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Abstract

We study that a solution of the initial value problem associated for the coupled system of equations of Korteweg - de Vries type which appears as a model to describe the strong interaction of weakly nonlinear long waves, has analyticity in time and smoothing effect up to real analyticity if the initial data only has a single point singularity at $x = 0$.

Keywords and phrases: Evolution equations, Gevrey class, Bourgain space, smoothing effect.

Mathematics Subject Classification: 35Q53

1 Introduction

We consider the following coupled system of equations of Korteweg - de Vries type

$$\widetilde{u}_t + \widetilde{u}_{xxx} + a_3\,\widetilde{v}_{xxx} + \widetilde{u}\,\widetilde{u}_x + a_1\,\widetilde{v}\,\widetilde{v}_x + a_2\,(\widetilde{u}\,\widetilde{v})_x = 0, \quad x,\, t \in \mathbb{R} \tag{1.1}$$

$$b_1\,\widetilde{v}_t + \widetilde{v}_{xxx} + b_2\,a_3\,\widetilde{u}_{xxx} + \widetilde{v}\,\widetilde{v}_x + b_2\,a_2\,\widetilde{u}\,\widetilde{u}_x + b_2\,a_1\,(\widetilde{u}\,\widetilde{v})_x = 0, \tag{1.2}$$

$$\widetilde{u}(x,\,0) = \widetilde{u}_0(x), \quad \widetilde{v}(x,\,0) = \widetilde{v}_0(x). \tag{1.3}$$

where $\widetilde{u} = \widetilde{u}(x,\,t)$, $\widetilde{v} = \widetilde{v}(x,\,t)$ are real-valued functions of the variables x and t and a_1, a_2, a_3, b_1, b_2 are real constants with $b_1 > 0$ and $b_2 > 0$. The original coupled system is

$$\widetilde{u}_t + \widetilde{u}_{xxx} + a_3\,\widetilde{v}_{xxx} + \widetilde{u}^p\,\widetilde{u}_x + a_1\,\widetilde{v}^p\,\widetilde{v}_x + a_2\,(\widetilde{u}^p\,\widetilde{v})_x = 0, \quad x,\, t \in \mathbb{R} \tag{1.4}$$

$$b_1\,\widetilde{v}_t + \widetilde{v}_{xxx} + b_2\,a_3\,\widetilde{u}_{xxx} + \widetilde{v}^p\,\widetilde{v}_x + b_2\,a_2\,\widetilde{u}^p\,\widetilde{u}_x + b_2\,a_1\,(\widetilde{u}\,\widetilde{v}^p)_x = 0 \tag{1.5}$$

$$\widetilde{u}(x,\,0) = \widetilde{u}_0(x), \quad \widetilde{v}(x,\,0) = \widetilde{v}_0(x) \tag{1.6}$$

where $\widetilde{u} = \widetilde{u}(x, t)$, $\widetilde{v} = \widetilde{v}(x, t)$ are real-valued functions of the variables x and t and a_1, a_2, a_3, b_1, b_2 are real constants with $b_1 > 0$ and $b_2 > 0$. The power p is an integer larger than or equal to one. The system (1.4)-(1.6) has the structure of a pair of Korteweg - de Vries equations coupled through both dispersive and nonlinear effects. In the case $p = 1$, the system (1.4)-(1.6) was derived by Gear and Grimshaw [9] as a model to describe the strong interaction of weakly nonlinear, long waves. Mathematical results on the system (1.4)-(1.6) were given by J. Bona *et al.* [5]. They proved that (1.4)-(1.6) is globally well posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s \geq 1$ provided $|a_3| < 1/\sqrt{b_2}$. The system (1.4)-(1.6) has been intensively studied by several authors (see [2, 3, 5, 7, 23] and the references therein). We have the following conservation laws

$$\mathbb{E}_1(\widetilde{u}) = \int_{\mathbb{R}} \widetilde{u}\, dx \quad , \quad \mathbb{E}_2(\widetilde{v}) = \int_{\mathbb{R}} \widetilde{v}\, dx \quad , \quad \mathbb{E}_3(\widetilde{u}, \widetilde{v}) = \int_{\mathbb{R}} (b_2\widetilde{u}^2 + b_1\widetilde{v}^2) dx \tag{1.7}$$

The time-invariance of the functionals $\mathbb{E}_1$ and $\mathbb{E}_2$ expresses the property that the mass of each mode separately is conserved during interaction, while that of $\mathbb{E}_3$ is an expression of the conservation of energy for the system of two models taken as a whole. The solutions of (1.4)-(1.6) satisfy an additional conservation law which is revealed by the time-invariance of the functional

$$\mathbb{E}_4 = \int_{\mathbb{R}} \left(b_2\, \widetilde{u}_x^2 + \widetilde{v}_x^2 + 2b_2a_3\widetilde{u}_x\widetilde{v}_x - b_2\frac{\widetilde{u}^3}{3} - b_2a_2\widetilde{u}^2\widetilde{v} - b_2a_2\widetilde{u}^2\widetilde{v} - b_2a_1\widetilde{u}\widetilde{v}^2 - \frac{\widetilde{v}^3}{3} \right) dx$$

The functional $\mathbb{E}_4$ is a Hamiltonian for the system (1.4)-(1.6) and if $b_2a_3^2 < 1$, ϕ_4 will be seen to provide an a priori estimate for the solutions $(\widetilde{u}, \widetilde{v})$ of (1.4)-(1.6) in the space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Furthermore, the linearization of (1.1)-(1.3) about the rest state can be reduced to two, linear Korteweg - de Vries equations by a process of diagonalization. Using this remark and the smoothing properties (in both the temporal and spatial variables) for the linear Korteweg - de Vries derived by Kato [13, 15], Kenig, Ponce and Vega [18, 19] it will be shown that (1.4)-(1.6) is locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s \geq 1$ whenever $\sqrt{b_2}a_3 \neq 1$. This result was improved by J. M. Ash *et al.* [1] showing that the system (1.1)-(1.3) is globally well-posed in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ provided that $\sqrt{b_2}a_3 \neq 1$. In 2004, F. Linares and M. Panthee [21] improve this result showing that the system (1.1)-(1.3) is locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > -3/4$ and globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > -3/10$ under some conditions on the coefficients, indeed for $a_3 = 0$ and $b_1 = b_2$. Following the idea W. Craig *et al.* [6], it is shown in [23] that C^∞ solutions $(\widetilde{u}(\,\cdot\,, t), \widetilde{v}(\,\cdot\,, t))$ to (1.1)-(1.3) are obtained for $t > 0$ if the initial data $(\widetilde{u}(x, 0), \widetilde{v}(x, 0))$ belong to a suitable Sobolev space satisfying resonable conditions as $|x| \to \infty$. Since (1.1)-(1.3) is a coupled system of Korteweg-de Vries equations, it is natural to ask whether it has a smoothing effect up to real analyticity if the initial data only has a single point singularity at $x = 0$ as the known results for the scalar case of a single Korteweg -de Vries equation. Using the scaling argument we can have an insight to this question. In this paper our purpose is to prove the analyticity in time of solutions to (1.1)-(1.3) without regularity assumption on the initial data improving those obtained in [23]. Our main tool is the generator of dilation $P = 3\,t\,\partial_t + x\,\partial_x$. which almost commutes with the linear Korteweg-de Vries operator $L = \partial_t + \partial_x$. Indeed $[L, P] = 3\,L$. A typical example of initial data satisfying the assumption of the above theorem is the Dirac delta measure, since $(x^k\,\partial_x)^k\delta(x) = (-1)^k\,k!\,\delta(x)$. The other example of the data is p. v. $\frac{1}{x}$, where

p. v. denotes the Cauchy principal value. Linear combination of those distributions with analytic H^s data satisfying the assumption is also possible. In this sense, the Dirac delta measure adding the soliton initial data can be taken as an initial datum. Using the operator $K = x \cdot \bigtriangledown + 2\,i\,t\,\partial_t$ it was proved the Gevrey smoothing effect in space variable [8]. Indeed, it was shown that, if the initial data belongs to a Gevrey class of order 2, then solutions of some nonlinear Schrödinger equations become analytic in the space variable for $t \neq 0$. For the Korteweg-de Vries equations version of the generator of dilation is also useful to study the analyticity in time and the Gevrey effect in the space variables for solutions [8].
This paper is organized as follows: In section 2 we have the reduction of the problem and we outline briefly the notation, terminology to be used subsequently and results that will be used several times. In section 3 we prove a theorem of existence and well-posedness of the solutions. In section 4 we prove the following theorem:

Theorem 1.1. *Suppose that the initial data* $(\widetilde{u}_0,\, \widetilde{v}_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$ *and* $A_0,\, A_1 > 0$ *such that*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!}\, ||(x\,\partial_x)^k \widetilde{u}_0||_{H^s(\mathbb{R})} < +\infty \qquad : \qquad \sum_{k=0}^{\infty} \frac{A_1^k}{k!}\, ||(x\,\partial_x)^k \widetilde{u}_0||_{H^s(\mathbb{R})} < +\infty. \tag{1.8}$$

Then for some $b \in (1/2,\, 7/12)$, *there exist* $T = T(||\widetilde{u}_0||_{H^s(\mathbb{R})},\, ||\widetilde{v}_0||_{H^s(\mathbb{R})})$ *and a unique solution of* (1.1)-(1.3) *in a certain time* $(-T,\, T)$ *and the solution* $(\widetilde{u},\, \widetilde{v})$ *is time locally well-posed, i. e., the solution continuously depends on the initial data. Moreover, the solution* $(\widetilde{u},\, \widetilde{v})$ *is analytic at any point* $(x,\, t) \in \mathbb{R} \times \{(-T,\, 0) \cup (0,\, T)\}$.

Corollary 1.1. *Let* $s > -3/4$, $b \in (1/2,\, 7/12)$. *Suppose that the initial data* $(\widetilde{u}_0,\, \widetilde{v}_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, *and* $A_0,\, A_1 > 0$ *such that*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{(k!)^3}\, ||(x\,\partial_x)^k \widetilde{u}_0||_{H^s(\mathbb{R})} < +\infty \qquad : \qquad \sum_{k=0}^{\infty} \frac{A_1^k}{(k!)^3}\, ||(x\,\partial_x)^k \widetilde{u}_0||_{H^s(\mathbb{R})} < +\infty. \tag{1.9}$$

Then there exists a unique solution $(\widetilde{u},\, \widetilde{v}) \in C((-T,\, T),\, H^s(\mathbb{R})) \cap X_b^s \times C((-T,\, T),\, H^s(\mathbb{R})) \cap X_b^s$ *to the coupled system of Korteweg- de Vries equation* (1.1)-(1.3) *for a certain* $(-T,\, T)$ *and for any* $t \in (-T,\, 0) \cup (0,\, T)$, *the pair* $(\widetilde{u},\, \widetilde{v})$ *are analytic functions in the space variable and for* $x \in \mathbb{R}$, $\widetilde{u}(x,\, \cdot\,)$ *and* $\widetilde{v}(x,\, \cdot\,)$ *are Gevrey 3 as function of the time variable.*

Remark 1.1. In Theorem 1.1 and Corollary 1.2, the assumption on the initial data implies analyticity and Gevrey 3 regularity except at the origin respectively. In this sense, those results state that the singularity at the origin immediately disappears after $t > 0$ or $t < 0$, up to analyticity.

Remark 1.2. The crucial part for obtaining a full regularity is to gain the $L^2(\mathbb{R}^2)$ regularity of the solutions $(u_k,\, v_k)$ from the negative order Sobolev space. This part is obtained in Proposition 4.1 in Section 4. We utilize a three steps recurrence argument for treating the nonlinearity appearing in the right hand side of

$$t\,\partial_x^3 u_k = -\frac{1}{3}\,P u_k + \frac{1}{3}\,x\,\partial_x u_k + t\,B_k^1(u,\, u) + t\,B_k^2(v,\, v) + t\,B_k^3(u,\, v) \tag{1.10}$$

$$t\,\partial_x^3 v_k = -\frac{1}{3}\,P v_k + \frac{1}{3}\,x\,\partial_x v_k + t\,C_k^1(u,\, u) + t\,C_k^2(v,\, v) + t\,C_k^3(u,\, v). \tag{1.11}$$

Then step by step, we obtain the pointwise analytic estimates

$$\sup_{t\in[t_0-\epsilon,\,t_0+\epsilon]} ||\partial_t^m\partial_x^l u||_{H^1(x_0-\epsilon,\,x_0+\epsilon)} \le c\,A_1^{m+l}\,(m+l)!, \qquad l,\,m=0,\,1,\,2,\,\ldots \tag{1.12}$$

$$\sup_{t\in[t_0-\epsilon,\,t_0+\epsilon]} ||\partial_t^m\partial_x^l v||_{H^1(x_0-\epsilon,\,x_0+\epsilon)} \le c\,A_2^{m+l}\,(m+l)!, \qquad l,\,m=0,\,1,\,2,\,\ldots \tag{1.13}$$

Since initially we do not know whether the solution belong to even $L^2(\mathbb{R}^2)$ we should mention that the local well-posedness is essentially important for our argument and therefore it merely satisfies the coupled system equations in the sense of distribution.

