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PENALIZING A $BES(d)$ PROCESS $(0 < d < 2)$ WITH A FUNCTION OF ITS LOCAL TIME, V

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Abstract. We describe the limit laws, as $t \to \infty$, of a Bessel process $(R_s, s \leq t)$ of dimension $d \in (0,2)$ penalized by an integrable function of its local time L_t at 0, thus extending our previous work of this kind, relative to Brownian motion.

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1 Introduction

1) Let $(\Omega, (X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, P_0)$ denote the canonical real-valued Brownian motion, starting from 0. We denote by $(L_t)_{t\geq 0}$ its local time at 0.
Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel function such that : $\displaystyle\int_0^\infty h(x)dx = 1$.
Define

$$H(x) = \int_0^x h(y)dy \,, \qquad x \geq 0 \,, \tag{1.1}$$

the primitive of h such that $H(0) = 0$.
For any $t \geq 0$, we introduce the probability $P_0^{(t)}$ on $\mathcal{F}_t$, which is defined by :

$$P_0^{(t)}(\Lambda_t) = \frac{E_0(1_{\Lambda_t} h(L_t))}{E_0[h(L_t)]}, \quad \Lambda_t \in \mathcal{F}_t. \tag{1.2}$$

We have shown, in [19] that the limit, as $t \to \infty$, of $P_0^{(t)}(\Lambda_s)$, for $\Lambda_s \in \mathcal{F}_s$, and s fixed, exists :

$$Q_0^{(h)}(\Lambda_s) := \lim_{t\to\infty} P_0^{(t)}(\Lambda_s) \qquad (s \geq 0,\, \Lambda_s \in \mathcal{F}_s). \tag{1.3}$$

This is a kind of "Brownian Gibbs measure", which induces a probability on $(\Omega, \mathcal{F}_\infty)$; in [19], we described precisely the process $(X_t)_{t\geq 0}$ under $Q_0^{(h)}$; the pair $(X_t, L_t)_{t\geq 0}$ is Markov under $Q_0^{(h)}$, while $(X_t)_{t\geq 0}$ is not, in general, Markov on its own.

2) The aim of the present work is to extend the above result for a d-dimensional Bessel process, $0 < d < 2$. Denote by $\nu = \frac{d}{2} - 1$ the index of this Bessel process, and let $\alpha = -\nu = 1 - \dfrac{d}{2} \in]0, 1[$.

More precisely, let $\left(\Omega_+, (R_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, (P_r^{(-\alpha)})_{r\geq 0}\right)$ denote the canonical Bessel process of dimension d, or index $\nu = -\alpha$, with $\alpha \in]0, 1[$. Ω_+ denotes the set of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, $(R_t, t \geq 0)$ is the coordinate process on Ω_+, and $(\mathcal{F}_t, t \geq 0)$ its natural filtration.
Finally, we denote : $\mathcal{F}_\infty = \vee_{s\geq 0} \mathcal{F}_s$.
The probability $P_r^{(-\alpha)}$ makes the coordinate process $(R_t, t \geq 0)$ a Bessel process with index $(-\alpha)$, starting from r. We denote by $(L_t^x; t \geq 0, x \in \mathbb{R}_+)$ the jointly continuous family of local times of the process $(R_t, t \geq 0)$. We choose the normalization of this family such that $(R_t^{2\alpha} - L_t^0, t \geq 0)$ is a martingale.
We note simply $(L_t)_{t\geq 0}$ for $(L_t^0)_{t\geq 0}$, and we consider a probability density $h : \mathbb{R}_+ \to \mathbb{R}_+$. Similarly as in (1.3), we are interested in the limit, as $t \to \infty$, of :

$$P_0^{(t)}(\Lambda_s) = \frac{E_0^{(-\alpha)}(1_{\Lambda_s} h(L_t))}{E_0^{(-\alpha)}(h(L_t))}, \quad \Lambda_s \in \mathcal{F}_s \; ; \; s \text{ fixed.} \tag{1.4}$$

Since, throughout this paper, the process of reference shall be the d-dimensional Bessel process with index $(-\alpha)$, we shall almost never again mention $(-\alpha)$ in our symbols, e.g., we shall write E_0 for $E_0^{(-\alpha)}$.
We shall prove :

Theorem 1.1 *Let h as in 1).*

1. *For every $s \geq 0$, and $\Lambda_s \in \mathcal{F}_s$,*

$$Q_0^{(h)}(\Lambda_s) := \lim_{t\to\infty} \frac{E_0(1_{\Lambda_s} h(L_t))}{E_0(h(L_t))} \quad \text{exists.} \tag{1.5}$$

2. *$Q_0^{(h)}$ satisfies :*

$$Q_0^{(h)}(\Lambda_s) = E_0(1_{\Lambda_s} M_s^h), \tag{1.6}$$

with

$$M_s^h := h(L_s) R_s^{2\alpha} + 1 - H(L_s). \tag{1.7}$$

The process $(M_s^h)_{s\geq 0}$ is a $((\mathcal{F}_s), P_0)$ positive martingale, which converges to 0, as $s \to \infty$. In particular, it is not a uniformly integrable martingale.

3. *The formula (1.6) induces a probability $Q_0^{(h)}$ on $(\Omega_+, \mathcal{F}_\infty)$. Under $Q_0^{(h)}$, the canonical process $(R_t, t \geq 0)$ satisfies :*

   (a) The random variable L_∞ is finite a.s., and it admits h as its probability density.

   (b) Let $g = \sup\{t \geq 0 : R_t = 0\}$. Then, $Q_0^{(h)}(0 < g < \infty) = 1$.

   (c) i. The two processes $(R_t, t \leq g)$ and $(R_{g+t}, t \geq 0)$ are independent;

   ii. The process $(R_{g+t}, t \geq 0)$ is a Bessel process with dimension $(4 - d)$, starting from 0;

   iii. Conditionally on $L_\infty = l$, the process $(R_t, t \leq g)$ is a Bessel process of dimension d, starting from 0, stopped at $\tau_l := \inf\{t > 0 : L_t > l\}$.

4. *Let :*

$$A_t := 4\alpha^2 \int_0^t R_s^{2(2\alpha-1)} ds \quad t \geq 0\,, \tag{1.8}$$

and denote its inverse by :

$$\rho(u) := \inf\{t \geq 0 : A_t > u\}. \tag{1.9}$$

Then, under $Q_0^{(h)}$*, the process* $(R_{\rho_u}^{2\alpha} + L_{\rho_u}, u \geq 0)$ *is a* 3*-dimensional Bessel process, starting from* 0*, which is independent from the random variable* L_∞*.*

Remark 1.2 1. *We now remark that, for* $d = 1$*, i.e :* $\alpha = 1/2$*, part 4. of Theorem 1.1 may be presented as follows :*

$$(R_t + L_t, t \geq 0) \text{ is a 3-dimensional Bessel process, independent from } L_\infty. \tag{1.10}$$

2. *Via Lévy's theorem* $\big($*: if* $(B_t, t \geq 0)$ *denotes a Brownian motion, starting from* 0*, and if :* $S_t^B = \sup_{s \leq t} B_s$*, then the two processes* $(S_t^B - B_t, S_t^B; t \geq 0)$ *and* $(|B_t|, L_t; t \geq 0)$ *have the same law* $\big)$*, the result (1.10) has already been obtained in [20] : thus, point 4. of Theorem 1.1 appears as a generalization of Pitman's theorem which asserts that :*

$$(2S_t^B - B_t; t \geq 0) \overset{(d)}{\equiv} (|B_t| + L_t, t \geq 0). \tag{1.11}$$

is a 3*-dimensional Bessel process.*

3) Just as we did in [18] concerning the 1-dimensional case, the above Theorem 1.1 invites to study the penalization with a function of the local time, not for the Bessel process itself, but for its "long bridges".
Precisely, we shall be interested to show the existence of the limit, as $t \to \infty$, of :

$$P_0(\Lambda_s | L_t = y), \quad y \geq 0, \Lambda_s \in \mathcal{F}_s, \tag{1.12}$$

and even of :

$$P_0(\Lambda_s | R_t = a, L_t = y), \quad a \geq 0, y \geq 0, \Lambda_s \in \mathcal{F}_s. \tag{1.13}$$

We obtain the following :

Theorem 1.3 1. *The limit*

$$Q_0^{(y)}(\Lambda_s) := \lim_{t \to \infty} P_0(\Lambda_s | L_t = y) \tag{1.14}$$

(with $\Lambda_s \in \mathcal{F}_s$*) exists and satisfies :*

$$Q_0^{(y)}(\Lambda_s) = p_{L_s}(y) E_0[1_{\Lambda_s} R_s^{2\alpha} | L_s = y] + E_0[1_{\Lambda_s} 1_{(L_s < y)}] \tag{1.15}$$

where p_{L_s} *is the density of* L_s*.*

2. *The preceding formula (1.15) induces a probability* $Q_0^{(y)}$ *on* $(\Omega, \mathcal{F}_\infty)$*. The probability* $Q_0^{(h)}$ *defined in Theorem 1.1 admits the following disintegration :*

$$Q_0^{(h)}(\cdot) = \int_0^\infty h(y) Q_0^{(y)}(\cdot) dy. \tag{1.16}$$

Consequently, for any $\Lambda \in \mathcal{F}_\infty$ *:*

$$Q_0^{(h)}(\Lambda | L_\infty = y) = Q_0^{(y)}(\Lambda). \tag{1.17}$$

Thus, the conditional law of $Q_0^{(h)}$ *given* $L_\infty = y$ *does not depend on* h*.*

3. *For every $s \geq 0, \Lambda_s \in \mathcal{F}_s$ and every $x, y \geq 0$,*

$$Q_0^{(x,y)}(\Lambda_s) := \lim_{t\to\infty} P_0(\Lambda_s | R_t = x, L_t = y) \tag{1.18}$$

exists, and satisfies :

$$\begin{aligned} Q_0^{(x,y)}(\Lambda_s) &= \frac{x}{x + yx^{1-2\alpha}} p_{L_s}(y) E_0(1_{\Lambda_s} R_s^{2\alpha} | L_s = y) \\ &+ \frac{1}{x + yx^{1-2\alpha}} E_0 \left[1_{\Lambda_s} 1_{(L_s < y)} \left\{ x + (y - L_s + R_s^{2\alpha}) x^{1-2\alpha} \right\} \right]. \end{aligned} \tag{1.19}$$

4. *For every $x, y \geq 0$,*

$$Q_0^{(x,y)}(\cdot) = \frac{x}{x + yx^{1-2\alpha}} Q_0^{(y)}(\cdot) + \frac{x^{1-2\alpha}}{x + yx^{1-2\alpha}} \int_0^y Q_0^{(z)}(\cdot) dz. \tag{1.20}$$

Note that formula (1.20) simplifies, in the case $\alpha = 1/2$, to yield formula (1.13) of Theorem 1.3 in [18] (via Lévy's Theorem) :

$$Q_0^{(x,y)}(\cdot) = \frac{x}{x+y} Q_0^{(y)}(\cdot) + \frac{1}{x+y} \int_0^y Q_0^{(z)}(\cdot) dz. \tag{1.21}$$

4) As a Corollary of Theorem 1.3, we now present Theorem 1.4, which describes the penalization of "long Bessel bridges" by an integrable function of their local times at 0 (see formula (1.23) below).

Theorem 1.4 *Let $\big(\Omega_+, (R_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, P_0\big)$ denote the canonical Bessel process starting from 0, with dimension $d = 2(1-\alpha), 0 < \alpha < 1$.*

1. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel function such that $\displaystyle\int_0^\infty h(y) dy = 1$. Denote, for $x > 0$,*

$$\overline{h}_x := \int_0^\infty h(y)(x + yx^{1-2\alpha}) dy, \tag{1.22}$$

assumed to be finite, and $h_x^* = 1/\overline{h}_x$.
Then, for every $s > 0$, and $\Lambda_s \in \mathcal{F}_s$,

$$\lim_{t\to\infty} \frac{E_0(\big[1_{\Lambda_s} h(L_t) | R_t = x\big]}{E_0\big[h(L_t) | R_t = x\big]} \quad \textit{exists and is equal to } Q_0^{(h_x)}(\Lambda_s), \tag{1.23}$$

where :

$$h_x(y) = h_x^* \{ x h(y) + x^{1-2\alpha}(1 - H(y)) \} \qquad (y \geq 0). \tag{1.24}$$

2. *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel function such that :*

$$\overline{f} := \int_{\mathbb{R}_+ \times \mathbb{R}_+} f(x, y)(x + yx^{1-2\alpha}) dx\, dy < \infty. \tag{1.25}$$

Then, for every $s \geq 0$, and $\Lambda_s \in \mathcal{F}_s$,

$$\lim_{t\to\infty} \frac{E_0\big[1_{\Lambda_s} f(R_t, L_t)\big]}{E_0\big[f(R_t, L_t)\big]} \quad \textit{exists and is equal to } Q_0^{(\tilde{f})}(\Lambda_s), \tag{1.26}$$

with :

$$\widetilde{f}(y) := f^* \Big\{ \int_0^\infty x f(x, y) dx + \int_0^\infty x^{1-2\alpha} dx \int_y^\infty f(x, z) dz \Big\} \tag{1.27}$$

and $f^* = 1/\overline{f}$.

Note that, for both points 1. and 2. of Theorem 1.4, the main properties of the canonical process $(R_t)_{t\geq 0}$ under the limit probabilities $Q_0^{(h_x)}$ and $Q_0^{(\widetilde{f})}$ are given by Theorem 1.1 : it suffices, in this Theorem 1.1, to replace h resp. by h_a and $\widetilde{f}$ $\big($and to note that : $\displaystyle\int_0^\infty h_a(y)dy = \int_0^\infty \widetilde{f}(y)dy = 1$ $\big)$.

