned} \lambda(t)g\lambda(t^{-1}) &= g.(g^{-1}\lambda(t)g)\lambda(t^{-1}) \\ &= g(u_2^{-1}\lambda_2(t)u_2).(u_1^{-1}\lambda_1(t^{-1})u_1) \\ &= gu_2^{-1}(\lambda_1(t)(u_2u_1^{-1})\lambda_1(t^{-1}))u_1 \end{aligned} \quad \text{since } \lambda_1 = \lambda_2,$$

which tends to  $gu_2^{-1}u$  when  $t$  tends to 0 since  $u_2u_1^{-1} \in U(\lambda_1) = U(\lambda_2)$ . In particular,  $\lambda(t)g\lambda(t^{-1})$  has a limit when  $t$  tends to 0; so,  $g$  belongs to  $P(\lambda)$ . This ends the proof of the theorem.  $\square$

### 9.3 Application

Using Proposition 12, one can make Proposition 11 more precise in the case when  $X = Y \times G/B$ .

**Theorem 7** *Let us fix a maximal torus  $T$  of  $G$  and a Borel subgroup  $B$  containing  $T$ . Let  $X = Y \times G/B$  where  $Y$  is a smooth  $G$ -variety. Let  $\Lambda$  be an abundant subgroup of  $\text{Pic}^G(X)$ . We assume that  $\dim(\mathcal{C}_\Lambda^G(X)) = \dim(\Lambda_\mathbb{Q})$ .*

1. *Let  $\mathcal{F}$  be a face of codimension  $d$  of  $\mathcal{C}_\Lambda^G(X)$ .*

*Then, there exists a generic closed isotropy  $H$  of  $\mathcal{F}$  such that:*

- (a)  *$H$  is a diagonalizable subgroup of  $T$  of dimension  $d$ ,*
- (b) *the interior of  $\mathfrak{h}_X^+ \cap \mathfrak{t}^+$  in  $\mathfrak{h}_\mathbb{R}$  is not empty,*

*Moreover, if  $C$  is an irreducible components of  $Y_{\mathcal{F}}$ , we have:*

- (c)  *$H = G_C$ .*
- (d) *for all  $\lambda \in X_*(H) \cap \mathfrak{h}_X^+$  the pair  $(C, \lambda)$  is well covering,*
- (e) *The face  $\mathcal{F}$  is the intersection of  $\mathcal{C}_\Lambda^G(X)$  and the linear subspace of  $\Lambda_\mathbb{Q}$  spanned by the kernel of  $r_\Lambda^{C,H}$ .*
- (f) *For all  $\mathcal{L}$  in the relative interior of  $\mathcal{F}$ ,  $C$  contains stable points for the action of the group  $G^{H^\circ}/H$  and the  $G^{H^\circ}/H$ -linearized line bundle  $\mathcal{L}|_C$  on  $C$ .*

2. *Conversely, let  $(C, \lambda)$  be a well covering pair with a dominant one parameter subgroup  $\lambda$  of  $T$  such that  $H := G_C^\circ$  is a torus of dimension  $d$ . We assume that there exists  $\mathcal{L} \in \Lambda_\mathbb{Q}^+$  such that  $H$  acts trivially on  $\mathcal{L}|_C$  and  $\mathcal{L}|_C$  belongs to the interior of  $\mathcal{C}^{G^{H^\circ}/H}(C)$  in  $\text{Pic}^{G^{H^\circ}/H}(C)_\mathbb{Q}$ .*

*Then, the intersection of  $\mathcal{C}_\Lambda^G(X)$  and the subspace spanned by the kernel of  $r_\Lambda^{C,H}$  is a face of  $\mathcal{C}_\Lambda^G(X)$  of codimension  $d$ .*

**Proof.** Let  $\mathcal{F}$  be a face of codimension  $d$  of  $\mathcal{C}_\Lambda^G(X)$ . Let  $H$  be a generic closed isotropy of  $\mathcal{F}$ . Since  $H$  is reductive and contained in a Borel subgroup of  $G$ , it is diagonalizable; so, changing  $H$  be a conjugate if necessary, we may assume that  $H$  is contained in  $T$ . But,  $\mathfrak{t}^+$  is a fundamental domain of the

action of  $W$  on  $\mathfrak{t}$ . So, conjugating  $H$  by an element of  $W$  if necessary, we obtain a  $H$  which satisfies Assertions 1a and 1b.

Let  $\mathcal{L}$  be a point in the relative interior of  $\mathcal{F}$ . By construction and by Proposition 7, each irreducible component of  $Y_{\mathcal{F}}$  contains semistable points for the action of  $N_G(H)$ . But, by Theorem 6,  $Y^{\text{ss}}(\mathcal{L})//N_G(H)$  is irreducible. We deduce that  $N_G(H)$  acts transitively on the set irreducible component of  $Y_{\mathcal{F}}$ . Now, Assertions 1c, 1e and 1f follow immediately from Theorem 6.

Proposition 12 implies Assertion 1d and ends the proof of the first part of the theorem.

Conversely, let  $(C, \lambda)$  be a well covering pair and  $\mathcal{L} \in \Lambda_{\mathbb{Q}}^+$  as in Assertion 2 of the theorem. Consider the restriction morphism  $r : \Lambda \rightarrow \text{Pic}^{G^\lambda}(C)$  and the morphism  $\chi_C^\bullet : \text{Pic}^{G^\lambda}(C) \rightarrow X(H)$  given by the action of  $H$  on the fibers of the line bundles. The kernel of  $\chi_C^\bullet$  canonically identifies with  $\text{Pic}^{G^\lambda/H}(C)$ . Let  $\mathcal{K}$  denote the kernel of  $r^{C,H}$  and  $\mathcal{K}_{\mathbb{Q}}$  the subspace spanned by  $\mathcal{K}$ . So,  $r$  induces a morphism:

$$\bar{r} : \Lambda \cap \mathcal{K} \rightarrow \text{Pic}^{G^\lambda}(C).$$

Moreover, by Lemma 2, the cone  $\mathcal{C}_{r(\Lambda)}^{G^\lambda}(C)$  is contained in the subspace generated by the kernel of  $\chi_C^\bullet$ . So, the  $\mathbb{Q}$ -linear map  $\bar{r}_{\mathbb{Q}}$  induced by  $\bar{r}$  maps  $\mathcal{C}_{\Lambda}^G(X)$  on  $\mathcal{C}_{r(\Lambda)}^{G^\lambda/H}(C)$ .