2 Reduction of the Problem and Preliminary Results

As mentioned in the introduction we consider the following coupled system of equations of Korteweg - de Vries type (1.1)-(1.3). If $a_3 = 0$ there is no coupling in the dispersive terms. Let us assume that $a_3 \neq 0$. We are interested in decoupling the dispersive terms in the system (1.1)-(1.3). For this, let $a_3^2\,b_2 \neq 1$. We consider the associated linear system

$$W_t + A\,W_{xxx} = 0, \qquad W(x,\,0) = W_0(x) \tag{2.1}$$

where

$$W = \begin{bmatrix} u \\ v \end{bmatrix} \qquad , \qquad A = \begin{bmatrix} 1 & a_3 \\ \frac{a_3\,b_2}{b_1} & \frac{1}{b_1} \end{bmatrix}.$$

The eigenvalues of A are given by

$$\alpha_+ = \frac{1}{2}\left(1 + \frac{1}{b_1} + \sqrt{\left(1-\frac{1}{b_1}\right)^2 + \frac{4\,b_2\,a_3^2}{b_1}}\right) \tag{2.2}$$

$$\alpha_- = \frac{1}{2}\left(1 + \frac{1}{b_1} - \sqrt{\left(1-\frac{1}{b_1}\right)^2 + \frac{4\,b_2\,a_3^2}{b_1}}\right) \tag{2.3}$$

which are distinct since $b_1 > 0$, $b_2 > 0$ and $a_3 \neq 0$. Our assumption $a_3^2\,b_2 \neq 1$ guarantees that $\alpha_\pm \neq 0$. Thus we can write the system (1.1)-(1.3) in a matrix form as in [21]. After we make the change of scale

$$\widetilde{u}(x,\,t) = u(\alpha_+^{-1/3}\,x,\,t) \quad \text{and} \quad \widetilde{v}(x,\,t) = v(\alpha_-^{-1/3}\,x,\,t).$$

Then we obtain the system of equations

$$u_t + u_{xxx} + a\,u\,u_x + b\,v\,v_x + c\,(u\,v)_x = 0, \quad x,\,t\in\mathbb{R} \tag{2.4}$$

$$v_t + v_{xxx} + \widetilde{a}\,u\,u_x + \widetilde{b}\,v\,v_x + \widetilde{c}\,(u\,v)_x = 0, \tag{2.5}$$

$$u(x,\,0) = u_0(x), \quad v(x,\,0) = v_0(x) \tag{2.6}$$

where a, b, c and $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ are constant.

Remark 2.1. Notice that the nonlinear terms involving the functions u and v are not evaluated at the same point. Therefore those terms are not local anymore.

For $s,\, b \in \mathbb{R}$ define the spaces X_b^s and X_{b-1}^s to be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with respect to the norms

$$||u||_{X_b^s} = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |\tau - \xi^3|)^{2b} \, (1 + |\xi|)^{2s} \, |\widehat{u}(\xi,\, \tau)|^2 \, d\xi \, d\tau \right)^{1/2}$$

and

$$||u||_{X_{b-1}^s} = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} (1 + |\tau - \xi^3|)^{2(b-1)} \, (1 + |\xi|)^{2s} \, |\widehat{u}(\xi,\, \tau)|^2 \, d\xi \, d\tau \right)^{1/2}$$

where $X_b^s = \{u \in \mathcal{S}'(\mathbb{R}^2) :\ ||u||_{X_b^s} < \infty\}$. Let $\mathcal{F}_x$ and $\mathcal{F}_{x,\,t}$ be the Fourier transform in the x and $(x,\, t)$ variables respectively. The Riesz operator D_x is defined by $D_x = \mathcal{F}_\xi^{-1} \, |\xi| \, \mathcal{F}_x$. The fractional derivative is defined by

$$\begin{aligned} < D_x >^s &= \mathcal{F}_\xi^{-1} \, < \xi >^s \, \mathcal{F}_x = \mathcal{F}_\xi^{-1} \, (1 + |\xi|^2)^{s/2} \, \mathcal{F}_x \\ < D_{x,\,t} >^s &= \mathcal{F}_{\xi,\,\tau}^{-1} \, < |\xi| + |\tau| >^s \, \mathcal{F}_{x,\,t} \end{aligned}$$

For $< \cdot >= (1 + |\cdot|^2)^{1/2}$, we have

i) $|| \cdot ||_{H^b(\mathbb{R}:\, H^r(\mathbb{R}))} = || < D_t >^b < D_x >^r \cdot ||_{L^2_{x,\,t}(\mathbb{R}^2)}$.
ii) $H^s(\mathbb{R}) = \{u \in \mathcal{S}'(\mathbb{R}) : < D_x >^s \, u \in L^2(\mathbb{R})\}$.
iii) $|| \cdot ||_{H^s(\mathbb{R})} = || < D_x >^s \cdot ||_{L^2(\mathbb{R})}$.

Remark 2.2. With the above notation we obtain

a) $||u||_{H^s_x(\mathbb{R})} = || < \xi >^s \widehat{u} \, ||_{L^2(\mathbb{R})}$.
b) $||u||_{L^2_t(\mathbb{R}:\, H^r_x(\mathbb{R}))} = || < \xi >^r \widehat{u} \, ||_{L^2(\mathbb{R}^2)}$.
c) $|| < D_x >^s u||_{L^2(\mathbb{R})} = ||u||_{H^s(\mathbb{R})}$.
d) $|| < D_t >^b < D_x >^r u||_{L^2_{x,\,t}(\mathbb{R}^2)} = ||u||_{H^b_t(\mathbb{R}:\, H^r_x(\mathbb{R}))}$.
e) $|| < D_{x,\,t} >^s u||_{L^2_t(\mathbb{R}:\, H^r_x(\mathbb{R}))} = || < \xi >^r < |\xi| + |\tau| >^s \widehat{u}(\xi,\, \tau)||_{L^2(\mathbb{R}^2)}$.

We consider the following operators: $L = \partial_t + \partial_x^3$ and $J = x - 3\,t\,\partial_x^2$ then $[L,\, J] \equiv L\,J - J\,L = 0$. We introduce the *"generator of dilation"* $P = 3\,t\,\partial_t + x\,\partial_x$ for the linear part of the coupled system (2.4)-(2.6) and the *"localized dilation operator"* $P_0 = 3\,t_0\,\partial_t + x_0\,\partial_x$. By employing a localization argument, we look at the operator P as a vector field $P_0 = 3\,t_0\,\partial_t + x_0\,\partial_x$ near a fixed point $(x_0,\, t_0) \in \mathbb{R} \times \{(-T,\, 0) \cup (0,\, T)\}$. Since P_0 is a directional derivative toward to $(x_0,\, t)$, we introduce another operator $\mathcal{L}_0^3 = t_0\,\partial_x^3$ which plays the role of a non-tangential vector field to P_0. Since P_0 and $\mathcal{L}_0$ are linearly independent, the space and time derivative can be covered by those operator. The main reason why we choose $\mathcal{L}_0$ is because the corresponding variable coefficients operator $\mathcal{L}^3 = t\,\partial_x^3$ can be treated via the equations (1.10)-(1.11) and a cut-off procedure enables us to handle the right hand side of those.

Remark 2.3. For L and P we have the following properties:

a) $[L,\, P] \equiv L\,P = (P + 3)L$.
b) $L\,P^k = (P + 3)^k L$.

c) $(P+3)^k\partial_x = \partial_x(P+2)^k$.
d) $(P+3)^k\partial_x^3 = \partial_x^3 P^k$.
e) $P_0\,P = P\,P_0 + 3\,P_0 - 2\,x_0\,\partial_x$.

Notation. The summation $\sum_{\substack{k=k_1+k_2+k_3\\ 0\le k_1,\,k_2,\,k_3\le k}}$ is simply abbreviated by $\sum_{k=k_1+k_2+k_3}$.

Let $P^k u = u_k$, then

$$\begin{aligned}
\partial_t(P^k u) + \partial_x^3(P^k u) &= L\,P^k u = (P+3)^k Lu = (P+3)^k(\partial_t u + \partial_x^3 u)\\
&= -(P+3)^k\left[\frac{a}{2}\,\partial_x(u^2) + \frac{b}{2}\,\partial_x(v^2) + c\,\partial_x(u\,v)\right]\\
&= -\frac{a}{2}\,(P+3)^k\partial_x(u^2) - \frac{b}{2}\,(P+3)^k\partial_x(v^2) - c\,(P+3)^k\partial_x(u\,v)\\
&= -\frac{a}{2}\,\partial_x(P+2)^k(u^2) - \frac{b}{2}\,\partial_x(P+2)^k(v^2) - c\,\partial_x(P+2)^k(u\,v).
\end{aligned}$$

Noting that $(P+2)^k u = \sum_{j=0}^{k}\binom{k}{j}2^{k-j}P^j u$. Hence

$$\begin{aligned}
B_k^1(u,\,u) &= -\frac{a}{2}\,\partial_x(P+2)^k(u^2)\\
&= -\frac{a}{2}\,\partial_x\sum_{m=0}^{k}\binom{k}{m}\,(P+2)^m u\cdot P^{k-m}u\\
&= -\frac{a}{2}\,\partial_x\sum_{m=0}^{k}\sum_{j=0}^{m}\binom{k}{m}\binom{m}{j}\,2^{m-j}\;P^j u\cdot P^{k-m}u\\
&= -\frac{a}{2}\,\partial_x\sum_{m=0}^{k}\sum_{j=0}^{m}\frac{k!}{(m-j)!\,j!\,(k-m)!}\,2^{m-j}\;P^j u\cdot P^{k-m}u\\
&= -\frac{a}{2}\sum_{k=k_1+k_2+k_3}\frac{k!}{k_1!\,k_2!\,k_3!}\,2^{k_1}\;\partial_x\left(u_{k_2}\cdot u_{k_3}\right). \qquad (2.7)
\end{aligned}$$

In a similar way

$$B_k^2(v,\,v) = -\frac{b}{2}\,\partial_x(P+2)^k(v^2) = -\frac{b}{2}\sum_{k=k_1'+k_2'+k_3'}\frac{k!}{k_1'!\,k_2'!\,k_3'!}\,2^{k_1'}\,\partial_x\left(v_{k_2'}\cdot v_{k_3'}\right). \quad (2.8)$$

$$B_k^3(u,\,v) = c\,\partial_x(P+2)^k(u\,v) = -\,c\sum_{k=k_1''+k_2''+k_3''}\frac{k!}{k_1''!\,k_2''!\,k_3''!}\,2^{k_1''}\,\partial_x\left(u_{k_2''}\cdot v_{k_3''}\right). \quad (2.9)$$

Therefore

$$\begin{aligned}
&\partial_t(P^k u)+\partial_x^3(P^k u)\\
&= -\frac{a}{2}\sum_{k=k_1+k_2+k_3}\frac{k!}{k_1!\,k_2!\,k_3!}\,2^{k_1}\,\partial_x\left(u_{k_2}\cdot u_{k_3}\right)-\frac{b}{2}\sum_{k=k_1'+k_2'+k_3'}\frac{k!}{k_1'!\,k_2'!\,k_3'!}\,2^{k_1'}\,\partial_x\left(v_{k_2'}\cdot v_{k_3'}\right)\\
&\quad -c\sum_{k=k_1''+k_2''+k_3''}\frac{k!}{k_1''!\,k_2''!\,k_3''!}\,2^{k_1''}\,\partial_x\left(u_{k_2''}\cdot v_{k_3''}\right)\\
&= B_k^1(u,\,u)+B_k^2(v,\,v)+B_k^3(u,\,v). \qquad (2.10)
\end{aligned}$$

Performing similar calculations as above we obtain

$$\begin{aligned}
&\partial_t(P^k v)+\partial_x^3(P^k v)\\
&= -\frac{\widetilde{a}}{2}\sum_{k=k_1+k_2+k_3}\frac{k!}{k_1!\,k_2!\,k_3!}\,2^{k_1}\,\partial_x\left(u_{k_2}\cdot u_{k_3}\right)-\frac{\widetilde{b}}{2}\sum_{k=k_1'+k_2'+k_3'}\frac{k!}{k_1'!\,k_2'!\,k_3'!}\,2^{k_1'}\,\partial_x\left(v_{k_2'}\cdot v_{k_3'}\right)\\
&\quad -\widetilde{c}\sum_{k=k_1''+k_2''+k_3''}\frac{k!}{k_1''!\,k_2''!\,k_3''!}\,2^{k_1''}\,\partial_x\left(u_{k_2''}\cdot v_{k_3''}\right)\\
&= C_k^1(u,\,u)+C_k^2(v,\,v)+C_k^3(u,\,v). \qquad (2.11)
\end{aligned}$$

The above nonlinear terms maintain the bilinear structure like that of the original coupled system of equations of KdV type, since Leibniz's rule can be applied for operations of P. Now, each u_k and v_k satisfies the following system of equations

$$\begin{aligned}
\partial_t u_k+\partial_x^3 u_k &= B_k^1(u,\,u)+B_k^2(v,\,v)+B_k^3(u,\,v)\equiv B_k \qquad (2.12)\\
\partial_t v_k+\partial_x^3 v_k &= C_k^1(u,\,u)+C_k^2(v,\,v)+C_k^3(u,\,v)\equiv C_k \qquad (2.13)\\
u_k(x,\,0)=(x\,\partial_x)^k u_0(x)\equiv u_0^k(x), &\qquad v_k(x,\,0)=(x\,\partial_x)^k v_0(x)\equiv v_0^k(x). \qquad (2.14)
\end{aligned}$$

In order to obtain a well-posedness result for the system (2.12)-(2.14) we use Duhamel's principle and we study the following system of integral equations equivalent to the system (2.12)-(2.14)

$$\begin{aligned}
\psi(t)\,u_k &= \psi(t)\,V(t)\,u_0^k-\psi(t)\int_0^t V(t-t')\,\psi_T(t')\,B_k(t')\,dt' \qquad (2.15)\\
\psi(t)\,v_k &= \psi(t)\,V(t)\,v_0^k-\psi(t)\int_0^t V(t-t')\,\psi_T(t')\,C_k(t')\,dt' \qquad (2.16)
\end{aligned}$$

where $V(t)=e^{-t\,\partial_x^3}$ is the unitary group associated with the linear problem and $\psi(t)\in C_0^\infty(\mathbb{R})$, $0\le\psi\le 1$ is a cut-off function such that

$$\psi(t)=\begin{cases}1, & \text{if} \quad |t|<1\\ 0, & \text{if} \quad |t|>2\end{cases} \qquad \text{and} \qquad \psi_T(t)=\psi(t/T)$$

The following results are going to be used several times in the rest of this paper.

