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Modeling Unbounded Wave Propagation Problems In Terms Of Transverse Fields Using 2D Mixed Finite Elements

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Abstract -We present in this paper an approach for the modeling of open boundary microwave problems in the frequency domain by using high order 2D mixed elements conforming in $H(\text{curl})$. The Galerkin formulation for the vector wave equation in two dimensions is used to discretize the problem. The analysis region is truncated using a 2D vector Absorbing Boundary Condition that satisfies the Sommerfeld radiation condition at infinity. This modeling is applied to scattering problems or to open ended waveguides.

I. INTRODUCTION

The finite element method (FEM) is used with success to modelize open boundary electromagnetic problems. The coupling with Absorbing Boundary Conditions (A.B.C's) preserves the sparsity of the F.E. matrix. Usually methods used are nodal-based F.E. formulations with the component of the field perpendicular to the studied domain as an unknown: Hz in the case of a transverse electric (T.E.) wave for example [9]. In such a case, due to the numerical derivation, the computation of the transverse field from the scalar field is inaccurate. Furthermore, singularities of the transverse field at sharp, perfectly conducting edges are difficult to modelize.

To avoid these difficulties, we have developed a formulation written in terms of the transverse field, \mathbf{E} for a T.E. case, and \mathbf{H} for a T.M. case. Mixed elements conforming in $H(\text{curl})$ of degree two, R2 and P2 on the triangle [1, 2], are used to enforce tangential continuity and to allow the normal discontinuity of transverse fields at interfaces of materials. It is this floating of normal continuity that allows the mixed-based elements to handle objects with sharp edges [3].

This vector formulation, with transverse field as unknown, is coupled with 2D vector A.B.C's, based on the second-order scalar Bayliss-Turkel (BT) condition applied on the circular boundary [4], and on the Engquist-Majda (EM) condition applied on a rectangular one [5]. These boundary conditions are modified to preserve the symmetry of the finite-element matrix [6-7].

II. THE MIXED FINITE ELEMENT FORMULATION

In this section we consider the problem of finding an electromagnetic field \mathbf{E} , \mathbf{H} in inhomogeneous medium with perfect electric conductors (pec), Ω_{pec} , obeying the time-harmonic Maxwell equations and the Silver-Müller radiation

condition. To formulate the problem via the FEM, it is necessary to enclose the objects with an artificial outer boundary Γ on which the radiation condition at infinity is enforced via an exterior boundary operator \mathcal{T} (see Fig. 1).

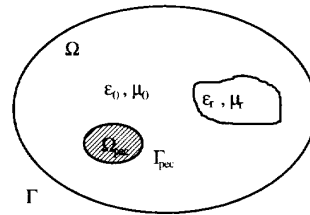


Fig. 1. Schematic configuration of a problem.

The problem in the interior domain, Ω , can be mathematically formulated in terms of \mathbf{E} , for example, as

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mu_r\mathbf{H} \quad \text{in } \Omega / \Omega_{\text{pec}}, \quad (1)$$

$$\nabla \times \mu_r^{-1}\nabla \times \mathbf{E} - k_0^2\epsilon_r\mathbf{E} = -j\omega\mu_0\mathbf{J}_e, \quad \text{in } \Omega / \Omega_{\text{pec}}, \quad (2)$$

$$\mathbf{n}_e \times \mathbf{E} = \mathbf{0} \quad \text{on } \Gamma_{\text{pec}}, \quad (3)$$

$$\mathbf{n} \times \nabla \times \mathbf{E} = \mathcal{T}(\mathbf{E}) \quad \text{on } \Gamma, \quad (4)$$

where \mathbf{n} is the unit outward normal to the contour Γ and \mathbf{n}_e is the unit normal to the pec boundary Γ_{pec} .

The Galerkin weighted residual method applied on the inhomogeneous vector wave equation (2) yields the formulation :

$$\int_{\Omega} [\mu_r^{-1}(\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{W}) - k_0^2\epsilon_r\mathbf{E} \cdot \mathbf{W}] d\Omega + \int_{\Gamma} \mathcal{T}(\mathbf{E}) \cdot \mathbf{W} d\tau = -j\omega\mu_0 \int_{\Omega} \mathbf{J}_e \cdot \mathbf{W} d\Omega \quad (5)$$

where \mathbf{J}_e denote the electric source term and \mathbf{W} is an arbitrary vector weighting function generated by the mixed elements.

