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# FINITE REPRESENTATIONS OF THE BRAID GROUP COMMUTATOR SUBGROUP

ABDELOUAHAB AROUCHE

ABSTRACT. We study the representations of the commutator subgroup  $K_n$  of the braid group  $B_n$  into a finite group  $\Sigma$ . This is done through a symbolic dynamical system. Some experimental results enable us to compute the number of subgroups of  $K_n$  of a given (finite) index, and, as a by-product, to recover the well known fact that every representation of  $K_n$  into  $S_r$ , with  $n > r$ , must be trivial.

## 1. INTRODUCTION

In [SiWi1], D. Silver and S. Williams exploited the structure of the kernel subgroup  $K$  of an epimorphism  $\chi : G \rightarrow \mathbb{Z}$ , where  $G$  is a finitely presented group, to show that the set  $Hom(K, \Sigma)$  of representations of  $K$  into a finite group  $\Sigma$  has a structure of a subshift of finite type (SFT), a symbolic dynamical system described by a graph  $\Gamma$ ; namely, there is a one to one correspondence between representations  $\rho : K \rightarrow \Sigma$  and bi-infinite paths in  $\Gamma$ .

We apply this method to the group  $B_n$  of braids with  $n$ -strands, with  $\chi$  being the abelianization homomorphism and  $\Sigma$  the symmetric group  $S_r$  of degree  $r$  or the special linear group over a finite field  $SL_2(F_q)$ . The subgroup  $K_n = \ker \chi$  is then the commutator subgroup of  $B_n$ .

It is a well known fact that for a given group  $K$ , there is a finite to one correspondence between its subgroups of index no greater than  $r$  and representations  $\rho : K \rightarrow S_r$ . This correspondence can be described by

$$\rho \longmapsto \{g \in K : \rho(g)(1) = 1\}.$$

The pre-image of a subgroup of index exactly  $r$  consists of  $(r-1)!$  transitive representations  $\rho$ . ( $\rho$  is said to be transitive if  $\rho(K)$  operates transitively on  $\{1, 2, \dots, r\}$ ). This will allow us to draw some conclusions about the subgroups of finite index of  $K_n$ , which motivates the choice of  $S_r$ ; we motivate that of  $SL_2(F_q)$  by the fact that any representation of  $K_n$  into any group  $\Sigma$  has range in the commutator subgroup  $[\Sigma, \Sigma]$ , for  $n \geq 6$ .

We give an algorithm to compute  $Hom(K_n, \Sigma)$ , for  $n \geq 5$ . Some experimental results enable us to compute the number of subgroups of  $K_n$  of a given (finite) index, and, as a by-product, to recover the well known fact that  $Hom(K_n, S_r)$  is trivial for  $n \geq 5$  and  $r < n$ . Since every representation in  $Hom(B_n, \Sigma)$  restricts to

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an element of  $\text{Hom}(K_n, \Sigma)$ , we enhance the given algorithm in order to compute  $\text{Hom}(B_n, \Sigma)$ .

## 2. GENERALITIES

Let  $B_n$  be the braid group given by the presentation:

$$\langle \sigma_1, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i; \quad |i-j| \geq 2 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j; \quad |i-j| = 1 \end{array} \rangle,$$

(see [BuZi] for additional background). Let  $\beta \in B_n$  be a braid. Then  $\beta$  can be written as :

$$\beta = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_k}^{\varepsilon_k},$$

with  $i_1, \dots, i_k \in \{1, \dots, n-1\}$  and  $\varepsilon_i = \pm 1$ . Define the exponent sum of  $\beta$  (in terms of the  $\sigma_i$ 's) denoted by  $\text{exp}(\beta)$ , as:

$$\text{exp}(\beta) = \varepsilon_1 + \cdots + \varepsilon_k.$$

Then  $\text{exp}(\beta)$  is an invariant of the braid group, that is, it doesn't depend on the writing of  $\beta$ . Moreover,  $\text{exp}(\beta) : B_n \rightarrow \mathbb{Z}$  is a homomorphism. Let  $H_n$  denote its kernel. Then the Reidemeister-Schreier theorem [LySc] enables us to find a presentation for  $H_n$ . We choose the set

$$\{\dots, \sigma_1^{-m}, \sigma_1^{-m+1}, \dots, \sigma_1^{-1}, 1, \sigma_1, \sigma_1^2, \dots, \sigma_1^m, \dots\}$$

as a Schreier system of right coset representatives of  $H_n$  in  $B_n$ . Putting  $z_m = \sigma_1^m (\sigma_2 \sigma_1^{-1}) \sigma_1^{-m}$  for  $m \in \mathbb{Z}$ , and  $x_i = \sigma_i \sigma_1^{-1}$  for  $i = 3, \dots, n-1$ , we get the following presentation of  $H_n$  :

$$H_n = \left\langle \begin{array}{l} z_m, \quad m \in \mathbb{Z} \\ x_i, \quad i = 3, \dots, n-1 \end{array} \mid \begin{array}{l} x_i x_j = x_j x_i, \quad |i-j| \geq 2; \\ x_i x_j x_i = x_j x_i x_j, \quad |i-j| = 1; \\ z_m z_{m+2} = z_{m+1}, \quad m \in \mathbb{Z}; \\ z_m x_3 z_{m+2} = x_3 z_{m+1} x_3, \quad i = 4, \dots, n-1; \\ z_m x_i = x_i z_{m+1}, \quad m \in \mathbb{Z}. \end{array} \right\rangle$$

**Example 1.** We have:

$$H_3 = \langle z_m \mid z_m z_{m+2} = z_{m+1}; \forall m \in \mathbb{Z} \rangle$$

is a free group on two generators  $z_0 = \sigma_2 \sigma_1^{-1}$  and  $z_{-1} = \sigma_1^{-1} \sigma_2$ , and

$$H_4 = \langle z_m, t \mid \begin{array}{l} z_m z_{m+2} = z_{m+1}, \\ z_m t z_{m+2} = t z_{m+1} t, \end{array} m \in \mathbb{Z} \rangle.$$

Note that  $H_2 = \{1\}$ , since  $B_2 = \langle \sigma_1 \mid - \rangle \cong \mathbb{Z}$ .