Lemma 2.1 ([16]). . *Let $s \in \mathbb{R}$, a, $a' \in (0,\, 1/2)$, $b \in (1/2,\, 1)$ and $\delta < 1$. Then for any $k = 0,\, 1,\, 2, \ldots$, we have*

$$||\psi_\delta \phi_k||_{X^s_{-a}} \leq c\, \delta^{(a-a')/4(1-a')}\, ||\phi_k||_{X^s_{-a'}}, \tag{2.17}$$

$$||\psi_\delta\, V(t)\, \phi_k||_{X^s_b} \leq c\, \delta^{1/2-b} ||\phi_k||_{H^s(\mathbb{R})}, \tag{2.18}$$

$$\left|\left|\psi_\delta \int_0^t V(t-t')\, F_k(t')\, dt'\right|\right|_{X^s_b} \leq c\, \delta^{1/2-b}\, ||F_k||_{X^s_{b-1}}. \tag{2.19}$$

Lemma 2.2 ([16]). . *Let $s > -3/4$, $b,\, b' \in (1/2,\, 7/12)$ with $b < b'$. Then for any $k,\, l = 0,\, 1,\, 2, \ldots$ we have*

$$||\partial_x(u_k\, v_l)||_{X^s_{b'-1}} \leq c\, ||v_k||_{X^s_b}\, ||v_l||_{X^s_b}. \tag{2.20}$$

Lemma 2.3 ([12]). . *Let $s < 0$, $b \in (1/2,\, 7/12)$ and $\psi = \psi(x,\, t)$ be a smooth cut-off function such that the support of ψ is in $\mathbb{B}_2(0)$ and $\psi = 1$ on $\mathbb{B}_1(0)$. We set $\psi_\epsilon = \psi((x-x_0)/\epsilon,\, (t-t_0)/\epsilon)$. Then for $f \in X^s_b$, we have*

$$||\psi_\epsilon\, f||_{X^s_b} \leq c\, \epsilon^{-|s|-5|b|} ||\psi_\epsilon||_{X^{|s|+2\,|b|}_{|b|}}\, ||f||_{X^{s+2\,|b|}_b}, \tag{2.21}$$

where the constant c is independent of ϵ and f.

Lemma 2.4 ([12]). . *Let P be the generator of the dilation and $D_{x,\,t}$ be an operator defined by $\mathcal{F}^{-1}_{\xi,\,\tau} < |\tau| + |\xi| > \mathcal{F}_{x,\,t}$. We fix an arbitrary point $(x_0,\, t_0) \in \mathbb{R} \times \{(-T,\, 0) \cup (0,\, T)\}$. Then*
1) Suppose that $b \in (0,\, 1]$, $r \in (-\infty,\, 0]$ and $g \in X^r_{b-1}$ with supp $g \subset \mathbb{B}_{2\epsilon}(x_0,\, t_0)$ and $t\partial^3_x g$, $P^3 g \in X^r_{b-1}$. If $\epsilon > 0$ is sufficiently small, then we have

$$|| < D_{x,\,t} >^{3b} g||_{L^2(\mathbb{R}:\, H^r(\mathbb{R}))} \leq c \left(||g||_{X^r_{b-1}} + ||t\partial^3_x g||_{X^r_{b-1}} + ||P^3 g||_{X^r_{b-1}} \right) \tag{2.22}$$

where the constant $c = c(x_0,\, t_0,\, \epsilon)$.
2) If $g \in H^{\mu-3}(\mathbb{R}^2)$ with supp $g \subset \mathbb{B}_{2\epsilon}(x_0,\, t_0)$ and $t\partial^3_x g$, $P^3 g \in H^{\mu-3}(\mathbb{R}^2)$. Then for small ϵ, we have

$$|| < D_{x,\,t} >^{\mu} g||_{L^2(\mathbb{R}^2)} \leq c \left(||g||_{H^{\mu-3}(\mathbb{R}^2)} + ||t\partial^3_x g||_{H^{\mu-3}(\mathbb{R}^2)} + ||P^3 g||_{H^{\mu-3}(\mathbb{R}^2)} \right) \tag{2.23}$$

where the constant $c = c(x_0,\, t_0,\, \epsilon)$.

Lemma 2.5 ([12]). . *Let $0 \leq s,\, r \leq n/2$ with $n/2 \leq s + r$ and suppose that $f \in H^s(\mathbb{R}^n)$ and $g \in H^r(\mathbb{R}^n)$. Then for any $\sigma < s + r - n/2$, we have $f\, g \in H^\sigma(\mathbb{R}^n)$ and*

$$||f\, g||_{H^\sigma(\mathbb{R}^n)} \leq c(\epsilon)\, ||f||_{H^s(\mathbb{R}^n)} ||g||_{H^r(\mathbb{R}^n)}, \tag{2.24}$$

where $\epsilon = s + r - n/2 - \sigma$.

Corollary 2.1 ([12]). . *For $1/2 < b < 1$ and $-3/4 < s < 0$, we have*

$$||\psi\, f||_{X^{s-1}_{b-1}} \leq c\, ||f||_{X^s_{b-1}} \tag{2.25}$$

where $\psi \in C^\infty_0(\mathbb{R}^2)$ and c is independent of f.

Lemma 2.6 ([12]). . *Let $\psi(x)$ be a smooth cut-off function in $C_0^\infty((-2,\,2))$ with $\psi(x)=1$ on $(-1,\,1)$. We set $\psi_\epsilon=\psi(x/\epsilon)$ for $0<\epsilon<1$. Then for $r\le 0$, and $f\in H^r$, we have*

$$||\psi_\epsilon\, f||_{H^r(\mathbb{R})} \le \begin{cases} c\,\epsilon^{-\delta}||f||_{H^r(\mathbb{R})} & \text{if} \quad -1/2\le r\le 0 \\ c\,\epsilon^{1/2+r}||f||_{H^r(\mathbb{R})} & \text{if} \quad r<-1/2 \end{cases}$$

where $\delta>0$ is an arbitrary small constant and c is independent of ϵ.

Throughout this paper c is a generic constant, not necessarily the same at each occasion (it will change from line to line), which depends in an increasing way on the indicated quantities.

3 Existence and Well-Posedness

We firstly solve the following (slightly general) system of equations

$$\begin{aligned} \partial_t u_k+\partial_x^3 u_k &= B_k^1(u,\,u)+B_k^2(v,\,v)+B_k^3(u,\,v)\equiv B_k \qquad (3.1)\\ \partial_t v_k+\partial_x^3 v_k &= C_k^1(u,\,u)+C_k^2(v,\,v)+C_k^3(u,\,v)\equiv C_k \qquad (3.2)\\ u_k(x,\,0)=(x\,\partial_x)^k u_0(x)\equiv u_0^k(x) \quad &, \quad v_k(x,\,0)=(x\,\partial_x)^k v_0(x)\equiv v_0^k(x) \qquad (3.3) \end{aligned}$$

where B_k and C_k are as above.

Definition 3.1. Let $f=(f_0,\,f_1,\,\dots,\,f_k)$ denotes the infinity series of distributions and define

$$\mathcal{A}_{A_0}(X_b^s)\equiv\Big\{f=(f_0,\,f_1,\,\dots,\,f_k),\,f_i\in X_b^s,\,(i=0,\,1,\,2\dots) \text{ such that } ||f||_{\mathcal{A}_{A_0}(X_b^s)}<+\infty\Big\}$$

where

$$||f||_{\mathcal{A}_{A_0}(X_b^s)}\equiv\sum_{k=0}^{\infty}\frac{A_0^k}{k!}\,||f_k||_{X_b^s}.$$

Similarly, for $u_0=\{u_0^0,\,u_0^1,\,\dots,\,u_0^k,\,\dots\}$ and $v_0=\{v_0^0,\,v_0^1,\,\dots,\,v_0^k,\,\dots\}$ we set

$$||u_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))}\equiv\sum_{k=0}^{\infty}\frac{A_0^k}{k!}\,||u_0^k||_{H^s(\mathbb{R})} \quad\text{and}\quad ||v_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))}\equiv\sum_{k=0}^{\infty}\frac{A_0^k}{k!}\,||v_0^k||_{H^s(\mathbb{R})}$$

respectively.

Remark 3.1. Each solution of the coupled system of Korteweg de Vries equations is accompanied by the following estimate

$$||P^k u||_{X_b^s}\le c\,A_0^k\,k!, \qquad\text{and}\qquad ||P^k v||_{X_b^s}\le c\,A_1^k\,k!, \qquad k=0,\,1,\,2,\dots$$

Theorem 3.1. *Let $-3/4<s,\,b\in(1/2,\,7/12)$. Suppose that $u_0^k,\,v_0^k\in H^s(\mathbb{R})(k=0,\,1,\,2,\,\dots)$ and satisfies*

$$||u_0||_{\mathcal{A}_{A_0}(X_b^s)}=\sum_{k=0}^{\infty}\frac{A_0^k}{k!}\,||u_0^k||_{H^s(\mathbb{R})}<+\infty \quad \textit{and} \quad ||v_0||_{\mathcal{A}_{A_0}(X_b^s)}=\sum_{k=0}^{\infty}\frac{A_0^k}{k!}\,||v_0^k||_{H^s(\mathbb{R})}<+\infty.$$

Then there exist $T = T(||u_0^k||_{H^s(\mathbb{R})}, ||v_0^k||_{H^s(\mathbb{R})})$ *and a unique solution* $u = (u_0, u_1, \ldots)$ *and* $v = v(v_0, v_1, \ldots)$ *of the system* (3.1)-(3.3) *with* $u_k, v_k \in C((-T, T) : H^s(\mathbb{R})) \cap X_b^s$ *and*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} ||u_k||_{X_b^s(\mathbb{R})} < +\infty, \qquad \sum_{k=0}^{\infty} \frac{A_0^k}{k!} ||v_k||_{X_b^s(\mathbb{R})} < +\infty.$$

Moreover, the map $(u_0^k, v_0^k) \to (u(t), v(t))$ *is Lipschitz continuous, i. e.,*

$$||u(t) - \widetilde{u}(t)||_{\mathcal{A}_{A_0}(X_b^s)} + ||u(t) - \widetilde{u}(t)||_{C((-T,T):\, H^s(\mathbb{R}))} \leq c(T)\, ||u_0 - \widetilde{u}_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))}$$

and

$$||v(t) - \widetilde{v}(t)||_{\mathcal{A}_{A_0}(X_b^s)} + ||v(t) - \widetilde{v}(t)||_{C((-T,T):\, H^s(\mathbb{R}))} \leq c(T)\, ||v_0 - \widetilde{v}_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))}.$$

Proof. For given $(u_0, v_0) \in \mathcal{A}_{A_0}(H^s(\mathbb{R})) \times \mathcal{A}_{A_0}(H^s(\mathbb{R}))$ and $b > 1/2$, let us define,

$$\mathbb{H}_{R_1,R_2} = \Big\{(u, v) \in \mathcal{A}_{A_0}(X_b^s) \times \mathcal{A}_{A_0}(X_b^s) : \quad ||u||_{\mathcal{A}_{A_0}(X_b^s)} \leq R_1, \quad ||v||_{\mathcal{A}_{A_0}(X_b^s)} \leq R_2\Big\}$$

where $R_1 = 2\, c_0\, ||u_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))}$ and $R_2 = 2\, c_0\, ||v_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))}$. Then $\mathbb{H}_{R_1,R_2}$ is a complete metric space with norm

$$||(u, v)||_{\mathbb{H}_{R_1,R_2}} = ||u||_{\mathcal{A}_{A_0}(X_b^s)} + ||v||_{\mathcal{A}_{A_0}(X_b^s)}.$$

Without loss of generality, we may assume that that $R_1 > 1$ and $R_2 > 1$. For $(u, v) \in \mathbb{H}_{R_1,R_2}$, let us define the maps,

$$\Phi_{u_0}^k(u, v) = \psi(t)\, V(t)\, u_0^k - \psi(t) \int_0^t V(t - t')\, \psi_T(t')\, B_k(t')\, dt' \tag{3.4}$$

$$\Psi_{v_0}^k(u, v) = \psi(t)\, V(t)\, v_0^k - \psi(t) \int_0^t V(t - t')\, \psi_T(t')\, C_k(t')\, dt'. \tag{3.5}$$

We prove that $\Phi \times \Psi$ maps $\mathbb{H}_{R_1,R_2}$ into $\mathbb{H}_{R_1,R_2}$ and it is a contraction. In fact, using lemma 2.1 and lemma 2.2 we have

$$\begin{aligned}
||\Phi_{u_0}^k(u, v)||_{X_b^s} &= ||\psi(t)\, V(t)\, u_0^k||_{X_b^s} + \left|\left|\psi(t) \int_0^t V(t - t')\, \psi_T(t')\, B_k(t')\, dt'\right|\right|_{X_b^s} \\
&\leq c_0\, ||u_0^k||_{H^s(\mathbb{R})} + c_1\, T^\mu\, ||B_k||_{X_{b'-1}^s} \\
&\leq c_0\, ||u_0^k||_{H^s(\mathbb{R})} + c_1\, T^\mu\, \frac{a}{2} \sum_{k=k_1+k_2+k_3} \frac{k!}{k_1!\, k_2!\, k_3!}\, 2^{k_1}\, ||u_{k_2}||_{X_b^s}\, ||u_{k_3}||_{X_b^s} \\
&\quad + c_1\, T^\mu\, \frac{b}{2} \sum_{k=k_1'+k_2'+k_3'} \frac{k!}{k_1'!\, k_2'!\, k_3'!}\, 2^{k_1'}\, ||v_{k_2'}||_{X_b^s}\, ||v_{k_3'}||_{X_b^s} \\
&\quad + c_1\, T^\mu\, c \sum_{k=k_1''+k_2''+k_3''} \frac{k!}{k_1''!\, k_2''!\, k_3''!}\, 2^{k_1''}\, ||u_{k_2''}||_{X_b^s}\, ||v_{k_3''}||_{X_b^s}.
\end{aligned}$$