Since  $\mathcal{L}|_C$  belongs to the interior of  $\mathcal{C}_{r(\Lambda)}^{G^\lambda/H}(C)$ , there exists an open neighborhood  $\Omega$  of  $\mathcal{L}$  in the intersection of  $\mathcal{C}_{\Lambda}^G(X)$  and  $\mathcal{K}_{\mathbb{Q}}$  such that  $C^{\text{ss}}(\mathcal{M}|_C)$  for the action of  $G^\lambda$  is not empty for all  $\mathcal{M} \in \Omega$ . By Proposition 7, this implies that  $\Omega \subset \mathcal{C}_{\Lambda}^G(X)$ . In particular, the intersection of  $\mathcal{C}_{\Lambda}^G(X)$  and  $\mathcal{K}_{\mathbb{Q}}$  has codimension  $d$ .

For any (rational) one parameter subgroup  $\lambda'$  of  $H$  closed to  $\lambda$ ,  $C$  is again an irreducible component of  $X^{\lambda'}$  and  $C^+ = \{x \in X : \lim_{t \rightarrow 0} \lambda'(t)x \in C\}$ . Since  $G.C^+$  is dense in  $X$ , this implies that  $\mathcal{C}_{\Lambda}^G(X)$  is contained in  $\mu^\bullet(C, \lambda') \leq 0$ . Since the images of such  $\lambda'$  generate  $G_C^\lambda$ , we deduce that the intersection of  $\mathcal{C}_{\Lambda}^G(X)$  and  $\mathcal{K}_{\mathbb{Q}}$  is a face of  $\mathcal{C}_{\Lambda}^G(X)$ .  $\square$

## 10 The case $X = \hat{G}/\hat{B} \times G/B$

### 10.1 Notation

From now on, we assume that  $G$  is a reductive subgroup of a semisimple group  $\hat{G}$ . Let us fix maximal tori  $T$  (resp.  $\hat{T}$ ) and Borel subgroups  $B$  (resp.



$\hat{B}$ ) of  $G$  (resp.  $\hat{G}$ ) such that  $T \subset B \subset \hat{B} \supset \hat{T} \supset T$ .

From now on,  $X$  denote the variety  $\hat{G}/\hat{B} \times G/B$  endowed with the diagonal action of  $G$ . We will apply Theorem 7 in this case and with  $\Lambda = \text{Pic}^G(X)$ . The cone  $\mathcal{C}_\Lambda^G(X)$  is simply denoted by  $\mathcal{C}^G(X)$ . Let us introduce some notation before.

The homomorphism  $X(\hat{T}) \times X(T) \longrightarrow \text{Pic}^G(X)$ ,  $(\hat{\nu}, \nu) \longmapsto \mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_\nu$  induces an isomorphism  $X(\hat{T})_{\mathbb{Q}} \times X(T)_{\mathbb{Q}} \longrightarrow \text{Pic}^G(X)_{\mathbb{Q}}$ . Let  $\lambda$  be a one parameter subgroup of  $T$  and so of  $\hat{T}$ . We denote with a  $\hat{\cdot}$  the objects associated to  $\hat{G}$ ;  $\hat{W}$  is the Weyl group of  $\hat{G}$ ,  $\hat{R} \subset X(\hat{T})$  is the set of roots of  $\hat{G}$ , etc. . .

## 10.2 Fix points of one parameter subgroups

Let  $\lambda$  be a one parameter subgroup of  $T$  and so of  $\hat{T}$ . We can describe the fix point set  $X^\lambda$ :

$$X^\lambda = \bigcup_{\substack{\hat{w} \in \hat{W}_\lambda \backslash \hat{W} \\ w \in W_\lambda \backslash W}} \hat{G}^\lambda \hat{w} \hat{B} / \hat{B} \times G^\lambda w B / B.$$

We will denote by  $C(\hat{w}, w)$  the component  $\hat{G}^\lambda \hat{w} \hat{B} / \hat{B} \times G^\lambda w B / B$  of  $X^\lambda$ .

Let  $Z_\lambda$  denote the connected center of  $G^\lambda$ ; it is a subtorus of  $T$  and it acts trivially on each  $C(\hat{w}, w)$ . So, the action of  $Z_\lambda$  on the fibers define a morphism

$$\chi_C^\bullet : \text{Pic}^{G^\lambda}(C(\hat{w}, w)) \longrightarrow X(Z_\lambda)$$

of cofinite image. On the other hand, we have the following exact sequence:

$$1 \longrightarrow X(G^\lambda) \longrightarrow \text{Pic}^{G^\lambda}(C(\hat{w}, w)) \longrightarrow \text{Pic}(C(\hat{w}, w)),$$

where the last map is of cofinite image (see [FHT84]). Since the restriction map  $X(G^\lambda)_{\mathbb{Q}} \longrightarrow X(Z_\lambda)_{\mathbb{Q}}$  is an isomorphism, we deduce that the preceding morphism induces a linear isomorphism:

$$\text{Pic}^{G^\lambda}(C(\hat{w}, w))_{\mathbb{Q}} \longrightarrow X(Z_\lambda)_{\mathbb{Q}} \times \text{Pic}(C(\hat{w}, w))_{\mathbb{Q}}. \quad (1)$$

**Lemma 5** *For any  $(\hat{w}, w)$ , the restriction morphism induces a surjective linear map  $\text{Pic}^G(X)_{\mathbb{Q}} \longrightarrow \text{Pic}^{G^\lambda}(C(\hat{w}, w))_{\mathbb{Q}}$ .*

*Moreover, for  $\chi_C^{(\mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_\nu)|_{C(\hat{w}, w)}} = (\hat{w}\hat{\nu} + w\nu)|_{Z_\lambda}$ .*

**Proof.** The torus  $\hat{T} \times T$  acts on the fiber  $(\hat{w}\hat{B}/\hat{B}, wB/B)$  in  $\mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_{\nu}$  by the character  $(\hat{w}, w)$ . The second assertion follows. Let  $\mathcal{M} \in \text{Pic}^{G^\lambda}(\mathbb{C}(\hat{w}, w))$ . Changing  $\mathcal{M}$  by a power if necessary, one obtains a  $(\hat{G}^\lambda \times G^\lambda)$ -linearization  $\tilde{\mathcal{M}}$  on  $\mathcal{M}$ . Since the restriction to the diagonal  $X(\hat{G}^\lambda) \times X(G^\lambda) \rightarrow X(G^\lambda)$  is surjective, changing this linearization if necessary, one can assume that it extends to those of  $G^\lambda$ . But, we have:  $X(\hat{B}) \times X(B) \simeq \text{Pic}^{\hat{G} \times G}(X) \simeq \text{Pic}^{\hat{G}^\lambda \times G^\lambda}(\mathbb{C}(\hat{w}, w)) \simeq X(\hat{B}^\lambda) \times X(B^\lambda)$ . So,  $\tilde{\mathcal{M}}$  can be extended to an element  $\mathcal{L}$  of  $\text{Pic}^{\hat{G} \times G}(X)$ . By restricting the action on  $\mathcal{L}$  to  $G$  we obtain an element of  $\text{Pic}^G(X)$  which maps to  $\mathcal{M}$ . The lemma is proved.  $\square$