Now, every commutator in  $B_n$  has exponent sum zero. Conversely, every generator of  $H_n$  is a product of commutators. Hence, we have  $H_n = K_n$ , and  $\text{exp}$  is the abelianization homomorphism.

Each  $K_n$  fits into a split exact sequence:

$$1 \rightarrow K_n \rightarrow B_n \rightarrow \mathbb{Z} \rightarrow 0,$$

and there are "natural" inclusions  $B_n \subset B_{n+1}$  and  $K_n \subset K_{n+1}$ .

**Remark 1.** We have the following consequences of relations in  $K_n$  :

- (1)  $z_{m+1} = x_j^{-1} z_m x_j$ , hence  $z_m = x_j^{-m} z_0 x_j^m$ ; for  $j \geq 4$ . where  $z_0 = \sigma_2 \sigma_1^{-1}$ .
- (2) The  $z_m$ 's are conjugate in  $B_n$  for  $n \geq 3$  (by  $\sigma_1$ ) and in  $K_n$  for  $n \geq 5$ .
- (3) The restriction of conjugation by  $\sigma_1$  in  $B_n$  to  $K_n$  induces an action of  $\mathbb{Z}$  on  $K_n$ . This presentation of  $K_n$  is said to be  $\mathbb{Z}$ -dynamical.

### 3. THE REPRESENTATION SHIFT

This work is essentially experimental. It aims to describe the set of representations of  $K_n$  into finite group  $\Sigma$ . We start with  $n = 3$  and describe  $Hom(K_3, \Sigma)$  by means of a graph  $\Gamma$  that we will construct in a step by step fashion [SiWi2]. A representation  $\rho : K_3 \rightarrow \Sigma$  is a function  $\rho$  from the set of generators  $z_m$  to  $\Sigma$  such that for each  $m \in \mathbb{Z}$ , the relation:

$$\rho(z_m) \rho(z_{m+2}) = \rho(z_{m+1})$$

holds in  $\Sigma$ . Any such function can be constructed as follows, beginning with step 0 and proceeding to steps  $\pm 1, \pm 2, \dots$

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(step -1) Choose  $\rho(z_{-1})$  if possible such that  $\rho(z_{-1}) \rho(z_1) = \rho(z_0)$ .

(step 0) Choose values  $\rho(z_0)$  and  $\rho(z_1)$ .

(step 1) Choose  $\rho(z_2)$  if possible such that  $\rho(z_0) \rho(z_2) = \rho(z_1)$ .

(step 2) Choose  $\rho(z_3)$  if possible such that  $\rho(z_1) \rho(z_3) = \rho(z_2)$ .

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This process leads to a bi-infinite graph whose vertices are the maps  $\rho : \{z_0, z_1\} \rightarrow \Sigma$ , each of which can be regarded as an ordered pair  $(\rho(z_0), \rho(z_1))$ . There is a directed edge from  $\rho$  to  $\rho'$  if and only if  $\rho(z_1) = \rho'(z_0)$  and  $\rho(z_0) \rho'(z_1) = \rho(z_1)$ . In such a case, we can extend  $\rho : \{z_0, z_1\} \rightarrow \Sigma$  by defining  $\rho(z_2)$  to be equal to  $\rho'(z_1)$ . Now if there is an edge from  $\rho'$  to  $\rho''$ , we can likewise extend  $\rho$  by defining  $\rho(z_3)$  to be  $\rho''(z_1)$ . We implement this process by starting with an ordered pair  $(a_0, a_1)$  of elements of  $\Sigma$ , and computing at each step a new ordered pair from the old one, so that every edge in the graph looks like:

$$(a_m, a_{m+1}) \rightarrow (a_{m+1}, a_{m+2}),$$

with

$$a_{m+2} = a_m^{-1} a_{m+1}.$$

In our case, since the group  $\Sigma$  is finite, the process must end, and the graph  $\Gamma$  we obtain consists necessarily of disjoint cycles. This gives an algorithm for finding  $Hom(K_3, \Sigma)$ . Observe that  $Hom(K_3, \Sigma)$  is endowed with a shift map

$$\sigma : \rho \mapsto \sigma(\rho)$$

defined by

$$\sigma(\rho) : x \mapsto \rho(\sigma_1 x \sigma_1^{-1}).$$

If we regard  $\rho$  as a bi-infinite path in the graph  $\Gamma$ , then  $\sigma$  correspond to the shift map  $(a_m, a_{m+1}) \mapsto (a_{m+1}, a_{m+2})$ , since  $z_{m+1} = \sigma_1 z_m \sigma_1^{-1}$ . Any cycle in the graph  $\Gamma$  with length  $p$  corresponds to  $p$  representations having least period  $p$ . These are the iterates of some representation  $\rho \in Hom(K_3, \Sigma)$  satisfying  $\rho(z_m) = a_m$  and  $\sigma^p(\rho) = \rho$ , since  $a_{m+p} = a_m$ .