Applying a sum over k we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{A_0^k}{k!}\, ||\Phi_{u_0}^k(u,\, v)||_{X_b^s} \\
&\quad\leq\ c_0 \sum_{k=0}^{\infty} \frac{A_0^k}{k!}\, ||u_0^k||_{H^s(\mathbb{R})} + c_1\, T^\mu\, \frac{a}{2} \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \sum_{k=k_1+k_2+k_3} \frac{k!}{k_1!\, k_2!\, k_3!}\, 2^{k_1}\, ||u_{k_2}||_{X_b^s}\, ||u_{k_3}||_{X_b^s} \\
&\qquad + c_1\, T^\mu\, \frac{b}{2} \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \sum_{k=k_1'+k_2'+k_3'} \frac{k!}{k_1'!\, k_2'!\, k_3'!}\, 2^{k_1'}\, ||v_{k_2'}||_{X_b^s}\, ||v_{k_3'}||_{X_b^s} \\
&\qquad + c_1\, T^\mu\, c \sum_{k=0}^{\infty} \frac{A_0^k}{k!} \sum_{k=k_1''+k_2''+k_3''} \frac{k!}{k_1''!\, k_2''!\, k_3''!}\, 2^{k_1''}\, ||u_{k_2''}||_{X_b^s}\, ||v_{k_3''}||_{X_b^s} \\
&\quad\leq\ c_0\, ||u_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))} + c_1\, T^\mu\, \frac{a}{2} \sum_{k=0}^{\infty} \sum_{k=k_1+k_2+k_3} 2^{k_1}\, \frac{A_0^{k_1}}{k_1!}\, \frac{A_0^{k_2}}{k_2!}\, ||u_{k_2}||_{X_b^s}\, \frac{A_0^{k_3}}{k_3!}\, ||u_{k_3}||_{X_b^s} \\
&\qquad + c_1\, T^\mu\, \frac{b}{2} \sum_{k=0}^{\infty} \sum_{k=k_1'+k_2'+k_3'} 2^{k_1'}\, \frac{A_0^{k_1'}}{k_1'!}\, \frac{A_0^{k_2'}}{k_2'!}\, ||v_{k_2'}||_{X_b^s}\, \frac{A_0^{k_3'}}{k_3'!}\, ||v_{k_3'}||_{X_b^s} \\
&\qquad + c_1\, T^\mu\, c \sum_{k=0}^{\infty} \sum_{k=k_1''+k_2''+k_3''} 2^{k_1''}\, \frac{A_0^{k_1''}}{k_1'!}\, \frac{A_0^{k_2''}}{k_2''!}\, ||u_{k_2''}||_{X_b^s}\, \frac{A_0^{k_3''}}{k_3''!}\, ||v_{k_3''}||_{X_b^s} \\
&\quad\leq\ c_0\, ||u_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))} + c_1\, T^\mu\, \frac{a}{2} \sum_{k_1=0}^{\infty} 2^{k_1}\, \frac{A_0^{k_1}}{k_1!} \sum_{k_2=0}^{\infty} \frac{A_0^{k_2}}{k_2!}\, ||u_{k_2}||_{X_b^s} \sum_{k_3=0}^{\infty} \frac{A_0^{k_3}}{k_3!}\, ||u_{k_3}||_{X_b^s} \\
&\qquad + c_1\, T^\mu\, \frac{b}{2} \sum_{k_1'=0}^{\infty} 2^{k_1'}\, \frac{A_0^{k_1'}}{k_1'!} \sum_{k_2'=0}^{\infty} \frac{A_0^{k_2'}}{k_2'!}\, ||v_{k_2'}||_{X_b^s} \sum_{k_3'=0}^{\infty} \frac{A_0^{k_3'}}{k_3'!}\, ||v_{k_3'}||_{X_b^s} \\
&\qquad + c_1\, T^\mu\, c \sum_{k_1''=0}^{\infty} 2^{k_1''}\, \frac{A_0^{k_1''}}{k_1''!} \sum_{k_2''=0}^{\infty} \frac{A_0^{k_2''}}{k_2''!}\, ||u_{k_2''}||_{X_b^s} \sum_{k_3''=0}^{\infty} \frac{A_0^{k_3''}}{k_3''!}\, ||v_{k_3''}||_{X_b^s} \\
&\quad=\ c_0\, ||u_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))} + c_1\, T^\mu\, \frac{a}{2}\, e^{2\,A_0}\, ||u||^2_{\mathcal{A}_{A_0}(X_b^s)} \\
&\qquad + c_1\, T^\mu\, \frac{b}{2}\, e^{2\,A_0}\, ||v||^2_{\mathcal{A}_{A_0}(X_b^s)} + c_1\, T^\mu\, c\, e^{2\,A_0}\, ||u||_{\mathcal{A}_{A_0}(X_b^s)}\, ||v||_{\mathcal{A}_{A_0}(X_b^s)}.
\end{aligned}$$

Hence, choosing $d = \max\{a/2,\, b/2,\, c\}$ we have

$$\begin{aligned}
&||\Phi_{u_0}(u,\, v)||_{\mathcal{A}_{A_0}(X_b^s)} \leq c_0\, ||u_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))} \\
&\qquad + c_1\, T^\mu\, d\, e^{2\,A_0}\, \left[||u||^2_{\mathcal{A}_{A_0}(X_b^s)} + ||v||^2_{\mathcal{A}_{A_0}(X_b^s)} + ||u||_{\mathcal{A}_{A_0}(X_b^s)}\, ||v||_{\mathcal{A}_{A_0}(X_b^s)}\right] \\
&\quad\leq\ c_0\, ||u_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))} + \frac{3}{2}\, c_1\, d\, T^\mu\, e^{2\,A_0}\, \left[||u||^2_{\mathcal{A}_{A_0}(X_b^s)} + ||v||^2_{\mathcal{A}_{A_0}(X_b^s)}\right]. \qquad (3.6)
\end{aligned}$$

In a similar way, choosing $\widetilde{d} = \max\{\widetilde{a}/2,\, \widetilde{b}/2,\, \widetilde{c}\}$ we have

$$||\Psi_{v_0}(u,\, v)||_{\mathcal{A}_{A_0}(X_b^s)} \leq c_0\, ||v_0||_{\mathcal{A}_{A_0}(H^s(\mathbb{R}))} + \frac{3}{2}\, c_2\, \widetilde{d}\, T^\mu\, e^{2\,A_0}\, \left[||u||^2_{\mathcal{A}_{A_0}(X_b^s)} + ||v||^2_{\mathcal{A}_{A_0}(X_b^s)}\right] \qquad (3.7)$$

If we choose T such that

$$T^{\mu} \leq \frac{1}{3\ \max\{c_1,\, c_2\}\ (R_1+R_2)^2}$$

Then we obtain in (3.6) and (3.7)

$$||\Phi_{u_0}(u,\, v)||_{\mathcal{A}_{A_0}(X_b^s)} \leq R_1 \qquad \text{and} \qquad ||\Psi_{v_0}(u,\, v)||_{\mathcal{A}_{A_0}(X_b^s)} \leq R_2.$$

Therefore, $(\Phi_{u_0},\, \Psi_{v_0}) \in \mathbb{H}_{R_1,\, R_2}$. We show that $\Phi_{u_0} \times \Psi_{v_0} : (u,\, v) \to (\Phi_{u_0}(u,\, v),\, \Psi_{v_0}(u,\, v))$ is a contraction.
Let $(u,\, v),\ (\widetilde{u},\, \widetilde{v}) \in \mathbb{H}_{R_1,\, R_2}$, then as above we get for $d = \max\{a/2,\, b/2,\, c\}$

$$\begin{aligned} &||\Phi_{u_0}(u,\, v) - \Phi_{u_0}(\widetilde{u},\, \widetilde{v})||_{\mathcal{A}_{A_0}(X_b^s)} \\ &\quad \leq \ \frac{3}{2}\, c_1\, d\, T^{\mu}\, e^{2\,A_0}\ (R_1+R_2)\ \left[||u-\widetilde{u}||_{\mathcal{A}_{A_0}(X_b^s)} + ||v-\widetilde{v}||_{\mathcal{A}_{A_0}(X_b^s)}\right]. \end{aligned} \tag{3.8}$$

In a similar way, choosing $\widetilde{d} = \max\{\widetilde{a}/2,\, \widetilde{b}/2,\, \widetilde{c}\}$ we have

$$\begin{aligned} &||\Psi_{v_0}(u,\, v) - \Psi_{v_0}(\widetilde{u},\, \widetilde{v})||_{\mathcal{A}_{A_0}(X_b^s)} \\ &\quad \leq \ \frac{3}{2}\, c_2\, \widetilde{d}\, T^{\mu}\, e^{2\,A_0}\ (R_1+R_2)\ \left[||u-\widetilde{u}||_{\mathcal{A}_{A_0}(X_b^s)} + ||v-\widetilde{v}||_{\mathcal{A}_{A_0}(X_b^s)}\right]. \end{aligned} \tag{3.9}$$

Choosing T^{μ} small enough, such that

$$T^{\mu} \leq \frac{1}{6\ \max\{c_1,\, c_2\}\,(R_1+R_2)^2}$$

we obtain

$$||\Phi_{u_0}(u,\, v) - \Phi_{u_0}(\widetilde{u},\, \widetilde{v})||_{\mathcal{A}_{A_0}(X_b^s)} \leq \frac{1}{4}\left[||u-\widetilde{u}||_{\mathcal{A}_{A_0}(X_b^s)} + ||v-\widetilde{v}||_{\mathcal{A}_{A_0}(X_b^s)}\right]. \tag{3.10}$$

In a similar way

$$||\Psi_{v_0}(u,\, v) - \Psi_{v_0}(\widetilde{u},\, \widetilde{v})||_{\mathcal{A}_{A_0}(X_b^s)} \leq \frac{1}{4}\left[||u-\widetilde{u}||_{\mathcal{A}_{A_0}(X_b^s)} + ||v-\widetilde{v}||_{\mathcal{A}_{A_0}(X_b^s)}\right]. \tag{3.11}$$

Therefore the map $\Phi_{u_0} \times \Psi_{v_0}$ is a contraction and we obtain a unique fixed point $(u,\, v)$ which solves the initial value problem (3.1)-(3.3) for $T < T^{\mu}$. The rest of the proof follows a standard argument.

Corollary 3.1. *Let $-3/4 < s,\ b \in (1/2,\, 7/12)$. Suppose that $(x\,\partial_x)^k u_0,\ (x\,\partial_x)^k v_0 \in H^s(\mathbb{R})(k = 0,\, 1,\, 2,\, \ldots)$ and that*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!}\ ||u_0^k||_{H^s(\mathbb{R})} < +\infty \quad \textit{and} \quad \sum_{k=0}^{\infty} \frac{A_0^k}{k!}\ ||v_0^k||_{H^s(\mathbb{R})} < +\infty.$$

Then there exist $T = T(||u_0^k||_{H^s(\mathbb{R})},\, ||v_0^k||_{H^s(\mathbb{R})})$ and a unique solution $(u,\, v)$ of the coupled system equations KdV type (1.1)-(1.3) with $u,\, v \in C((-T,\, T) :\ H^s(\mathbb{R})) \cap X_b^s$ and

$$\sum_{k=0}^{\infty} \frac{A_1^k}{k!}\ ||P^k u||_{X_b^s(\mathbb{R})} < +\infty, \qquad \sum_{k=0}^{\infty} \frac{A_1^k}{k!}\ ||P^k v||_{X_b^s(\mathbb{R})} < +\infty.$$

Moreover, the map $(u_0,\, v_0) \to (u(t),\, v(t))$ *is Lipschitz continuous in the following sense:*

$$||P^k u(t) - P^k \widetilde{u}(t)||_{X_b^s} + ||P^k u(t) - P^k \widetilde{u}(t)||_{C((-T,\,T):\, H^s(\mathbb{R}))} \leq c(T) \sum_{k=0}^{\infty} \frac{A_0^k}{k!}\, ||(x\,\partial_x)^k (u_0 - \widetilde{u}_0)||_{H^s(\mathbb{R})}$$

and

$$||v(t) - \widetilde{v}(t)||_{X_b^s} + ||v(t) - \widetilde{v}(t)||_{C((-T,\,T):\, H^s(\mathbb{R}))} \leq c(T) \sum_{k=0}^{\infty} \frac{A_0^k}{k!}\, ||(x\,\partial_x)^k (v_0 - \widetilde{v}_0)||_{H^s(\mathbb{R})}.$$

4 The main result

In this section we prove the analyticity of the solution obtained in the previous section. We treat the solution $u_k \equiv P^k u$ and $v_k \equiv P^k v$ as if they satisfy the coupled system of equations (3.1)-(3.3) in the classical sense. This can be justified by a proper approximation procedure. The following results are going to be used in this section. Let $(x_0,\, t_0)$ be arbitrarily taken in $\mathbb{R} \times \{(-T,\, 0) \cup (0,\, T)\}$. By $\psi(x,\, t)$ we denote a smooth cut-off function in $C_0^\infty(\mathbb{B}_1(0))$ and $\psi_\epsilon = \psi((x - x_0)/\epsilon,\, (t - t_0)/\epsilon)$.

Let ψ be a smooth cut-off function around the freezing point $(x_0,\, t_0)$ with $supp\, \psi \subset C_0^\infty(\mathbb{B}_\epsilon(x_0,\, t_0))$.