In the scattering case, the source term \mathbf{J}_e represents the source induced by the incident wave inside domain filled with dielectric and eventually on Γ_{pec} (for a scattered field formulation) [8]. For open ended waveguides, it is an equivalent current line [9].

The required continuity of tangential \mathbf{E} at material interfaces is automatically ensured by mixed-based elements; however, the boundary condition (3) must be imposed. This is simply done by setting the degrees of freedom associated with edges located on pec boundary to zero.

A same type of formulation may be obtained with the magnetic field \mathbf{H} as unknown.

III. DERIVATION OF 2D VECTOR ABC'S

ABC's have been derived to approximate the global operator \mathbf{T} and their use in FEM solutions of open region two-dimensional electromagnetic problems has been extensive [8-9]. However these scalar 2D ABC's are not directly suitable for use with our vector formulation. But, as \mathbf{E} is *a priori* a radiation vector field (obeying (2) and Silver-Müller condition), it follows that the cartesian components of \mathbf{E} are scalar radiation fields (obeying Helmholtz scalar equation and Sommerfeld condition) and then they verify scalar 2D ABC's. Thus \mathbf{E} must satisfy, in cylindrical coordinates (ρ, ϕ) [4],

$$\frac{\partial \mathbf{E}}{\partial \rho} = \alpha(\rho)\mathbf{E} + \beta(\rho)\frac{\partial^2 \mathbf{E}}{\partial \phi^2} \quad (6)$$

where α and β , for the second order BT condition, are given by

$$\alpha(\rho) = \left(-jk - \frac{1}{2\rho} - \frac{j}{8k\rho^2} + \frac{1}{8k^2\rho^3} \right) \quad (7-a)$$

$$\beta(\rho) = \left(-\frac{j}{2k\rho^2} + \frac{1}{2k^2\rho^3} \right) \quad (7-b)$$

Using the vector identity

$$(\mathbf{e}_\rho \cdot \nabla)\mathbf{E} = \nabla(\mathbf{e}_\rho \cdot \mathbf{E}) - \mathbf{e}_\rho \times \nabla \times \mathbf{E} \quad (8)$$

and considering the tangential parts of equations (6) and (8), it can be shown that \mathbf{E} satisfies [10] the vector BT condition, on the circular boundary

$$\mathbf{T}(\mathbf{E}) = \alpha' \mathbf{E}_\tau + \beta' \nabla_\tau (\nabla \cdot \mathbf{E}_\tau) + \gamma \nabla_\tau (\mathbf{n} \cdot \mathbf{E}) \quad (9)$$

where \mathbf{E}_τ and ∇_τ denote tangential parts on the exterior boundary Γ . The coefficients α' , β' and γ are given by

$$\alpha'(\rho) = -\left(\alpha(\rho) - \beta(\rho) + \frac{1}{\rho} \right)$$

$$\beta'(\rho) = -\rho^2 \beta(\rho)$$

$$\gamma(\rho) = (1 - 2\rho\beta(\rho))$$

The second-order Enquist-Majda condition applied on the rectangular boundary is obtained from (9) assuming that ρ tends to infinity (this supposition is only formal) in the above coefficients.

The ABC is incorporated into the integral formulation (5) by substituting operator \mathbf{T} with its approximation (9). The line integral term in (5) becomes after integration by parts:

$$\begin{aligned} \oint_\Gamma \mathbf{T}(\mathbf{E}) \cdot \mathbf{W} \, d\tau &= \alpha' \oint_\Gamma \mathbf{E}_\tau \cdot \mathbf{W}_\tau \, d\tau - \beta' \oint_\Gamma (\nabla \cdot \mathbf{E}_\tau) (\nabla \cdot \mathbf{W}_\tau) \, d\tau \\ &\quad + \gamma \oint_\Gamma \mathbf{W} \cdot \nabla_\tau (\mathbf{n} \cdot \mathbf{E}) \, d\tau \end{aligned} \quad (10)$$

The first-order derivative contained in (9) makes the matrix

resulting from the discretization of (10) non-symmetric. To deal with this problem and to preserve the symmetry of the matrix, we use an approach suggested in [7]. We obtain the following approximation

$$\nabla_\tau (\mathbf{n} \cdot \mathbf{E}) \approx -\frac{j}{k} \nabla_\tau (\nabla \cdot \mathbf{E}_\tau) \quad (11)$$