**Example 2.** A cycle of length 2 has the form  $(a, a^2) \rightleftharpoons (a^2, a)$ , with  $a^3 = 1$ .

**Remark 2.** Since  $K_3$  is a free group of rank 2,  $\text{Hom}(K_3, \Sigma)$  has a simple description; namely, there is a one to one correspondence between  $\text{Hom}(K_3, \Sigma)$  and  $\Sigma^2$ . The interest of our approach, beside the dynamical structure, is that it allows to go further and describe  $\text{Hom}(K_n, \Sigma)$ , for  $n \geq 3$ . As a consequence of the dynamical approach, we get a partition of  $\text{Hom}(K_3, \Sigma)$  into orbits, hence the identity  $|\Sigma|^2 = 1 + \sum_{p \geq 2} p.n_p$ , where  $n_p$  is the number of orbits of (least) period  $p$ .

**Proposition 1.** If a cycle has length  $p$ , then the identity  $a_0 a_1 \dots a_{p-1} = 1$  holds.

In order to minimize calculations, we extract some foreseeable behaviour for various choices of the ordered pair  $(a_0, a_1)$  in the previous algorithm.

First, the dynamics of ordered pairs  $(a_0, a_1)$  such that  $a_0 = 1$  or  $a_1 = 1$  or  $a_0 = a_1$  is entirely known. To be precise, we get a cycle of length 6 unless  $a = a^{-1}$ , in which case it is of length 3 (or 1 if and only if  $a = 1$ ).

$$\begin{aligned} (1, a) &\rightarrow (a, a) \rightarrow (a, 1) \rightarrow (1, a^{-1}) \\ &\rightarrow (a^{-1}, a^{-1}) \rightarrow (a^{-1}, 1) \rightarrow (1, a). \end{aligned}$$

This sort of dynamics will be generalized later to representations into abelian groups. Second, when we proceed to a new step, we do not need to take a pair we have already got in a previous cycle, since we would get indeed the same cycle. The following dichotomy will prove useful in the sequel:

**Definition 1.** If a vertex of a cycle in  $\Gamma$  has equal components, then the cycle is said to be of type I. Otherwise, it is of type II.

Note that a cycle is determined by any of its vertices. Furthermore, the type I cycles are determined by elements of  $\Sigma$ .

Now let us proceed to compute  $\text{Hom}(K_4, \Sigma)$ . Since  $K_3 \subset K_4$ , every representation  $\rho \in \text{Hom}(K_4, \Sigma)$  restricts to a representation  $\rho|_{K_3} \in \text{Hom}(K_3, \Sigma)$ , the latter being described by a cycle. All we have to do is then to check which representation in  $\text{Hom}(K_3, \Sigma)$  does extend to  $K_4$ . To this end, observe that  $K_4$  is gotten from  $K_3$  by adjunction of a generator  $x_3$  subject to the relations

$$z_m x_3 z_{m+2} = x_3 z_{m+1} x_3; m \in \mathbb{Z}.$$

Hence we may proceed as follows. Take a cycle in  $\text{Hom}(K_3, \Sigma)$  (by abuse of language, i.e. identify each representation with its orbit, since a representation in  $\text{Hom}(K_3, \Sigma)$  extends to  $K_4$  if and only if every element in its orbit does), and choose if possible a value  $b_3 \in \Sigma$  for  $\rho(x_3)$ . This value must satisfy the relations

$$a_m b_3 a_{m+2} = b_3 a_{m+1} b_3;$$

for  $m = 0, \dots, p-1$ , where  $p$  is the cycle's length and the indexation is  $\text{mod } p$ . Observe that the choice  $b_3 = 1$  is convenient, so all cycles extend to  $K_4$ . However, this is the only possibility for type I cycles to extend, for if  $b_3$  commute with some  $a_m$ , then  $b_3 = 1$ .

**Proposition 2.** Let  $\rho \in \text{Hom}(K_4, \Sigma)$  be encoded by a cycle  $(a_m)_{m=0, \dots, p-1}$  and an element  $b_3 \in \Sigma$ . Then  $b_3^p = 1$ .

As a consequence, the order of  $b_3$  divides  $p$ ; hence, if  $\gcd(p, |\Sigma|) = 1$ , then  $b_3 = 1$ .

Before giving the general procedure, let us proceed one further step to show that all type I cycles will vanish for  $n \geq 5$ . Take a cycle in  $\text{Hom}(K_3, \Sigma)$ , along with a convenient value  $b_3$  of  $\rho(x_3)$ . We look for an element  $b_4 \in \Sigma$  satisfying :

$$\begin{aligned} a_m b_4 &= b_4 a_{m+1}, \quad m = 0, \dots, p-1; \text{ and} \\ b_3 b_4 b_3 &= b_4 b_3 b_4 \end{aligned}$$

Hence, if

$$b_3 b_4 = b_4 b_3,$$

then

$$b_3 = b_4.$$

Using

$$a_m b_3 a_{m+2} = b_3 a_{m+1} b_3,$$

and

$$a_m b_4 = b_4 a_{m+1},$$

we get

$$b_3 = b_4 = 1,$$

and

$$a_m = a_{m+1}, \forall m = 0, \dots, p-1,$$

so that the representation is trivial. So, except for the trivial cycle, only type II cycles with  $b_3 \neq 1$  possibly extend to  $K_5$ , with  $b_4$  not commuting with  $b_3$ , in particular,  $b_4 \neq 1$ . Note that  $b_3$  and  $b_4$  are conjugate.

