



**HAL**  
open science

# Navier-Stokes equations with periodic boundary conditions and pressure loss

Chérif Amrouche, Macaire Batchi, Jean Batina

► **To cite this version:**

Chérif Amrouche, Macaire Batchi, Jean Batina. Navier-Stokes equations with periodic boundary conditions and pressure loss. 2007. hal-00141973

**HAL Id: hal-00141973**

**<https://hal.science/hal-00141973>**

Preprint submitted on 17 Apr 2007

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Navier-Stokes equations with periodic boundary conditions and pressure loss

Chérif Amrouche<sup>a</sup>, Macaire Batchi<sup>a,b</sup>, Jean Batina<sup>b</sup>.

<sup>a</sup>*Laboratoire de Mathématiques Appliquées, CNRS UMR 5142*

<sup>b</sup>*Laboratoire de Thermique Energétique et Procédés*

*Université de Pau et des Pays de l'Adour*

*Avenue de l'Université 64000 Pau, France*

## Abstract

We present in this note the existence and uniqueness results for the Stokes and Navier-Stokes equations which model the laminar flow of an incompressible fluid inside a two-dimensional channel of periodic sections. The data of the pressure loss coefficient enables us to establish a relation on the pressure and to thus formulate an equivalent problem.

**Keywords :** Navier-Stokes equations, incompressible fluid, bidimensional channel, periodic boundary conditions, pressure loss.

## 1 Introduction

The problem which one proposes to study here is that modelling a laminar flow inside a two-dimensional plane channel with periodic section. Let  $\Omega$  be an open bounded connected lipschtzian of  $\mathbb{R}^2$  (see figure hereafter), where  $\Gamma_0 = \{0\} \times ]-1, 1[$  and  $\Gamma_1 = \{1\} \times ]-1, 1[$ .

One defines the space

$$V = \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega); \operatorname{div} \mathbf{v} = 0, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_2, \mathbf{v}|_{\Gamma_0} = \mathbf{v}|_{\Gamma_1} \right\}$$

and for  $\lambda \in \mathbb{R}$  given, one considers the problem

$$(\mathcal{S}) \left\{ \begin{array}{l} \text{Find } \mathbf{u} \in V, \text{ such that} \\ \forall \mathbf{v} \in V, \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x} = \lambda \int_{-1}^{+1} v_1(1, y) \, dy. \end{array} \right.$$

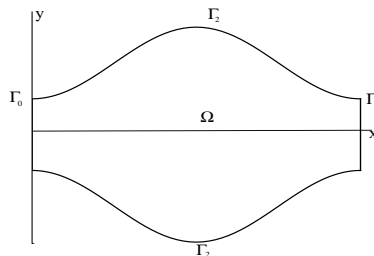


Figure 1. Geometry of channel

## 2 Resolution of the problem $(\mathcal{S})$

Initially one proposes to study the problem  $(\mathcal{P})$ . One has it

**Theorem 2.1** *Problem (S) has an unique solution  $\mathbf{u} \in V$ . Moreover, there is a constant  $C(\Omega) > 0$  such that:*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \leq \lambda C(\Omega). \quad (1)$$

*Proof:* Let us note initially that space  $V$  provided the norm  $H^1(\Omega)^2$  being a closed subspace of  $H^1(\Omega)^2$  is thus an Hilbert space. Let us set

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x}, \quad l(\mathbf{v}) = \lambda \int_{-1}^{+1} v_1(1, y) \, dy.$$

It is clear, thanks to the Poincaré inequality, that the bilinear continuous form is  $V$ -coercive. It is easy to also see that  $l \in V'$ . One deduces from Lax-Milgram Theorem the existence and uniqueness of  $\mathbf{u}$  solution of (S). Moreover,

$$\int_{\Omega} |\nabla \mathbf{u}|^2 \, d\mathbf{x} \leq \lambda \sqrt{2} \left( \int_{-1}^{+1} |u_1(1, y)|^2 \, dy \right)^{1/2},$$