### 10.3 Closure of the ample cone

We denote by  $\overline{\mathcal{C}}^G(X)$  the closure in  $\text{Pic}^G(X)_{\mathbb{Q}}$  of  $\mathcal{C}^G(X)$ . The convex cone  $\overline{\mathcal{C}}^G(X)$  has a simple representation theoretic interpretation:

**Lemma 6** *Assume that  $\mathcal{C}^G(X)$  is not empty.*

*A point  $(\hat{\nu}, \nu) \in X(\hat{T})_{\mathbb{Q}} \times X(T)_{\mathbb{Q}} \simeq \text{Pic}^G(X)_{\mathbb{Q}}$  belongs to  $\overline{\mathcal{C}}^G(X)$  if and only if  $\hat{\nu}$  and  $\nu$  are dominant and for  $n$  big enough  $V_{n\hat{\nu}} \otimes V_{n\nu}$  contains non zero  $G$ -invariant vectors.*

**Proof.** Let  $\mathcal{E}$  denote the set of points  $(\hat{\nu}, \nu)$  in the dominant chamber of  $X(\hat{T})_{\mathbb{Q}} \times X(T)_{\mathbb{Q}}$  such that there exists  $n$  such that  $V_{n\hat{\nu}} \otimes V_{n\nu}$  contains non zero  $G$ -invariant vectors. For any dominant weight  $\hat{\nu}$  (resp.  $\nu$ ), let  $P(\hat{\nu})$  (resp.  $P(\nu)$ ) denote the maximal standard parabolic subgroup of  $G$  such that  $\hat{\nu}$  (resp.  $\nu$ ) can be extended to  $P(\hat{\nu})$  (resp.  $P(\nu)$ ). The line bundle  $\mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_{\nu}$  is the pullback of an ample line bundle (also denoted by  $\mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_{\nu}$ ) by the  $G$ -equivariant map  $\pi : \hat{G}/\hat{B} \times G/B \rightarrow \hat{G}/P(\hat{\nu}) \times G/P(\nu)$ . We denote by  $X^{\text{ss}}(\mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_{\nu})$  the set of the points in  $X$  such that  $\pi(x)$  is semistable for  $\mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_{\nu}$ . Notice that  $X^{\text{ss}}(\mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_{\nu})$  is the open subset of the points  $x \in X$  such that  $\mu^{\mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_{\nu}}(x, \lambda) \leq 0$  for all one parameter subgroup  $\lambda$  of  $G$ . Moreover,  $\mathcal{E}$  is the set of the points  $(\hat{\nu}, \nu)$  in the dominant chamber of  $X(\hat{T})_{\mathbb{Q}} \times X(T)_{\mathbb{Q}}$  such that there exists  $n$  such that  $X^{\text{ss}}(\mathcal{L}_{n\hat{\nu}} \otimes \mathcal{L}_{n\nu})$  is non empty.

Since  $\hat{G} \times G$  contains only a finite number of parabolic subgroups up to conjugacy, and by Theorem 1.3.9 of [DH98] there exists  $x$  in the intersection of the non empty subsets of the form  $X^{\text{ss}}(\mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_{\nu})$ . One easily checks that  $\mathcal{E}$  is the set of points  $(\hat{\nu}, \nu)$  in the dominant chamber of  $X(\hat{T})_{\mathbb{Q}} \times X(T)_{\mathbb{Q}}$  such that  $\mu^{\mathcal{L}_{\hat{\nu}} \otimes \mathcal{L}_{\nu}}(x, \lambda) \leq 0$  for all one parameter subgroup  $\lambda$  of  $G$ . In particular,  $\mathcal{E}$  is a closed convex cone. Since,  $\mathcal{C}^G(X)$  is contained in  $\mathcal{E}$ , so is  $\overline{\mathcal{C}}^G(X)$ .

Let  $p \in \mathcal{E}$  and  $q \in \mathcal{C}^G(X)$ . Each point of the segment  $[p, q]$  except eventually  $p$  belongs to the strictly dominant chamber. Since,  $\mathcal{E}$  is convex these points belong to  $\mathcal{C}^G(X)$ . Then,  $p$  belongs to the closure of  $\mathcal{C}^G(X)$ .  $\square$

#### 10.4 Faces of the ample cone

Now, we can prove a kind of converse of Theorem 7:

**Theorem 8** *Let  $X = \hat{G}/\hat{B} \times G/B$  with above notation. Let  $H$  be subtorus of  $T$  and  $C$  be an irreducible component of  $X^H$  such that  $H = G_C^\circ$ , and for a  $\lambda \in X_*(T) \cap \mathfrak{h}_X^+$  the pair  $(C, \lambda)$  is well covering. Denote by  $d$  the rank of  $H$ .*

*We assume that the interior in  $\text{Pic}^{G^\lambda/H}(C)$  of the cone  $\mathcal{C}^{G^\lambda/H}(C)$  is non empty.*

*Then, the intersection of  $\overline{\mathcal{C}}^G(X)$  and the subspace spanned by the kernel of  $r_\Lambda^{C,H}$  is a face of codimension  $d$  of  $\overline{\mathcal{C}}^G(X)$ .*

**Proof.** Let us consider the restriction map:

$$r : \text{Pic}^G(X) \longrightarrow \text{Pic}^{G^\lambda}(C), \mathcal{L} \longmapsto \mathcal{L}|_C.$$

The cone  $\mathcal{C}^{G^\lambda}(C)$  is contained in the kernel of  $r^{C,H}$  and spans this linear subspace by assumption on  $\mathcal{C}^{G^\lambda/H}(C)$ .

We claim that there exists  $\mathcal{L} \in \overline{\mathcal{C}}^G(X)$  such that  $r(\mathcal{L})$  belongs to the relative interior of  $\mathcal{C}^{G^\lambda}(C)$ .