**Algorithm 1.** *The general procedure for  $\text{Hom}(K_n, \Sigma)$ ,  $n \geq 5$  is to consider only type II cycles along with convenient non trivial values  $b_3, \dots, b_{n-2}$ , which correspond to representations in  $\text{Hom}(K_{n-1}, \Sigma)$  and find a non trivial element  $b_{n-1} \in \Sigma$  such that the following relations are satisfied:*

$$\begin{aligned} a_m b_{n-1} &= b_{n-1} a_{m+1}, & m &= 0, \dots, p-1; \\ b_{n-1} b_i &= b_i b_{n-1} & i &= 3, \dots, n-3; \\ b_{n-1} b_{n-2} b_{n-1} &= b_{n-2} b_{n-1} b_{n-2} \end{aligned}$$

The element  $b_{n-1}$  has to be non trivial, otherwise the representation is trivial.

Using the various relations between the  $a_m$ 's and the  $b_i$ 's, we get the following ones :

**Proposition 3.** *Let  $\rho \in \text{Hom}(K_n, \Sigma)$ ,  $n \geq 5$  be encoded by a type II cycle  $(a_m)_{m=0}^{p-1}$  of length  $p$ , along with elements  $b_3, \dots, b_{n-1} \in \Sigma$  as previously. Then we have the following relations :*

- (1) The  $b_i$ 's are non trivial and conjugate for  $i = 3, \dots, n-1$  and  $[b_i^p, a_m] = 1$ , for  $m = 0, \dots, p-1$ ;  $i = 4, \dots, n-1$ .
- (2)  $\begin{cases} [b_i, b_j] = 1; & |i-j| \geq 2; \\ [b_i, b_j] \neq 1; & |i-j| = 1. \end{cases}$
- (3)  $b_i \neq b_j$ , for  $i, j = 3, \dots, n-1$  and  $i \neq j$ , except for the possibility  $b_3 = b_5$ , in which case  $\rho$  doesn't extend to  $\text{Hom}(K_7, \Sigma)$ .

As a consequence of the first relation,  $p$  divides the order of  $b_i$ ,  $i = 4, \dots, n-1$ ; hence that of  $\Sigma$ .

## 4. THE ABELIAN CASE

Starting with  $n = 3$ , we see that the relation between the  $a_m$ 's, written additively, becomes :

$$a_{m+2} = a_{m+1} - a_m.$$

Implementing our algorithm gives a matrix  $A$  :

$$(a, b) \xrightarrow{A} (b, b - a),$$

with  $A^3 = -I$ . All cycles have length dividing 6. More precisely, the length may be 1 ( $a = b = 0$ ) or 2 ( $a = -b$ , with  $3a = 0$ ) or 3 ( $2a = 2b = 0$ ) or 6 (otherwise). Moving to  $n = 4$ , we find that all cycles extend to  $K_4$  with  $b_3 = 1$  (only). No non trivial cycle extends to  $K_5$ , that is  $Hom(K_n, \Sigma) = 0$ , for  $\Sigma$  abelian and  $n \geq 5$ . This latter fact can also be seen from :

$$\frac{K_n}{[K_n, K_n]} = 1; n \geq 5.$$

## 5. EXTENSION TO THE BRAID GROUP

In this section, we address the question of extending representations

$$\rho \in Hom(K_n, \Sigma)$$

to representations

$$\tilde{\rho} \in Hom(B_n, \Sigma).$$

Applying [SiWil (3.5)], the extension is possible if and only if there is an element  $c \in \Sigma$  such that

$$\begin{aligned} a_m c &= c a_{m+1}, & m &= 0, \dots, p-1; \\ c b_i &= b_i c & i &= 3, \dots, n-1; \end{aligned}$$

Observe that such an element  $c$  must satisfy  $c \neq 1$ , unless  $\rho$  is trivial; and  $[c^p, a_m] = 1$ , for  $m = 0, \dots, p-1$ . In particular,  $p$  divides the order of  $c$ . Hence, if  $p \nmid |\Sigma|$ , then  $\rho$  doesn't extend to  $B_n$ . Observe also that a necessary condition for a representation  $\rho \in Hom(K_n, \Sigma)$  to extend to  $\tilde{\rho} \in Hom(B_n, \Sigma)$  is that  $\rho \in Hom(K_n, [\Sigma, \Sigma])$ . A sufficient condition is that  $\rho$  be the restriction of some representation  $\hat{\rho} \in Hom(K_{n+2}, \Sigma)$  for if this is the case, the choice  $c = b_{n+1}$  will do. In this case, since  $\hat{\rho}$  maps  $K_n$  into  $[\Sigma, \Sigma]$ ; it also maps  $K_{n+2}$  into  $[\Sigma, \Sigma]$ , for the  $b_i$ 's are conjugate (if  $n \geq 4$ ) and  $[\Sigma, \Sigma]$  is normal in  $\Sigma$ . As a result, we get the following

**Proposition 4.** *for  $n \geq 6$ ,  $Hom(K_n, \Sigma) = Hom(K_n, \Sigma^{(r)})$ , where  $r \geq 0$  is the smallest integer such that  $\Sigma^{(r+1)} = \Sigma^{(r)}$ .*

As a consequence, we get the fact that  $n \geq 6$ ,  $Hom(K_n, \Sigma)$  is trivial for any solvable group  $\Sigma$ . This generalizes the abelian case. In this case, every  $\rho \in Hom(B_n, \Sigma)$  has a cyclic image generated by  $\rho(\sigma_1) = \dots = \rho(\sigma_1)$ , i.e.  $|Hom(B_n, \Sigma)| = |\Sigma|$ .