*i.e.*

$$\begin{aligned} \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 &\leq \lambda \sqrt{2} \|\mathbf{u}\|_{L^2(\Gamma)} \leq \lambda \sqrt{2} \|\mathbf{u}\|_{H^{1/2}(\Gamma)} \\ &\leq \lambda C_1(\Omega) \|\mathbf{u}\|_{H^1(\Omega)} \end{aligned}$$

Thus there is the estimate (1).  $\square$

We now will give an interpretation of the problem (S). One introduces the space

$$\mathcal{V} = \left\{ \mathbf{v} \in \mathcal{D}(\Omega)^2; \quad \operatorname{div} \mathbf{v} = 0 \right\}.$$

Let  $\mathbf{u}$  be the solution of (S). Then, for all  $\mathbf{v} \in \mathcal{V}$ , one has :

$$\langle -\Delta \mathbf{u}, \mathbf{v} \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = 0.$$

So that thanks to De Rham Theorem, there exists  $p \in \mathcal{D}'(\Omega)$  such that

$$-\Delta \mathbf{u} + \nabla p = 0 \text{ in } \Omega. \quad (2)$$

Moreover, since  $\nabla p \in H^{-1}(\Omega)^2$ , it is known that there exists  $q \in L^2(\Omega)$  such that (see [1])

$$\nabla q = \nabla p \quad \text{in } \Omega. \quad (3)$$

The open  $\Omega$  being connected, there exists  $C \in \mathbb{R}$  such that  $p = q + C$ , what means that  $p \in L^2(\Omega)$ . Let us recall that (see [1])

$$\inf_{K \in \mathbb{R}} \|p + K\|_{L^2(\Omega)} \leq C \|\nabla p\|_{H^{-1}(\Omega)^2}.$$

One deduces from the estimate (1) and from (2) that

$$\inf_{K \in \mathbb{R}} \|p + K\|_{L^2(\Omega)} \leq C \|\Delta \mathbf{u}\|_{H^{-1}(\Omega)^2} \leq C \|\mathbf{u}\|_{H^1(\Omega)^2} \leq \lambda C(\Omega).$$

Since  $\mathbf{u} \in H^1(\Omega)^2$  and  $\mathbf{0} = -\Delta \mathbf{u} + \nabla p \in L^2(\Omega)^2$ , it is shown that  $-\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + p\mathbf{n} \in H^{-1/2}(\Gamma)^2$  and one has the Green formula: for all  $\mathbf{v} \in V$

$$\int_{\Omega} (-\Delta \mathbf{u} + \nabla p) \cdot \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x} + \left\langle -\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + p\mathbf{n}, \mathbf{v} \right\rangle, \quad (4)$$

where the bracket represents the duality product  $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ . Moreover, as  $p \in L^2(\Omega)$  and  $\Delta p = 0$  in  $\Omega$ , one has  $p \in H^{-1/2}(\Gamma)$ . Consequently, one has therefore  $\frac{\partial \mathbf{u}}{\partial \mathbf{n}} \in H^{-1/2}(\Gamma)^2$ . The function  $\mathbf{u}$  being solution of  $(\mathcal{S})$ , for all  $\mathbf{v} \in V$  one has:

$$\left\langle \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n}, \mathbf{v} \right\rangle = \lambda \int_{-1}^{+1} v_1(1, y) \, dy, \quad (5)$$

*i.e.*

$$\left\langle \frac{\partial \mathbf{u}}{\partial x} - p\mathbf{e}_1, \mathbf{v} \right\rangle_{\Gamma_1} + \left\langle -\frac{\partial \mathbf{u}}{\partial x} + p\mathbf{e}_1, \mathbf{v} \right\rangle_{\Gamma_0} = \langle \lambda \mathbf{e}_1, \mathbf{v} \rangle_{\Gamma_1}. \quad (6)$$

where  $\mathbf{e}_1 = (1, 0)$ .