Let  $\mathcal{M}$  be a  $G^\lambda$ -linearized ample line bundle on  $C$  belonging to the relative interior of  $\mathcal{C}^{G^\lambda}(C)$ . Let  $\sigma$  be a non zero  $G^\lambda$ -invariant section of  $\mathcal{M}$ . By Lemma 5, changing  $\mathcal{M}$  by a power if necessary, we may assume that there exists  $\mathcal{L} \in \text{Pic}^G(X)$  such that  $r(\mathcal{L}) = \mathcal{M}$ .

Let us recall that  $p_\lambda : C^+ \longrightarrow C$  is a vector bundle. Consider also the  $G$ -linearized line bundle  $G \times_{P(\lambda)} p_\lambda^*(\mathcal{M})$  on  $G \times_{P(\lambda)} C^+$ . Since  $\eta^*(\mathcal{L})$  and  $G \times_{P(\lambda)} p_\lambda^*(\mathcal{M})$  have the same restriction to  $C$ , Propositions 1 and 6 show that  $\eta^*(\mathcal{L}) = G \times_{P(\lambda)} p_\lambda^*(\mathcal{M})$ . Moreover, since  $\mu^{\mathcal{M}}(C, \lambda) = 0$ , Proposition 6 shows that  $\sigma$  admits a unique  $P(\lambda)$ -invariant extension to a section  $\sigma'$  of  $p_\lambda^*(\mathcal{M})$ . On the other hand, Proposition 1 shows that  $\sigma'$  admits a unique  $G$ -invariant extension  $\tilde{\sigma}$  from  $C^+$  to  $G \times_{P(\lambda)} C^+$ . So, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{L} & \longleftarrow & \eta^*(\mathcal{L}) = G \times_{P(\lambda)} p_\lambda^*(\mathcal{M}) & \longleftarrow & p_\lambda^*(\mathcal{M}) & \longrightarrow & \mathcal{M} \\
\downarrow & & \downarrow & \uparrow & \tilde{\sigma} & & \downarrow & \uparrow & \sigma \\
X & \xleftarrow{\eta} & G \times_{P(\lambda)} C^+ & \xleftarrow{\quad} & C^+ & \xrightarrow{p_\lambda} & C & \uparrow & \sigma
\end{array}$$

Since  $\eta$  is birational,  $\tilde{\sigma}$  descends to a rational  $G$ -invariant section  $\tau$  of  $\mathcal{L}$ . Write  $\text{div}(\tau) = \sum n_i D_i - \sum m_i E_i$ , where  $D_i$  and  $E_i$  are prime  $G$ -invariant divisors of  $X$  and  $n_i$  and  $m_i$  are positive integers. Consider the  $G$ -linearized line bundle  $\mathcal{O}(\sum m_i E_i)$  associated to the divisor  $\sum m_i E_i$ . The section  $\tau$  induces a regular  $G$ -invariant section  $\tau'$  of  $\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}(\sum m_i E_i)$ . By Lemma 6, the claim will be proved if  $r(\mathcal{L}')$  belongs to the relative interior of  $\mathcal{C}^{G^\lambda}(C)$ .

The  $G$ -linearized line bundle  $\mathcal{O}(\sum m_i E_i)$  has a regular  $G$ -invariant section which is non zero at any point of  $X - \cup_i E_i$ . Since  $(C, \lambda)$  is well covering no  $E_i$  contains  $C$ . It follows that  $r(\mathcal{O}(\sum m_i E_i))$  contains non zero  $G^\lambda$ -invariant sections; and by Lemma 6, that  $r(\mathcal{O}(\sum m_i E_i))$  belongs to the closure of  $\mathcal{C}^{G^\lambda}(C)$ . By an obvious argument of convexity it follows that  $r(\mathcal{L}') = \mathcal{M} \otimes r(\mathcal{O}(\sum m_i E_i))$  belongs to the relative interior of  $\mathcal{C}^{G^\lambda}(C)$ .

The claim proves that  $\overline{\mathcal{C}}^G(X)$  intersects the kernel of  $r^{C,H}$ . An analogous proof shows that the intersection of  $\overline{\mathcal{C}}^G(X)$  and the kernel of  $r^{C,H}$  spans this kernel. Indeed, since  $r(\mathcal{L}')$  belongs to the relative interior of  $\mathcal{C}^{G^\lambda}(C)$ , there exists  $G$ -linearized line bundles  $\mathcal{L}_j$  on  $X$  which span  $r^{-1}(\mathcal{C}^{G^\lambda}(C))$  and such that the  $\mathcal{M}_j := r(\mathcal{L}_j)$  belong to the relative interior of  $\mathcal{C}^{G^\lambda}(C)$ . As above, we prove that there exists a  $G$ -stable divisor  $\sum m_i E_i$  such that no  $E_i$  contains  $C$ , and  $\mathcal{L}'_j := \mathcal{L}_j \otimes \mathcal{O}(\sum m_i E_i)$  belongs to  $\overline{\mathcal{C}}^G(X)$  for all  $j$ . Moreover, the  $\mathcal{L}'_j$ 's belong to the kernel of  $r^{C,H}$ . Finally, the intersection of  $\overline{\mathcal{C}}^G(X)$  and the kernel of  $r^{C,H}$  is of codimension  $d$ .  $\square$

## 10.5 Well covering pairs

In this subsection, we explain how to find the well covering pairs in the case when  $X = \hat{G}/\hat{B} \times G/B$ .

### 10.5.1 Birationality of $\eta$

Let  $\lambda$  be a dominant one parameter subgroup of  $T$ . We consider the cohomology ring  $H^*(G/P(\lambda), \mathbb{Z})$  of  $G/P(\lambda)$ . Here, we use simplicial cohomology

with integers coefficients. If  $w \in W/W_\lambda$ , we set  $\Lambda_w = BwP(\lambda)/P(\lambda)$  and denote by  $[\Lambda_w]$  the class in  $H^*(G/P(\lambda), \mathbb{Z})$  of the closure in  $G/P(\lambda)$  of  $\Lambda_w$ . Let us recall that

$$H^*(G/P(\lambda), \mathbb{Z}) = \bigoplus_{w \in W/W_\lambda} \mathbb{Z}[\Lambda_w].$$

We use similar notation for  $\hat{G}/\hat{P}(\lambda)$ . Since  $P(\lambda) = G \cap \hat{P}(\lambda)$ ,  $G/P(\lambda)$  identifies with the orbit by  $G$  of  $\hat{P}(\lambda)/\hat{P}(\lambda)$  in  $\hat{G}/\hat{P}(\lambda)$ ; let  $\iota : G/P(\lambda) \rightarrow \hat{G}/\hat{P}(\lambda)$  denote this closed immersion. The map  $\iota$  induces a map  $\iota^*$  in cohomology:

$$\iota^* : H^*(\hat{G}/\hat{P}(\lambda), \mathbb{Z}) \rightarrow H^*(G/P(\lambda), \mathbb{Z}).$$

Let  $\hat{w} \in \hat{W}$  and  $w \in W$ . Consider the map:

$$\eta : G \times_{P(\lambda)} C^+(\hat{w}, w) \rightarrow X.$$

The following proposition gives a criterion in terms of cohomology for  $\eta$  being birational.