5) Point 2. of Theorem 1.5 invites to study the penalization of $(R_t)_{t\geq 0}$ by a function of L_t and R_t which is not integrable, i.e. which does not satisfy (1.25). This led us to the following :

Theorem 1.5 *Let $\lambda > 0$, and $h : \mathbb{R}_+ \to \mathbb{R}_+$ a Borel function such that :*

$$\int_0^\infty h(y)dy < \infty \quad \text{and} \quad \int_0^\infty h(y)e^{-\sigma_\lambda y}dy = 1\,, \tag{1.28}$$

with

$$\sigma_\lambda := \left(\frac{\lambda}{2}\right)^{2\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}. \tag{1.29}$$

1. *For every $s \geq 0$, and $\Lambda_s \in \mathcal{F}_s$,*

$$\lim_{t\to\infty} \frac{E_0\left[1_{\Lambda_s} h(L_t)\exp(\lambda R_t)\right]}{E_0\left[h(L_t)\exp(\lambda R_t)\right]} \quad \text{exists} \tag{1.30}$$

and is equal to :

$$Q_0^{(\lambda,\widetilde{h})}(\Lambda_s) := E_0\big[1_{\Lambda_s} M_s^{\lambda,\widetilde{h}}\big] \tag{1.31}$$

with

$$\begin{aligned} M_s^{\lambda,\widetilde{h}} \quad &:= \quad e^{-\lambda^2 s/2} R_s^\alpha \Big[\widetilde{h}(L_s)\left(\frac{2}{\lambda}\right)^{\alpha} \Gamma(1+\alpha) I_\alpha\left(\lambda R_s\right) \\ &\qquad + \Big(1-\widetilde{H}(L_s)\Big)\left(\frac{2}{\lambda}\right)^{-\alpha} \Gamma(1-\alpha) I_{-\alpha}\left(\lambda R_s\right)\Big], \end{aligned} \tag{1.32}$$

where I_ν denotes the modified Bessel function with index ν (cf [9]), and

$$\widetilde{h}(y) := h(y) - \sigma_\lambda e^{\sigma_\lambda y} \int_y^\infty h(z)e^{-\sigma_\lambda z}dz, \tag{1.33}$$

$$1 - \widetilde{H}(y) := e^{\sigma_\lambda y} \int_y^\infty h(z)e^{-\sigma_\lambda z}dz = \int_y^\infty \tilde{h}(z)dz. \tag{1.34}$$

2. *$(M_s^{\lambda,\widetilde{h}}\,, s \geq 0)$ is a positive martingale, which tends to 0 a.s. as $s \to \infty$.*

3. *Formula (1.31) induces a probability $Q_0^{(\lambda,\widetilde{h})}$ on the canonical space $(\Omega_+, \mathcal{F}_\infty)$, with respect to which the canonical process $(R_t, t \geq 0)$ satisfies :*

   (a) *L_∞ is finite a.s. and its distribution function is :*

$$Q_0^{(\lambda,\widetilde{h})}(L_\infty < c) = 1 - \big(1-\widetilde{H}(c)\big)e^{-c\sigma_\lambda} \tag{1.35}$$

   with σ_λ given by (1.29).

   (b) *Let $g = \inf\{t \geq 0 : L_t = L_\infty\} = \sup\{t \geq 0 : R_t = 0\}$. Then :*

$$Q_0^{(\lambda,\widetilde{h})}(0 < g < \infty) = 1. \tag{1.36}$$

(c) i. *The processes* $(R_t, t < g)$ *and* $(R_{g+t}, t \geq 0)$ *are independent.*

ii. *The process* $(R_{g+t}, t \geq 0)$ *is a diffusion process starting from* 0*, whose infinitesimal generator* $\mathcal{L}^{\uparrow}$ *satisfies :*

$$\mathcal{L}^{\uparrow} f(r) = \frac{1}{2} f''(r) + \left\{ \frac{1-2\alpha}{2r} + \lambda \frac{I_{\alpha-1}(\lambda r)}{I_\alpha(\lambda r)} \right\} f'(r). \tag{1.37}$$

iii. *Conditionally on* $L_\infty = l$*, the process* $(R_t, t \leq g)$ *is a diffusion process starting from* 0*, whose infinitesimal generator* $\mathcal{L}^{\downarrow}$ *satisfies :*

$$\mathcal{L}^{\downarrow} f(r) = \frac{1}{2} f''(r) + \left\{ \frac{1-2\alpha}{2r} - \lambda \frac{K_{\alpha-1}(\lambda r)}{K_\alpha(\lambda r)} \right\} f'(r) \tag{1.38}$$

stopped when its local time at 0 *reaches level* l*.*

Remark 1.6 1. *Let* $h_0 : \mathbb{R}_+ \to \mathbb{R}_+$ *a Borel function such that* $\int_0^\infty h_0(y)dy < \infty$*. Note that* $h = h_0/c$ *verifies (1.28) where* $c = \int_0^\infty h(y)e^{-\sigma_\lambda y}dy$*.*

2. *It is not difficult to check that, as* $\lambda \to 0$*, Theorem 1.5 yields precisely Theorem 1.1, because* $I_\nu(z) \underset{z\to 0}{\sim} \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu$ *and* $K_\nu(z) \underset{z\to 0}{\sim} \frac{1}{2}\Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu}$ *(cf [9]).*

3. *Recall that the diffusions whose infinitesimal generators* $\mathcal{L}^{\uparrow}$ *and* $\mathcal{L}^{\downarrow}$ *are given in (1.37) and (1.38) are the Bessel processes with dimension* $d = 2(1-\alpha)$*, and drift* $\lambda \uparrow$ *and* $\lambda \downarrow$ *respectively. These processes have been studied by Watanabe [21] and Pitman-Yor [14]. They play an important role in Matsumoto-Yor ([12], [13]).*

5) Organization of the paper.

- In Section 2, we define precisely the normalization of the continuous family of the local times $(L_t^x; t \geq 0; x \geq 0)$ of the Bessel process $(R_t, t \geq 0)$ of dimension $d \in]0, 2[$, which we use throughout this paper.
- In Section 3, we prove Theorem 1.1.
- In Section 4 we prove Theorem 1.3, and we deduce Theorem 1.4 from Theorem 1.3 .
- Finally, Section 5 is devoted to the proof of Theorem 1.5.

6) An overview of some penalization results. In our paper [17], we propose a survey-without proofs- of most of the results obtained in our previous works [18], [19], [20] on the subject.

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2 Definition and properties of the local time at 0

1) The Bessel process $(R_t, t \geq 0)$, with dimension $d = 2(1-\alpha) \in (0, 2)$ which is being considered throughout this paper, is an $\mathbb{R}_+$-valued diffusion whose infinitesimal generator $\mathcal{L}$ is defined as :

$$\mathcal{L}f(r) = \frac{1}{2}\frac{d^2 f}{dr^2} + \frac{1-2\alpha}{2r}\frac{df}{dr}, \tag{2.1}$$

on the domain

$$\mathcal{D} = \left\{ f : \mathbb{R}_+ \to \mathbb{R}\, ;\, \mathcal{L}f \in \mathcal{C}_b(\mathbb{R}_+); \lim_{r\to 0} r^{1-2\alpha} f'(r) = 0 \right\} \tag{2.2}$$

2) The normalization we shall use for the local time at 0 , $(L_t, t \geq 0)$ of $(R_t, t \geq 0)$ is such that :

$$(N_t := R_t^{2\alpha} - L_t\,,\, t \geq 0) \qquad \text{is a martingale} \tag{2.3}$$

We note that the bracket of (N_t) equals :

$$A_t :=< N >_t = 4\alpha^2 \int_0^t R_s^{2(2\alpha-1)} ds, \tag{2.4}$$

and that there exists a reflecting Brownian motion $(\gamma_u, u \geq 0)$ such that

$$R_t^{2\alpha} = \gamma_{A_t} \quad ; \quad L_t = \ell_{A_t}\,, \tag{2.5}$$

where $(\ell_u, u \geq 0)$ is the local time at 0 of γ, chosen such that :

$$(\gamma_u - \ell_u, u \geq 0) \text{ is a } (\mathcal{F}_u^\gamma := \sigma\{\gamma_s, s \leq u\}, u \geq 0) \text{ martingale.} \tag{2.6}$$

Note that the finiteness of A_t, especially for $\alpha < 1/2$, follows from :

$$E_0\big[R_s^{2(2\alpha-1)}\big] = \frac{2^\alpha}{\Gamma(1-\alpha)} s^{\alpha-1} \int_0^\infty x^{2\alpha-1} e^{-\frac{x^2}{2s}} dx = \frac{2^{2\alpha-1}\Gamma(\alpha)}{\Gamma(1-\alpha)} s^{2\alpha-1},$$

which implies that $E_0[A_t] < \infty$.
3) With this normalization of $(L_t, t \geq 0)$ (cf [3]) , the occupation density formula writes :

$$\int_0^t g(R_s) ds = \frac{1}{\alpha} \int_0^\infty g(x) L_t^x x^{1-2\alpha} dx\,, \tag{2.7}$$

for every Borel function $g : \mathbb{R}_+ \to \mathbb{R}_+$, and $\{L_t^x\}$ a jointly continuous family of local times, such that $L_t^0 = L_t$.
4) Under P_0,

$$\text{the variable } L_t \text{ is distributed as } t^\alpha L_1, \tag{2.8}$$

and the law of L_1 is the Mittag-Leffler distribution of index α (see for instance [2]; for details see [3] and [10], p. 142), with density p_{L_1} :

$$p_{L_1}(l) = \frac{1}{\pi\alpha l} \sum_{k=1}^\infty \frac{(-1)^{k+1}\Gamma(\alpha k+1)}{k!} \Big(\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\, 2^{-\alpha}\, l\Big)^k \sin(k\pi\alpha) \quad (l \geq 0). \tag{2.9}$$

In particular it satisfies :

$$p_{L_1}(0) = \lim_{l\to 0} p_{L_1}(l) = \frac{2^{-\alpha}}{\Gamma(1+\alpha)}. \tag{2.10}$$

5) Define the right continuous inverse of L :

$$\tau_l := \inf\{t \geq 0 : L_t > l\}\,. \tag{2.11}$$

Then, $(\tau_l, l \geq 0)$ is a stable subordinator with index α; more precisely, its Laplace transform is given by :

$$E\left[\exp(-\lambda\tau_l)\right] = \exp\Big(-l\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\, 2^{-\alpha}\lambda^\alpha\Big) \qquad \lambda, l \geq 0. \tag{2.12}$$

6) Let $T_0 = \inf\{t \geq 0 : R_t = 0\}$ denote the first hitting time of 0 for the process $(R_t, t \geq 0)$.

Then, T_0 is, under P_r, distributed as :

$$T_0 \overset{(d)}{=} r^2/2\gamma_\alpha\,, \tag{2.13}$$

where γ_α is a standard gamma variable with index α.
Consequently,

$$P_r(T_0 \in dt) = \frac{2^{-\alpha}}{\Gamma(\alpha)}\, t^{-\alpha-1} r^{2\alpha} \exp(-\frac{r^2}{2t})\, 1_{[0,\infty[}(t)dt, \tag{2.14}$$

$$E_r\big[e^{-\frac{\lambda^2}{2}T_0}\big] = \frac{2}{\Gamma(\alpha)}\left(\frac{\lambda r}{2}\right)^\alpha K_\alpha(\lambda r). \tag{2.15}$$

Identity (2.15) is found in [8], see also Proposition (2.3) in [14]. For (2.14), which also extends to $\alpha = 1$ (that is : $d = 0$, when (R_t) is the 0-dimensional Bessel process), see [4], and e.g ([16], ex 4.16, p321). In [3], the reader will find a more detailed discussion of the various normalisations of the local time process (L_t) at level 0 for a Bessel process of dimension $d \in (0, 2)$ which have been used in the literature. The results presented in this section may be considered as standard knowledge; see, e.g. Borodin-Salminen [1] for a more general presentation of diffusion local times.
7) In Section 5, the role of the following martingale will be crucial :

$$M_t^{\frac{\lambda^2}{2}\downarrow} := R_t^\alpha K_\alpha(\lambda R_t) \exp\big(\sigma_\lambda L_t - \frac{\lambda^2 t}{2}\big), \quad t \geq 0, \tag{2.16}$$

where σ_λ is defined in (1.29).
That this process is indeed a martingale follows from the computation relative to a general diffusion (R_t), its local time (L_t), and inverse local time (τ_l) :

$$E_0\big[e^{-\mu\tau_l}\big|\mathcal{F}_t\big] = e^{-\mu t} f(R_t, l - L_t), \quad \text{ on } \{t < \tau_l\} = \{L_t < l\}, \tag{2.17}$$

where $f(r,\lambda) = E_r\big[A(\lambda)\big]$, with $A(\lambda) = e^{-\mu\tau_\lambda}$.
Using the strong Markov property we obtain :

$$f(r,\lambda) = E_r\big[e^{-\mu T_0(R)}\big]e^{-\lambda\psi(\mu)},$$

where $\psi(\mu)$ denotes the Lévy exponent for (τ_l).
Using the Laplace transform given in (2.15), for $\mu = \lambda^2/2$, and the fact that : $\psi(\frac{\lambda^2}{2}) = \sigma_\lambda$ in this particular case (cf (2.12) above), we get :

$$R_t^\alpha K_\alpha(\lambda R_t) \exp\big(\sigma_\lambda L_t - \frac{\lambda^2 t}{2}\big) = C_\lambda e^{\sigma_\lambda l} E_0\big[e^{-\mu\tau_l}\big|\mathcal{F}_t\big], \;\text{ on } \{t < \tau_l\}, \tag{2.18}$$

where C_λ is a positive constant.
Now property (2.16) is a direct consequence of (2.18) together with the following calculations :

$$\begin{aligned} E_0\big[M_t^{\frac{\lambda^2}{2}\downarrow} 1_{\Lambda_s}\big] &= \lim_{l\to\infty} E_0\big[M_t^{\frac{\lambda^2}{2}\downarrow} 1_{\Lambda_s} 1_{\{s<\tau_l\}}\big] &= \lim_{l\to\infty} E_0\Big[C_\lambda e^{\sigma_\lambda l} E_0\big[e^{-\mu\tau_l}\big|\mathcal{F}_s\big] 1_{\Lambda_s} 1_{\{s<\tau_l\}}\Big] \\ &= \lim_{l\to\infty} E_0\big[M_s^{\frac{\lambda^2}{2}\downarrow} 1_{\Lambda_s} 1_{\{s<\tau_l\}}\big] &= E_0\big[M_s^{\frac{\lambda^2}{2}\downarrow} 1_{\Lambda_s}\big], \end{aligned}$$

for any $s \leq t$ and $\Lambda_s \in \mathcal{F}_s$.