Proposition 4.1. *For the cut-off function* ψ *defined above, there exists a positive constant* c *and* A *such that*

$$||\psi\, P^k u||_{L^2_{x,\,t}(\mathbb{R}^2)} \leq c\, A^k\, (k!)^2, \qquad k = 0,\, 1,\, 2,\, \ldots \tag{4.1}$$

$$||\psi\, P^k v||_{L^2_{x,\,t}(\mathbb{R}^2)} \leq c\, A^k\, (k!)^2, \qquad k = 0,\, 1,\, 2,\, \ldots \tag{4.2}$$

Proof. Using (2.22) with $r = s - 1$, we obtain

$$|| < D_{x,\,t} >^{3b} \psi P^k u||_{L^2_t(\mathbb{R}:\, H^{s-1}_x(\mathbb{R}))} \leq c \left(||\psi u_k||_{X^{s-1}_{b-1}} + ||t\, \partial_x^3 (\psi u_k)||_{X^{s-1}_{b-1}} + ||P^3(\psi u_k)||_{X^{s-1}_{b-1}} \right) \tag{4.3}$$

Each term in (4.3) is estimated separately. For the first term in the right hand side we use Lemma 2.3. Indeed,

$$||\psi\, u_k||_{X^{s-1}_{b-1}} \leq ||\psi\, u_k||_{X^s_{b-1}} \leq c\, ||\psi||_{X^{|s|+2|b-1|}_{|b-1|}} ||u_k||_{X^s_b} \leq c(\psi) A_1^k\, k!. \qquad k = 0,\, 1,\, 2,\, \ldots \tag{4.4}$$

The third term is estimated again using Corollary 2.6.

$$\begin{aligned}
||P^3(\psi\, u_k)||_{X^{s-1}_{b-1}} &\leq \sum_{l=0}^{3} \frac{3!}{l\,(l-3)!}\, ||(P^{3-l}\psi)\, P^l u_k||_{X^s_{b-1}} \\
&\leq c(\psi) \sum_{l=0}^{3} \frac{3!}{l\,(l-3)!}\, ||P^l u_k||_{X^s_b} \\
&\leq c \sum_{l=0}^{3} \frac{3!}{l\,(l-3)!}\, ||P^{k+l} u||_{X^s_b} \\
&= c \sum_{l=0}^{3} A_1^{k+l}\, (k+l)! \\
&\leq c\, A_2^k\, k!. \qquad k = 0,\, 1,\, 2,\, \ldots
\end{aligned} \tag{4.5}$$

For the second term, we use (3.1) to reduce the third derivative in space to the dilation operator P. Since the generator of dilation is $Pu_k = 3\,t\,\partial_t u_k + x\,\partial_x u_k$ we obtain

$$t\,\partial_t u_k = \frac{1}{3}\,Pu_k - \frac{1}{3}\,x\,\partial_x u_k. \tag{4.6}$$

Multiplying (3.1) by $\psi\,t$, we have

$$\psi\,t\,\partial_t u_k + \psi\,t\,\partial_x^3 u_k = \psi\,t\,B_k. \tag{4.7}$$

Replacing (4.6) in (4.7) we obtain

$$\psi\,t\,\partial_x^3 u_k = -\frac{1}{3}\,\psi\,Pu_k + \frac{1}{3}\,\psi\,x\,\partial_x u_k + \psi\,t\,B_k. \tag{4.8}$$

hence

$$\begin{aligned} ||\psi\,t\,\partial_x^3 u_k||_{X_{b-1}^{s-1}} &= \frac{1}{3}\,||\psi\,Pu_k||_{X_{b-1}^{s-1}} + \frac{1}{3}\,||\psi\,x\,\partial_x u_k||_{X_{b-1}^{s-1}} + ||\psi\,t\,B_k||_{X_{b-1}^{s-1}} \\ &= F_1 + F_2 + F_3. \end{aligned} \tag{4.9}$$

Using the assumption in the Theorem, we have

$$\begin{aligned} F_1 = \frac{1}{3}\,||\psi\,Pu_k||_{X_{b-1}^{s-1}} &\leq c\,||\psi||_{X_{1-b}^{-s}}||P^{k+1}u||_{X_{b-1}^{s}} \leq c\,||P^{k+1}u||_{X_b^s} \\ &\leq c\,A_3^{k+1}(k+1)! \;\leq\; c\,A_4^k\,k!. \end{aligned} \tag{4.10}$$

Similarly, we obtain

$$\begin{aligned} F_2 = \frac{1}{3}\,||\psi\,x\,\partial_x u_k||_{X_{b-1}^{s-1}} &\leq \frac{1}{3}\,||\partial_x(\psi\,x\,u_k)||_{X_{b-1}^{s-1}} + \frac{1}{3}\,||\partial_x(\psi\,x)\,u_k)||_{X_{b-1}^{s-1}} \\ &\leq \frac{1}{3}\,||\partial_x(\psi\,x\,v_k)||_{X_{b-1}^{s}} + c\,||\partial_x(\psi\,x)||_{X_{1-b}^{-s}}||u_k||_{X_{b-1}^{s}} \\ &\leq \frac{1}{3}\,||\psi\,x||_{X_b^s}||u_k||_{X_b^s} + c\,||\partial_x(\psi\,x)||_{X_{1-b}^{-s}}||u_k||_{X_{b-1}^{s}} \\ &\leq c\left(||\psi\,x||_{X_b^s} + ||\partial_x(\psi\,x)||_{X_{1-b}^{-s}}\right) A_5^k k! \;\leq\; c\,A_6^k k!. \end{aligned} \tag{4.11}$$

Using Lemma 2.3 and 2.2, we have

$$\begin{aligned} F_3 = ||\psi\,t\,B_k||_{X_{b-1}^{s-1}} &\leq c\,||\psi||_{X_{b-1}^{-s}}\,||B_k^1 + B_k^2 + B_k^3||_{X_{b-1}^{s}} \\ &\leq c\left(||B_k^1||_{X_{b-1}^{s}} + ||B_k^2||_{X_{b-1}^{s}} + ||B_k^3||_{X_{b-1}^{s}}\right) \end{aligned}$$

Then replacing B_k^1, B_k^2 and B_k^3 in (2.7), (2.8) and (2.9) we deduce

$$\begin{aligned}
F_3 &\leq c\sum_{k=k_1+k_2+k_3}\frac{k!}{k_1!\,k_2!\,k_3!}\,2^{k_1}\,||u_{k_2}||_{X_b^s}\,||u_{k_3}||_{X_b^s}+c\sum_{k=k_1'+k_2'+k_3'}\frac{k!}{k_1'!\,k_2'!\,k_3'!}\,2^{k_1'}\,||v_{k_2'}||_{X_b^s}\,||v_{k_3'}||_{X_b^s}\\
&\quad+c\sum_{k=k_1''+k_2''+k_3''}\frac{k!}{k_1''!\,k_2''!\,k_3''!}\,2^{k_1''}\,||u_{k_2''}||_{X_b^s}\,||v_{k_3''}||_{X_b^s}\\
&\leq c\sum_{k=k_1+k_2+k_3}\frac{k!}{k_1!\,k_2!\,k_3!}\,2^{k_1}\,A_7^{k_2}\cdot k_2!\,A_7^{k_3}\cdot k_3!+c\sum_{k=k_1'+k_2'+k_3'}\frac{k!}{k_1'!\,k_2'!\,k_3'!}\,2^{k_1'}\,A_8^{k_2'}\cdot k_2'!\,A_8^{k_3'}\cdot k_3'!\\
&\quad+c\sum_{k=k_1''+k_2''+k_3''}\frac{k!}{k_1''!\,k_2''!\,k_3''!}\,2^{k_1''}\,A_9^{k_2''}\cdot k_2''!\,A_{10}^{k_3''}\cdot k_3''!\\
&\leq c\sum_{k=k_1+k_2+k_3}\frac{k!}{k_1!}\,2^{k_1}\,A_7^{k_2+k_3}+c\sum_{k=k_1'+k_2'+k_3'}\frac{k!}{k_1'!}\,2^{k_1'}\,A_8^{k_2'+k_3'}\\
&\quad+c\sum_{k=k_1''+k_2''+k_3''}\frac{k!}{k_1''!}\,2^{k_1''}\,A_9^{k_2''}\cdot A_{10}^{k_3''}\\
&\leq c\,k!\,A_7^k\sum_{k_1=0}^{k}\sum_{k_2=0}^{k-k_1}\frac{1}{k_1!}\,2^{k_1}\,A_7^{-\,k_1}+c\,k!\,A_8^k\sum_{k_1'=0}^{k}\sum_{k_2'=0}^{k-k_1'}\frac{1}{k_1'!}\,2^{k_1'}\,A_8^{-\,k_1'}\\
&\quad+c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,A_9^{k_2''}\cdot A_{10}^{k_3''}\\
&\leq c\,k!\,A_7^k\sum_{k_1=0}^{k}\sum_{k_2=0}^{k-k_1}\frac{\left(\frac{2}{A_7}\right)^{k_1}}{k_1!}+c\,k!\,A_8^k\sum_{k_1'=0}^{k}\sum_{k_2'=0}^{k-k_1'}\frac{\left(\frac{2}{A_8}\right)^{k_1'}}{k_1'!}+c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,A_9^{k_2''}\cdot A_{10}^{k_3''}\\
&\leq c\,e^{2/A_7}\,A_7^k\cdot k!+c\,e^{2/A_8}\,A_8^k\cdot k!+c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,A_9^{k_2''}\cdot A_{10}^{k_3''}\\
&\leq c\left(e^{2/A_7}+e^{2/A_8}\right)A_{11}\cdot k!+c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,A_9^{k_2''}\cdot A_{10}^{k_3''}.\qquad k=0,\,1,\,2,\,\ldots \qquad (4.12)
\end{aligned}$$

Hence, from (4.10), (4.11) and (4.12) in (4.9) we obtain that there exists a positive constant c and A_{11} such that

$$||\psi\,t\,\partial_x^3 u_k||_{X_{b-1}^{s-1}}\leq c\,A_{11}\cdot k!+c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,A_9^{k_2''}\cdot A_{10}^{k_3''},\qquad k=0,\,1,\,2,\,. \qquad (4.13)$$

On the other hand, using $\partial_x^3(\psi\cdot f)=\psi\cdot\partial_x^3 f+3\,\partial_x^2(\partial_x\psi\cdot f)-3\,\partial_x(\partial_x^2\psi\cdot f)+\partial_x^3\psi\cdot f$ we have that

$$\begin{aligned}
||t\,\partial_x^3(\psi\cdot u_k)||_{X_{b-1}^{s-1}} &\leq ||t\,\psi\cdot\partial_x^3 u_k||_{X_{b-1}^{s-1}}+3\,||\partial_x^2(t\,\partial_x\psi\cdot u_k)||_{X_{b-1}^{s-1}}\\
&\quad+3\,||\partial_x(t\,\partial_x^2\psi\cdot u_k)||_{X_{b-1}^{s-1}}+||t\,\partial_x^3\psi\cdot u_k||_{X_{b-1}^{s-1}}. \qquad (4.14)
\end{aligned}$$

Using Lemma 2.2 and Lemma 2.3 we obtain

$$\begin{aligned} ||\partial_x^2(t\,\partial_x\psi\cdot u_k)||_{X_{b-1}^{s-1}} &\leq ||\partial_x(t\,\partial_x\psi\cdot u_k)||_{X_{b-1}^{s}} \leq c\,||t\,\partial_x\psi||_{X_b^s}||u_k||_{X_b^s} \\ &\leq c\,A_{10}^k\,k! \end{aligned} \tag{4.15}$$

$$\begin{aligned} ||\partial_x(t\,\partial_x^2\psi\cdot u_k)||_{X_{b-1}^{s-1}} &\leq ||\partial_x(t\,\partial_x^2\psi\cdot u_k)||_{X_{b-1}^{s}} \leq c\,||t\,\partial_x^2\psi||_{X_b^s}||u_k||_{X_b^s} \\ &\leq c\,A_{11}^k\,k! \end{aligned} \tag{4.16}$$

$$\begin{aligned} ||t\,\partial_x^3\psi\cdot u_k||_{X_{b-1}^{s-1}} &\leq c\,||<D_{x,t}>^{3/2}\,t\,\partial_x^3\psi||_{X_{1-b}^{|s|+2|b-1|}}||u_k||_{X_{b-1}^{s}} \leq c\,||u_k||_{X_b^s} \\ &\leq c\,A_{12}^k\,k!. \end{aligned} \tag{4.17}$$

Hence, replacing (4.13), (4.15),(4.16) and (4.17) in (4.14) we obtain that there exists a constant c and A_{14} such that

$$||t\,\partial_x^3(\psi u_k)||_{X_{b-1}^{s-1}} \leq c\,A_{14}^k\cdot k! + c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,A_9^{k_2''}\cdot A_{10}^{k_3''}, \qquad k=0,\,1,\,2,\,. \tag{4.18}$$

Therefore, replacing (4.4), (4.5) and (4.18) in (4.3) we obtain that there exists a constant c and A_{15} such that

$$\begin{aligned} &||<D_{x,t}>^{3b}\,\psi\,u_k||_{L_t^2(\mathbb{R}:\,H_x^{s-1}(\mathbb{R}))} \\ &\leq c\,A_{15}^k\cdot k! + c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,A_9^{k_2''}\cdot A_{10}^{k_3''}, \qquad k=0,\,1,\,2,\,\ldots \end{aligned} \tag{4.19}$$

In a similar way, we obtain that there exists a constant c and A_{16} such that

$$\begin{aligned} &||<D_{x,t}>^{3b}\,\psi\,v_k||_{L_t^2(\mathbb{R}:\,H_x^{s-1}(\mathbb{R}))} \\ &\leq c\,A_{16}^k\cdot k! + c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,A_9^{k_2''}\cdot A_{10}^{k_3''}, \qquad k=0,\,1,\,2,\,\ldots \end{aligned} \tag{4.20}$$

Adding (4.19) and (4.20) we have

$$\begin{aligned} &||<D_{x,t}>^{3b}\,\psi\,u_k||_{L_t^2(\mathbb{R}:\,H_x^{s-1}(\mathbb{R}))} + ||<D_{x,t}>^{3b}\,\psi\,v_k||_{L_t^2(\mathbb{R}:\,H_x^{s-1}(\mathbb{R}))} \\ &\leq c\,A_{15}^k\cdot k! + c\,A_{16}^k\cdot k! + c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,2\cdot A_9^{k_2''}\cdot A_{10}^{k_3''} \\ &\leq c\,(A_{15}^k + A_{16}^k)\cdot k! + c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,2\cdot A_9^{k_2''}\cdot A_{10}^{k_3''} \\ &\leq c\,A_{17}^k\cdot k! + c\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{1}{k_1''!}\,2^{k_1''}\,2\cdot A_9^{k_2''}\cdot A_{10}^{k_3''}. \end{aligned} \tag{4.21}$$

We estimate the last term on the right hand side of (4.21)

$$\begin{aligned}
\sum_{k=k_1''+k_2''+k_3''} \frac{1}{k_1''!}\, 2^{k_1''}\, 2\cdot A_9^{k_2''}\cdot A_{10}^{k_3''} &= \sum_{m=0}^{k}\sum_{j=0}^{m} \frac{1}{(m-j)!}\, 2^{(m-j)}\, 2\cdot A_9^{j}\cdot A_{10}^{k-m} \\
&\le A_{10}^{k}\sum_{m=0}^{k}\sum_{j=0}^{m} \frac{1}{(m-j)!}\, 2\cdot \left(\frac{A_9}{2}\right)^{j}\cdot \left(\frac{2}{A_{10}}\right)^{m} \\
&\le A_{10}^{k}\sum_{m=0}^{k}\sum_{j=0}^{m} \left[\left(\frac{A_9^2}{4}\right)^{j} + \left(\frac{4}{A_{10}^2}\right)^{m}\right] \\
&\le A_{10}^{k}\sum_{m=0}^{k}\sum_{j=0}^{m} j!\,\frac{\left(\frac{A_9^2}{4}\right)^{j}}{j!} + A_{10}^{k}\sum_{m=0}^{k}\sum_{j=0}^{m} m!\,\frac{\left(\frac{4}{A_{10}^2}\right)^{m}}{m!} \\
&\le A_{10}^{k}\,k!\sum_{m=0}^{k}\sum_{j=0}^{m} \frac{\left(\frac{A_9^2}{4}\right)^{j}}{j!} + A_{10}^{k}\,k!\sum_{m=0}^{k}\sum_{j=0}^{m} \frac{\left(\frac{4}{A_{10}^2}\right)^{m}}{m!} \\
&\le e^{A_9^2/4}\, A_{10}^{k}\, k! + e^{4/A_{10}^2}\, A_{10}^{k}\, k! \\
&\le c\, A_{10}^{k}\, k!. \qquad (4.22)
\end{aligned}$$

Replacing (4.22) in (4.21) we obtain

$$\begin{aligned}
&|| < D_{x,t} >^{3b} \psi\, u_k||_{L_t^2(\mathbb{R}:\, H_x^{s-1}(\mathbb{R}))} + || < D_{x,t} >^{3b} \psi\, v_k||_{L_t^2(\mathbb{R}:\, H_x^{s-1}(\mathbb{R}))} \\
&\le c\, A_{17}^{k}\cdot k! + c\, A_{19}^{k}\cdot (k!)^2 \\
&\le c\, A_{17}^{k}\cdot (k!)^2 + c\, A_{19}^{k}\cdot (k!)^2 \\
&\le c\, A_{20}^{k}\cdot (k!)^2 \qquad (4.23)
\end{aligned}$$

and the result follows.