Then, substituting (11) in (9), we have an alternative symmetric version of (9)

$$\mathbf{T}(\mathbf{E}) \approx \alpha_s(\rho)\mathbf{E}_\tau + \beta_s(\rho)\nabla_\tau (\nabla \cdot \mathbf{E}_\tau) \quad (12)$$

where α_s and β_s are given by

$$\alpha_s(\rho) = \alpha'(\rho) + O(\rho^{-3})$$

$$\beta_s(\rho) = -\beta'(\rho) + O(\rho^{-1})$$

Finally, the contour integral term (10) coupled with (11) becomes

$$\begin{aligned} \oint_\Gamma \mathbf{T}(\mathbf{E}) \cdot \mathbf{W} \, d\tau &= \alpha_s \oint_\Gamma \mathbf{E}_\tau \cdot \mathbf{W}_\tau \, d\tau \\ &\quad - \beta_s \oint_\Gamma (\nabla \cdot \mathbf{E}_\tau) (\nabla \cdot \mathbf{W}_\tau) \, d\tau \end{aligned} \quad (13)$$

IV. DISCRETIZATION

As was mentioned previously, finite elements conforming in space $H(\text{curl})$ i.e. having tangential continuity property, are used to discretize the integral formulations (5) and (13). The mixed elements of order two, R2 and P2, on the triangle shown in Fig. 2 have been implemented. In [1-2] these finite elements are defined by a list of space of interpolation and corresponding degrees of freedom.

The unknown vector field \mathbf{E} is expanded as

$$\mathbf{E} = \sum_{i=1}^N E_i \mathbf{W}_i \quad \text{and} \quad \mathbf{H} = \mathbf{j} / (\omega \mu_0 \mu_r) \sum_{i=1}^N E_i \nabla \times \mathbf{W}_i,$$

where $E_i = \sigma_i(\mathbf{E})$.

The linear form σ_i is the degree of freedom associated with vector shape function \mathbf{W}_i and N is the total number of degrees of freedom associate with the mesh.

The details for deriving an explicit expression of each vector basis function can be found in [10]. Here, we rewrite only the final results.

A. Mixed elements R2

- Vector basis functions \mathbf{W}_{ij}^m , $1 \leq m \leq 2$, associated with edge $C = \{ a_i, a_j \}$

$$\mathbf{W}_{ij}^1 = (-3 + 4\lambda_i + 4\lambda_j)\lambda_i \nabla \lambda_j - (-3 + 4\lambda_i + 4\lambda_j)\lambda_j \nabla \lambda_i$$

$$\mathbf{W}_{ij}^2 = (-1 + 4\lambda_i - 4\lambda_j)\lambda_i \nabla \lambda_j - (1 + 4\lambda_i - 4\lambda_j)\lambda_j \nabla \lambda_i$$

have degrees of freedom σ_{ij}^m , $1 \leq m \leq 2$,

$$\sigma_{ij}^1(\mathbf{p}) = \int_C \mathbf{p} \cdot \boldsymbol{\tau}_{ij} \, ds \quad \text{and} \quad \sigma_{ij}^2(\mathbf{p}) = \int_C \mathbf{p} \cdot \boldsymbol{\tau}_{ij} (\lambda_i - \lambda_j) \, ds$$

where

- λ_i , $1 \leq i \leq 3$, are barycentric coordinates within a triangle associated with vertices $\{a_i\}$, $1 \leq i \leq 3$,
- τ_{ij} is a unit tangent vector to edge C and orientates it,
- \mathbf{p} is an arbitrary vector defined inside the triangle.
- Vector basis functions \mathbf{W}_K^m , $1 \leq m \leq 2$, associated with the volume K, can be written in the base $(\nabla\lambda_1, \nabla\lambda_2)$ as

$$\mathbf{W}_K^1 = 8(-1 + \lambda_1 + 2\lambda_2)\lambda_1\nabla\lambda_2 - 8(-2 + \lambda_1 + 2\lambda_2)\lambda_2\nabla\lambda_1$$

$$\mathbf{W}_K^2 = 8(2 - 2\lambda_1 - \lambda_2)\lambda_1\nabla\lambda_2 - 8(1 - 2\lambda_1 - \lambda_2)\lambda_2\nabla\lambda_1$$

and have degrees of freedom σ_K^m , $1 \leq m \leq 2$,

$$\sigma_K^1(\mathbf{p}) = \iint_K \mathbf{p} \cdot \nabla_t \lambda_2 dx \text{ and } \sigma_K^2(\mathbf{p}) = \iint_K \mathbf{p} \cdot \nabla_t \lambda_1 dx$$

where

$$-\nabla_t \lambda_i = \nabla \lambda_i \times \mathbf{e}_z, \quad 1 \leq i \leq 3.$$

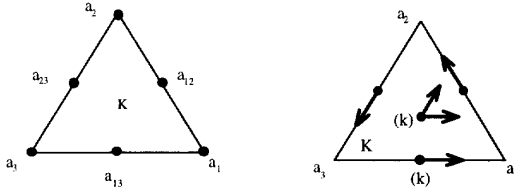


Fig. 2. Node-based (left) and mixed-based (right) second order triangular elements. For mixed elements, k degrees of freedom associated with a simplex are symbolized by a point located at its center of gravity and a local vector basis.