Actually we can enhance our algorithm to one which gives for fixed  $n \geq 5$  the sets  $Hom(K_n, \Sigma)$  and  $Hom(B_n, \Sigma)$ .

*Step one: find all cycles of both types. This gives  $Hom(K_3, \Sigma)$ .*

*Step two: For the trivial cycle, take any  $c \in \Sigma$  to be arbitrary. For a type II cycle  $C$  (with length  $p$  dividing  $|\Sigma|$ ), find  $c \neq 1$  such that  $a_m c = c a_{m+1}$ , for  $m = 0, \dots, p-1$ . This gives  $Hom(B_3, \Sigma)$ .*

*Step three: For a cycle of any type, take  $b_3 = 1$ . For a type II cycle  $C$  (with length  $p$  such that  $\gcd(p, |\Sigma|) \neq 1$ ), find  $b_3 \neq 1$  such that  $a_m b_3 a_{m+2} = b_3 a_{m+1} b_3$ , for  $m = 0, \dots, p-1$ . This gives  $Hom(K_4, \Sigma)$ .*

*Step four: Beside the trivial cycle, take a type II cycle  $C$  along with a convenient  $b_3$  (with length  $p$  dividing  $|\Sigma|$ ). If this cycle occurs in  $\text{Hom}(B_3, \Sigma)$  with some convenient  $c$  then :*

*if  $cb_3 = b_3c$  (which need not be checked if  $b_3 = 1$ ), then the representation  $[C, b_3]$  moves up to a representation  $[C, b_3, c]$  in  $\text{Hom}(B_4, \Sigma)$ ;*

*if  $cb_3c = b_3cb_3$  (which is impossible if  $b_3 = 1$ ), then the representation  $[C, b_3]$  moves up to a representation  $[C, b_3, b_4]$  in  $\text{Hom}(K_5, \Sigma)$  by taking  $b_4 = c$ .*

*·*  
*·*  
*·*

*Step i: take a representation  $\rho$  in  $\text{Hom}(K_i, \Sigma)$ , encoded by a type II cycle  $C$  along with convenient values  $b_3, \dots, b_{i-1}$ . if  $[C, b_3, b_4, \dots, b_{i-2}, c]$  occurs in  $\text{Hom}(B_{i-1}, \Sigma)$  with some convenient  $c$  then:*

*if  $cb_{i-1} = b_{i-1}c$ , then the representation  $[C, b_3, b_4, \dots, b_{i-1}]$  moves up to a representation  $[C, b_3, b_4, \dots, b_{i-1}, c]$  in  $\text{Hom}(B_i, \Sigma)$ ;*

*if  $cb_{i-1}c = b_{i-1}cb_{i-1}$ , then the representation  $[C, b_3, b_4, \dots, b_{i-1}]$  moves up to a representation  $[C, b_3, b_4, \dots, b_{i-1}, b_i]$  in  $\text{Hom}(K_{i+1}, \Sigma)$  by taking  $b_i = c$ .*

## 6. PERMUTATION REPRESENTATIONS

Our goal in this section is to study representations of  $K_n$  into the symmetric group  $S_r$ . Note that there is a natural homomorphism:

$$\pi : B_n \rightarrow S_n,$$

for all  $n \geq 2$ , given by  $\sigma_i \mapsto (ii + 1)$ . This restricts to  $K_n$  to give a non trivial homomorphism:

$$\begin{aligned} x_i &\mapsto (12)(ii + 1) \quad i = 3, \dots, n - 1 \\ z_m &\mapsto \begin{cases} (132), & m \text{ is even} \\ (123), & m \text{ is odd} \end{cases} \end{aligned}$$

We start, as usual, with  $n = 3$ , and describe  $\text{Hom}(K_3, S_r)$ . Note that it contains  $\pi|_{K_3}$ , for  $r \geq 3$ . Recall that a non trivial element  $a \in S_r$  has order two if and only if it is a product of disjoint transpositions. Let  $n_r$  be the number of such elements. This gives us a means to compute the number of type I cycles to be  $\frac{1}{2}(1 + n_r + r!)$  and of representations coming from them to be  $3r! - 2$ .

Since  $S_2 = \{1, (12)\}$ ,  $\text{Hom}(K_3, S_2)$  consists only of the following type I cycle:

$$(1, (12)) \rightarrow ((12), (12)) \rightarrow ((12), 1) \rightarrow (1, (12)),$$

along with the trivial representation. So

**Claim 1.**  $|\text{Hom}(K_3, S_2)| = 4$ .

As for  $\text{Hom}(K_3, S_3)$ , there are three type I cycles of length 3 corresponding to transpositions and one type I cycle of length 6 corresponding to the 3-cycle (123) (and its inverse). Looking at type II cycles, we find two cycles of length 9 corresponding to the pairs  $((23), (12))$  and  $((23), (123))$  and one cycle of length 2 corresponding to the pair  $((123), (132))$ . This last one is exactly the orbit (under the shift map  $\sigma$ ) of  $\pi|_{K_3}$ . All by all, we have

**Claim 2.**  $|\text{Hom}(K_3, S_3)| = 36$ .

In the last section, we present among other things, the results of computer calculations of type II cycles in the graph of  $\text{Hom}(K_3, S_4)$  using Maple. Moving to  $n = 4$ , we find that all (type I) cycles in  $\text{Hom}(K_3, S_2)$  extend to  $K_4$  with  $b_3 = 1$ . So

**Claim 3.**  $|\text{Hom}(K_4, S_2)| = 4$ .

No cycle in  $\text{Hom}(K_3, S_3)$  extends to  $K_4$  with non trivial  $b_3$  :