Remark 4.1. a) For simplicity, we only illustrate the conclusion for the case $s \ge -1/2-\delta$ with $b = 1/2+\delta/3$ (for small $\delta > 0$) and the case $s = -3/4+\delta$ and $b = 7/12-\delta/3$. If $s = -1/2-\delta$ with $b = 1/2+\delta/3$, the initial data can involve Dirac's delta measure δ_0 and the latter is the critical case of the local well-posedness.
b) The following inequality is simple to verify in both cases,

$$||\psi\, u_k||_{L^2_{x,\,t}(\mathbb{R}^2)} \le || < D_x >^{3\,b} (\psi\, u_k)||_{L_t^2(\mathbb{R}:\, H_x^{s-1}(\mathbb{R}))} \le c\, || < D_{x,\,t} >^{3\,b} (\psi\, u_k)||_{L_t^2(\mathbb{R}:\, H_x^{s-1}(\mathbb{R}))}.$$

Proposition 4.2. *Under the same assumptions as in Proposition 4.1, there exist positive constants c and A such that*

$$||\psi\, P^k u||_{H^{7/2}(\mathbb{R}^2)} \le c\, A^k\, (k!)^2, \quad k = 0,\, 1,\, 2, \ldots \qquad (4.24)$$

$$||\psi\, P^k v||_{H^{7/2}(\mathbb{R}^2)} \le c\, A^k\, (k!)^2, \quad k = 0,\, 1,\, 2, \ldots \qquad (4.25)$$

Proof. We apply Lemma 2.4 to $\psi\, u_k \equiv \psi\, P^k u$ with $b = 1$ and $r = 0$.

$$\begin{aligned}
&|| < D_{x,\,t} >^{3}\, \psi\, P^k u||_{L^2(\mathbb{R}:\, L_x^2(\mathbb{R}))} \\
&\quad \le c\, \left(||\psi\, u_k||_{L^2(\mathbb{R}:\, L_x^2(\mathbb{R}))} + ||t\partial_x^3(\psi\, u_k)||_{L^2(\mathbb{R}:\, L_x^2(\mathbb{R}))} + ||P^3(\psi\, u_k)||_{L^2(\mathbb{R}:\, L_x^2(\mathbb{R}))}\right) \qquad (4.26)
\end{aligned}$$

Therefore, if we wish to estimate the second term in the right hand side of (4.26) with the aid of the equation (2.12)

$$\psi\, t\, \partial_x^3 u_k = -\frac{1}{3}\,\psi\, P u_k + \frac{1}{3}\,\psi\, x\, \partial_x u_k + t\,\psi\, B_k$$

it is necessary to estimate $||\psi\, u_k||_{L^2_t(\mathbb{R}:\, H^1_x(\mathbb{R}))}$ which is not yet obtained. Hence, we start from the lower regularity setting, i. e., applying (2.23) in Lemma 2.4 to $\psi\, u_k$ with $\mu = 1/2$. Let ψ_1 be a smaller size of smooth cut-off function with $\psi_1 \leq \psi$ and $\psi_1 = 1$ around $(x_0,\, t_0)$. Applying (2.23) a $\psi\, u_k = \psi\, P^k u$ with $\mu = 1/2$ we have

$$\begin{aligned} ||<D_{x,\,t}>^3\, \psi_1\, P^k u||_{H^{-\,5/2}(\mathbb{R}^2)} &\leq c\,||<D_{x,\,t}>^3\, \psi_1\, P^k u||_{L^2(\mathbb{R}^2)} \\ &\leq c\,\left(||\psi_1 u_k||_{H^{-\,5/2}(\mathbb{R}^2)} + ||t\partial_x^3(\psi_1 u_k)||_{H^{-\,5/2}(\mathbb{R}^2)} + ||P^3(\psi_1 u_k)||_{H^{-\,5/2}(\mathbb{R}^2)}\right). \quad (4.27) \end{aligned}$$

The first term on the right hand side of (4.27) has already been estimated. For the third term we have

$$\begin{aligned} ||P^3(\psi_1 u_k)||_{H^{-\,5/2}(\mathbb{R}^2)} &\leq ||P^3(\psi_1 u_k)||_{L^2_{x,\,t}(\mathbb{R}^2)} \\ &= \sum_{l=0}^{3} \frac{3!}{l!(3-l)!}||(P^{3-l}\psi_1)(P^l u_k)||_{L^2_{x,\,t}(\mathbb{R}^2)} \\ &\leq \sum_{l=0}^{3} \frac{3!}{l!(3-l)!}||P^{3-l}\psi_1||_{L^\infty_{x,\,t}(\mathbb{R}^2)}||P^l u_k||_{L^2_{x,\,t}(\mathbb{R}^2)} \\ &\leq c\sum_{l=0}^{3} \frac{3!}{l!(3-l)!}||P^{k+l} u||_{L^2_{x,\,t}(\mathbb{R}^2)} \\ &\leq c\sum_{l=1}^{3} A_1^{k+l} k! \;\leq\; c\, A_2^k k! \;\leq\; c\, A_2^k (k!)^2. \quad (4.28) \end{aligned}$$

For the second term on the right side hand we use the same idea of the remark above, using the dilation operator P. Indeed,

$$\begin{aligned} ||t\,\partial_x^3(\psi_1\, u_k)||_{H^{-5/2}} &\leq ||\psi_1\, t\,\partial_x^3 u_k||_{H^{-5/2}(\mathbb{R}^2)} + 3\,||\partial_x^2(t\,\partial_x\psi_1 \cdot u_k)||_{H^{-5/2}(\mathbb{R}^2)} \\ &+ 3\,||\partial_x(t\,\partial_x^2\psi_1 \cdot u_k)||_{H^{-5/2}(\mathbb{R}^2)} + ||t\,(\partial_x^3\psi_1)u_k||_{H^{-5/2}(\mathbb{R}^2)}. \quad (4.29) \end{aligned}$$

The last three term are bounded by the following:

$$\begin{aligned} &c\left(||\partial_x\psi_1||_{L^\infty_{x,\,t}(\mathbb{R}^2)} + ||\partial_x^2\psi_1||_{L^\infty_{x,\,t}(\mathbb{R}^2)} + ||\partial_x^3\psi_1||_{L^\infty_{x,\,t}(\mathbb{R}^2)}\right)||\psi\, u_k||_{L^2_{x,\,t}(\mathbb{R}^2)} \\ &\leq c\, A_3^k\, k! \;\leq\; c\, A_3^k\,(k!)^2. \quad (4.30) \end{aligned}$$

On the other hand, using

$$\begin{aligned} ||\psi_1\, t\,\partial_x^3 u_k||_{H^{-5/2}(\mathbb{R}^2)} &\leq \frac{1}{3}\,||\psi_1\, P u_k||_{L^2(\mathbb{R}:\, L^2_x(\mathbb{R}))} + \frac{1}{3}\,||x\,\psi_1\,\partial_x u_k||_{H^{-5/2}(\mathbb{R}^2)} + ||t\,\psi_1\, B_k||_{H^{-5/2}(\mathbb{R}^2)} \\ &= F_1 + F_2 + F_3. \quad (4.31) \end{aligned}$$

Thus

$$\begin{aligned} F_1 &\leq c\,||\psi_1||_{L^\infty_{x,\,t}(\mathbb{R}^2)}\,||\psi\, P^{k+1} u||_{L^2_{x,\,t}(\mathbb{R}^2)} \;\leq\; c\,||\psi\, P^{k+1} v||_{L^2_{x,\,t}(\mathbb{R}^2)} \\ &\leq c\, A_4^{k+1}(k+1)! \;\leq\; c\, A_5^k\, k! \;\leq\; c\, A_5^k\,(k!)^2, \quad (4.32) \end{aligned}$$

$$
\begin{aligned}
F_2 &\le ||x\,\psi_1\,\partial_x v_k||_{L^2(\mathbb{R}:\,H_x^{-1}(\mathbb{R}))} \\
&\le ||\partial_x(x\,\psi_1\,v_k)||_{L^2(\mathbb{R}:\,H_x^{-1}(\mathbb{R}))} + ||\partial_x(x\,\psi_1)\psi\,v_k||_{L^2(\mathbb{R}:\,H_x^{-1}(\mathbb{R}))} \\
&\le ||x\,\psi_1\,v_k||_{L^2_{x,\,t}(\mathbb{R}^2)} + ||\partial_x(x\,\psi_1)||_{L^\infty_{x,\,t}(\mathbb{R}^2)}||\psi\,v_k||_{L^2_{x,\,t}(\mathbb{R}^2)} \\
&\le \Big(||x\,\psi_1||_{L^\infty_{x,\,t}(\mathbb{R}^2)} + ||\partial_x(x\,\psi_1))||_{L^\infty_{x,\,t}(\mathbb{R}^2)}\Big)\,||\psi\,v_k||_{L^2_{x,\,t}(\mathbb{R}^2)} \\
&\le c\,A_6^k\,k! \;\le\; c\,A_6^k\,(k!)^2. \qquad (4.33)
\end{aligned}
$$

Using Lemma 2.5(case $\sigma = -5/2$, $s = 5$, $r = -5/2$)

$$F_3 = ||t\,\psi_1\,B_k||_{H^{-5/2}(\mathbb{R}^2)} \le c_1\,||\psi_1||_{H^5(\mathbb{R}^2)}||\psi^2\,B_x||_{H^{-5/2}(\mathbb{R}^2)}$$

and replacing B_k by (2.10), we have

$$
\begin{aligned}
F_3 &\le c_1\,\frac{|a|}{2}\sum_{k=k_1+k_2+k_3}\frac{k!}{k_1!\,k_2!\,k_3!}2^{k_1}||\psi\,u_{k_2}\psi\,u_{k_3}||_{H^{-3/2}(\mathbb{R}^2)} \\
&\quad + c_1\,\frac{|b|}{2}\sum_{k=k_1'+k_2'+k_3'}\frac{k!}{k_1'!\,k_2'!\,k_3'!}2^{k_1'}||\psi\,v_{k_2'}\psi\,v_{k_3'}||_{H^{-3/2}(\mathbb{R}^2)} \\
&\quad + c_1\,|c|\sum_{k=k_1''+k_2''+k_3''}\frac{k!}{k_1''!\,k_2''!\,k_3''!}2^{k_1''}||\psi\,u_{k_2''}\psi\,v_{k_3''}||_{H^{-3/2}(\mathbb{R}^2)} \\
&\le c_1\,\frac{|a|}{2}\sum_{k=k_1+k_2+k_3}\frac{k!}{k_1!\,k_2!\,k_3!}2^{k_1}||\psi\,u_{k_2}||_{L^2(\mathbb{R}^2)}||\psi\,u_{k_3}||_{L^2(\mathbb{R}^2)} \\
&\quad + c_1\,\frac{|b|}{2}\sum_{k=k_1'+k_2'+k_3'}\frac{k!}{k_1'!\,k_2'!\,k_3'!}2^{k_1'}||\psi\,v_{k_2'}||_{L^2(\mathbb{R}^2)}||\psi\,v_{k_3'}||_{L^2(\mathbb{R}^2)} \\
&\quad + c_1\,|c|\sum_{k=k_1''+k_2''+k_3''}\frac{k!}{k_1''!\,k_2''!\,k_3''!}2^{k_1''}||\psi\,u_{k_2''}||_{L^2(\mathbb{R}^2)}||\psi\,v_{k_3''}||_{L^2(\mathbb{R}^2)} \\
&\le c_1\,\frac{|a|}{2}\sum_{k=k_1+k_2+k_3}\frac{k!}{k_1!\,k_2!\,k_3!}2^{k_1}\,A_7^{k_2}k_2!A_7^{k_3}k_3! \\
&\quad + c_1\,\frac{|b|}{2}\sum_{k=k_1'+k_2'+k_3'}\frac{k!}{k_1'!\,k_2'!\,k_3'!}2^{k_1'}\,A_8^{k_2'}k_2'!A_8^{k_3'}k_3'! \\
&\quad + c_1\,|c|\sum_{k=k_1''+k_2''+k_3''}\frac{k!}{k_1''!\,k_2''!\,k_3''!}2^{k_1''}\,A_9^{k_2''}k_2''!A_{10}^{k_3''}k_3''! \\
&\le c_1\,\frac{|a|}{2}\,k!\sum_{k=k_1+k_2+k_3}\frac{2^{k_1}}{k_1!}\,A_7^{k_2+k_3} + c_1\,\frac{|b|}{2}\,k!\sum_{k=k_1'+k_2'+k_3'}\frac{2^{k_1'}}{k_1'!}\,A_8^{k_2'+k_3'} \\
&\quad + c_1\,|c|\,k!\sum_{k=k_1''+k_2''+k_3''}\frac{2^{k_1''}}{k_1''!}\,A_9^{k_2''}\,A_{10}^{k_3''},
\end{aligned}
$$

and then

$$\begin{aligned} F_3 &\leq c_1\,\frac{|a|}{2}\,k!\,A_7^k \sum_{k_1=0}^{k}\sum_{k_2=0}^{k-k_1}\frac{2^{k_1}}{k_1!}\,A_7^{-\,k_1} + c_1\,\frac{|b|}{2}\,k!\,A_8^k \sum_{k_1=0}^{k}\sum_{k_2'=0}^{k-k_1'}\frac{2^{k_1'}}{k_1'!}\,A_8^{-\,k_1'} \\ &\quad + c_1\,|c|\,k! \sum_{k=k_1''+k_2''+k_3''}\frac{2^{k_1''}}{k_1''!}\,A_9^{k_2''}A_{10}^{k_3''} \\ &\leq c_1\,\frac{|a|}{2}\,e^{2/A_7}\,A_7^k\,(k+1)! + c_1\,\frac{|b|}{2}\,e^{3/A_8}\,A_8^k\,(k+1)! \\ &\quad + c_1\,|c|\,k! \sum_{k=k_1''+k_2''+k_3''}\frac{2^{k_1''}}{k_1''!}\,A_9^{k_2''}A_{10}^{k_3''}. \end{aligned} \tag{4.34}$$