**Proposition 13** *With above notation, the following are equivalent:*

1. the map  $\eta$  is birational,
2.  $\iota^*([\Lambda_{\hat{w}^{-1}}]) \cdot [\Lambda_{w^{-1}}] = [\Lambda_e]$ .

**Proof.** Set  $P = P(\lambda)$  and  $\hat{P} = \hat{P}(\lambda)$ . Since the characteristic of  $\mathbb{K}$  is zero,  $\eta$  is birational if and only if for  $x$  in an open subset of  $X$ ,  $\eta^{-1}(x)$  is reduced to a point. Consider the projection  $\pi : G \times_P C^+(\hat{w}, w) \rightarrow G/P$ . For any  $x$  in  $X$ ,  $\pi$  induces an isomorphism from  $\eta^{-1}(x)$  onto the following locally closed subvariety of  $G/P$ :  $F_x := \{gP \in G/P : g^{-1}x \in C^+(\hat{w}, w)\}$ .

Let  $(\hat{g}, g) \in \hat{G} \times G$  and set  $x = (\hat{g}\hat{B}/\hat{B}, gB/B) \in X$ . We have:

$$\begin{aligned} F_x &= \{hP/P \in G/P : h^{-1}gB/B \in PwB/B \text{ and } h^{-1}\hat{g}\hat{B}/\hat{B} \in \hat{P}\hat{w}\hat{B}/\hat{B}\} \\ &= \{hP/P \in G/P : h^{-1} \in PwBg^{-1} \text{ and } h^{-1} \in \hat{P}\hat{w}\hat{B}\hat{g}^{-1}\} \\ &= \{hP/P \in G/P : h \in (gBw^{-1}P) \cap (\hat{g}\hat{B}\hat{w}^{-1}\hat{P})\} \\ &= g \cdot \Lambda_{w^{-1}} \cap \hat{g} \hat{\Lambda}_{\hat{w}^{-1}}, \end{aligned}$$

where  $g \cdot \Lambda_{w^{-1}}$  is identified to a part of  $\hat{G}/\hat{P}$  using  $\iota$ .

Let us fix  $g$  arbitrarily. By Kleiman's Theorem (see [Kle74]), there exists an open subset of  $\hat{g}$ 's in  $\hat{G}$  such that the intersection  $g \cdot \overline{\Lambda_{w^{-1}}} \cap \hat{g} \hat{\Lambda}_{\hat{w}^{-1}}$  is transversal. Moreover (see for example [BK06]), one may assume that  $g \cdot \Lambda_{w^{-1}} \cap \hat{g} \hat{\Lambda}_{\hat{w}^{-1}}$  is dense in  $g \cdot \overline{\Lambda_{w^{-1}}} \cap \hat{g} \hat{\Lambda}_{\hat{w}^{-1}}$ . We deduce that the following are equivalent:

1. for generic  $\hat{g}$ ,  $F_x$  is reduced to a point,
2.  $\iota^*([\Lambda_{\hat{w}^{-1}}]).[\Lambda_{w^{-1}}] = [\Lambda_e]$ .

Since  $\eta$  is  $G$ -equivariant, the above Condition 1 is clearly equivalent to the fact that  $\eta$  is birational.  $\square$

### 10.5.2 The rank of $G_{C(\hat{w},w)}$

As above,  $\lambda$  is a dominant one parameter subgroup of  $T$ ,  $\hat{w} \in \hat{W}$  and  $w \in W$ . By Theorem 7, the subgroup  $G_{C(\hat{w},w)}$  play an important role in the description of the faces of  $\mathcal{C}_\Lambda^G(X)$ . In Lemma 7 bellow, we will describe the Lie algebra of this group.

Let  $r_T : X(\hat{T}) \rightarrow X(T)$  denote the restriction morphism. We have the following:

**Lemma 7** *With above notation, the Lie algebra of  $G_{C(\hat{w},w)}$  is the intersection in  $\mathfrak{t}$  of the Kernels of the  $r_T(\hat{\alpha})$  for  $\hat{\alpha} \in \hat{R}$  such that  $\langle \lambda, r_T(\hat{\alpha}) \rangle = 0$ . In particular, it only depends on  $\lambda$ ; this Lie subalgebra of  $\mathfrak{t}$  will be denoted by  $\mathfrak{t}_\lambda$ .*

**Proof.** Since  $C(\hat{w}, w) = \hat{G}^\lambda \hat{w} \hat{B} / \hat{B} \times G^\lambda w B / B$ ,  $G_{C(\hat{w},w)}$  is the intersection of the centers of  $G^\lambda$  and  $\hat{G}^\lambda$ . In particular, it is contained in the torus  $T$ . Moreover, its Lie algebra equals

$$\bigcap_{\substack{\alpha \in R \\ \langle \lambda, \alpha \rangle = 0}} \ker(\alpha) \cap \bigcap_{\substack{\hat{\alpha} \in R \\ \langle \lambda, \hat{\alpha} \rangle = 0}} \ker(\hat{\alpha}),$$

where  $\ker(\alpha)$  (resp.  $\ker(\hat{\alpha})$ ) is the Lie subalgebra of  $\mathfrak{t}$  (resp.  $\hat{\mathfrak{t}}$ ) of the Kernel of  $\alpha$  (resp.  $\hat{\alpha}$ ). Since  $r_T(\hat{R})$  contains  $R$ , this intersection equals those of the lemma.  $\square$

### 10.5.3 The tangent map to $\eta$

Let us fix again a dominant one parameter subgroup  $\lambda$  of  $T$  and  $\hat{w} \in \hat{W}$  and  $w \in W$ .