3 Proof of Theorem 1.1

Proof of Theorem 1.1
The proof of Theorem 1.1 will be divided into eleven steps.
1) We prove the existence of : $\lim_{t\to\infty} P_0^{(t)}(\Lambda_s)$.
Let $s \geq 0$, and $\Lambda_s \in \mathcal{F}_s$. By conditioning $h(L_t)$ with respect to $\mathcal{F}_s$, we get :

$$\frac{E_0\big[1_{\Lambda_s} h(L_t)\big]}{E_0[h(L_t)]} = \frac{E_0\big[1_{\Lambda_s}\theta(R_s, L_s, t-s)\big]}{\theta(0,0,t)}, \tag{3.1}$$

with

$$\theta(r,y,u) := E_r[h(y + L_u)], \quad r, y, u \geq 0. \tag{3.2}$$

Thus we are led to estimate $\theta(r,y,u)$ when u tends to $+\infty$.
We denote by T_0 the first hitting time of 0 by the process $(R_t, t \geq 0)$:

$$T_0 := \inf\{t \geq 0, R_t = 0\}. \tag{3.3}$$

Thus, we obtain :

$$\theta(r,y,t) = \theta_1(r,y,t) + \theta_2(r,y,t), \tag{3.4}$$

with :

$$\theta_1(r,y,t) = h(y)P_r(T_0 > t), \quad \theta_2(r,y,t) = E_r\big[1_{\{T_0<t\}}\, h(y + L_{T_0+(t-T_0)})\big]. \tag{3.5}$$

We examine separately the two terms $\theta_1(r,y,t)$ and $\theta_2(r,y,t)$ featured in (3.5).
From (2.13), the first term $\theta_1(r,y,t)$ is equal to :

$$\begin{aligned} \theta_1(r,y,t) &= h(y)P\big(\gamma_\alpha < \frac{r^2}{2t}\big) \\ &= \frac{h(y)}{\Gamma(\alpha)}\int_0^{\frac{r^2}{2t}} x^{\alpha-1}e^{-x}dx \underset{t\to\infty}{\sim} \frac{h(y)}{\Gamma(\alpha+1)}\left(\frac{r^2}{2t}\right)^{\alpha}. \end{aligned} \tag{3.6}$$

As to the second term $\theta_2(r,y,t)$, we find it to be equal, thanks to the scaling property (2.8), and after conditioning with respect to $\mathcal{F}_{T_0}$, to :

$$\theta_2(r,y,t) = E_r[1_{\{T_0<t\}}\theta_3(y, t-T_0)],$$

with $\theta_3(y,u) = E_0\big[h(y + u^\alpha L_1))\big]$.
Hence, denoting by p_{L_1} the density of L_1, under P_0 :

$$\theta_2(r,y,t) = \int_0^\infty E_r\big[1_{\{T_0<t\}}\, h(y + (t-T_0)^\alpha x)\big]p_{L_1}(x)dx,$$

so that, after making the change of variable : $(t-T_0)^\alpha x = z$, we get :

$$\theta_2(r,y,t) = E_r\Big[1_{\{T_0<t\}}\frac{1}{(t-T_0)^\alpha}\int_0^\infty h(y+z)p_{L_1}\big(\frac{z}{(t-T_0)^\alpha}\big)dz\big]. \tag{3.7}$$

Consequently (2.10) implies :

$$\theta_2(r,y,t) \underset{t\to\infty}{\sim} \frac{p_{L_1}(0)}{t^\alpha}\int_0^\infty h(y+z)dz = \frac{2^{-\alpha}}{t^\alpha\Gamma(1+\alpha)}[1 - H(y)]. \tag{3.8}$$

Bringing together (3.6), (3.8) and (3.4), we get :

$$\theta(r,y,t) \underset{t\to\infty}{\sim} \frac{1}{t^{\alpha}} \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \left(h(y)r^{2\alpha} + 1 - H(y)\right). \tag{3.9}$$

We then deduce from (3.1) and (3.9) :

$$\lim_{t\to\infty} \frac{E_0\left[1_{\Lambda_s} h(L_t)\right]}{E_0[h(L_t)]} = E_0\Big[1_{\Lambda_s}\left(h(L_s)R_s^{2\alpha} + 1 - H(L_s)\right)\Big]. \tag{3.10}$$

We note that, in (3.10), exchanging the order of taking either the limit or the expectation does not make any problem, since it is justified by Lebesgue's dominated convergence theorem, once it has been noted that, with the help of (3.6) and (3.7) :

$$t^{\alpha}\theta(r,y,t) \leq C[h(y)+1]r^{2\alpha}.$$

2) We now show that $(M_s^h := h(L_s)R_s^{2\alpha} + 1 - H(L_s), s \geq 0)$ is a martingale.
For h in C^1, it easily follows from Itô's formula using (2.3), that $(M_s^h, s \geq 0)$ is a local martingale. Moreover it writes :

$$M_s^h = 1 + \int_0^s h(L_u)dN_u, \quad s \geq 0.$$

Now, to obtain the general case, it remains to apply the monotone class theorem. We might also have used a balayage argument, see e.g. [16], Chap. VI. Thus, in particular,

$$M_t^h = 1 + \int_0^t h(L_s)dN_s, \quad t \geq 0, \tag{3.11}$$

where $(N_t, t \geq 0)$ is the martingale defined by (2.3) is a local martingale.
Since $M_t^h \geq 0$, for any $t \geq 0$, (M_t^h) is a positive supermartingale. In order to prove that $(M_t^h, t \geq 0)$ is a martingale, it suffices to show :

$$E_0(M_t^h) = 1, \text{ for every } \ t \geq 0. \tag{3.12}$$

Now, for $n \in \mathbb{N}$, let $h_n(x) = (h(x) \wedge n)1_{(x \leq n)}$. It is clear that $(M_t^{h_n})$ is a martingale, therefore (3.12) implies that : $E_0\left[M_t^{h_n}\right] = 1$. Then, with the help of Beppo-Levi's Theorem, we obtain :

$$1 = \lim_{n\to\infty}\Big(E_0\big[h_n(L_t)R_t^{2\alpha} + (1 - H_n(L_t)\big]\Big) = E_0\big[h(L_t)R_t^{2\alpha} + 1 - H(L_t)\big] = E_0[M_t^h].$$

3) We now prove that : $M_t^h \to 0$ as $t \to \infty$.
Since $(M_t^h, t \geq 0)$ is a positive martingale, it converges a.s. as $t \to \infty$. Let $\tau_l = \inf\{s > 0 : L_s > l\}$ denote the inverse local time. Then :

$$M_{\tau_l}^h = h(L_{\tau_l})R_{\tau_l}^{2\alpha} + 1 - H(L_{\tau_l}) = 1 - H(l) \to 0, \quad \text{as } l \to \infty. \tag{3.13}$$

Hence $M_t^h \underset{t\to\infty}{\rightarrow} 0$ a.s. In particular, the martingale $(M_t^h, t \geq 0)$ is not uniformly integrable.

4) We now establish that $Q_0^{(h)}(L_\infty \in dl) = h(l)dl$.
Indeed, for every $t \geq 0$, using (1.6), Doob's optional stopping theorem and (3.13), we have :

$$Q_0^{(h)}(L_t > c) = Q_0^{(h)}(t > \tau_c) = E_0[1_{\{\tau_c < t\}}M_t^h] = E_0[1_{\{\tau_c < t\}}M_{\tau_c}^h] = \big(1 - H(c)\big)P_0(t > \tau_c).$$

Consequently, letting $t \to +\infty$, we obtain :

$$Q_0^{(h)}(L_\infty > c) = 1 - H(c).$$

5) An auxiliary result.
In the sequel, we shall use a general result about continuous positive martingales, which is stated and proven in [19], and which we shall then apply to $M = M^h$. Thus, we present this result without proof.

Proposition 3.1 *(Theorem 4.2 in [19])*
Let $\big(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, \mathbb{P}\big)$ denote a given filtered probability space, and consider a strictly positive continuous martingale (M_t), with respect to $\big((\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$, such that $M_0 = 1$ and $M_\infty = 0$ a.s. We then define the probability $\mathbb{Q}$ on $(\Omega, \mathcal{F}_\infty)$ via :

$$\mathbb{Q}(\Lambda_t) = E\big[1_{\Lambda_t} M_t\big] \quad t \geq 0, \Lambda_t \in \mathcal{F}_t. \tag{3.14}$$

We also define :

$$\underline{M}_{\,t} = \inf_{s\leq t} M_s. \tag{3.15}$$

Then, under $\mathbb{Q}$, the following holds :

1. *$\underline{M}_{\,\infty}$ is uniformly distributed on $[0,1]$.*
2. *Let $g := \sup\{t \geq 0, M_t = \underline{M}_{\,\infty}\}$. Then,*
$$\mathbb{Q}(0 < g < \infty) = 1. \tag{3.16}$$
3. *Let :*
$$Z_t = \mathbb{Q}(g > t|\mathcal{F}_t). \tag{3.17}$$
Then :
   (a) *$Z_t = \underline{M}_t / M_t$.*
   (b) *$(Z_t, t \geq 0)$ is a $\big((\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}\big)$ positive supermartingale with additive decomposition :*
$$Z_t = 1 - \int_0^t \frac{\underline{M}_u}{M_u^2} d\widetilde{M}_u + \ln(\underline{M}_t). \tag{3.18}$$
where $\widetilde{M}_t := M_t - \displaystyle\int_0^t \frac{d < M >_u}{M_u}$ is the martingale part of (M_t) under $\mathbb{Q}$, from Girsanov's theorem.

6) <u>We now remark that :</u>

$$\underline{M}_{\,t}^{\,h} = 1 - H(L_t). \tag{3.19}$$

Indeed,
$$M_s^h = h(L_s) R_s^{2\alpha} + 1 - H(L_s) \geq 1 - H(L_s) \geq 1 - H(L_t), \quad \text{for any } 0 \leq s \leq t.$$
Moreover $M_{g_t}^h = 1 - H(L_t)$ where $g_t = \sup\{s \leq t; R_s = 0\}$. This implies (3.19).
We also note that (3.19), together with point 1. of Proposition 3.1, allows to rediscover the fact, obtained in **4)**, that $Q_0^{(h)}(L_\infty \in dl) = h(l)dl$. Indeed, we get :

$$\begin{aligned} Q_0^{(h)}(L_\infty < c) &= Q_0^{(h)}\big(H(L_\infty) < H(c)\big) = Q_0^{(h)}\big(1 - H(L_\infty) > 1 - H(c)\big) \\ &= Q_0^{(h)}\big(\underline{M}_\infty^h > 1 - H(c)\big) = H(c). \end{aligned}$$

7) <u>Another definition of g.</u>
Let g be defined as in point 2. of Proposition 3.1, but we now replace M by M^h, ie :

$$g = \sup\left\{t \geq 0; M_t^h = \underline{M}_{\,\infty}^{\,h}\right\}. \tag{3.20}$$

Then, under $Q_0^{(h)}$:

$$g = \sup\{s \geq 0, R_s = 0\} = \inf\{s \geq 0, L_s = L_\infty\}. \tag{3.21}$$

Indeed, (3.21) follows from (3.19) and the fact that $\underline{M}_t^h = 1 - H(L_t)$ is constant after g (hence, so is L_t).

8) A preliminary step to prove point 3. (c) of Theorem 1.1.

We shall use the technique of progressive enlargement of filtrations (see [6], [22], [7] and[11]). We denote by $(\mathcal{G}_t, t \geq 0)$ the smallest filtration which contains $(\mathcal{F}_t, t \geq 0)$, and which makes g, defined by (3.21), a $(\mathcal{G}_t, t \geq 0)$ stopping time.

a) Recall that (N_t) is the $\big((\mathcal{F}_t)_{t\geq 0}, P_0\big)$ martingale defined by (2.3), whose bracket is given by (2.4). Hence, from Girsanov's theorem, and (3.11), the process :

$$\widetilde{N}_t := N_t - \int_0^t \frac{h(L_u)}{M_u^h}\, d < N >_u \tag{3.22}$$

is a $\big((\mathcal{F}_t)_{t\geq 0}, Q_0^{(h)}\big)$ martingale, so that :

$$R_t^{2\alpha} = L_t + \widetilde{N}_t + \int_0^t \frac{h(L_u)}{M_u^h}\, d < N >_u . \tag{3.23}$$

b) From Proposition 3.1, 3. (b), we have :

$$Z_t = Q_0^{(h)}(g > t|\mathcal{F}_t) = 1 - \int_0^t \frac{\underline{M}_u^h}{(M_u^h)^2}\, d\widetilde{M}_u^h + \ln(\underline{M}_t^h), \tag{3.24}$$

with

$$\widetilde{M}_t^h = M_t^h - \int_0^t \frac{d < M^h >_u}{M_u^h}. \tag{3.25}$$

a $\big((\mathcal{F}_t)_{t\geq 0}, Q_0^{(h)}\big)$ martingale.

Due to (3.22), (3.25) and (3.11) we have :

$$< \widetilde{N}, \int_0^{\cdot} \frac{\underline{M}_u^h}{(M_u^h)^2}\, d\widetilde{M}_u^h >_t = < N, \int_0^{\cdot} \frac{\underline{M}_u^h}{(M_u^h)^2}\, dM_u^h >_t = \int_0^t \frac{\underline{M}_u^h}{(M_u^h)^2} h(L_u) d < N >_u .$$

We deduce, after Jeulin [6] and Yor [23], that in the filtration $(\mathcal{G}_t)_{t\geq 0}$, under $Q_0^{(h)}$:

$$\widetilde{N}_t = \widetilde{N}_t^{(2)} - \int_0^{t\wedge g} \frac{1}{Z_u} \frac{\underline{M}_u^h}{(M_u^h)^2} h(L_u) d < N >_u + \int_{t\wedge g}^t \frac{1}{1 - Z_u} \frac{\underline{M}_u^h}{(M_u^h)^2} h(L_u) d < N >_u . \tag{3.26}$$

where $(\widetilde{N}_t^{(2)}, t \geq 0)$ is a $\big((\mathcal{G}_t)_{t\geq 0}, Q_0^{(h)}\big)$ local martingale.