B. Mixed elements P2

- Vector basis functions \mathbf{W}_{ij}^m , $1 \leq m \leq 3$, associated with edge C,

$$\mathbf{W}_{ij}^m = \left(\alpha_1^m N_i + \beta_3^m (N_{ij} - N_{ji}) \right) \nabla \lambda_j + \left(\alpha_2^m N_j + \alpha_3^m (N_{ji} - N_{ij}) \right) \nabla \lambda_i,$$

have degrees of freedom σ_{ij}^m , $1 \leq m \leq 3$,

$$\sigma_{ij}^m(\mathbf{p}) = \int_C \mathbf{p} \cdot \tau_{ij} \varphi^m(s) ds,$$

where

$$\begin{cases} \alpha_1^1 = 1 \\ \alpha_2^1 = 0 \\ \alpha_3^1 = \beta_3^1 = 1/8 \end{cases}, \begin{cases} \alpha_1^2 = 0 \\ \alpha_2^2 = -1 \\ \alpha_3^2 = \beta_3^2 = -1/8 \end{cases}, \begin{cases} \alpha_1^3 = 0 \\ \alpha_2^3 = 0 \\ \alpha_3^3 = -\beta_3^3 = -1/4 \end{cases}$$

$$\varphi^m(s) = \varphi_1^m N_i(s) + \varphi_2^m N_j(s) + \varphi_3^m N_{ij}(s),$$

the constants φ_1^m , φ_2^m and φ_3^m verify

$$\sigma_{ij}^l(\mathbf{W}_{ij}^m) = \delta_{lm} = \begin{cases} 1, & \text{si } l=m \\ 0, & \text{si } l \neq m \end{cases},$$

N_i , $1 \leq i \leq 3$, and N_{ij} , $1 \leq i < j \leq 3$, are Lagrange shape

functions of order two within a triangle associated with nodes $\{a_i\}$, $1 \leq i \leq 3$, and $\{a_{ij}\}$, $1 \leq i < j \leq 3$.

- Vector basis functions \mathbf{W}_K^m , $1 \leq m \leq 3$, associated with the volume K, can be written as

$$\mathbf{W}_K^1 = N_{12} \nabla \lambda_3, \quad \mathbf{W}_K^2 = N_{13} \nabla \lambda_2, \quad \mathbf{W}_K^3 = N_{23} \nabla \lambda_1,$$

and have degrees of freedom σ_K^m , $1 \leq m \leq 3$,

$$\sigma_K^m(\mathbf{p}) = \iint_K \mathbf{p} \cdot \mathbf{q}^m dx,$$

where

$$\mathbf{q}^m(x) = (s_1^m + t^m \lambda_1) \nabla_t \lambda_2 + (s_2^m + t^m \lambda_2) \nabla_t \lambda_1,$$

the constants s_1^m , s_2^m and t^m verify:

$$\sigma_K^l(\mathbf{W}_K^m) = \delta_{lm}.$$

V. NUMERICAL RESULTS

In order to valid the vector formulation, a study of numerical error as a function of the mesh density is done for a scattering case. A cylinder of radius 0.5λ , centered in a $4\lambda \times 3\lambda$ rectangular domain which is submitted to the second order vector EM-ABC, is illuminated by a plane wave incident at 30° . The frequency is 3 Ghz. The global error, over a centered $3\lambda \times 2\lambda$ window, is evaluated as

$$\text{err} = \frac{\int_S |\epsilon| dS}{S} \text{ with } \epsilon = \frac{\psi_c - \psi_a}{\max(\psi_a)}$$

ψ_c : computed field at each node of the domain

ψ_a : analytical field.

From Fig.3 and Fig.4, it appears that the transverse field is more accurate when computed by a mixed-based approximation while the dual field is more accurate when computed by a single scalar formulation. However when the mesh becomes more refined, the error runs low and the two formulations concur regarding global error. Fig.4 points out an interesting property of $H(\text{curl})$ elements R2 and P2; their curls belong to the same space of interpolation [2].