To simplify notation, we set  $P = P(\lambda)$ ,  $C = C(\hat{w}, w)$  and  $C^+ = C^+(\hat{w}, w)$ . Consider

$$\eta : G \times_P C^+ \rightarrow X = \hat{G} / \hat{B} \times G / B.$$

Consider the restriction of  $T\eta$  to  $C^+$ :

$$T\eta|_{C^+} : \mathcal{T}(G \times_P C^+)|_{C^+} \longrightarrow \mathcal{T}(X)|_{C^+},$$

and the restriction of  $\mathcal{D}et\eta$  to  $C^+$ :

$$\mathcal{D}et\eta|_{C^+} : \mathcal{D}et(G \times_P C^+)|_{C^+} \longrightarrow \mathcal{D}et(X)|_{C^+}.$$

Since  $\eta$  is  $G$ -equivariant, the morphism  $\mathcal{D}et\eta|_{C^+}$  is  $P$ -equivariant; it can be thought as a  $P$ -invariant section of the line bundle  $\mathcal{D}et(G \times_P C^+)^*|_{C^+} \otimes \mathcal{D}et(X)|_{C^+}$  over  $C^+$ . We denote by  $\mathcal{L}_{P,\hat{w},w}$  this last  $P$ -linearized line bundle on  $C^+$ .

To study the line bundle  $\mathcal{L}_{P,\hat{w},w}$ , we need to introduce notation:

$$\chi_{w^{-1}} = \rho - w\rho + 2\rho_\lambda \text{ and } \chi_{\hat{w}^{-1}} = \hat{\rho} - \hat{w}\hat{\rho} + 2\hat{\rho}_\lambda.$$

**Lemma 8** *With above notation, we have:*

1. *If  $\eta$  is birational then  $\mathcal{D}et\eta|_{C^+}$  is a non zero  $P$ -invariant section of  $\mathcal{L}_{P,\hat{w},w}$ .*
2. *We assume that  $(\hat{w}, w) \in \hat{W}^\lambda \times W^\lambda$ . The torus  $T$  acts on the fiber over the point  $(\hat{w}\hat{B}/\hat{B}, wB/B)$  in  $\mathcal{L}_{P,\hat{w},w}$  by the character  $\rho - w\rho + r_T(\hat{\rho} - \hat{w}\hat{\rho} + 2\hat{\rho}_\lambda) = r_T(\chi_{\hat{w}^{-1}}) + \chi_{w^{-1}} - \chi_1$ .*

**Proof.** If  $\eta$  is birational,  $\mathcal{D}et\eta$  is  $G$ -equivariant and non zero. It follows that  $\mathcal{D}et\eta|_{C^+}$  is  $P$ -equivariant and non zero. This proves the first assertion.

We claim that

$$\sum_{\alpha \in \text{St}_T(\mathfrak{p}/\mathfrak{wb})} \alpha = \rho - w\rho - 2\rho_\lambda. \quad (2)$$

Indeed,

$$\begin{aligned} \gamma_w := \sum_{\alpha \in \text{St}_T(\mathfrak{p}/\mathfrak{wb})} \alpha &= \sum_{\alpha \in (R^+ \cup -R_\lambda^+) \cap (-wR^+)} \alpha \\ &= \sum_{\alpha \in R^+ \cap (-wR^+)} \alpha - \sum_{\alpha \in R_\lambda^+ \cap (wR^+)} \alpha. \end{aligned}$$

But, by Lemma 1.3.2.2 of [Kum02] we have  $\sum_{\alpha \in R^+ \cap (-wR^+)} \alpha = \rho - w\rho$ . Moreover, our assumption about  $w$  implies that  $R_\lambda^+ \subset wR^+$ , and  $\sum_{\alpha \in R_\lambda^+ \cap (wR^+)} \alpha = \sum_{\alpha \in R_\lambda^+} \alpha = 2\rho_\lambda$ . Formula (2) follows.

Set  $x = (\hat{w}\hat{B}/\hat{B}, wB/B)$ . Let  $\chi$  denote the character of the action of  $T$  on the fiber over  $x$  in  $\mathcal{L}_{P,\hat{w},w}$ . As a  $T$ -module, the tangent space  $T_x(G \times_P C^+)$

is isomorphic to  $\mathfrak{g}/\mathfrak{p} \oplus T_x C^+$ ; and  $T_x X$  is isomorphic to  $T_x X/T_x C^+ \oplus T_x C^+$ . Thus,  $\chi$  equal the sum of the weights of  $T$  on  $\mathfrak{g}/\mathfrak{p}$  minus the sum of the weights of  $T$  on  $T_x X/T_x C^+$ . So, we obtain that:

$$\begin{aligned} \chi &= \sum_{\alpha \in \text{St}_T(T_x X)} \alpha - \sum_{\alpha \in \text{St}_T(T_x C^+)} \alpha - \sum_{\alpha \in R^+} \alpha + \sum_{\alpha \in R_\lambda^+} \alpha \\ &= \sum_{\alpha \in wR^+} \alpha + r_T \left( \sum_{\hat{\alpha} \in \hat{w}\hat{R}^+} \alpha - \sum_{\hat{\alpha} \in \text{St}_T(\hat{\mathfrak{p}}/\hat{w}\hat{\mathfrak{b}})} \hat{\alpha} \right) - \sum_{\alpha \in \text{St}_T(\mathfrak{p}/w\mathfrak{b})} \alpha - 2\rho + 2\rho_\lambda. \end{aligned}$$

Now, Assertion 2 follows immediately from Formula (2).  $\square$

#### 10.5.4 Well covering pairs

The following theorem describes the well covering pairs:

**Theorem 9** *Let  $\lambda$  be a dominant one parameter subgroup of  $T$ . Let  $(\hat{w}, w) \in \hat{W}^\lambda \times W^\lambda$ .*

*The following are equivalent:*

1. *The pair  $(C(\hat{w}, w), \lambda)$  is well covering.*
2.  *$\iota^*([\Lambda_{\hat{w}^{-1}}]) \cdot [\Lambda_{w^{-1}}] - [\Lambda_e] = 0$  and the restriction of  $r_T(\chi_{\hat{w}^{-1}}) + \chi_{w^{-1}} - \chi_1$  to  $\mathfrak{t}_\lambda$  is trivial.*

**Proof.** Let us assume that  $(C(\hat{w}, w), \lambda)$  is well covering. Since  $\eta$  is birational, Proposition 13 shows that  $\iota^*([\Lambda_{\hat{w}^{-1}}]) \cdot [\Lambda_{w^{-1}}] = [\Lambda_e]$ . Moreover,  $\text{Det}\eta|_C$  is a non zero  $G^\lambda$ -invariant section of the restriction of  $\mathcal{L}_{P, \hat{w}, w}$  to  $C$ . By Lemmas 2 and 8, the restriction of  $r_T(\chi_{\hat{w}^{-1}}) + \chi_{w^{-1}} - \chi_1$  to  $G_C^\circ$  must be trivial.