Plugging (3.26) in (3.23), we obtain :

$$\begin{aligned} R_t^{2\alpha} = L_t \quad &+ \quad \int_0^t \frac{h(L_u)}{M_u^h}\, d < N >_u + \widetilde{N}_t^{(2)} - \int_0^{t\wedge g} \frac{1}{Z_u} \frac{\underline{M}_u^h}{(M_u^h)^2} h(L_u) d < N >_u \\ &+ \quad \int_{t\wedge g}^t \frac{1}{1 - Z_u} \frac{\underline{M}_u^h}{(M_u^h)^2} h(L_u) d < N >_u . \end{aligned} \tag{3.27}$$

Using 3. (a) of Proposition 3.1 the relation (3.27) simplifies, and becomes :

$$R_t^{2\alpha} = L_t + \widetilde{N}_t^{(2)} + \int_{t\wedge g}^t \frac{1}{M_u^h - \underline{M}_u^h} h(L_u) d < N >_u . \tag{3.28}$$

But, since, from (3.19), we have :

$$M_u^h - \underline{M}_u^h = h(L_u) R_u^{2\alpha} + 1 - H(L_u) - (1 - H(L_u)) = h(L_u) R_u^{2\alpha},$$

then (2.4) implies :

$$R_t^{2\alpha} = L_t + \widetilde{N}_t^{(2)} + 4\alpha^2 \int_{t\wedge g}^t R_u^{2(\alpha-1)} du. \tag{3.29}$$

We note that, despite the different changes of probability, or of filtration, which we have made, the brackets of N and $\widetilde{N}^{(2)}$ are equal, hence :

$$< \widetilde{N}^{(2)} >_t = 4\alpha^2 \int_0^t R_u^{2(2\alpha-1)} du. \tag{3.30}$$

9) <u>Description of the $Q_0^{(h)}$ process, after g.</u>
From (3.29) and because $R_g = 0$ and $L_{t+g} = 0,\ t \geq 0$, we have :

$$R_{g+t}^{2\alpha} = \widetilde{N}_t^{(3)} + 4\alpha^2 \int_0^t R_{g+s}^{2(\alpha-1)} ds, \tag{3.31}$$

where

$$\widetilde{N}_t^{(3)} = \widetilde{N}_{g+t}^{(2)} - \widetilde{N}_g^{(2)},\ t \geq 0. \tag{3.32}$$

Note that g is a $(\mathcal{G}_t)_{t\geq 0}$ stopping time, therefore $(\widetilde{N}_t^{(3)})$ is a $\left((\mathcal{G}_{g+t})_{t\geq 0}, Q_0^{(h)}\right)$ continuous local martingale.
We then apply Itô's formula to compute $f(R_{g+t}^{2\alpha})$, with $f(x) := x^{1/2\alpha}$; we get, from (3.30) :

$$\begin{aligned} R_{g+t} &= \frac{1}{2\alpha} \int_0^t R_{g+s}^{1-2\alpha} d\widetilde{N}_s^{(3)} + 2\alpha \int_0^t R_{g+s}^{1-2\alpha} R_{g+s}^{2(\alpha-1)} ds + \frac{1-2\alpha}{2} \int_0^t R_{g+s}^{1-4\alpha} R_{g+s}^{2(2\alpha-1)} ds \\ &= \frac{1}{2\alpha} \int_0^t R_{g+s}^{1-2\alpha} d\widetilde{N}_s^{(3)} + \frac{1+2\alpha}{2} \int_0^t \frac{ds}{R_{g+s}}. \end{aligned} \tag{3.33}$$

But, from (3.30), the $\left((\mathcal{G}_{g+t})_{t\geq 0}, Q_0^{(h)}\right)$ local martingale $\left(B_t := \dfrac{1}{2\alpha} \displaystyle\int_0^t R_{g+s}^{1-2\alpha} d\widetilde{N}_s^{(3)}, t \geq 0\right)$ admits as bracket :

$$\frac{1}{4\alpha^2} \int_0^t R_{g+s}^{2(1-2\alpha)} (4\alpha^2) R_{g+s}^{2(2\alpha-1)} ds = t.$$

This implies that $(B_t, t \geq 0)$ is a $\left((\mathcal{G}_{g+t})_{t\geq 0}, Q_0^{(h)}\right)$ Brownian motion and is therefore independent from $\mathcal{G}_g$.
Finally (R_{g+t}) solves :

$$R_{g+t} = B_t + \frac{1+2\alpha}{2} \int_0^t \frac{ds}{R_{g+s}}. \tag{3.34}$$

This proves that $(R_{g+t}, t \geq 0)$ is a Bessel process starting from 0, with dimension $\delta = 2 + 2\alpha = 4 - d$. The solution of (3.34) being strong, the processes $(R_t, t \leq g)$ and $(R_{g+t}, t \geq 0)$ are independent under $Q_0^{(h)}$.
10) <u>Description of the $Q_0^{(h)}$ process before g.</u>
Before g, we have, from (3.29) and (3.30) :

$$R_{t\wedge g}^{2\alpha} = L_{t\wedge g} + \widetilde{N}_{t\wedge g}^{(2)}, \quad \text{with} \quad < \widetilde{N}^{(2)} >_t = 4\alpha^2 \int_0^t R_s^{2(2\alpha-1)} ds. \tag{3.35}$$

Let us introduce :

$$\beta_t := \frac{1}{2\alpha} \int_0^t \frac{1}{R_u^{2\alpha-1}} d\widetilde{N}_u^{(2)}, \quad t \geq 0.$$

It is clear that (3.35) implies that $(\beta_t, t \geq 0)$ is a $\big((\mathcal{G}_t)_{t\geq 0}, Q_0^{(h)}\big)$ Brownian motion and :

$$R_{t\wedge g}^{2\alpha} = L_{t\wedge g} + 2\alpha \int_0^{t\wedge g} R_s^{2\alpha-1} d\beta_s. \tag{3.36}$$

Then applying Itô's formula to (3.36), to compute $R_{t\wedge g}^2 = (R_{t\wedge g}^{2\alpha})^{1/\alpha}$, we obtain (since $1/\alpha > 1$) :

$$R_{t\wedge g}^2 = \int_0^{t\wedge g} \frac{1}{\alpha} R_s^{2(1-\alpha)} 2\alpha R_s^{2\alpha-1} d\beta_s + \frac{1-\alpha}{2\alpha^2} \int_0^{t\wedge g} R_s^{2(1-2\alpha)} (4\alpha^2) R_s^{2(2\alpha-1)} ds.$$

(note that, because $\frac{1}{\alpha} > 1$, the term in dL_s disappears) hence :

$$R_{t\wedge g}^2 = 2\int_0^{t\wedge g} \sqrt{R_s^2} d\beta_s + 2(1-\alpha)(t\wedge g). \tag{3.37}$$

which proves that $R_{t\wedge g}^2$ is the square of a Bessel process with dimension $d = 2(1-\alpha)$ stopped at time $g = \inf\{t \geq 0; L_t = L_\infty\}$.

11) <u>We now prove point 4. of Theorem 1.1.</u>

We first show that $(R_{\rho(u)}^{2\alpha} + L_{\rho(u)}, u \geq 0)$ is a 3-dimensional Bessel process, starting from 0 (recall that $\rho(u)$ is defined by (1.9)).

a) Let us start by studying the process (R_t) before g.

It is clear that (3.36) implies :

$$R_{\rho(u)}^{2\alpha} = L_{\rho(u)} - W_u, \quad u \leq A_g, \tag{3.38}$$

where $\big(W_u := -2\alpha \int_0^{\rho(u)} R_s^{2\alpha-1} d\beta_s,\ u \geq 0\big)$ is $\big((\mathcal{G}_{\rho(u)})_{u\geq 0}, Q_0^{(h)}\big)$ Brownian motion.

From Skorokhod's reflection lemma ([16], Chap. VI) we have : $L_{\rho(u)} = \sup_{s\leq u} W_s,\ u \geq 0$.

According to Pitman's theorem (cf [15]), the process $\big(2\sup_{u\leq t} W_u - W_t,\ t \geq 0\big)$ is a 3-dimensional Bessel process, started at 0.

Finally $\big(R_{\rho(u)}^{2\alpha} + L_{\rho(u)},\ u \leq A_g\big)$ is a three dimensional Bessel process, started at 0, stopped at the stopping time A_g.

b) We consider now (R_{g+t}).

We first observe :

$$\widetilde{R}_u^{2\alpha} := R_{\rho(u+A_g)}^{2\alpha} = R_{g+\widetilde{\rho}(u)}^{2\alpha}, \quad u \geq 0,$$

where $(\widetilde{\rho}(u))_{u\geq 0}$ is the right-inverse of :

$$\widetilde{A}_t = A_{g+t} - A_g = 4\alpha^2 \int_0^t R_{s+g}^{2(2\alpha-1)} ds, \quad t \geq 0,$$

(recall that $(A_t)_{t\geq 0}$ is the process defined by (1.8)).

Then (3.31) may be written as :

$$\widetilde{R}_u^{2\alpha} = \widetilde{N}_{\tilde{\rho}(u)}^{(3)} + \int_0^{\tilde{\rho}(u)} \frac{d\widetilde{A}_s}{R_{s+g}^{2\alpha}}.$$

Since $\big(W'_u := \widetilde{N}_{\tilde{\rho}(u)}^{(3)},\ u \geq 0\big)$ is a $\Big((\mathcal{G}_{\widetilde{\rho}(u)})_{u\geq 0}, Q_0^{(h)}\Big)$ martingale, with bracket $\widetilde{A}_{\tilde{\rho}(u)} = u$, we obtain after making the change of variables $\widetilde{A}_s = v$:

$$\widetilde{R}_t^{2\alpha} = W'_t + \int_0^t \frac{dv}{\widetilde{R}_v^{2\alpha}}.$$

Note that $L_{\rho(t+A_g)} = L_{g+\tilde{\rho}(t)} = L_g$ is a $\mathcal{G}_{\widetilde{\rho}(0)}$ measurable r.v. Consequently, $\big(\widetilde{R}_t^{2\alpha} + L_{\rho(t+A_g)},\ t \geq 0\big)$ is a 3-dimensional Bessel process, starting from L_g.
This result, together with point a) proves that $(R^{2\alpha}_{\rho(t)} + L_{\rho(t)} t \geq 0)$ is a three-dimensional Bessel process started at 0.
The independence of L_∞ and of $(R^{2\alpha}_{\rho(u)} + L_{\rho(u)}\,;\, u \geq 0)$ follows from the fact that the law of $(R^{2\alpha}_{\rho(u)} + L_{\rho(u)}\,;\, u \geq 0)$, conditionally on $L_\infty = y$, does not depend on y.

■

Remark 3.2 *1. Replacing in step 4) of the above proof, the event $\{L_t > c\}$ by $\{L_{t_1} > c_1, \dots, L_{t_n} > c_n\}$ with $t_1 > \dots > t_n > 0$ and $c_1 \geq \dots \geq c_n > 0$, proves that the law of the process $(L_t, t \geq 0)$ under $Q_0^{(h)}$ is the same as that of the process $\big(L_t \wedge \xi, t \geq 0\big)$ under P_0, where ξ is a random variable of density h and independent from $(L_t, t \geq 0)$ (under P_0).*

2. *We now present a heuristic method to obtain the distribution of L_∞ under $Q_0^{(h)}$. We write, for every function g, bounded and continuous :*

$$\begin{aligned}
\frac{E_0\big[g(L_t)h(L_t)\big]}{E_0[h(L_t)]} &= \frac{E_0\big[g(t^\alpha L_1)h(t^\alpha L_1)\big]}{E_0[h(t^\alpha L_1)]} \quad \text{(by scaling)}\\
&= \frac{\displaystyle\int_0^\infty (gh)(t^\alpha x) p_{L_1}(x)dx}{\displaystyle\int_0^\infty h(t^\alpha x) p_{L_1}(x)dx}\\
&= \frac{\displaystyle\int_0^\infty gh(y) p_{L_1}(\frac{y}{t^\alpha})dy}{\displaystyle\int_0^\infty h(y) p_{L_1}(\frac{y}{t^\alpha})dy}.
\end{aligned}$$

Property (2.10) implies :

$$\lim_{t\to\infty} \frac{E_0\big[g(L_t)h(L_t)\big]}{E_0[h(L_t)]} = \frac{p_{L_1}(0)\displaystyle\int_0^\infty (gh)(y)dy}{p_{L_1}(0)\displaystyle\int_0^\infty h(y)dy} = \int_0^\infty g(y)h(y)dy.$$

However, this computation is not, at least without any further justification, "licit". The correct manner to obtain the law of L_∞ under $Q_0^{(h)}$ is to first study

$$\frac{E_0(g(L_s)h(L_t))}{E_0(h(L_t))} \qquad \textit{for a fixed } s < t,$$

then to first let t tend to $+\infty$, and finally to let s tend to ∞.

4 Proofs of Theorems 1.3 and 1.4

Recall that under P_0, (R_t) is a d-dimensional Bessel process started at 0, with $d = 2(1-\alpha)$ and $\alpha \in]0,1[$.
To prove Theorems 1.3 and 1.4, it is convenient to introduce the following notation :

1. $p_{L,t}(l)$ is the density function of L_t, under P_0,
2. $p_{R,L,t}(r,l)$ is the density function of the couple (R_t, L_t), under P_0,
3. $\{\Pi_t\}$ denotes the semigroup of (R_t, L_t),
4. $p_t^{(\mu)}(x,y)$ denotes the density of the transition semigroup of the Bessel process with index μ, at time $t > 0$.

4.1 Proof of Theorem 1.3

We begin the proof of Theorem 1.3 with two preliminary results : Lemma 4.1 and 4.2 below, in which we compute the conditional expectation of an event Λ_s in $\mathcal{F}_s$, given L_t, resp. given X_t, L_t with $t > s$.

Lemma 4.1 *For every s and t such that $0 \leq s \leq t$, $\Lambda_s \in \mathcal{F}_s$, and $y \geq 0$, one has :*

$$\begin{aligned} E_0[1_{\Lambda_s}|L_t = y] &= \frac{p_{L,s}(y)}{p_{L,t}(y)} E_0[1_{\Lambda_s}\varphi_1(t-s, R_s)|L_s = y] \\ &+ \frac{1}{p_{L,t}(y)} E_0\big[1_{\Lambda_s} 1_{(L_s<y)} \varphi_2(t-s, R_s, y - L_s)\big], \end{aligned} \tag{4.1}$$

with

$$\varphi_1(u,r) := P_r(T_0 > u), \quad r, u \geq 0, \tag{4.2}$$

$$\varphi_2(u,r,l) := E_r\big[1_{\{T_0<u\}} p_{L,u-T_0}(l)\big], \quad r, l, u \geq 0, \tag{4.3}$$

and $T_0 := \inf\{s \geq 0, R_s = 0\}$.

Proof. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive, Borel function. We compute in two different manners the quantity $E_0[1_{\Lambda_s} f(L_t)]$.
On one hand, by conditioning with respect to $L_t = y$, we obtain :

$$E_0(1_{\Lambda_s} f(L_t)) = \int_0^\infty E_0(1_{\Lambda_s}|L_t = y) f(y) p_{L,t}(y) dy. \tag{4.4}$$

On the other hand, by conditioning with respect to $\mathcal{F}_s$, we obtain :

$$E_0\big[1_{\Lambda_s} f(L_t)\big] = E_0\big[1_{\Lambda_s} E(f(L_t)|\mathcal{F}_s)\big] = E_0\big[1_{\Lambda_s} \Pi_{t-s} f(R_s, L_s)\big]. \tag{4.5}$$

Let us introduce :

$$\psi(t,l) = E_0[f(l + L_t)] = \int_0^\infty f(y) 1_{\{y>l\}} p_{L,t}(y - l) dy. \tag{4.6}$$

Then, using the strong Markov property at time T_0, we get :

$$\begin{aligned} \Pi_u f(r,l) &= E_r[f(l + L_u)] = f(l) P_r(T_0 > u) + E_r\big[1_{\{T_0<u\}} \psi(u - T_0, l)\big] \\ &= f(l)\varphi_1(u,r) + \int_0^\infty f(y)\varphi_2(u, r, y - l) 1_{\{y>l\}} dy. \end{aligned} \tag{4.7}$$

Now, plugging (4.7) into (4.5), and then comparing (4.4) and (4.5), for an arbitrary function f, yields Lemma 4.1.