Replacing (4.30), (4.35) and (4.29) in (4.31) we obtain

$$\begin{aligned} &||\psi_1\,t\,\partial_x^3 u_k||_{H^{-5/2}(\mathbb{R}^2)} \\ &\quad\leq c_2\,A_{11}^k\,k! + c_1\,|c|\,k! \sum_{k=k_1''+k_2''+k_3''}\frac{2^{k_1''}}{k_1''!}\,A_9^{k_2''}A_{10}^{k_3''}, \quad k=0,\,1,\,2,\,\ldots \end{aligned} \tag{4.35}$$

Replacing (4.30) and (4.35) in (4.29)

$$\begin{aligned} &||t\,\partial_x^3(\psi_1\,u_k)||_{H^{-5/2}(\mathbb{R}^2)} \\ &\quad\leq c_3\,A_{12}^k\,k! + c_1\,|c|\,k! \sum_{k=k_1''+k_2''+k_3''}\frac{2^{k_1''}}{k_1''!}\,A_9^{k_2''}A_{10}^{k_3''}, \quad k=0,\,1,\,2,\,\ldots \end{aligned} \tag{4.36}$$

Now replacing (4.28) and (4.36) in (4.27) we obtain

$$\begin{aligned} &||<D_{x,\,t}>^3\,\psi\,u_k||_{H^{-\,5/2}(\mathbb{R}^2)} \\ &\quad\leq c_4\,A_{13}^k\,k! + c_1\,|c|\,k! \sum_{k=k_1''+k_2''+k_3''}\frac{2^{k_1''}}{k_1''!}\,A_9^{k_2''}A_{10}^{k_3''}, \quad k=0,\,1,\,2,\,\ldots \end{aligned} \tag{4.37}$$

In particular

$$\begin{aligned} &||\psi\,u_k||_{H^{1/2}(\mathbb{R}^2)} \\ &\quad\leq c_5\,A_{14}^k\,k! + c_1\,|c|\,k! \sum_{k=k_1''+k_2''+k_3''}\frac{2^{k_1''}}{k_1''!}\,A_9^{k_2''}A_{10}^{k_3''}, \quad k=0,\,1,\,2,\,\ldots \end{aligned} \tag{4.38}$$

Using a similar argument as above for $||<D_{x,\,t}>^3\,\psi\,P^k u||_{H^{-\,3/2}(\mathbb{R}^2)}$ with $\mu = 3/2$ in (2.23) and replacing the support of the cut-off function ψ_ϵ we obtain

$$\begin{aligned} &||\psi\,u_k||_{H^{3/2}(\mathbb{R}^2)} \\ &\quad\leq c_5\,A_{14}^k\,k! + c_1\,|c|\,k! \sum_{k=k_1''+k_2''+k_3''}\frac{2^{k_1''}}{k_1''!}\,A_9^{k_2''}A_{10}^{k_3''}, \quad k=0,\,1,\,2,\,\ldots \end{aligned} \tag{4.39}$$

In a similar way we have

$$\begin{aligned} &||\psi\, v_k||_{H^{3/2}(\mathbb{R}^2)} \\ &\leq \quad c_5\, A_{15}^k\, k! + c_1\, |\widetilde{c}|\, k! \sum_{k=k_1''+k_2''+k_3''} \frac{2^{k_1''}}{k_1''!}\, A_9^{k_2''} A_{10}^{k_3''}, \quad k = 0,\, 1,\, 2,\, \ldots \end{aligned} \tag{4.40}$$

Adding (4.39) with (4.40) and performing straightforward calculations as (4.22) we obtain

$$||\psi\, u_k||_{H^{3/2}(\mathbb{R}^2)} + ||\psi\, v_k||_{H^{3/2}(\mathbb{R}^2)} \leq C\, A^k\, (k!)^2, \quad k = 0,\, 1,\, 2,\, \ldots \tag{4.41}$$

To obtain the estimate for $||\psi\, P^k u||_{H^{7/2}(\mathbb{R}^2)}$ and $||\psi\, P^k v||_{H^{7/2}(\mathbb{R}^2)}$ we repeat the above method with $\mu = 7/2$.

Proposition 4.3. *Suppose that*

$$||\psi\, u_k||_{H^{7/2}(\mathbb{R}^2)} \leq c\, A_1^k\, (k!)^2, \quad k = 0,\, 1,\, 2,\, \ldots \tag{4.42}$$

$$||\psi\, v_k||_{H^{7/2}(\mathbb{R}^2)} \leq c\, A_2^k\, (k!)^2, \quad k = 0,\, 1,\, 2,\, \ldots \tag{4.43}$$

then we have

$$\sup_{t\in[t_0-\epsilon,\, t_0+\epsilon]} ||(t^{1/3}\partial_x)P^k u||_{H^1(x_0-\epsilon,\, x_0+\epsilon)} \leq c_1\, A_3^{k+l}\, [\,(k+l)!]^2, \quad k,\, l = 0,\, 1,\, 2,\, \ldots \tag{4.44}$$

$$\sup_{t\in[t_0-\epsilon,\, t_0+\epsilon]} ||(t^{1/3}\partial_x)P^k v||_{H^1(x_0-\epsilon,\, x_0+\epsilon)} \leq c_1\, A_4^{k+l}\, [\,(k+l)!\,]^2, \quad k,\, l = 0,\, 1,\, 2,\, \ldots \tag{4.45}$$

where $\epsilon > 0$ is so small that $\psi \equiv 1$ near $I = (x_0 - \epsilon,\, x_0 + \epsilon) \times (t_0 - \epsilon,\, t_0 + \epsilon)$.

Proof. Let $I_{t_0} = (t_0 - \epsilon,\, t_0 + \epsilon)$ and $I_{x_0} = (x_0 - \epsilon,\, x_0 + \epsilon)$, then we have $I = I_{x_0} \times I_{t_0}$. For any fixed $t \in I_{x_0}$, let $\mathcal{L} = t^{1/3}\partial_x$. We show that for some positive constants c and A_0 the following inequality holds

$$||\mathcal{L}^l P^k u||_{H_x^1(I_{x_0})} \leq c\, A_0^{k+l} [\,(k+l)!\,]^2, \quad \forall\, k,\, \forall\, l = 0,\, 1,\, 2,\, \ldots \tag{4.46}$$

Now, let use induction over l. By the trace theorem, we have

$$\begin{aligned} ||\mathcal{L}^l P^k u||_{H_x^1(I_{x_0})} &\leq ||t^{l/3}\, \partial_x^l P^k u(t)||_{H_x^1(I_{x_0})} \leq (t_0+\epsilon)^{l/3} ||\partial_x^l P^k u||_{H^{3/2}(I_{x_0}\times I_{t_0})} \\ &\leq (t_0+\epsilon)^{l/3} ||P^k u||_{H^{7/2}(I_{x_0}\times I_{t_0})} \leq (t_0+\epsilon)^{l/3} ||\psi\, P^k u||_{H^{7/2}(\mathbb{R}^2)} \\ &\leq (t_0+\epsilon)^{l/3}\, c_1\, A_1^k\, k! \leq (t_0+\epsilon)^{l/3}\, c_1\, A_0^{k+l}\, (k+l) \\ &\leq (t_0+\epsilon)^{l/3}\, c_1\, A_0^{k+l}\, [\,(k+l)!\,]^2. \end{aligned} \tag{4.47}$$

where we take $c = (t_0 + \epsilon)^{l/3} c_1$ and $A_0 = \max\{1,\, A_1\}$. Hence, in the case $l = 0,\, 1,\, 2$, it is easy to show that (4.46) follows directly from the assumption.

Now, we assume that (4.46) is true to $l \geq 2$. Applying P^k to the equation (2.4), we have

$$\begin{aligned}
\partial_t(P^k u) + \partial_x^3(P^k u) &= L\,P^k u \\
&= (P+3)^k L u \\
&= (P+3)^k(\partial_t u + \partial_x^3 u) \\
&= -(P+3)^k\left[\frac{a}{2}\,\partial_x(u^2) + \frac{b}{2}\,\partial_x(v^2) + c\,\partial_x(u\,v)\right] \\
&= -\frac{a}{2}\,(P+3)^k\partial_x(u^2) - \frac{b}{2}\,(P+3)^k\partial_x(v^2) - c\,(P+3)^k\partial_x(u\,v) \\
&= -\frac{a}{2}\,\partial_x(P+2)^k(u^2) - \frac{b}{2}\,\partial_x(P+2)^k(v^2) - c\,\partial_x(P+2)^k(u\,v)
\end{aligned}$$

such that

$$t\,\partial_t(P^k u) + t\,\partial_x^3(P^k u) = -\frac{a}{2}\,t\,\partial_x(P+2)^k(u^2) - \frac{b}{2}\,t\,\partial_x(P+2)^k(v^2) - c\,t\,\partial_x(P+2)^k(u\,v) \quad (4.48)$$

Moreover, $P = 3\,t\,\partial_t + x\,\partial_x$. Then

$$t\,\partial_t(P^k u) = \frac{1}{3}\,P^{k+1}u - \frac{1}{3}\,x\,\partial_x(P^k u). \quad (4.49)$$

Replacing (4.49) in (4.48) we obtain

$$\begin{aligned}
\mathcal{L}^3 P^k u &= t\,\partial_x^3(P^k u) = -\frac{1}{3}\,P^{k+1}u + \frac{1}{3}\,x\,\partial_x(P^k u) \\
&\quad - \frac{a}{2}\,t\,\partial_x(P+2)^k(u^2) - \frac{b}{2}\,t\,\partial_x(P+2)^k(v^2) - c\,t\,\partial_x(P+2)^k(u\,v). \quad (4.50)
\end{aligned}$$

Hence, applying $\mathcal{L}^{l-2}$ we have

$$\begin{aligned}
||\mathcal{L}^{l+1}P^k u||_{H_x^1(I_{x_0})} &= ||\mathcal{L}^{l-2}\mathcal{L}^3 P^k u||_{H_x^1(I_{x_0})} \\
&\leq \frac{1}{3}\,||\mathcal{L}^{l-2}\,P^{k+1}u||_{H_x^1(I_{x_0})} + \frac{1}{3}\,||\mathcal{L}^{l-2}\,x\,\partial_x(P^k u)||_{H_x^1(I_{x_0})} \\
&\quad + \frac{|a|}{2}\,||t\,\mathcal{L}^{l-2}\,\partial_x(P+2)^k(u^2)||_{H_x^1(I_{x_0})} + \frac{|b|}{2}\,||t\,\mathcal{L}^{l-2}\,\partial_x(P+2)^k(v^2)||_{H_x^1(I_{x_0})} \\
&\quad + |c|\,||t\,\mathcal{L}^{l-2}\,\partial_x(P+2)^k(u\,v)||_{H_x^1(I_{x_0})} \\
&= F_1 + F_2 + F_3 + F_4 + F_5. \quad (4.51)
\end{aligned}$$

Using the induction assumption, we obtain

$$F_1 \leq \frac{1}{3}\,c_1\,A_{14}^{k+l+1}(k+l+1)!. \quad (4.52)$$

We estimate the term $\mathcal{L}^{l-2}(x\,\partial_x)$ for $l \geq 3$. Let $r = l-2$, then we estimate $\mathcal{L}^r(x\,\partial_x)$ for $r \geq 1$.

$$\partial_x^r(x\,\partial_x) = \sum_{k=0}^{r}\binom{r}{k}\,\partial_x^{r-k}(\,x\,)\cdot\partial_x^k(\,\partial_x\,). \quad (4.53)$$

But

$$\partial_x^{r-k}(\,x\,) = \begin{cases} 1 & \text{if} \quad k = r-1 \\ 0 & \text{if} \quad k \leq r-2 \end{cases}$$

then in (4.53) we obtain

$$\begin{aligned}\partial_x^r(x\,\partial_x) &= r\,\partial_x^{r-1}(\,\partial_x\,) + x\,\partial_x^r(\,\partial_x\,) \;=\; r\,\partial_x^r + x\,\partial_x(\,\partial_x^r\,)\\ &= (l-2)\,\partial_x^{(l-2)} + x\,\partial_x(\,\partial_x^{(l-2)}\,),\end{aligned}$$

that is, $\mathcal{L}^{l-2}(x\,\partial_x) = x\,\partial_x\mathcal{L}^{l-2} + (l-2)\,\mathcal{L}^{l-2}$, for $l \geq 3$. For F_2 we have

$$\begin{aligned}F_2 &\leq ||x\,\partial_x\mathcal{L}^{l-2}P^k u||_{H_x^1(I_{x_0})} + (l-2)\,||\mathcal{L}^{l-2}P^k u||_{H_x^1(I_{x_0})}\\ &\leq ||x\,t^{-1/3}\mathcal{L}^{l-1}P^k u||_{H_x^1(I_{x_0})} + (l-2)\,||\mathcal{L}^{l-2}P^k u||_{H_x^1(I_{x_0})}\\ &\leq c\,(t_0-\epsilon)\,(|x_0|+\epsilon+1)\,||\mathcal{L}^{l-1}P^k u||_{H_x^1(I_{x_0})} + (l-2)\,||\mathcal{L}^{l-2}P^k u||_{H_x^1(I_{x_0})}\\ &\leq (t_0-\epsilon)^{-1/3}(|x_0+\epsilon+1)\,c_1\,A_{14}^{k+l-1}(k+l-1)! + c_1\,A_{14}^{k+l-1}(l-2)\,(k+l-1)!\\ &\leq \frac{1}{3}\,c_1\,A_{14}^{k+l+1}(k+l+1)! \qquad (4.54)\end{aligned}$$

where we take A_{14} larger than $(t_0-\epsilon)^{-1/3}(|x_0|+\epsilon+1)$ and 3. Using that ($\mathcal{L} = t^{1/3}\,\partial_x^3$)

$$t\,\mathcal{L}^{l-2}\partial_x = t\,t^{(l-2)/3}\partial_x^{(l-2)}\partial_x = t\,t^{-1/3}\,t^{(l-1)/3}\partial_x^{(l-1)} = t^{2/3}\,\mathcal{L}^{l-1},$$

we have

$$\begin{aligned}F_3 &= \frac{|a|}{2}\,||t^{2/3}\,\mathcal{L}^{l-1}\,(P+2)^k(u^2)||_{H_x^1(I_{x_0})}\\ &\leq \frac{|a|}{2}\,(t_0+\epsilon)^{2/3}\sum_{l-1=l_1+l_2}\;\sum_{k=k_1+k_2+k_3}\frac{(l-1)!}{l_1!l_2!}\,\frac{k!}{k_1!k_2!k_3!}\,2^{k_3}\\ &\quad\times c_2\,||\mathcal{L}^{l_1}P^{k_1}u||_{H_x^1(I_{x_0})}\,||\mathcal{L}^{l_2}P^{k_2}u||_{H_x^1(I_{x_0})}.\end{aligned}$$