Conversely, let us assume that Condition 2 is fulfilled. By Proposition 13,  $\eta$  is birational and  $G$ -invariant. In particular,  $\mathcal{D}\text{et}\eta|_{C^+}$  is a non-zero  $P$ -invariant section of  $\mathcal{L}_{P, \hat{w}, w}$ . With the assumption about  $r_T(\chi_{\hat{w}^{-1}}) + \chi_{w^{-1}} - \chi_1$ , Lemma 8 implies that  $\mu^{\mathcal{L}_{P, \hat{w}, w}}(C, \lambda) = 0$ . So, Proposition 6 shows that  $\mathcal{D}\text{et}\eta|_C$  is a non-zero section of  $\mathcal{L}_{P, \hat{w}, w}$ . Since  $X$  is smooth, this implies that  $(C(\hat{w}, w), \lambda)$  is well covering.  $\square$

## 10.6 Description of $\overline{\mathcal{C}}^G(\hat{G}/\hat{B} \times G/B)$

In this section, we apply our results to the description of  $\overline{\mathcal{C}}^G(\hat{G}/\hat{B} \times G/B)$ .

**Theorem 10** *We assume that the interior of  $\mathcal{C}_\lambda^G(X)$  in  $\text{Pic}^G(X)_\mathbb{Q}$  is not empty.*



1. A dominant weight  $(\hat{\nu}, \nu)$  belongs to  $\bar{\mathcal{C}}^G(\hat{G}/\hat{B} \times G/B)$  if and only if

$$\langle \lambda, \hat{w}\hat{\nu} + w\nu \rangle \geq 0,$$

for all dominant one parameter subgroup of  $T$  and for all  $(\hat{w}, w) \in \hat{W}^\lambda \times W^\lambda$  such that

(a)  $\mathfrak{Im}(\lambda) = \mathfrak{t}_\lambda$ ,

(b)  $\iota^*([\Lambda_{\hat{w}^{-1}}]).[\Lambda_{w^{-1}}] = [\Lambda_e] \in H^*(G/P(\lambda), \mathbb{Z})$ , and

(c)  $\langle \lambda, r_T(\chi_{\hat{w}^{-1}}) + \chi_{w^{-1}} - \chi_1 \rangle = 0$ .

2. Let  $\lambda$  be a dominant one parameter subgroup of  $T$  and  $(\hat{w}, w) \in \hat{W}^\lambda \times W^\lambda$  such that:

(a)  $\iota^*([\Lambda_{\hat{w}^{-1}}]).[\Lambda_{w^{-1}}] = [\Lambda_e] \in H^*(G/P(\lambda), \mathbb{Z})$ ,

(b) the restriction of  $r_T(\chi_{\hat{w}^{-1}}) + \chi_{w^{-1}} - \chi_1$  to  $\mathfrak{t}_\lambda$  is trivial, and

(c) the interior of  $\mathcal{C}^{G^\lambda/\mathbb{Z}^\lambda}(\hat{G}^\lambda/\hat{B}^\lambda \times G^\lambda/B^\lambda)$  in  $\text{Pic}^{G^\lambda/\mathbb{Z}^\lambda}(\hat{G}^\lambda/\hat{B}^\lambda \times G^\lambda/B^\lambda)_\mathbb{Q}$  is non empty.

The set of  $(\hat{\nu}, \nu) \in \Lambda_\mathbb{Q}$  such that the restriction of  $\hat{w}\hat{\nu} + w\nu$  to  $\mathfrak{t}_\lambda$  is trivial is a linear subspace  $F(\lambda, \hat{w}, w)$  of the same dimension  $d$  as  $\mathfrak{t}_\lambda$ . Moreover, the intersection of  $F(\lambda, \hat{w}, w)$  and  $\bar{\mathcal{C}}_\Lambda^G(X)$  is a face of  $\bar{\mathcal{C}}_\Lambda^G(X)$  of dimension  $d$ .

3. Conversely, for any face  $\mathcal{F}$  of  $\mathcal{C}^G(\hat{G}/\hat{B} \times G/B)$  there exists  $\lambda$  and  $(\hat{w}, w) \in \hat{W}^\lambda \times W^\lambda$  as in Assertion 2 such that the subspace spanned by  $\mathcal{F}$  is  $F(\lambda, \hat{w}, w)$ .

**Proof.** The first assertion is a simple rephrasing of Proposition 11 using Lemmas 5 and 7 and Theorem 9. The last one is a rephrasing of the first assertion of Theorem 7 using the same results as above. The second assertion is a consequence of Theorem 8.  $\square$

**Remark.** In [BS00], Berenstein and Sjamaar gives the linear inequalities of Assertion 1 of Theorem 10 among a lot of others inequalities of the form  $\mu^\bullet(x, \lambda) \leq 0$ . In other words, Assertion 1 of Theorem 10 selects some of the inequalities of Berenstein and Sjamaar.

## 11 Application to the tensor product

In this section, we fix a integer  $s \geq 2$  and set  $\hat{G} = G^s$ ,  $\hat{T} = T^s$  and  $\hat{B} = B^s$ . We embed  $G$  diagonally in  $\hat{G}$ . Then  $\bar{\mathcal{C}}^G(X)$  identifies with the  $(s+1)$ -uple  $(\nu_1, \dots, \nu_{s+1}) \in X(T)^{s+1}$  such that the  $\nu_i$ 's are dominant and  $V_{n\nu_1} \otimes \dots \otimes V_{n\nu_{s+1}}$  contains a non zero  $G$ -invariant vector for  $n$  big enough.

We will apply Theorem 10 at this situation. Let us start by studying the objects used in this statement in this particular case.