■

Lemma 4.2 *For every $a, y \geq 0$, $s \geq 0$, $\Lambda_s \in \mathcal{F}_s$, and $t \geq s$,*

$$\begin{aligned} E_0(1_{\Lambda_s}|R_t = x, L_t = y) &= \frac{p_{L,s}(y) x^{-2\alpha}}{p_{R,L,t}(x,y)} E_0\big[1_{\Lambda_s} R_s^{2\alpha} p_{t-s}^{(\alpha)}(R_s, x)|L_s = y\big] \\ &+ \frac{1}{p_{R,L,t}(x,y)} E_0\big[1_{\Lambda_s} \varphi_3(t-s, R_s, x, y - L_s) 1_{\{y>L_s\}}\big] \end{aligned} \tag{4.8}$$

with

$$\varphi_3(u,r,x,y) := E_r\big[1_{\{T_0<u\}} p_{R,L,u-T_0}(x,y)\big], \quad r, u, x, y \geq 0, \tag{4.9}$$

Proof. Let $g : \mathbb{R}_+ \times \mathbb{R}_+$ be a positive Borel function. We shall compute in two different manners the quantity $E_0[1_{\Lambda_s} g(R_t, L_t)]$. First, by conditioning with respect to $R_t = x$ and $L_t = y$, we obtain :

$$E_0[1_{\Lambda_s} g(R_t, L_t)] = \int_{\mathbb{R}_+^2} E_0[1_{\Lambda_s} | R_t = x, L_t = y] g(x, y) p_{R,L,t}(x, y) dx dy. \tag{4.10}$$

Secondly, by conditioning with respect to $\mathcal{F}_s$, we obtain :

$$E_0\big[1_{\Lambda_s} g(R_t, L_t)\big] = E_0\big[1_{\Lambda_s} \Pi_{t-s} g(R_s, L_s)\big]. \tag{4.11}$$

We note that :

$$\Pi_u g(r, l) = E_r[g(R_u, L_u + l)). \tag{4.12}$$

We proceed as in the proof of Lemma 4.1, decomposing the right-hand side of (4.12) in two parts A_1, resp. A_2 depending upon whether u is smaller, or greater than $T_0 = \inf\{s \geq 0; R_s = 0\}$. Thus, we obtain :

$$\Pi_u g(r, l) = A_1 + A_2, \tag{4.13}$$

where :

$$A_1 := E_r[g(R_u, l) 1_{(T_0 > u)}], \quad A_2 := E_r[g(R_u, L_u + l) 1_{(T_0 < u)}]. \tag{4.14}$$

We shall study successively A_1 and A_2.
a) Recall the absolute continuity relationship between : $P_r^{-\alpha}\ _{|\mathcal{F}_u \cap \{u < T_0\}}$ and $P_r^{(+\alpha)}|_{\mathcal{F}_u}$ (see, [16], Chap. XI or [5], section 1.2):

$$P_r^{-\alpha}\ _{|\mathcal{F}_u \cap \{u < T_0\}} = \Big(\frac{r}{R_u}\Big)^{2\alpha} P_r^{\alpha}\ _{|\mathcal{F}_u}. \tag{4.15}$$

Consequently :

$$A_1 = E_r^{(\alpha)}\Big[\frac{r^{2\alpha}}{R_u^{2\alpha}} g(R_u, l)\Big] = r^{2\alpha} \int_0^\infty \frac{g(x, l)}{x^{2\alpha}} p_u^{(\alpha)}(r, x) dx. \tag{4.16}$$

b) Next we compute A_2. Conditioning with respect to $\mathcal{F}_{T_0}$, we get :

$$A_2 = E_r\big[1_{\{T_0 < u\}} \psi_2(u - T_0, l)\big],$$

with

$$\psi_2(v, l) = E_0\big[g(R_v, L_v + l)\big] = \int_{\mathbb{R}_+^2} g(x, y) p_{R,L,v}(x, y - l) 1_{\{y > l\}} dx dy.$$

Using (4.9) we have :

$$A_2 = \int_{\mathbb{R}_+^2} g(x, y) \varphi_3(u, r, x, y - l) 1_{\{y > l\}} dx dy. \tag{4.17}$$

Combining (4.13), (4.14), (4.16) and (4.17), we get :

$$\begin{aligned} E_0\big[1_{\Lambda_s} g(R_t, L_t)\big] \quad &= \quad E_0\Big[1_{\Lambda_s} R_s^{2\alpha} \int_0^\infty \frac{g(x, L_s)}{x^{2\alpha}} p_{t-s}^{(\alpha)}(R_s, x) dx\Big] \\ &\quad + \int_{\mathbb{R}_+^2} g(x, y) E_0\big[1_{\Lambda_s} \varphi_3(t - s, R_s, x, y - L_s) 1_{\{y > L_s\}}\big] dx dy. \end{aligned}$$

It is then easy to conclude since the function g is arbitrary.

■

To prove the existence of the limit (1.14) (resp. (1.18)) of Theorem 1.3, we need to obtain an asymptotic estimate of $p_{L,t}(y)$ (resp. $p_{R,L,t}(x,y)$) as $t \to \infty$. The first result may be obtained directly. As for $p_{R,L,t}(x,y)$, we first prove in Lemma 4.3 below that this function can be written as a convolution of two functions, having a decay rate of the type $Ct^{-(1+\alpha)}$ ($t \to \infty$). Then Lemma 4.4 allows to prove in Lemma 4.5 that $t \mapsto p_{R,L,t}(x,y)$ enjoys an analogous polynomial decay.

Lemma 4.3 *Let γ_1 denote the density of τ_1.*

1. *For every $x, y, q \geq 0$ there is the identity :*

$$\int_0^\infty e^{-qt} p_{R,L,t}(x,y)dt = \frac{2^{-\alpha}}{\Gamma(1+\alpha)} \left(\int_0^\infty e^{-qt-\frac{x^2}{2t}} \frac{x}{t^{1+\alpha}} dt \right) \left(\int_0^\infty e^{-qt} \gamma_1 \big(\frac{t}{y^{1/\alpha}}\big) \frac{dt}{y^{1/\alpha}} \right). \quad (4.18)$$

2. *Let $\beta_1, \beta_2 :]0,\infty[\times]0,\infty[\to \mathbb{R}$ be the two functions :*

$$\beta_1(x,t) := \frac{2^{-\alpha}}{\Gamma(1+\alpha)} \frac{x}{t^{\alpha+1}} e^{-\frac{x^2}{2t}}, \qquad x, t > 0, \quad (4.19)$$

and

$$\beta_2(y,t) := \frac{1}{y^{1/\alpha}} \gamma_1 \left(\frac{t}{y^{1/\alpha}} \right), \qquad y, t > 0. \quad (4.20)$$

Then :

$$p_{R,L,t}(x,y) = \left(\beta_1(x,\cdot) * \beta_2(y,\cdot)\right)(t). \quad (4.21)$$

Proof. Let Θ denote an exponential variable with parameter $q > 0$, independent from $(R_t, t \geq 0)$. Let χ_Θ denote the last zero of $(R_t, t \geq 0)$ before Θ. It is well known, from the last exit decomposition results, that $(R_t, t \leq \chi_\Theta)$ and $(R_{\chi_\Theta + u}, u \leq \Theta - \chi_\Theta)$ are two independent processes. Since $R_{\chi_\Theta} = 0$, and $L_\Theta = L_{\chi_\Theta}$, it follows that R_Θ and L_Θ are independent. As a consequence, we obtain, for every pair f, g of $\mathbb{R}_+$ valued Borel functions :

$$E_0\left[f(R_\Theta)g(L_\Theta)\right] = \int_{\mathbb{R}_+^3} q e^{-qt} f(\lambda) g(y) p_{R,L,t}(\lambda, y) d\lambda dy dt = E_0\left[f(R_\Theta)\right] E_0\left[g(L_\Theta)\right]. \quad (4.22)$$

a) We first compute $E_0\left[f(R_\Theta)\right]$.
Recall :

$$p_t^{(-\alpha)}(\lambda) = \frac{2^\alpha}{\Gamma(1-\alpha)} t^{\alpha-1} \lambda^{1-2\alpha} e^{-\frac{\lambda^2}{2t}} 1_{\{\lambda>0\}}, \quad (4.23)$$

then :

$$\begin{aligned}
E_0\left[f(R_\Theta)\right] &= \int_0^\infty q e^{-qt} dt \int_0^\infty f(\lambda) p_t^{(-\alpha)}(\lambda) d\lambda \\
&= q \int_0^\infty dt \int_0^\infty f(\lambda) \frac{2^\alpha}{\Gamma(1-\alpha)} t^{\alpha-1} \lambda^{1-2\alpha} \exp\big\{-\frac{\lambda^2}{2t} - qt\big\} d\lambda \\
&= \frac{2^\alpha q}{\Gamma(1-\alpha)} \int_0^\infty f(\lambda) \lambda^{1-2\alpha} d\lambda \int_0^\infty t^{\alpha-1} \exp\big\{-\frac{\lambda^2}{2t} - qt\big\} dt. \quad (4.24)
\end{aligned}$$

Setting $t = \lambda^2/(2qs)$ we obtain :

$$\begin{aligned}
E_0[f(R_\Theta)] &= \frac{q^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^\infty f(\lambda) d\lambda \int_0^\infty \frac{\lambda}{s^{\alpha+1}} \exp\big\{-\frac{\lambda^2}{2s} - qs\big\} ds \quad (4.25) \\
&= \frac{2^{1+\alpha/2} q^{1-\alpha/2}}{\Gamma(1-\alpha)} \int_0^\infty f(\lambda) K_{-\alpha}(\lambda\sqrt{2q}) \frac{d\lambda}{\lambda^\alpha}
\end{aligned}$$

b) As for the computation of $E_0\left[g(L_\Theta)\right]$, we observe that L_Θ is exponentially distributed with parameter $\dfrac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\Big(\dfrac{q}{2}\Big)^\alpha$, since from (2.12) we have :

$$P_0(L_\Theta > l) = P_0(\Theta > \tau_l) = E_0(e^{-q\tau_l}) = \exp\Big\{-l\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\Big(\frac{q}{2}\Big)^\alpha\Big\}, \quad l>0.$$

Hence :

$$E_0[g(L_\Theta)] = \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\left(\frac{q}{2}\right)^\alpha \int_0^\infty g(l)\exp\Big\{-l\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\left(\frac{q}{2}\right)^\alpha\Big\}dl. \tag{4.26}$$

Applying (2.12) with (l,λ) changed into $(1, ql^{1/\alpha})$ yields to :

$$\exp\Big\{-l\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\left(\frac{q}{2}\right)^\alpha\Big\} = E[e^{-ql^{1/\alpha}\tau_1}] = \int_0^\infty e^{-ql^{1/\alpha}s}\gamma_1(s)ds = \int_0^\infty e^{-qt}\gamma_1\big(\frac{t}{l^{1/\alpha}}\big)\frac{dt}{l^{1/\alpha}}.$$

Consequently :

$$E_0[g(L_\Theta)] = \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\left(\frac{q}{2}\right)^\alpha \int_0^\infty g(l)dl\int_0^\infty e^{-qt}\gamma_1\big(\frac{t}{l^{1/\alpha}}\big)\frac{dt}{l^{1/\alpha}}. \tag{4.27}$$

c) Since f and g are arbitrary, it is clear that (4.22), (4.25) and (4.27) imply (4.18).
d) (4.21) follows directly from (4.18).

■

Lemma 4.4 *Let β_1^0 and β_2^0 be two integrable functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, such that :*

$$\beta_1^0(t) \underset{t\to\infty}{\sim} \frac{k_1}{t^{1+\alpha}} \quad , \qquad \beta_2^0(t) \underset{t\to\infty}{\sim} \frac{k_2}{t^{1+\alpha}}. \tag{4.28}$$

Then :

$$\beta_1^0 * \beta_2^0(t) \underset{t\to\infty}{\sim} \frac{1}{t^{1+\alpha}}\left\{k_1\int_0^\infty \beta_2^0(u)du + k_2\int_0^\infty \beta_1^0(u)du\right\}. \tag{4.29}$$

Proof. Let us write :

$$\begin{aligned}
\beta_1^0 * \beta_2^0(t) &= \int_0^t \beta_1^0(u)\beta_2^0(t-u)du \\
&= \int_0^{\varepsilon t}\beta_1^0(u)\beta_2^0(t-u)du + \int_{\varepsilon t}^{(1-\varepsilon)t}\cdots du + \int_{(1-\varepsilon)t}^t \cdots du \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For t large enough, one has :

$$\begin{aligned}
I_2 &= t\int_\varepsilon^{1-\varepsilon}\beta_1^0(tu)\beta_2^0(t(1-u))du \\
&\le \frac{k_1'k_2'}{t^{1+2\alpha}}\int_\varepsilon^{1-\varepsilon}\frac{1}{u^{1+\alpha}}\frac{1}{(1-u)^{1+\alpha}}dv.
\end{aligned}$$

This implies that $I_2 = o\Big(\dfrac{1}{t^{1+\alpha}}\Big)$, $t\to\infty$ and this term does not contribute to the limit.
On the other hand, for any $0<\delta<k_2$, there exists $r_0>0$ such that :

$$\frac{k_2-\delta}{r^{1+\alpha}} \le \beta_2^0(r) \le \frac{k_2+\delta}{r^{1+\alpha}}, \quad r\ge r_0.$$

Let $0 < \varepsilon < 1/2$ and $t \geq 2r_0$. For any $u \leq \varepsilon t$ we have : $t \geq t - u \geq t(1-\varepsilon) \geq t/2 \geq r_0$. Therefore replacing r by $t-u$ in the previous inequality we get :

$$\frac{k_2 - \delta}{t^{1+\alpha}} \leq \beta_2^0(t-u) \leq \frac{k_2+\delta}{t^{1+\alpha}} \frac{1}{(1-\varepsilon)^{1+\alpha}}, \quad t \geq 2r_0. \tag{4.30}$$

Integrating (4.30) over $[0, \varepsilon t]$ with respect to $\beta_1^0(u)du$, we obtain :

$$(k_2 - \delta) \int_0^{\varepsilon t} \beta_1^0(u)du \leq t^{1+\alpha} I_1 \leq \frac{k_2+\delta}{(1-\varepsilon)^{1+\alpha}} \int_0^{\infty} \beta_1^0(u)du.$$

Taking the limit $t \to \infty$ we have :

$$(k_2 - \delta) \int_0^{\infty} \beta_1^0(u)du \leq \liminf_{t\to\infty} \left(t^{1+\alpha} I_1\right) \leq \limsup_{t\to\infty} \left(t^{1+\alpha} I_1\right) \leq \frac{k_2+\delta}{(1-\varepsilon)^{1+\alpha}} \int_0^{\infty} \beta_1^0(u)du.$$

Taking the limit $\delta, \varepsilon \to 0$ implies that $I_1 \underset{t\to\infty}{\sim} \frac{k_2}{t^{(1+\alpha)}} \int_0^{\infty} \beta_1^0(u)du$.
Since

$$I_3 = \int_{(1-\varepsilon)t}^{t} \beta_1^0(u)\beta_2^0(t-u)du = \int_0^{\varepsilon t} \beta_2^0(u)\beta_1^0(t-u)du,$$

we can apply the previous result, with β_1^0 and β_2^0 interchanged, to obtain :

$$I_3 \underset{t\to\infty}{\sim} \frac{k_1}{t^{(1+\alpha)}} \int_0^{\infty} \beta_2^0(u)du.$$

■

Thanks to Lemmas 4.3 and 4.4, we are able to determine the asymptotic behavior of $p_{R,L,t}(x,y)$ as $t \to \infty$. Observe that we may not deduce it from (4.18), since we do not know that $t \mapsto p_{R,L,t}(x,y)$ is monotone, hence the Tauberian theorem may not be applied.