i) Let  $\mu \in H_{00}^{1/2}(\Gamma_1)$  and let us set

$$\mu_2 = \begin{cases} \mu \text{ on } \Gamma_0 \cup \Gamma_1 \\ 0 \text{ on } \Gamma_2 \end{cases} \quad \text{and} \quad \boldsymbol{\mu} = \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix}$$

where (see [2])

$$H_{00}^{1/2}(\Gamma_1) = \left\{ \varphi \in \mathbf{L}^2(\Gamma_1); \exists \mathbf{v} \in \mathbf{H}^1(\Omega), \text{ with } \mathbf{v}|_{\Gamma_2} = \mathbf{0}, \mathbf{v}|_{\Gamma_0 \cup \Gamma_1} = \varphi \right\}.$$

It is checked easily that

$$\boldsymbol{\mu} \in H^{1/2}(\Gamma)^2 \quad \text{and} \quad \int_{\Gamma} \boldsymbol{\mu} \cdot \mathbf{n} \, d\sigma = 0.$$

So that there exists  $\mathbf{v} \in H^1(\Omega)^2$  satisfying (see [3])

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \mathbf{v} = \boldsymbol{\mu} \quad \text{on} \quad \Gamma.$$

In particular  $\mathbf{v} \in V$  and according to (6), one has

$$\left\langle \frac{\partial u_2}{\partial x}, \mu \right\rangle_{\Gamma_1} = \left\langle \frac{\partial u_2}{\partial x}, \mu \right\rangle_{\Gamma_0},$$

which means that

$$\frac{\partial u_2}{\partial x} \Big|_{\Gamma_1} = \frac{\partial u_2}{\partial x} \Big|_{\Gamma_0}. \quad (7)$$

One deduces now from (6) that for all  $\mathbf{v} \in V$ ,

$$\left\langle \frac{\partial u_1}{\partial x} - p, v_1 \right\rangle_{\Gamma_1} + \left\langle -\frac{\partial u_1}{\partial x} + p, v_1 \right\rangle_{\Gamma_0} = \langle \lambda, v_1 \rangle_{\Gamma_1}. \quad (8)$$

But,  $\operatorname{div} \mathbf{u} = 0$  and  $u_2|_{\Gamma_1} = u_2|_{\Gamma_0}$ , one thus has

$$\frac{\partial u_2}{\partial y} \Big|_{\Gamma_1} = \frac{\partial u_2}{\partial y} \Big|_{\Gamma_0} \quad \text{and} \quad \frac{\partial u_1}{\partial x} \Big|_{\Gamma_1} = \frac{\partial u_1}{\partial x} \Big|_{\Gamma_0}. \quad (9)$$

Consequently, thanks to (8) one has:

$$\langle -p, v_1 \rangle_{\Gamma_1} + \langle p, v_1 \rangle_{\Gamma_0} = \langle \lambda, v_1 \rangle_{\Gamma_1} \quad (10)$$

ii) Let  $\nu \in H_{00}^{1/2}(\Gamma_1)$  and let us set

$$\nu_1 = \begin{cases} \nu & \text{on } \Gamma_0 \cup \Gamma_1 \\ 0 & \text{on } \Gamma_2 \end{cases} \quad \text{and} \quad \boldsymbol{\nu} = \begin{pmatrix} \nu_1 \\ 0 \end{pmatrix}.$$

One easily checks that

$$\boldsymbol{\nu} \in H^{1/2}(\Gamma)^2 \quad \text{and} \quad \int_{\Gamma} \boldsymbol{\nu} \cdot \mathbf{n} \, d\sigma = 0.$$

So that there exists  $\mathbf{v} \in H^1(\Omega)^2$  satisfying

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{v} = \boldsymbol{\nu} \quad \text{on } \Gamma.$$