**Claim 4.**  $|\text{Hom}(K_4, S_3)| = |\text{Hom}(K_3, S_3)| = 36$ .

Out of 71 (type II) cycles in  $\text{Hom}(K_3, S_4)$  only ten do extend to  $K_4$ , each with three possibilities for  $b_3$  (the same for all; see the last section). So

**Claim 5.**  $|\text{Hom}(K_4, S_4)| = |\text{Hom}(K_3, S_4)| + 30$ .

For  $n = 5$ , we find that no type I cycle and no type II cycle with  $b_3 = 1$  extends to  $K_5$ , and that

**Claim 6.**  $|\text{Hom}(K_5, S_2)| = |\text{Hom}(K_5, S_3)| = 1$ .

As for type II cycles with  $b_3 \neq 1$ , none of the thirty cycles extends to  $K_5$ :

**Claim 7.**  $|\text{Hom}(K_5, S_4)| = 1$ .

We also find :

**Claim 8.**  $|\text{Hom}(K_6, S_5)| = |\text{Hom}(K_7, S_6)| = 1$ .

Experimental results recover the well known fact that that the process given by Algorithm 1 will stop at step  $n = r$ . That is:

**Proposition 5.**  $\text{Hom}(K_n, S_r)$  is trivial for  $r \geq 4$  and  $n \geq r + 1$ .

*Proof.* see Lin □

It is obvious that a cycle (of any type) can not extend to  $K_n$  if it doesn't extend to  $K_{n-1}$ , so Proposition 5 asserts exactly that  $\text{Hom}(K_{r+1}, S_r)$  is trivial. Recall that for  $n \leq r$ ,  $\text{Hom}(K_n, S_r)$  is not trivial since it contains the homomorphism  $\pi|_{K_n} : K_n \rightarrow S_n$ .

**Proposition 6.** for  $n \geq 6$ ,  $\text{Hom}(K_n, S_r) = \text{Hom}(K_n, A_r)$ .

*Proof.* Apply Proposition 4 to  $\Sigma = S_r$ . □

Note that according to Proposition 5, for  $n \geq r + 1$ , every representation  $\tilde{\rho} : B_n \rightarrow S_r$  factorizes through the abelianized group  $(B_n)_{ab}$ , and has a cyclic image. Hence, there are  $r!$  possible choices for  $\tilde{\rho}$ .

## 7. CONSEQUENCES

Regarding the correspondence between subgroups of finite index of a group  $K$  and its representations into symmetric groups, we investigate the subgroups of index  $r$  of  $K_n$  for low degrees  $r$ . The general principle is to compute the number of transitive representations of  $K$  into  $S_r$  to deduce the number of subgroups of  $K$  with index exactly  $r$ . We start with  $K_3$  as usual. Note that since  $K_3$  is freely generated by  $z_0$  and  $z_{-1}$ , it maps onto any symmetric group, and hence, has subgroups of every index. Now, if a representation in  $\text{Hom}(K_3, S_r)$  is transitive,

then so are the representations in its orbit. Consider a type I cycle in  $\text{Hom}(K_3, S_r)$ . Then the representations it defines are transitive if and only if the defining element  $a$  is (with respect to the action of  $S_r$  on  $\{1, \dots, r\}$ ). This exactly means that  $a$  is an  $r$ -cycle. If  $r > 2$  then  $a^2 \neq 1$  and the cycle has length 6.

**Claim 9.** *The number of transitive representations  $\rho \in \text{Hom}(K_3, S_r)$ ,  $r \geq 2$  coming from type I cycles is  $3(r-1)!$ .*

For  $r = 2$ , there are only type I cycles and there is only one 2-cycle, which has length 3; Hence, The number of transitive representations  $\rho \in \text{Hom}(K_3, S_2)$  is 3. The kernels of these representations give rise to subgroups of  $K_3$  with index 2.

**Claim 10.** *There are three subgroups of  $K_3$  with index 2.*

Now we compute the number of subgroups of  $K_3$  with index 3. Among all representations we have seen in example 2, there are six transitive representations coming from the only type I cycle and all representations coming from type II cycles are transitive. Hence:

**Claim 11.** *The number of transitive representations in  $\text{Hom}(K_3, S_3)$  is 26, consequently there are thirteen subgroups of  $K_3$  with index 3.*

We can proceed in this way for every degree  $r$ . To compute the number of transitive representations of  $K_3$  into  $S_r$ , we need only consider those coming from type II cycles, since we already know the number of those coming from type I cycles. This can be done using a computer algebra system, by taking any cycle  $C = (a_0, \dots, a_{p-1})$  and checking if the subgroup  $\langle a_0, \dots, a_{p-1} \rangle$  of  $S_r$  acts transitively on  $\{1, \dots, r\}$ . If so, this gives rise to  $p$  transitive representations in  $\text{Hom}(K_3, S_3)$ . Then we divide the total number by  $(r-1)!$  to find the number of subgroups of  $K_3$  of index  $r$ .