Lemma 4.5 *The following equivalence holds :*

$$p_{R,L,t}(x,y) \underset{t\to\infty}{\sim} \frac{2^{-\alpha}}{\Gamma(1+\alpha)} \frac{x + yx^{1-2\alpha}}{t^{1+\alpha}}, \quad x, y > 0. \tag{4.31}$$

Proof. Recall that β_1 and β_2 are defined resp. in (4.19), (4.20).
a) It is clear that :

$$\beta_1(x,t) \underset{t\to\infty}{\sim} \frac{2^{-\alpha}}{\Gamma(1+\alpha)} \frac{x}{t^{\alpha+1}}. \tag{4.32}$$

Recall that from (2.12), we have :

$$E_0(e^{-q\tau_1}) = \exp - \left(\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \left(\frac{q}{2}\right)^{\alpha} \right), \quad q \geq 0.$$

Then we deduce, by differentiating both sides of this identity with respect to q and using the Tauberian theorem that :

$$\gamma_1(t) \underset{t\to\infty}{\sim} \frac{\alpha 2^{-\alpha}}{\Gamma(1+\alpha)} \frac{1}{t^{1+\alpha}}.$$

Hence :

$$\beta_2(y,t) \underset{t\to\infty}{\sim} \frac{\alpha 2^{-\alpha}}{\Gamma(1+\alpha)} \frac{y}{t^{1+\alpha}}. \tag{4.33}$$

b) We have :

$$\int_0^\infty \beta_1(x,t)dt = \frac{x^{1-2\alpha}}{\Gamma(1+\alpha)}\Gamma(\alpha) = \frac{x^{1-2\alpha}}{\alpha} \tag{4.34}$$

$$\int_0^\infty \beta_2(y,t)dt = 1. \tag{4.35}$$

Finally, Lemma 4.5 follows from Lemma 4.4, together with (4.32)-(4.35). ■

Proof of Theorem 1.3

1) a) We first prove points 1. and 2. of Theorem 1.3.

We claim that the proofs of (1.14) and (1.15) follow immediately from Lemma 4.1.

From (2.13), we have :

$$P_r(T_0 > t-s) = P(\gamma_\alpha < \frac{r^2}{2(t-s)}) \underset{t\to\infty}{\sim} \frac{1}{\Gamma(1+\alpha)}\left(\frac{r^2}{2t}\right)^\alpha. \tag{4.36}$$

Relations (2.8) and (2.10) imply :

$$p_{L_t}(y) = \frac{1}{t^\alpha}p_{L_1}(\frac{y}{t^\alpha}) \underset{t\to\infty}{\sim} \frac{1}{t^\alpha}p_{L_1}(0) = \frac{1}{t^\alpha}\frac{2^{-\alpha}}{\Gamma(1+\alpha)}. \tag{4.37}$$

Taking the limit $t \to \infty$ in (4.1), using (4.2), (4.3) and the two estimates (4.36), (4.37) above demonstrate point 1. of Theorem 1.3.

b) We now prove (1.16).

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ as in Theorem 1.1 and $\Lambda_s \in \mathcal{F}_s$. Thanks to the definition (1.15) of $Q_0^{(y)}$ we have :

$$\begin{aligned}
\int_0^\infty Q_0^{(y)}(\Lambda_s)h(y)dy &= \int_0^\infty p_{L_s}(y)E_0\big[1_{\Lambda_s}R_s^{2\alpha}|L_s=y\big]h(y)dy + \int_0^\infty E_0\big[1_{\Lambda_s}1_{\{L_s<y\}}\big]h(y)dy \\
&= \int_0^\infty E_0\big[1_{\Lambda_s}h(L_s)R_s^{2\alpha}|L_s=y\big]p_{L_s}(y)dy + E_0\big[1_{\Lambda_s}\int_0^\infty h(y)1_{\{L_s<y\}}dy\big] \\
&= E_0\big[1_{\Lambda_s}\{h(L_s)R_s^{2\alpha} + 1 - H(L_s)\}\big] = E_0[1_{\Lambda_s}M_s^h] \\
&= Q_0^{(h)}(\Lambda_s) = \int_0^\infty Q_0^{(h)}(\Lambda_s|L_\infty = y)h(y)dy,
\end{aligned}$$

the latter relation following from the fact that L_∞ admits h as its probability density.

Therefore the two probability measures on $(\Omega, \mathcal{F}_\infty)$, $\int_0^\infty Q_0^{(y)}(\cdot)h(y)dy$ and $Q_0^{(h)}$ coincide on $\mathcal{F}_s$, for any $s \geq 0$, hence they are equal :

$$Q_0^{(h)}(\cdot) = \int_0^\infty Q_0^{(y)}(\cdot)h(y)dy.$$

On the other hand, from the definition (1.15) of $Q_0^{(y)}$, we easily deduce that $Q_0^{(y)}$ is carried by $L_\infty = y$. Indeed, for every $\varepsilon > 0$,

$$\begin{aligned}
Q_0^{(y)}(L_\infty \leq y-\varepsilon) &= \lim_{s\to\infty} Q_0^{(y)}(L_s \leq y-\varepsilon) \\
&= \lim_{s\to\infty} \{p_{L_s}(y)E_0\big[1_{\{L_s\leq y-\varepsilon\}}R_s^{2\alpha}|L_s=y\big] + P_0(L_s \leq y-\varepsilon)\} \\
&= \lim_{s\to\infty} P_0(L_s \leq y-\varepsilon) \\
&= 0 \quad \text{since } L_\infty = \infty, P_0 \text{ a.s.}
\end{aligned}$$

A similar computation shows that $Q_0^{(y)}(L_\infty \geq y+\varepsilon)=0$.
Consequently :

$$Q_0^{(h)}(\cdot|L_\infty = y) = Q_0^{(y)}(\cdot).$$

2) Proof of point 3. of Theorem 1.3.
To prove that the limit in (1.18) exists, we start with Lemma 4.2 :

$$E_0\left(\Lambda_s | R_t = x, L_t = y\right) = \Theta_1 + \Theta_2, \quad \Lambda \in \mathcal{F}_s, t > s,$$

where

$$\begin{aligned}
\Theta_1 &= \frac{p_{L_s}(y)x^{-2\alpha}}{p_{R,L,t}(x,y)} E_0\big[1_{\Lambda_s} R_s^{2\alpha} p_{t-s}^{(\alpha)}(R_s, x) | L_s = y\big] \\
\Theta_2 &= \frac{1}{p_{R,L,t}(x,y)} E_0\big[1_{\Lambda_s} \varphi_3(t-s, R_s, x, y-L_s) 1_{\{y>L_s\}}\big],
\end{aligned} \tag{4.38}$$

the function φ_3 being defined by (4.9).
We study successively the limits of Θ_1, Θ_2, as $t \to \infty$.
a) From ([16], Chap. 10), we have :

$$p_t^{(\alpha)}(r,a) = \frac{a}{t}\left(\frac{a}{r}\right)^\alpha I_\alpha\left(\frac{ar}{t}\right) \exp -\frac{a^2+r^2}{2t}.$$

Since $I_\alpha(z) \underset{z\to 0}{\sim} \dfrac{1}{\Gamma(\alpha+1)}\left(\dfrac{z}{2}\right)^\alpha$ an equivalent for $p_t^{(\alpha)}(r,a)$ as $t \to \infty$ is easily deduced :

$$p_t^{(\alpha)}(r,a) \underset{t\to\infty}{\sim} \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \frac{a^{1+2\alpha}}{t^{1+\alpha}}. \tag{4.39}$$

Consequently using moreover Lemma 4.5, we obtain :

$$\lim_{t\to\infty} \Theta_1 = \frac{p_{L_s}(y)x}{x + yx^{1-2\alpha}} E_0\left(1_{\Lambda_s} R_s^{2\alpha} | L_s = y\right). \tag{4.40}$$

b) Next we study the limit of Θ_2, as $t \to \infty$.
It is clear that (4.9) may be interpreted as :

$$\varphi_3(u,r,x,y) = \big(\mu(r,\cdot) \star p_{R,L,\cdot}(x,y)\big)(u), \tag{4.41}$$

where μ is the density function of T_0 under P_r.
Thanks to (2.14), we have :

$$\mu(r,t) \underset{t\to\infty}{\sim} \frac{\alpha 2^{-\alpha}}{\Gamma(1+\alpha)} \frac{r^{2\alpha}}{t^{1+\alpha}}.$$

Taking $q=0$ in (4.18) we have :

$$\int_0^\infty p_{R,L,t}(x,y)dt = \frac{x^{1-2\alpha}}{\alpha}.$$

Hence, applying (4.41) together with Lemmas 4.5 and 4.4 leads to :

$$\begin{aligned}
\varphi_3(u,r,x,y) &\underset{u\to\infty}{\sim} \left[\frac{\alpha 2^{-\alpha} r^{2\alpha}}{\Gamma(1+\alpha)} \frac{x^{1-2\alpha}}{\alpha} + \frac{2^{-\alpha}}{\Gamma(1+\alpha)}\left(x + yx^{1-2\alpha}\right)\right] \frac{1}{u^{1+\alpha}} \\
&\underset{u\to\infty}{\sim} \frac{2^{-\alpha}}{\Gamma(1+\alpha)}\left[x + (r^{2\alpha}+y)x^{1-2\alpha}\right] \frac{1}{u^{1+\alpha}}.
\end{aligned} \tag{4.42}$$

Plugging this expression in (4.38), and using again (4.31), we deduce that :

$$\lim_{t\to\infty}\Theta_2 = \frac{1}{x+yx^{1-2\alpha}} E_0\left[1_{\Lambda_s}1_{(L_s<y)}\left(x+\left(R_s^{2\alpha}+y-L_s\right)a^{1-2\alpha}\right]\right).$$

This result together with (4.40), proves that the limit in (1.18) exists and has the form given in (1.19).
3) <u>We end the proof of Theorem 1.3, by showing point 4.</u>
Following the definitions (1.19) and (1.15), of resp. $Q_0^{(x,y)}$ and $Q_0^{(y)}$, we have :

$$\begin{aligned}\left(x+yx^{1-2\alpha}\right)Q_0^{(x,y)}(\Lambda_s) &= x\left(Q_0^{(y)}(\Lambda_s) - E_0\left[1_{\Lambda_s}1_{\{L_s<y\}}\right]\right) \\ &\quad + E_0\left[1_{\Lambda_s}1_{\{L_s<y\}}\left(x+(y-L_s+R_s^{2\alpha})x^{1-2\alpha}\right)\right] \\ &= xQ_0^{(y)}(\Lambda_s) + x^{1-2\alpha}E_0\left[1_{\Lambda_s}1_{\{L_s<y\}}(y-L_s+R_s^{2\alpha})\right]. \end{aligned} \tag{4.43}$$

For a given $y>0$, let h^y be the function : $h^y(x) = \frac{1}{y}1_{[0,y]}(x),\ x\geq 0$, and H^y the primitive of h^y, vanishing at 0; hence :

$$1 - H^y(x) = \int_x^\infty h^y(z)dz = 1_{\{x\leq y\}}\left(1-\frac{x}{y}\right). \tag{4.44}$$

Thanks to (1.16), (1.7) and (1.6), we have :

$$\begin{aligned}\int_0^\infty Q_0^{(z)}(\Lambda_s)h^y(z)dz &= E_0\left[1_{\Lambda_s}M_s^{h^y}\right] \\ &= E_0\left[1_{\Lambda_s}\left(h^y(L_s)R_s^{2\alpha} + 1_{\{L_s<y\}}(1-\frac{L_s}{y})\right)\right] \\ &= \frac{1}{y}E_0\left[1_{\Lambda_s}\left(R_s^{2\alpha}+y-L_s\right)1_{\{L_s<y\}}\right]. \end{aligned} \tag{4.45}$$

Plugging (4.45) in (4.43), we get :

$$Q_0^{(x,y)}(\Lambda_s) = \frac{x}{x+yx^{1-2\alpha}}Q^{(y)}(\Lambda_s) + \frac{x^{1-2\alpha}}{x+yx^{1-2\alpha}}\int_0^y Q_0^{(z)}(\Lambda_s)dz.$$

This ends the proof of Theorem 1.3.

■

Remark 4.6 *Suppose that $\alpha = 1/2$ (i.e : $d=1$).*

1. *Several of the above computations become easier, in particular, that of the function φ_3 introduced in (4.9). From Lévy's theorem :*

$$\left(\left(S_t - X_t, S_t\right), t\geq 0\right) \stackrel{(d)}{=} \left(\left(R_t, L_t\right), t\geq 0\right) \tag{4.46}$$

where, on the left-hand side (X_t) is a standard Brownian motion started at 0, (S_t) its unilateral maximum, i.e. $S_t = \max_{u\leq t} X_u$, and the right-hand side $(R_t, t\geq 0)$ is a reflected Brownian motion (i.e. a Bessel process with index $(-1/2)$), and (L_t) its local time at level 0, and ([16] section III.3 p105), we have :

$$p_{R,L,t}(x,y) = \sqrt{\frac{2}{\pi t^3}}(x+y)e^{-\frac{(x+y)^2}{2t}}\left(=2\frac{P_0\left(T_{x+y}(X)\in dt\right)}{dt}\right), \quad x,y>0, \tag{4.47}$$

since :

$$P_0(T_r(X)\in dt) = \sqrt{\frac{1}{2\pi t^3}}re^{-\frac{r^2}{2t}}1_{\{t>0\}}dt, \quad r>0,$$

where $T_r(X)$ denotes the first hitting time of level r for the Brownian motion (X_t).
Since under P_r, T_0 is distributed as $T_r(X)$ and $T_r(X)+\widehat{T_{r'}(X)}$ is distributed as $T_{r+r'}(X)$, where $\widehat{T_{r'}(X)}$ is independent from $T_r(X)$ and $\widehat{T_{r'}(X)} \overset{(d)}{=} T_{r'}(X)$, then using (4.41), we have :

$$\varphi_3(u,r,x,y) = 2\frac{P(T_{x+y+r}(X) \in du)}{du}.$$

This implies that

$$\varphi_3(u,r,x,y) \underset{u\to\infty}{\sim} \sqrt{\frac{2}{\pi u^3}}\,(x+y+r).$$

We recover (4.42).