Let  $\lambda$  be a dominant one parameter subgroup of  $T$ . For  $(\hat{w}, w_{s+1}) = (w_1, \dots, w_{s+1}) \in \hat{W} \times W = W^{s+1}$ , and  $(\hat{\nu}, \nu) = (\nu_1, \dots, \nu_{s+1}) \in \text{Pic}^G(\mathbf{X})_{\mathbb{Q}} = X(T)_{\mathbb{Q}}^{s+1}$  we have:

- $Z_\lambda$  is the connected center of  $G^\lambda$ ,
- $r_T(\hat{w}\hat{\nu}) = \sum_{i=1}^s w_i \nu_i$ , and  $r_T(\chi_{\hat{w}^{-1}}) = \sum_{i=1}^s \chi_{w_i^{-1}}$ ,
- $\iota^*([\Lambda_{\hat{w}^{-1}}]) = [\Lambda_{w_1^{-1}}] \cdot \dots \cdot [\Lambda_{w_s^{-1}}]$ ,

In [BK06], Belkale and Kumar defined a new product denoted  $\odot_0$  on the cohomology groups  $H^*(G/P, \mathbb{Z})$  for any parabolic subgroup  $P$  of  $G$ . By Proposition 17 of [BK06], this product  $\odot_0$  has the following very interesting property:

For  $w_i \in W^\lambda$ , the following are equivalent:

1.  $[\Lambda_{w_1^{-1}}] \cdot \dots \cdot [\Lambda_{w_{s+1}^{-1}}] - [\Lambda_e] = 0$  and the restriction of  $\chi_{w_1^{-1}} + \dots + \chi_{w_{s+1}^{-1}} - \chi_1$  to  $Z(G^\lambda)^\circ$  is trivial;
2.  $[\Lambda_{w_1^{-1}}] \odot_0 \dots \odot_0 [\Lambda_{w_{s+1}^{-1}}] = [\Lambda_e]$ .

Using this result of Belkale and Kumar our Theorem 10 gives the following

**Theorem 11** *1. A point  $(\nu_1, \dots, \nu_{s+1}) \in X(T)_{\mathbb{Q}}^{s+1}$  belongs to the cone  $\bar{\mathcal{C}}^G((G/B)^{s+1})$  if and only if*

- (a) *each  $\nu_i$  is dominant; that is  $\langle \alpha^\vee, \nu_i \rangle \geq 0$  for all simple root  $\alpha$ .*
- (b) *for all simple root  $\alpha$ ; for all  $(w_1, \dots, w_{s+1}) \in (W/W_{\omega_{\alpha^\vee}})^{s+1}$  such that  $[\Lambda_{w_1^{-1}}] \odot_0 \dots \odot_0 [\Lambda_{w_{s+1}^{-1}}] = [\Lambda_e] \in H^*(G/P(\alpha), \mathbb{Z})$ , we have:*

$$\sum_i \langle \omega_{\alpha^\vee}, w_i \nu_i \rangle \geq 0.$$

2. In the above description of  $\overline{\mathcal{C}}^G((G/B)^{s+1})$ , one can omit no linear inequality in this list (neither in 1a nor 1b).

3. Let  $\mathcal{F}$  be a face of  $\mathcal{C}^G((G/B)^{s+1})$  of codimension  $d$ . There exists a subset  $I$  of  $d$  simple roots and  $(w_1, \dots, w_{s+1}) \in (W/W(I))^{s+1}$  such that:

(a)  $[\Lambda_{w_1^{-1}}] \odot_0 \cdots \odot_0 [\Lambda_{w_{s+1}^{-1}}] = [\Lambda_e] \in H^*(G/P(I), \mathbb{Z})$ ,

(b) the subspace spanned by  $\mathcal{F}$  is the set  $(\nu_1, \dots, \nu_{s+1}) \in X(T)_{\mathbb{Q}}^{s+1}$  such that:

$$\forall \alpha \in I \quad \sum_i \langle \omega_{\alpha^\vee}, w_i \nu_i \rangle = 0.$$

4. Conversely, let  $I$  be a subset of  $d$  simple roots and  $(w_1, \dots, w_{s+1}) \in (W/W(I))^{s+1}$  such that  $[\Lambda_{w_1^{-1}}] \odot_0 \cdots \odot_0 [\Lambda_{w_{s+1}^{-1}}] = [\Lambda_e] \in H^*(G/P(I), \mathbb{Z})$ .

Then, the set of  $(\nu_1, \dots, \nu_{s+1}) \in \overline{\mathcal{C}}^G((G/B)^{s+1})$  such that

$$\forall \alpha \in I \quad \sum_i \langle \omega_{\alpha^\vee}, w_i \nu_i \rangle = 0,$$

is a face of codimension  $d$  of  $\overline{\mathcal{C}}^G((G/B)^{s+1})$ .

**Proof.** Let  $\lambda$  be a dominant one parameter subgroup of  $T$  such that  $\text{Im}(\lambda)$  is the connected center of  $G^\lambda$ . Then there exists a positive integer  $n$  and a simple root  $\alpha$  such that  $n\omega_{\alpha^\vee} = \lambda$ . Now, the first assertion is a simple rephrasing of the first assertion of Theorem 10.

By for example Corollary 1 of [MR04], the interior of  $\mathcal{C}^G(X)$  is not empty in  $\text{Pic}^G(X)$ . Equations 1a are all different and are not repeated in Equations 1b. Moreover, by Proposition 7 of [MR04] they define codimension one faces of  $\overline{\mathcal{C}}^G(X)$ .

Let  $\lambda$  be any dominant one parameter subgroup. All the irreducible components  $C$  of  $X^\lambda$  are isomorphic to  $(G^\lambda/B^\lambda)^{s+1}$ . In particular, Corollary 1 of [MR04] shows that the interior of  $\mathcal{C}^{G^\lambda/Z^\lambda}(C)$  in  $\text{Pic}^{G^\lambda/Z^\lambda}(C)$  is non empty. Now, the rest of Theorem 11 is a direct consequence of Theorem 10.  $\square$

**Remark.**

1. The equations of Assertion 1 of Theorem 11 are the same as those obtained by Belkale and Kumar in their Theorem 22. The fact that no

equation is irredundant seems to be new in general, whereas some particular cases are known. Indeed, Knutson, Tao and Woodward shown in [KTW04] the case when  $s = 2$  and  $G = \mathrm{SL}_n$  by combinatorial tools. Using explicit calculation with the help of a computer, Kapovich, Kumar and Millson proves the case when  $s = 2$  and  $G = \mathrm{SO}(8)$  in [KKM06].

2. The description of the other faces of  $\mathcal{C}^G((G/B)^{s+1})$  gives an application of the Belkale-Kumar product  $\odot_0$  for all the complete homogeneous spaces.

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N. R.  
 Université Montpellier II  
 Département de Mathématiques  
 Case courrier 051-Place Eugène Bataillon  
 34095 Montpellier Cedex 5  
 France  
 e-mail: [ressayre@math.univ-montp2.fr](mailto:ressayre@math.univ-montp2.fr)