Using the induction assumption

$$\begin{aligned}F_3 &\leq \frac{|a|}{2}\,(t_0+\epsilon)^{2/3}\sum_{l-1=l_1+l_2}\;\sum_{k=k_1+k_2+k_3} c_2\,c_1^3\,k!\,(l-1)!\,\frac{2^{k_3}}{k_3!}\\ &\quad\times\frac{(l_1+k_1)!}{l_1!\,k_1!}\,\frac{(l_2+k_2)!}{l_2!\,k_2!}\,A_{14}^{k+l-1}\\ &\leq \frac{|a|}{2}\,(t_0+\epsilon)^{2/3}\,c_2\,c_1^3\,(l+k-1)!\,A_{14}^{k+l-1}\sum_{l-1=l_1+l_2}\;\sum_{k=k_1+k_2+k_3}\frac{2^{k_3}}{k_3!}\\ &\quad\times\frac{(l_1+k_1)!}{l_1!\,k_1!}\,\frac{(l_2+k_2)!}{l_2!\,k_2!}\,\frac{k!\,(l-1)!}{(l+k-1)!}.\end{aligned}$$

Using that

$$\sum_{l-1=l_1+l_2}\;\sum_{k=k_1+k_2+k_3}\frac{2^{k_3}}{k_3!}\,\frac{(l_1+k_1)!}{l_1!\,k_1!}\,\frac{(l_2+k_2)!}{l_2!\,k_2!}\,\frac{k!\,(l-1)!}{(l+k-1)!} \leq e^2(l+k)!$$

we obtain

$$F_3 \;\leq\; (t_0+\epsilon)^{2/3}\,c_2\,c_1^3\,e^2\,(l+k)!\,A_{14}^{k+l-1} \;\leq\; \frac{1}{3}\,c_1\,A_{14}^{k+l+1}\,(k+l+1)! \qquad (4.55)$$

where we take A_{14} larger than $(t_0-\epsilon)^{-1/3}\,c_2\,c_1^2\,e^2$, and 3. In a similar way

$$F_4 \leq \frac{1}{3}\,c_3\,A_{15}^{k+l+1}\,(k+l+1)! \qquad (4.56)$$

where we take A_{15} larger than $(t_0-\epsilon)^{-1/3}\,c_4\,c_3^2\,e^2$, and 3. Finally, in a similar way

$$F_5 \leq \frac{1}{3}\,c_6\,A_{16}^{k+l+1}\,(k+l+1)! \tag{4.57}$$

where we take A_{16} larger than $(t_0-\epsilon)^{-1/3}\,c_6\,c_5^2\,e^2$, and 3. Therefore, from (4.52), (4.54), (4.55), (4.56) and (4.57) we obtain

$$||\mathcal{L}^{l+1}P^k u||_{H_x^1(I_{x_0})} \leq c_7\,A_{17}^{k+l+1}\,(k+l+1)!. \tag{4.58}$$

In a similar way, we obtain

$$||\mathcal{L}^{l+1}P^k v||_{H_x^1(I_{x_0})} \leq c_7\,A_{17}^{k+l+1}\,(k+l+1)!, \tag{4.59}$$

and the result follows.

Proposition 4.4. *Suppose that there exists a positive constants c_1, c_2 and A_{14}, A_{15} such that*

$$\sup_{t\in[t_0-\epsilon,\,t_0+\epsilon]} ||\partial_x^l P^k u||_{H_x^1(x_0-\epsilon,\,x_0+\epsilon)} \leq c_1\,A_{14}^{k+l}\,[\,(k+l)!\,]^2, \quad k,\,l=0,\,1,\,2,\,\ldots \tag{4.60}$$

$$\sup_{t\in[t_0-\epsilon,\,t_0+\epsilon]} ||\partial_x^l P^k v||_{H_x^1(x_0-\epsilon,\,x_0+\epsilon)} \leq c_2\,A_{15}^{k+l}\,[\,(k+l)!\,]^2, \quad k,\,l=0,\,1,\,2,\,\ldots \tag{4.61}$$

Then we have respectively

$$\sup_{t\in[t_0-\epsilon,\,t_0+\epsilon]} ||\partial_t^m\partial_x^l u||_{H_x^1(x_0-\epsilon,\,x_0+\epsilon)} \leq c_3\,A_{16}^{m+l}\,[\,(m+l)!\,]^2, \quad m,\,l=0,\,1,\,2,\,\ldots \tag{4.62}$$

$$\sup_{t\in[t_0-\epsilon,\,t_0+\epsilon]} ||\partial_t^m\partial_x^l v||_{H_x^1(x_0-\epsilon,\,x_0+\epsilon)} \leq c_4\,A_{17}^{m+l}\,[\,(m+l)!\,]^2, \quad m,\,l=0,\,1,\,2,\,\ldots \tag{4.63}$$

where c_3, c_4 and A_{16}, A_{17} only depend on c_1, c_2 and A_{14}, A_{15}, respectively and ϵ, $(x_0,\,t_0)$.

Proof. Using the idea of Proposition 4.3, we fix $t \in I_{x_0}$. First we show that for some positive constants c_3, A_{16} and B_{16}

$$||(x\,\partial_x)^m\,\partial_x^l P^k v||_{H_x^1(I_{x_0})} \leq c_3\,A_{16}^{k+m+l}\,B_{16}^m(k+m+l)!, \quad k,\,m,\,l=0,\,1,\,2,\,\ldots \tag{4.64}$$

We use induction. Suppose that (4.64) is true for m.

$$\begin{aligned}
&||(x\,\partial_x)^{m+1}\,\partial_x^l P^k v||_{H_x^1(I_{x_0})}\\
&\quad= \;||(x\,\partial_x)\,(x\,\partial_x)^m\,\partial_x^l P^k v||_{H_x^1(I_{x_0})}\\
&\quad\leq\; (|x_0|+\epsilon+1)\,||(x\,\partial_x+I)^m\,\partial_x^{l+1}P^k v||_{H_x^1(I_{x_0})}\\
&\quad\leq\; c(|x_0|,\,\epsilon)\sum_{j=1}^{m}\binom{m}{j}||(x\,\partial_x)^j\,\partial_x^{l+1}P^k v||_{H_x^1(I_{x_0})}\\
&\quad\leq\; c\sum_{j=1}^{m}\binom{m}{j}c_3\,A_{16}^{k+l+j+1}\,B_{16}^j(k+l+j+1)!\\
&\quad\leq\; c_3\,A_{16}^{k+l+m+1}\,B_{16}^m(k+l+m+1)!\sum_{j=1}^{m}\frac{(A_{16}\,B_{16})^{-(m-j)}}{(m-j)!}\;\frac{m!}{j!}\;\frac{(k+l+j+1)!}{(k+l+m+1)!}\\
&\quad\leq\; e^{-\,A_{16}\,B_{16}}c_3\,A_{16}^{k+l+m+1}\,B_{16}^m(k+l+m+1)!
\end{aligned} \tag{4.65}$$

where we take B_{16} so large that $B_{16} \geq \max\{|x_0|+\epsilon+1,\, 1\}$. We show that for some positive constants c_4, A_{17} we have

$$||(t\,\partial_t)^m\,\partial_x^l u||_{H_x^1(I_{x_0})} \leq c_4\,A_{17}^{l+m}\,(l+m)!, \quad l,\, m = 0,\, 1,\, 2,\, \ldots$$

Using that $t\,\partial_t = \frac{1}{3}\,(P - x\,\partial_x)$, we obtain

$$\begin{aligned}
||(t\,\partial_t)^m\,\partial_x^l u||_{H_x^1(I_{x_0})} &= 3^{-m}\,||(P - x\,\partial_x)^m\,\partial_x^l u||_{H_x^1(I_{x_0})} \\
&\leq 3^{-m} \sum_{m=j_1+j_2} \frac{m!}{j_1!\,j_2!}\,||(x\,\partial_x)^{j_1}\,P^{j_2}\partial_x^l u||_{H_x^1(I_{x_0})} \\
&\leq 3^{-m} \sum_{m=j_1+j_2} \frac{m!}{j_1!\,j_2!}\,||(x\,\partial_x)^{j_1}\,\partial_x^l (P-l)^{j_2} u||_{H_x^1(I_{x_0})} \\
&\leq 3^{-m} \sum_{m=j_1+j_2+j_3} \frac{m!}{j_1!\,j_2!\,j_3!}\,l^{j_3}\,||(x\,\partial_x)^{j_1}\,\partial_x^l P^{j_2} u||_{H_x^1(I_{x_0})}.
\end{aligned}$$

where we replace j_2 into $j_2 + j_3$. Now, using the induction hypothesis we have (with $B_{17} \geq A_{16}\,B_{16}$)

$$\begin{aligned}
&||(t\,\partial_t)^m\,\partial_x^l u||_{H_x^1(I_{x_0})} \\
&\leq 3^{-m} \sum_{m=j_1+j_2+j_3} \frac{m!}{j_1!\,j_2!\,j_3!}\,l^{j_3}\,c_3\,B_{17}^{j_1+j_2+l}\,(j_1+j_2+l)! \\
&\leq 3^{-m}\,c_3\,B_{17}^{m+l}\,(m+l)! \sum_{m=j_1+j_2+j_3} B_{17}^{-j_3} \frac{m!}{j_1!\,j_2!\,j_3!}\,l^{j_3}\,\frac{(j_1+j_2+l)!}{(m+l)!}, \qquad (4.66)
\end{aligned}$$

Observing that $l^{j_3}\,\dfrac{(j_1+j_2+l)!}{(m+l)!} \leq 1$, we obtain in (4.66)

$$\begin{aligned}
||(t\,\partial_t)^m\,\partial_x^l u||_{H_x^1(I_{x_0})} &\leq 3^{-m}\,c_3\,(2 + B_{17}^{-1})^m\,B_{17}^{l+m}\,(l+m)! \\
&\leq c_4\,A_{17}^{l+m}\,(l+m)!
\end{aligned}$$

where we take $A_{17} = \max\{B_{17},\, 3^{-1}\,B_{17}\,(2 + B_{17}^{-1})\}$. We show that for some positive constants c_4, A_{18} and B_{18} we have

$$||(t\,\partial_t)^j\,\partial_t^m\partial_x^l u||_{H_x^1(I_{x_0})} \leq c_4\,A_{18}^{j+m+l}\,B_{18}(j+m+l)!, \quad j,\, l,\, m = 0,\, 1,\, 2,\, \ldots \qquad (4.67)$$

Induction in m.

$$\begin{aligned}
||(t\,\partial_t)^j\,\partial_t^{m+1}\partial_x^l u||_{H_x^1(I_{x_0})} &\leq ||\partial_t(t\,\partial_t - I)^m\,\partial_t^m\partial_x^l u||_{H_x^1(I_{x_0})} \\
&= t^{-1}\,||t\,\partial_t(t\,\partial_t - I)^j\,\partial_t^m\partial_x^l u||_{H_x^1(I_{x_0})} \\
&\leq (t_0-\epsilon)^{-1} \sum_{j_1=0}^{j} \binom{j}{j_1}\,||(t\,\partial_t)^{j_1+1}\,\partial_t^m\partial_x^l u||_{H_x^1(I_{x_0})}.
\end{aligned}$$

Using the induction hypothesis

$$
\begin{aligned}
&||(t\,\partial_t)^j\,\partial_t^{m+1}\partial_x^l u||_{H^1_x(I_{x_0})}\\
&\leq\ (t_0-\epsilon)^{-1}\sum_{j_1=0}^{j}\binom{j}{j_1}\,c_4\,A_{18}^{j_1+l+m+1}\,B_{18}^m\,(j_1+l+m+1)!\\
&=\ c_4\,(t_0-\epsilon)^{-1}\,A_{18}^{j_1+l+m+1}\,B_{18}^m\,(j_1+l+m+1)!\\
&\quad\times\sum_{j_1=0}^{j}\frac{A_{18}^{-(j-j_1}}{(j-j_1)!}\binom{j}{j_1}\,\frac{(j_1+m+l+1)!\,(j-j_1)!}{(j+m+l+1)!}\\
&=\ c_4\,(t_0-\epsilon)^{-1}\,e^{-A_{18}}\,A_{18}^{j+l+m+1}\,B_{18}^m\,(j+l+m+1)!\\
&\leq\ c_4\ A_{18}^{j+l+m+1}\,B_{18}^m\,(j+l+m+1)!
\end{aligned}
$$

where we take B_{18} larger than $(t_0-\epsilon)^{-1}\,e^{-A_{18}}$. Finally, we choose $j=0$ in (4.67) and take $c_2=c_4$ and $A_{15}=A_{18}\,B_{18}$. The result of analyticity follows.

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