Now let us consider  $\text{Hom}(K_4, S_r)$ . For  $r = 2$  we have, as previously:

**Claim 12.** *There are three subgroups of  $K_4$  with index 2.*

As for transitive representations in  $\text{Hom}(K_4, S_3)$ , since all cycles in  $\text{Hom}(K_3, S_3)$  extend to  $K_4$  with only  $b_3 = 1$ , we have:

**Claim 13.** *There are twenty six transitive representations in  $\text{Hom}(K_4, S_3)$ , hence thirteen subgroups of  $K_3$  with index 3.*

For  $r \geq 4$ , we have  $3(r-1)!$  transitive representations coming from type I cycles, and we must check which representation coming from a type II cycle is transitive. For a cycle  $C = (a_0, \dots, a_{p-1})$  such that  $\langle a_0, \dots, a_{p-1} \rangle$  failed to be transitive, we check if  $\langle a_0, \dots, a_{p-1}, b_3 \rangle$  (with  $b_3$  non trivial) is transitive. Indeed, if  $\langle a_0, \dots, a_{p-1} \rangle$  is transitive, then so is  $\langle a_0, \dots, a_{p-1}, b_3 \rangle$ . Finally, we divide the total number by  $(r-1)!$  to find the number of subgroups of  $K_4$  of index  $r$ .

Now, we consider  $n \geq 5$ , where we get rid of type I cycles. Suppose we have found the transitive representations in  $\text{Hom}(K_{n-1}, S_r)$ . We then take, for fixed  $r$ , a type II cycle  $C = (a_0, \dots, a_{p-1})$  along with values  $b_3, \dots, b_{n-1}$ , such that  $\langle a_0, \dots, a_{p-1}, b_3, \dots, b_{n-2} \rangle$  failed to be transitive and check if  $\langle a_0, \dots, a_{p-1}, b_3, \dots, b_{n-1} \rangle$  is transitive. We may enhance algorithm 1 by checking, each time we get a new type II cycle, if it is transitive, and if not, we re-check at each time the cycle extends from  $K_i$  to  $K_{i+1}$ ,  $i = 3, \dots, n-1$ , after having augmented it with  $b_i$ . Dividing by  $(r-1)!$  the number of transitive representations in  $\text{Hom}(K_n, S_r)$  we find the

number of subgroups of  $K_n$  with index  $r$ . As a consequence of Proposition 5, we get the following:

**Proposition 7.** *For  $n \geq 5$  and  $2 \leq r \leq n - 1$ , there are no subgroups of  $K_n$  with index  $r$ . Moreover, every nontrivial representation  $\rho$  of  $K_n$  into  $S_n$  is transitive.*

**Remark 3.** *We can likewise investigate the number of subgroups of  $B_n$  with a given index  $r$  by looking at transitive representations of  $B_n$  into  $S_r$ . Namely, according to Proposition 5, there is exactly one subgroup of index  $r$  in  $B_n$ , for  $1 \leq r \leq n - 1$ . Moreover, if  $\rho : B_n \rightarrow S_n$  is a representation, then  $\rho|_{K_n}$  is either trivial or transitive, according to Proposition 7. In the first case,  $\rho$  has a cyclic image and we know when it is transitive. In the second case,  $\rho$  is transitive.*

## 8. EXPERIMENTAL FACTS

In what follows, we list the type II cycles of  $Hom(K_n, S_r)$  for various (small)  $n$  and  $r$ . A word about the notation: each cycle  $B[a_0, a_1] = [a_2, a_3, \dots, a_{p-1}, a_0, a_1]$  is indexed by its first vertex  $(a_0, a_1)$  and is followed by its length  $p$ . Elements  $\tau \in S_r$  are ordered from 1 to  $r!$  with respect to the lexicographic order on the vectors  $(\tau(1), \dots, \tau(r))$ . It would have taken too much space to list the cycles for  $r = 5$ . We found that there were no (type II) cycles in  $Hom(K_5, S_4)$  nor in  $Hom(K_6, S_5)$ . Furthermore,  $Hom(K_4, S_3)$  contains no type II cycles with non trivial  $b_3$ , as predicted by Proposition 5.

$n = 3; r = 3 :$

B[2, 3] = [5, 6, 2, 5, 3, 6, 5, 2, 3]

9

B[2, 4] = [6, 3, 4, 2, 6, 4, 3, 2, 4]

9

B[4, 5] = [4, 5]

2

$n = 3; r = 4 :$

B[2, 3] = [5, 6, 2, 5, 3, 6, 5, 2, 3]

9

B[2, 4] = [6, 3, 4, 2, 6, 4, 3, 2, 4]

9

B[2, 7] = [8, 2, 7]

3

B[2, 8] = [7, 2, 8]

3

B[2, 9] = [11, 6, 16, 18, 3, 20, 19, 2, 9]

9

B[2, 10] = [12, 3, 23, 21, 6, 14, 13, 2, 10]

9

B[2, 11] = [9, 6, 18, 16, 3, 19, 20, 2, 11]

9

B[2, 12] = [10, 3, 21, 23, 6, 13, 14, 2, 12]

9

B[2, 13] = [19, 22, 4, 23, 15, 12, 11, 2, 13]