In particular  $\mathbf{v} \in V$  and according to (14), one has

$$\langle -p, \boldsymbol{\nu} \rangle_{\Gamma_1} + \langle p, \boldsymbol{\nu} \rangle_{\Gamma_0} = \langle \lambda, \boldsymbol{\nu} \rangle_{\Gamma_1}$$

*i.e.*

$$p|_{\Gamma_1} = p|_{\Gamma_0} - \lambda \tag{11}$$

where the equality takes place with the  $H^{1/2}$  sense. In short, if  $\mathbf{u} \in H^1(\Omega)^2$  is solution of  $(\mathcal{S})$ , then there exists  $p \in L^2(\Omega)$ , unique with an additive constant such that:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \Omega, \tag{12}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{13}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_2, \quad \mathbf{u}|_{\Gamma_1} = \mathbf{u}|_{\Gamma_0}, \tag{14}$$

$$\frac{\partial \mathbf{u}}{\partial x}|_{\Gamma_1} = \frac{\partial \mathbf{u}}{\partial x}|_{\Gamma_0}, \tag{15}$$

$$p|_{\Gamma_1} = p|_{\Gamma_0} - \lambda. \tag{16}$$

It is clear that if  $(\mathbf{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$  checks (12)-(16), then  $\mathbf{u}$  is solution

of  $(\mathcal{S})$ . Thus it

**Theorem 2.2** *The problem (12)-(16) has an unique solution  $(\mathbf{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$  up to an additive constant for  $p$ . Moreover,  $\mathbf{u}$  verifies  $(\mathcal{S})$  and*

$$\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leq \lambda C(\Omega). \quad \square$$

*Remark 1* *The pressure verifies the relation (16), which means that  $p$  satisfies the relation of Patankar et al.[5].*

### 3 Navier-Stokes Equations

One takes again the assumptions of the Stokes problem given above. For  $\lambda \in \mathbb{R}$  given, the one considers the following problem

$$(\mathcal{NS}) \left\{ \begin{array}{l} \text{Find } \mathbf{u} \in V, \text{ such that} \\ \forall \mathbf{v} \in V, \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\mathbf{x} + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \lambda \int_{-1}^{+1} v_1(1, y) \, dy \end{array} \right.$$

with

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x}$$

With an aim of establishing the existence of the solutions of the problem  $(\mathcal{NS})$ , one uses the Brouwer fixed point theorem (see [4], [6]). One will show it

**Theorem 3.1** *The problem  $(\mathcal{NS})$  has at least a solution  $\mathbf{u} \in V$ . Moreover,  $\mathbf{u}$  checks the estimate (1).*



*Proof:* To show the existence of  $\mathbf{u}$ , one constructs the approximate solutions of the problem  $(\mathcal{NS})$  by the Galerkin method and then thanks to the arguments of compactness, one makes a passage to the limit.

*i) For each fixed integer  $m \geq 1$ , one defines an approximate solution  $\mathbf{u}_m$  of  $(\mathcal{NS})$  by*

$$\mathbf{u}_m = \sum_{i=1}^m g_{im} \mathbf{w}_i, \quad \text{with } g_{im} \in \mathbb{R} \quad (17)$$

$$((\mathbf{u}_m, \mathbf{w}_i)) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{w}_i) = \langle \lambda \mathbf{n}, \mathbf{w}_i \rangle_{\Gamma_1}, i = 1, \dots, m$$

where  $V_m = \langle \mathbf{w}_1, \dots, \mathbf{w}_m \rangle$  vector spaces spanned by the vectors  $\mathbf{w}_1, \dots, \mathbf{w}_m$  and  $\{\mathbf{w}_i\}$  is an Hilbertian basis of  $V$  which is separable. Let us note that (17) is equivalent to:

$$\forall \mathbf{v} \in V_m, ((\mathbf{u}_m, \mathbf{v})) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}) = \lambda \int_{-1}^{+1} v_1(1, y) dy. \quad (18)$$