2. *We keep the notation relative to Brownian motion introduced above. We have proven in ([18], Theorems 1.2, 1.3 and the proof of Theorem 1.3) that, for $a<y$, $y \geq 0$ and $\Lambda_u \in \mathcal{F}_u$:*

$$\lim_{t\to\infty} E\big[1_{\Lambda_u}|X_t=a, S_t=y\big]$$
$$= \frac{y-a}{2y-a}p_{S_u}(y)E\big[1_{\Lambda_u}(y-X_u)|S_u=y\big] + \frac{1}{2y-a}E\big[1_{\Lambda_u}1_{\{S_u<y\}}(2y-a-X_u)\big],$$

where p_{S_u} denotes the density function of S_u.
Obviously, this result is equivalent to :

$$\lim_{t\to\infty} E\big[1_{\Lambda_u}|X_t=y-x, S_t=y\big]$$
$$= \frac{x}{x+y}p_{S_u}(y)E\big[1_{\Lambda_u}(y-X_u)|S_u=y\big] + \frac{1}{x+y}E\big[1_{\Lambda_u}1_{\{S_u<y\}}(x+y-X_u)\big].$$

Therefore, from Lévy's theorem (4.46), we obtain :

$$\lim_{t\to\infty} E_0\big[1_{\Lambda_u}|R_t=x, L_t=y\big]$$
$$= \frac{x}{x+y}p_{L_u}(y)E_0\big[1_{\Lambda_u}R_u|L_u=y\big] + \frac{1}{x+y}E_0\big[1_{\Lambda_u}1_{\{L_u<y\}}(x+y-L_u+R_u)\big].$$

which is indeed (1.19) of our Theorem 1.3 for $\alpha = 1/2$.

4.2 Proof of Theorem 1.4

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a probability density. Recall that $\bar{h}_x = \displaystyle\int_0^\infty h(y)(x+yx^{1-2\alpha})dy < \infty$ where $x>0$. Define $H(z) = \displaystyle\int_0^z h(y)dy,\ z\geq 0$.

1) We first prove (1.23).
It is clear that :

$$P_0(L_t\in dy|R_t=x) = \frac{p_{R,L,t}(x,y)}{p_t^{(-\alpha)}(0,x)}dy = 2^{-\alpha}\Gamma(1-\alpha)\frac{e^{x^2/2t}}{x^{1-2\alpha}}\frac{1}{t^{\alpha-1}}p_{R,L,t}(x,y)dy.$$

Applying Lemma 4.5, formula (4.31), we obtain :

$$\frac{P_0(L_t\in dy|R_t=x)}{dy} \underset{t\to\infty}{\sim} \frac{2^{-2\alpha}\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\frac{x+yx^{1-2\alpha}}{x^{1-2\alpha}}\frac{1}{t^{2\alpha}}. \tag{4.48}$$

Since

$$\frac{E_0\big[1_{\Lambda_s}h(L_t)|R_t=x\big]}{E_0\big[h(L_t)|R_t=x\big]} = \frac{\displaystyle\int_0^\infty E_0\left[1_{\Lambda_s}|R_t=a, L_t=y\right]h(y)P_0(L_t\in dy|R_t=x)}{\displaystyle\int_0^\infty h(y)P_0(L_t\in dy|R_t=x)},$$

we deduce from (4.48) and point 3. of Theorem 1.3 that :

$$\lim_{t\to\infty} \frac{E_0\big[1_{\Lambda_s} h(L_t)|R_t = x\big]}{E_0\big[h(L_t)|R_t = x\big]} = h_x^* \int_0^\infty Q_0^{(x,y)}(\Lambda_s) h(y)(x + yx^{1-2\alpha})dy,$$

where $h_x^* = 1/\bar{h}_x$.
According to (1.20), we may write the right-hand side as follows :

$$h_x^* \Big\{ \int_0^\infty Q_0^{(y)}(\Lambda_s) x h(y) dy + \int_0^\infty x^{1-2\alpha} h(y) dy \int_0^y Q_0^{(z)}(\Lambda_s) dz \Big\}.$$

Applying Fubini's theorem, the previous term equals :

$$h_x^* \Big\{ \int_0^\infty Q_0^{(y)}(\Lambda_s)\big(x h(y) + x^{1-2\alpha}(1 - H(y))\big) dy \Big\} = Q^{h_x}(\Lambda_s).$$

This proves (1.23).
2) Next we prove point 2. of Theorem 1.4.
Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be Borel and such that (1.25) holds.
We proceed as above. Using Lemma 4.5 and point 3. of Theorem 1.3 we have :

$$\frac{E_0\big[1_{\Lambda_s} f(R_t, L_t)\big]}{E_0\big[f(R_t, L_t)\big]} = \frac{\displaystyle\int_{\mathbb{R}_+\times\mathbb{R}_+} E_0\big[1_{\Lambda_s}|R_t = x, L_t = y\big] f(x,y) p_{R,L,t}(x,y) dx dy}{\displaystyle\int_{\mathbb{R}_+\times\mathbb{R}_+} f(x,y) p_{R,L,t}(x,y) dx dy}.$$

Consequently :

$$\lim_{t\to\infty} \frac{E_0\big[1_{\Lambda_s} f(R_t, L_t)\big]}{E_0\big[f(R_t, L_t)\big]} = f^* \int_{\mathbb{R}_+\times\mathbb{R}_+} Q_0^{(x,y)}(\Lambda_s)(x + yx^{1-2\alpha}) f(x,y)\, dy.$$

$$\frac{E_0\big[1_{\Lambda_s} f(R_t, L_t)\big]}{E_0\big[f(R_t, L_t)\big]} = \frac{\displaystyle\int_{\mathbb{R}_+\times\mathbb{R}_+} E_0\big[1_{\Lambda_s}|R_t = x, L_t = y\big] f(x,y) p_{R,L,t}(x,y) dx dy}{\displaystyle\int_{\mathbb{R}_+\times\mathbb{R}_+} f(x,y) p_{R,L,t}(x,y) dx dy}.$$

$$\lim_{t\to\infty} \frac{E_0\big[1_{\Lambda_s} f(R_t, L_t)\big]}{E_0\big[f(R_t, L_t)\big]} = f^* \int_{\mathbb{R}_+\times\mathbb{R}_+} Q_0^{(x,y)}(\Lambda_s)(x + yx^{1-2\alpha}) f(x,y)\, dy.$$

Hence, from (1.20) and Fubini's theorem, this limit equals :

$$\begin{aligned}
& f^* \Big\{ \int_{\mathbb{R}_+\times\mathbb{R}_+} \big(x Q_0^{(y)}(\Lambda_s) + x^{1-2\alpha} \int_0^y Q_0^{(z)}(\Lambda_s) dz\big) f(x,y)\, dx dy \Big\} \\
= \ & f^* \int_0^\infty Q_0^{(y)}(\Lambda_s) dy \Big\{ \int_0^\infty x f(x,y) dx + \int_0^\infty x^{1-2\alpha} dx \int_y^\infty f(x,z)\, dz \Big\} \\
= \ & \int_0^\infty Q_0^{(y)}(\Lambda_s) \tilde{f}(y) dy = Q_0^{(\tilde{f})}(\Lambda_s).
\end{aligned}$$

This ends the proof of Theorem 1.4.

5 Proof of Theorem 1.5

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy (1.28).
1) We first prove point 1. of Theorem 1.5.
Let $t > s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$.
The Markov property at time s allows to write :

$$\frac{E_0\left[1_{\Lambda_s} h(L_t) e^{\lambda R_t}\right]}{E_0\left[h(L_t) e^{\lambda R_t}\right]} = \frac{N(s,t)}{D(t)}, \tag{5.1}$$

where :

$$D(t) = E_0\big[h(L_t) e^{\lambda R_t}\big] = \int_{\mathbb{R}_+ \times \mathbb{R}_+} h(y) e^{\lambda x} p_{R,L,t}(x,y) dx dy, \tag{5.2}$$

$$N(s,t) = E_0\left[1_{\Lambda_s} h(L_t) e^{\lambda R_t}\right] = E_0\left[1_{\Lambda_s} N_1(R_s, L_s, t-s)\right], \tag{5.3}$$

$$N_1(x,y,u) = E_x\left[h(y + L_u) e^{\lambda R_u}\right], \quad x, y, u \geq 0. \tag{5.4}$$

Recall that $p_{R,L,t}(x,y)$ denotes the density function of (R_t, L_t) under P_0.
We study successively the asymptotic behaviors of $D(t)$ and of $N_1(x,y,t)$ as $t \to \infty$. Note that we cannot apply Lemma 4.5 since $f(x,y) = h(y)e^{\lambda x}$ does not satisfy (1.25).
1.a) Let us determine the rate of decay of $D(t)$, as $t \to \infty$.
Since $p_{R,L,t}$ satisfies (4.21), then

$$D(t) = \frac{2^{-\alpha}}{\Gamma(1+\alpha)} \int_0^t \frac{du}{(t-u)^{\alpha+1}} \int_0^\infty dy h(y) \beta_2(y,u) \int_0^\infty x e^{\lambda x - \frac{x^2}{2(t-u)}} dx.$$

Setting $x = \lambda(t-u) + z\sqrt{t-u}$ in the integral with respect to dx, we obtain :

$$\int_0^\infty x e^{\lambda x - \frac{x^2}{2(t-u)}} dx = (t-u)^{3/2} e^{\frac{\lambda^2 (t-u)}{2}} \int_{-\lambda\sqrt{t-u}}^\infty \big(\frac{z}{\sqrt{t-u}} + \lambda\big) e^{-z^2/2} dz \underset{t\to\infty}{\sim} \lambda\sqrt{2\pi}(t-u)^{3/2} e^{\frac{\lambda^2(t-u)}{2}}.$$

Consequently :

$$\begin{aligned} D(t) &\underset{t\to\infty}{\sim} \frac{2^{-\alpha}\lambda\sqrt{2\pi}}{\Gamma(1+\alpha)} \int_0^t \frac{e^{\frac{\lambda^2(t-u)}{2}}}{(t-u)^{\alpha-1/2}} du \int_0^\infty h(y)\beta_2(y,u) dy \\ &\underset{t\to\infty}{\sim} \frac{2^{-\alpha}\lambda\sqrt{2\pi}}{\Gamma(1+\alpha)} t^{\frac{1}{2}-\alpha} e^{\frac{\lambda^2 t}{2}} \int_0^t e^{-\frac{\lambda^2 u}{2}} du \int_0^\infty h(y)\beta_2(y,u) dy. \end{aligned}$$

Next, using the definition (4.20) of β_2, we get :

$$\begin{aligned} D(t) &\underset{t\to\infty}{\sim} \frac{2^{-\alpha}\lambda\sqrt{2\pi}}{\Gamma(1+\alpha)} t^{\frac{1}{2}-\alpha} e^{\frac{\lambda^2 t}{2}} \int_0^\infty \frac{h(y)}{y^{1/\alpha}} dy \int_0^t e^{-\frac{\lambda^2 u}{2}} \gamma_1\big(\frac{u}{y^{1/\alpha}}\big) du \\ &\underset{t\to\infty}{\sim} \frac{2^{-\alpha}\lambda\sqrt{2\pi}}{\Gamma(1+\alpha)} t^{\frac{1}{2}-\alpha} e^{\frac{\lambda^2 t}{2}} \int_0^\infty h(y) dy \int_0^{t/y^{1/\alpha}} \gamma_1(v) e^{-\frac{\lambda^2 y^{1/\alpha}}{2} v} dv. \\ &\underset{t\to\infty}{\sim} \frac{2^{-\alpha}\lambda\sqrt{2\pi}}{\Gamma(1+\alpha)} t^{\frac{1}{2}-\alpha} e^{\frac{\lambda^2 t}{2}} \int_0^\infty h(y) dy \int_0^\infty \gamma_1(v) e^{-\frac{\lambda^2 y^{1/\alpha}}{2} v} dv. \end{aligned}$$

γ_1 being the density function of τ_1, applying identity (2.12) (with $l = 1$ and λ replaced by $\dfrac{\lambda^2 y^{1/\alpha}}{2}$) leads to :

$$D(t) \underset{t\to\infty}{\sim} \frac{2^{-\alpha}\lambda\sqrt{2\pi}}{\Gamma(1+\alpha)} \Big(\int_0^\infty h(y) e^{-\sigma_\lambda y} dy \Big) t^{\frac{1}{2}-\alpha} e^{\frac{\lambda^2 t}{2}}, \tag{5.5}$$

since σ_λ is defined by (1.29).

We now consider the function $N_1(x,y,u)$ defined by (5.4), where $x, y \geq 0$ are fixed and $u \to \infty$. We decompose $N_1(x,y,u)$ as the sum of two terms :

$$N_1(x,y,u) = N_{1,1}(x,y,u) + N_{1,2}(x,y,u), \tag{5.6}$$

where :

$$\begin{aligned} N_{1,1}(x,y,u) &= E_x[h(y+L_u)e^{\lambda R_u}1_{\{u<T_0\}}] = h(y)E_x[e^{\lambda R_u}1_{\{u<T_0\}}] \\ N_{1,2}(x,y,u) &= E_x[h(y+L_u)e^{\lambda R_u}1_{\{u \geq T_0\}}]. \end{aligned} \tag{5.7}$$

1. b) <u>We look for an equivalent of $N_{1,1}(x,y,u)$, as $u \to \infty$.</u>
From the absolute continuity relationship between Bessel laws (cf ex 1.22, chap XI in [16])) we get :

$$N_{1,1}(x,y,u) = h(y)E_x^{(\alpha)}\left[\frac{x^{2\alpha}}{R_u^{2\alpha}}e^{\lambda R_u}\right],$$

where as in the Introduction, under $P_x^{(\alpha)}$, the process (R_t) is a Bessel process with index α, starting at x.
Since (cf for instance section 1. p446 of [16]) :

$$P_x^{(\alpha)}(R_u \in da) = \frac{a}{u}\left(\frac{a}{x}\right)^\alpha I_\alpha\left(\frac{ax}{u}\right)\exp -\frac{x^2+a^2}{2u}da, \tag{5.8}$$

we get :

$$N_{1,1}(x,y,u) = h(y)\frac{x^\alpha}{u}e^{-\frac{x^2}{2u}+\frac{\lambda^2 u}{2}}\int_0^\infty a^{1-\alpha}I_\alpha\left(\frac{ax}{u}\right)e^{-\frac{1}{2u}(a-\lambda u)^2}da.$$