9

$B[2, 14] = [20, 15, 18, 5, 22, 10, 9, 2, 14]$   
 9  
 $B[2, 15] = [21, 22, 2, 21, 15, 22, 21, 2, 15]$   
 9  
 $B[2, 16] = [22, 15, 16, 2, 22, 16, 15, 2, 16]$   
 9  
 $B[2, 17] = [23, 7, 24, 23, 2, 17]$   
 6  
 $B[2, 18] = [24, 7, 18, 17, 2, 18]$   
 6  
 $B[2, 19] = [13, 22, 23, 4, 15, 11, 12, 2, 19]$   
 9  
 $B[2, 20] = [14, 15, 5, 18, 22, 9, 10, 2, 20]$   
 9  
 $B[2, 23] = [17, 7, 23, 24, 2, 23]$   
 6  
 $B[2, 24] = [18, 7, 17, 18, 2, 24]$   
 6  
 $B[3, 7] = [13, 15, 3, 13, 7, 15, 13, 3, 7]$   
 9  
 $B[3, 8] = [14, 22, 17, 14, 3, 8]$   
 6  
 $B[3, 9] = [15, 7, 9, 3, 15, 9, 7, 3, 9]$   
 9  
 $B[3, 10] = [16, 7, 11, 5, 15, 23, 20, 3, 10]$   
 9  
 $B[3, 11] = [17, 22, 11, 8, 3, 11]$   
 6  
 $B[3, 12] = [18, 15, 4, 14, 7, 21, 19, 3, 12]$   
 9  
 $B[3, 14] = [8, 22, 14, 17, 3, 14]$   
 6  
 $B[3, 16] = [10, 7, 5, 11, 15, 20, 23, 3, 16]$   
 9  
 $B[3, 17] = [11, 22, 8, 11, 3, 17]$   
 6  
 $B[3, 18] = [12, 15, 14, 4, 7, 19, 21, 3, 18]$   
 9  
 $B[3, 22] = [24, 3, 22]$   
 3  
 $B[3, 24] = [22, 3, 24]$   
 3  
 $B[4, 5] = [4, 5]$   
 2  
 $B[4, 8] = [20, 16, 17, 4, 13, 8, 16, 20, 17, 13, 4, 8]$   
 12  
 $B[4, 9] = [21, 20, 4, 9]$   
 4

$B[4, 10] = [22, 13, 18, 6, 21, 11, 7, 4, 10]$   
 9  
 $B[4, 11] = [23, 12, 19, 11, 13, 23, 19, 4, 11]$   
 9  
 $B[4, 12] = [24, 9, 16, 8, 12, 4, 24, 16, 9, 8, 4, 12]$   
 12  
 $B[4, 16] = [12, 9, 4, 16]$   
 4  
 $B[4, 17] = [9, 20, 24, 4, 21, 17, 20, 9, 24, 21, 4, 17]$   
 12  
 $B[4, 18] = [10, 13, 11, 18, 21, 10, 11, 4, 18]$   
 9  
 $B[4, 19] = [14, 21, 18, 19, 12, 14, 18, 4, 19]$   
 9  
 $B[4, 20] = [13, 16, 4, 20]$   
 4  
 $B[4, 22] = [18, 13, 6, 11, 21, 7, 10, 4, 22]$   
 9  
 $B[5, 7] = [14, 16, 6, 23, 9, 22, 19, 5, 7]$   
 9  
 $B[5, 8] = [13, 21, 24, 5, 20, 8, 21, 13, 24, 20, 5, 8]$   
 12  
 $B[5, 9] = [17, 12, 21, 8, 9, 5, 17, 21, 12, 8, 5, 9]$   
 12  
 $B[5, 10] = [18, 9, 14, 10, 20, 18, 14, 5, 10]$   
 9  
 $B[5, 12] = [16, 13, 5, 12]$   
 4  
 $B[5, 13] = [20, 21, 5, 13]$   
 4  
 $B[5, 14] = [19, 16, 23, 14, 9, 19, 23, 5, 14]$   
 9  
 $B[5, 16] = [24, 13, 12, 17, 16, 5, 24, 12, 13, 17, 5, 16]$   
 12  
 $B[5, 19] = [7, 16, 14, 6, 9, 23, 22, 5, 19]$   
 9  
 $B[5, 21] = [9, 12, 5, 21]$   
 4  
 $B[5, 23] = [11, 20, 10, 23, 16, 11, 10, 5, 23]$   
 9  
 $B[6, 7] = [20, 22, 6, 20, 7, 22, 20, 6, 7]$   
 9  
 $B[6, 8] = [19, 15, 24, 19, 6, 8]$   
 6  
 $B[6, 10] = [24, 15, 10, 8, 6, 10]$   
 6  
 $B[6, 12] = [22, 7, 12, 6, 22, 12, 7, 6, 12]$   
 9

$B[6, 15] = [17, 6, 15]$   
 3  
 $B[6, 17] = [15, 6, 17]$   
 3  
 $B[6, 19] = [8, 15, 19, 24, 6, 19]$   
 6  
 $B[6, 24] = [10, 15, 8, 10, 6, 24]$   
 6  
 $B[8, 17] = [24, 8, 17]$   
 3  
 $B[8, 18] = [23, 8, 23, 18, 8, 18]$   
 6  
 $B[8, 24] = [17, 8, 24]$   
 3  
 $B[9, 13] = [9, 13]$   
 2  
 $B[9, 18] = [11, 16, 19, 18, 20, 11, 19, 9, 18]$   
 9  
 $B[10, 14] = [12, 23, 10, 21, 14, 23, 13, 10, 14]$   
 9  
 $B[10, 17] = [10, 19, 17, 19, 10, 17]$   
 6  
 $B[11, 14] = [24, 14, 11, 24, 11, 14]$   
 6  
 $B[12, 20] = [12, 20]$   
 2  
 $B[16, 21] = [16, 21]$   
 2

$n = 4; r = 4 :$

ten cycles of length 2 along with three values  $b_3 = 8, 17, 24$ .

$[8, [4, 5], [4, 9], [4, 16], [4, 20], [5, 12], [5, 13], [5, 21], [9, 13], [12, 20], [16, 21]]$   
 $[17, [4, 5], [4, 9], [4, 16], [4, 20], [5, 12], [5, 13], [5, 21], [9, 13], [12, 20], [16, 21]]$   
 $[24, [4, 5], [4, 9], [4, 16], [4, 20], [5, 12], [5, 13], [5, 21], [9, 13], [12, 20], [16, 21]]$

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