With an aim to establish the existence of the solutions of the problem  $\mathbf{u}_m$ , the operator as follows is considered

$$P_m : V_m \rightarrow V_m$$

$$\mathbf{u} \longmapsto P_m(\mathbf{u})$$

defined by

$$\forall \mathbf{u}, \mathbf{v} \in V_m, ((P_m(\mathbf{u}), \mathbf{v})) = ((\mathbf{u}, \mathbf{v})) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \lambda \int_{-1}^{+1} v_1(1, y) dy.$$

Let us note initially that  $P_m$  is continuous and

$$\forall \mathbf{u} \in V, b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = 0.$$

Indeed, thanks to the Green formula, one has

$$b(\mathbf{u}, \mathbf{u}, \mathbf{u}) = -\frac{1}{2} \int_{\Omega} |\mathbf{u}|^2 \operatorname{div} \mathbf{u} \, d\mathbf{x} + \frac{1}{2} \int_{\Gamma} (\mathbf{u} \cdot \mathbf{n}) |\mathbf{u}|^2 \, d\sigma = 0.$$

But,  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$  and

$$\int_{\Gamma} (\mathbf{u} \cdot \mathbf{n}) |\mathbf{u}|^2 \, d\sigma = \int_{\Gamma_0} (\mathbf{u} \cdot \mathbf{n}) |\mathbf{u}|^2 \, d\sigma + \int_{\Gamma_1} (\mathbf{u} \cdot \mathbf{n}) |\mathbf{u}|^2 \, d\sigma.$$

since the external normal to  $\Gamma_0$  is opposed to that of  $\Gamma_1$  and  $\mathbf{u} \in V$ .

Thanks to Brouwer Theorem, there exists  $\mathbf{u}_m$  satisfying (18) and

$$\|\mathbf{u}_m\|_{\mathbf{H}^1(\Omega)} \leq \lambda C(\Omega). \quad (19)$$

ii) We can extract a subsequence  $\mathbf{u}_\nu$  such that

$$\mathbf{u}_\nu \rightharpoonup \mathbf{u} \quad \text{weakly in } V,$$

and thanks to the compact imbedding of  $V$  in  $L^2(\Omega)^2$ , we obtain

$$\forall \mathbf{v} \in V, ((\mathbf{u}, \mathbf{v})) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \lambda \int_{-1}^{+1} v_1(1, y) \, dy.$$

As for the Stokes problem, one shows the existence of  $p \in L^2(\Omega)$ , unique except for an additive constant, such that

$$\left\{ \begin{array}{ll} -\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_2, \\ \mathbf{u}|_{\Gamma_1} = \mathbf{u}|_{\Gamma_0}. & \end{array} \right.$$

It is checked finally that

$$\frac{\partial \mathbf{u}}{\partial x} \Big|_{\Gamma_1} = \frac{\partial \mathbf{u}}{\partial x} \Big|_{\Gamma_0} ,$$

$$p|_{\Gamma_1} = p|_{\Gamma_0} - \lambda.$$

*Remark 2 i) Theorem 3.1 of problem ( $\mathcal{NS}$ ) takes place in three dimension.*

*ii) One can show that the solution  $(\mathbf{u}, p)$  belongs to  $H^2(\Omega)^2 \times H^1(\Omega)$ .*

## References

- [1] C. Amrouche et V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension*, Czechoslovak Mathematical Journal, **44 (119)**, Praha, pp.109-139, (1994).
- [2] R. Dautray et J. L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, tome 1-6, Masson, 1984.
- [3] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer Series SCM, 1986.
- [4] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Gauthier-Villars, 1969.
- [5] S. V. Patankar, C. H. Liu and E. M. Sparrow, *Fully developed flow and heat transfer in ducts having streamwise-periodic variations of cross sectional area*, *J. Heat Transfer*, **99**, pp.180-186, (1977).
- [6] R. Temam, *Navier-Stokes Equations. Theory and Analysis*, North-Holland, Amsterdam, 1985.