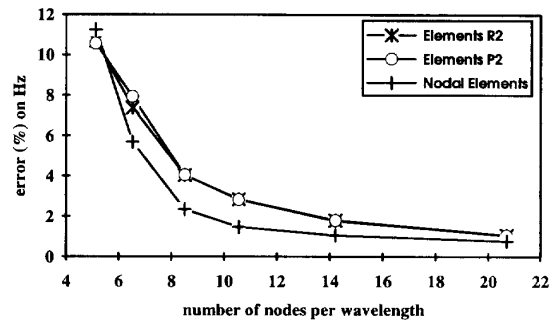


Fig. 3. global error on field component Hz, comparison analytical-computed

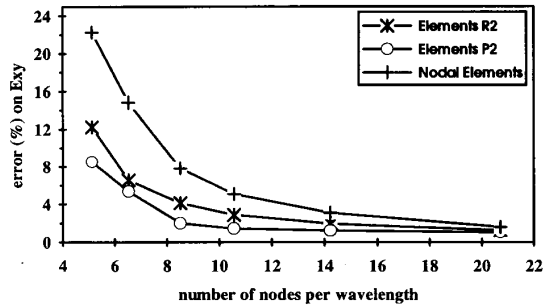


Fig. 4. global error on transverse field Exy, comparison analytical-computed

The behaviour of the vector formulation in problems with material discontinuities is evaluated. The test case, considered here, is the scattering of plane wave by a penetrable inhomogeneous cylinder. The frequency is 3 Ghz. The cylinder has relative magnetic permeability $\mu_r = 2$ and radius $a = 0.5\lambda$. The computational domain is bounded by a circle of radius $R = 2\lambda$ subjected to the second-order vector Bayliss-Turkel ABC.

In table I, we give some information on the computation. The mesh density is about 14 nodes per wavelength in each media. The global error is evaluated on a centered window $0.75\lambda \times 0.75\lambda$.

In Fig.5, we plot the transverse field near the interface

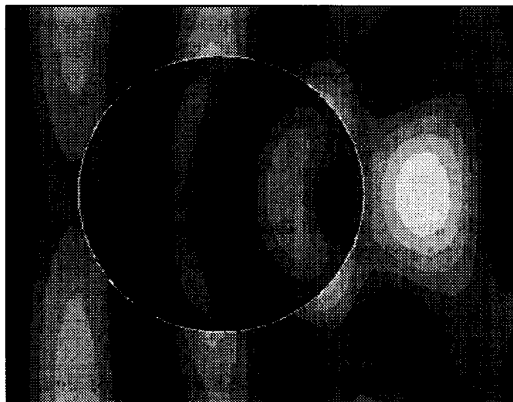


Fig. 5. Transverse magnetic field in magnetic cylinder.

TABLE I
PRACTICAL INFORMATION ON THE COMPUTATION
Solving by Nested Dissection Renumbering Algorithm on HP900/710.

	Elements R2	Elements P2	Nodal Elements
error on Ez	1.09 %	1.09 %	0.79 %
error on Hxy	1.42 %	1.22 %	2.13 %
Nb. Unk.	8152	12228	3317
Non-zeros	50916	111504	20381
CPU	101 s	178 s	78 s

In Fig.6 we show the direction of the transverse electric field at the upper extremity of an open ended waveguide. It illustrates the fact that edge-based elements allow the field to change direction abruptly at the sharp, perfectly conducting edge [3].

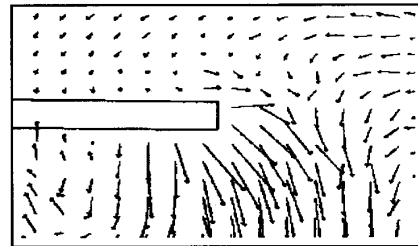


Fig. 6. Near transverse electric field at upper extremity of an open ended waveguide.

CONCLUSION

In this paper, we have shown that the finite element formulation, in terms of transverse fields, with 2D $H(\text{curl})$ vector basis functions, when coupled with ABC's, is an efficient method in modeling two-dimensional unbounded microwave problems. These 2D vector ABC's preserve the symmetry of the finite-element matrix. Further, the use of second-order $H(\text{curl})$ elements, allows the formulation to treat transverse fields discontinuities at interfaces and singularities at sharp conducting edges. The coupling of this formulation and the usual single scalar formulation may be the best procedure for solving two-dimensional electromagnetic problems.

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