Setting $a = \lambda u + \sqrt{u}b$, we get :

$$\begin{aligned} N_{1,1}(x,y,u) \quad &= \quad h(y)x^\alpha u^{\frac{1}{2}-\alpha}e^{-\frac{x^2}{2u}+\frac{\lambda^2 u}{2}}\int_{-\lambda\sqrt{u}}^\infty \big(\lambda + \frac{b}{\sqrt{u}}\big)^{1-\alpha}I_\alpha\big((\lambda+\frac{b}{\sqrt{u}})x\big)e^{-b^2/2}db \\ &\underset{u\to\infty}{\sim} \quad h(y)x^\alpha\lambda^{1-\alpha}I_\alpha(\lambda x)\sqrt{2\pi}\ u^{\frac{1}{2}-\alpha}e^{\frac{\lambda^2 u}{2}}. \end{aligned} \tag{5.9}$$

1. c) <u>We now find an equivalent of $N_{1,2}(x,y,u), u \to \infty$.</u>
Conditioning with respect to T_0, we get :

$$N_{1,2}(x,y,u) = E_x\big[\psi(u-T_0)1_{\{u \geq T_0\}}\big],$$

where :

$$\psi(v) = E_0\big[h(y+L_v)e^{\lambda R_v}\big].$$

Applying (5.5), we have :

$$\psi(v) \underset{v\to\infty}{\sim} \frac{2^{-\alpha}\sqrt{2\pi}\lambda}{\Gamma(1+\alpha)}\Big(\int_0^\infty h(y+z)e^{-\sigma_\lambda z}dz\Big)v^{\frac{1}{2}-\alpha}e^{\frac{\lambda^2 v}{2}}.$$

Consequently :

$$N_{1,2}(x,y,u) \underset{u\to\infty}{\sim} \frac{2^{-\alpha}\sqrt{2\pi}\lambda e^{\sigma_\lambda y}}{\Gamma(1+\alpha)}\Big(\int_y^\infty h(z)e^{-\sigma_\lambda z}dz\Big)E_x\big[e^{-\frac{\lambda^2}{2}T_0}\big]\ u^{\frac{1}{2}-\alpha}e^{\frac{\lambda^2 u}{2}}. \tag{5.10}$$

Using (2.15), we get finally :

$$N_{1,2}(x,y,u) \underset{u\to\infty}{\sim} \sqrt{2\pi}\,\frac{2^{1-2\alpha}\lambda^{\alpha+1}}{\Gamma(\alpha)\Gamma(\alpha+1)} x^{\alpha} K_{\alpha}(\lambda x) e^{\frac{\lambda^2 u}{2}} u^{\frac{1}{2}-\alpha}(1-\widetilde{H}(y)), \tag{5.11}$$

where we denoted :

$$1-\widetilde{H}(y) := e^{y\sigma_\lambda}\int_y^\infty h(z)e^{-\sigma_\lambda z}dz.$$

1. d) We now compute : $\lim\limits_{t\to\infty} \dfrac{E_0\left[1_{\Lambda_s} h(L_t)e^{\lambda R_t}\right]}{E_0\left[h(L_t)e^{\lambda R_t}\right]}$.

From (5.5), (5.6), (5.7), (5.9) and (5.11), we have :

$$\frac{E_0\left[1_{\Lambda_s} h(L_t)e^{\lambda R_t}\right]}{E_0\left[h(L_t)e^{\lambda R_t}\right]} \underset{t\to\infty}{\sim} \frac{A_t}{B_t}$$

where :

$$\begin{aligned} A_t \quad &:= \quad E_0\Bigg(1_{\Lambda_s}\Bigg[\sqrt{2\pi}\,h(L_s)R_s^{\alpha}\lambda^{1-\alpha}I_{\alpha}(\lambda R_s)(t-s)^{\frac{1}{2}-\alpha}e^{\frac{\lambda^2}{2}(t-s)} \\ &\qquad + \frac{\sqrt{2\pi}2^{1-2\alpha}}{\Gamma(\alpha)\Gamma(1+\alpha)}\lambda^{\alpha+1}R_s^{\alpha}K_{\alpha}(\lambda R_s)e^{\frac{\lambda^2}{2}(t-s)}(t-s)^{\frac{1}{2}-\alpha}\Bigg]\Bigg), \\ B_t \quad &= \quad \frac{2^{-\alpha}\sqrt{2\pi}}{\Gamma(1+\alpha)}\lambda e^{\frac{\lambda^2 t}{2}} t^{\frac{1}{2}-\alpha}\int_0^\infty h(\ell)e^{-\ell\sigma_\lambda}d\ell. \end{aligned} \tag{5.12}$$

Hence :

$$\lim_{t\to\infty}\frac{E_0\left[1_{\Lambda_s} h(L_t)e^{\lambda R_t}\right]}{E_0\left[h(L_t)e^{\lambda R_t}\right]} = E_0[1_{\Lambda_s}M_s], \tag{5.13}$$

with

$$M_s = e^{-\frac{\lambda^2 s}{2}}R_s^{\alpha}\Bigg\{\left(\frac{2}{\lambda}\right)^{\alpha}\Gamma(1+\alpha)h(L_s)I_{\alpha}(\lambda R_s) + \left(\frac{\lambda}{2}\right)^{\alpha}\frac{2}{\Gamma(\alpha)}(1-\widetilde{H}(L_s))K_{\alpha}(\lambda R_s)\Bigg\}. \tag{5.14}$$

2) We now prove that $M_s = M_s^{\lambda,\tilde{h}},\ s\geq 0$, where $(M_s^{\lambda,\tilde{h}})$ is the process defined by (1.32).
From (1.33) and (1.34), the function h can be written as a linear combination of $\tilde{h}$ and $1-\tilde{H}$:

$$h(y) = \widetilde{h}(y) + \sigma_\lambda e^{\sigma_\lambda y}\int_y^\infty h(z)e^{-\sigma_\lambda z}dz = \widetilde{h}(y) + \sigma_\lambda\big(1-\tilde{H}(y)\big).$$

Consequently :

$$M_s = e^{-\frac{\lambda^2 s}{2}}R_s^{\alpha}\Big\{\left(\frac{2}{\lambda}\right)^{\alpha}\Gamma(1+\alpha)\tilde{h}(L_s)I_{\alpha}(\lambda R_s) + \big(1-\widetilde{H}(L_s)\big)\xi_s\Big\},$$

where :

$$\xi_s = \left(\frac{2}{\lambda}\right)^{\alpha}\Gamma(1+\alpha)\sigma_\lambda I_{\alpha}(\lambda R_s) + \left(\frac{\lambda}{2}\right)^{\alpha}\frac{2}{\Gamma(\alpha)}K_{\alpha}(\lambda R_s).$$

Given the relations (see [9] p3 and p108) :

$$\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}\frac{1}{\Gamma(\alpha)} \quad , \quad K_{\alpha}(r) = \frac{\pi}{2\sin(\pi\alpha)}\left(I_{\alpha}(r) - I_{-\alpha}(r)\right) \tag{5.15}$$

then the definition (1.29) of σ_λ implies :

$$\begin{aligned}\xi_s &= \left(\frac{\lambda}{2}\right)^{\alpha}\Gamma(1-\alpha)I_\alpha(\lambda R_s)+\left(\frac{\lambda}{2}\right)^{\alpha}\Gamma(1-\alpha)\left(I_\alpha(\lambda R_s)-I_{-\alpha}(\lambda R_s)\right)\\ &= \left(\frac{\lambda}{2}\right)^{\alpha}\Gamma(1-\alpha)I_{-\alpha}(\lambda R_s).\end{aligned}$$

This proves $M_s = M_s^{\lambda,\tilde{h}}$.
The local behavior of $I_\beta(z), z\to 0$ is known (see [9] formula (5.7.1) p108) :

$$I_\beta(z)=\frac{1}{\Gamma(1+\beta)}\left(\frac{z}{2}\right)^{\beta}+0(z^{\beta+2})\quad (z\to 0), \tag{5.16}$$

In particular :

$$\lim_{r\to 0} r^\alpha I_\alpha(\lambda r)=0,\quad \text{and}\quad \lim_{r\to 0}\left(\frac{\lambda}{2}\right)^{\alpha}\Gamma(1-\alpha)r^\alpha I_{-\alpha}(\lambda r)=1. \tag{5.17}$$

This implies :

$$\lim_{s\to 0} R_s^\alpha I_\alpha(\lambda R_s)=0,\quad \lim_{s\to 0} R_s^\alpha \xi_s=1.$$

It is clear that (1.28) and (1.34] imply that $\widetilde{H}(0)=0$. As a result : $M_0^{\lambda,\tilde{h}}=1$.
This ends the proof of point 1. of Theorem 1.5.

3) We verify that $(M_s^{\lambda,\tilde{h}}, s\geq 0)$ is a martingale.

We shall show that $(M_t^{\lambda,\tilde{h}}, t\geq 0)$ is a local martingale. It will suffice to assume that $\tilde{h}$ is of class C^1 to prove that $(M_t^{\lambda,\tilde{h}})$ is a martingale (cf. point 2) of the proof of Theorem 1.1).
It is clear that $(M_t^{\lambda,\tilde{h}})$ can be decomposed as follows :

$$M_t^{\lambda,\tilde{h}}=e^{-\frac{\lambda^2 t}{2}}\left\{\left(\frac{2}{\lambda}\right)^{\alpha}\Gamma(1+\alpha)\tilde{h}(L_t)\Psi_1(R_t)+\left(\frac{\lambda}{2}\right)^{\alpha}\Gamma(1-\alpha)\big(1-\widetilde{H}(L_t)\big)\Psi_2(R_t)\right\}, \tag{5.18}$$

where $\Psi_1,\Psi_2:\mathbb{R}_+\to\mathbb{R}_+$ are defined by :

$$\Psi_1(r):=r^\alpha I_\alpha(\lambda r)\quad \Psi_2(r):=r^\alpha I_{-\alpha}(\lambda r). \tag{5.19}$$

3. a) In a first step, we prove that $(\Psi_i(R_t))$, $i=1,2$ are two semimartingales, and we determine their decompositions.
Using [9] p110, we have :

$$Y''(r)+\frac{1}{r}Y'(r)=\left(1+\frac{\alpha^2}{r^2}\right)Y(r), \tag{5.20}$$

where $Y(r)$ denotes either $I_\alpha(r)$ or $K_\alpha(r)$.
Then,

$$\mathcal{L}\Psi_i(r)=\frac{\lambda^2}{2}\Psi_i(r)\qquad (i=1,2), \tag{5.21}$$

where $\mathcal{L}$ denotes the infinitesimal generator of $(R_t, t\geq 0)$ (cf. (2.1)).
We deduce from property (5.16) (with $\beta=-\alpha$) and the definition (2.2) of the domain $\mathcal{D}$ of $\mathcal{L}$ that $\Psi_2\in\mathcal{D}$, and :

$$\begin{aligned}\Psi_2(R_t) &= R_t^\alpha I_{-\alpha}(\lambda R_t)=\int_0^t \mathcal{L}\Psi_2(R_s)ds+M_2(t)\\ &= \frac{\lambda^2}{2}\int_0^t R_s^\alpha I_{-\alpha}(\lambda R_s)ds+M_2(t),\end{aligned} \tag{5.22}$$

where $(M_2(t), t \geq 0)$ is a P_0 local martingale.
As $\Psi_1 \notin \mathcal{D}$, we write :

$$\Psi_1(r) = r^\alpha I_\alpha(\lambda r) = \left(\frac{\lambda}{2}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} r^{2\alpha} + \widetilde{\Psi}_1(r), \quad r > 0. \tag{5.23}$$

Relation (5.16) implies that $\widetilde{\Psi}_1 \in \mathcal{D}$. Moreover $\mathcal{L}\Psi_1 = \mathcal{L}\widetilde{\Psi}_1$, then $\big(\widetilde{\Psi}_1(R_t) - \int_0^t \mathcal{L}\Psi_1(R_s)ds\big)$ is a local martingale.
Since from (2.3), $(R_t^{2\alpha} - L_t, t \geq 0)$ is a martingale, then, with the help of (5.21), we get :

$$\Psi_1(R_t) = R_t^\alpha I_\alpha(\lambda R_t) = \left(\frac{\lambda}{2}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} L_t + \frac{\lambda^2}{2} \int_0^t R_s^\alpha I_\alpha(\lambda R_s) ds + M_1(t), \tag{5.24}$$

where $(M_1(t), t \geq 0)$ is a P_0 local martingale.
3. b) We are now able to prove that $(M_t^{\lambda,\tilde{h}})$ is a P_0 local martingale.
With the help of (1.34), (5.18),(5.22) and (5.24), we deduce, from Itô's formula, that :

$$\begin{aligned} dM_t^{\lambda,\tilde{h}} \quad = \quad & -\frac{\lambda^2}{2} M_t^{\lambda,\tilde{h}} dt + e^{-\frac{\lambda^2 t}{2}} \bigg\{ \left(\frac{2}{\lambda}\right)^\alpha \Gamma(1+\alpha) \tilde{h}'(L_t) \Psi_1(R_t) dL_t \\ & + \left(\frac{2}{\lambda}\right)^\alpha \Gamma(1+\alpha) \tilde{h}(L_t) \Big[\left(\frac{\lambda}{2}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} dL_t + \frac{\lambda^2}{2} R_t^\alpha I_\alpha(\lambda R_t) dt \Big] \bigg\} \\ & + e^{-\frac{\lambda^2 t}{2}} \left(\frac{\lambda}{2}\right)^\alpha \Gamma(1-\alpha) \Big\{ -\tilde{h}(L_t) \Psi_2(R_t)) dL_t + \big(1 - \widetilde{H}(L_t)\big) R_t^\alpha I_{-\alpha}(\lambda R_t) dt \Big\} \\ & + dM_3(t), \end{aligned} \tag{5.25}$$

where $(M_3(t))$ is a local martingale.
It is clear that (5.17) implies that :

$$\Psi_1(0) = 0, \quad \Psi_2(0) = \left(\frac{2}{\lambda}\right)^\alpha \frac{1}{\Gamma(1-\alpha)}. \tag{5.26}$$

Using moreover (5.14), it is easy to verify that, in (5.25), both the terms in (dt) and those in (dL_t) are equal to 0.
Note that the relations (5.26) and (5.18) force $M_0^{\lambda,\tilde{h}} = 1$.
4) We now prove point 3. (a) of Theorem 1.5.
Indeed, for every t and $c > 0$, one has :

$$Q_0^{(\lambda,\tilde{h})}(L_t > c) = Q_0^{(\lambda,\tilde{h})}(\tau_c < t) = E_0\big[1_{(\tau_c < t)} M_t^{\lambda,\tilde{h}}\big].$$

Using successively Doob's optional stopping theorem and the property : $M_{\tau_c}^{\lambda,\tilde{h}} = e^{-\frac{\lambda^2}{2}\tau_c}(1 - \widetilde{H}(c))$, we obtain :

$$Q_0^{(\lambda,\tilde{h})}(L_t > c) = E_0\big[1_{(\tau_c < t)} M_{\tau_c}^{\lambda,\tilde{h}}\big] = \big(1 - \widetilde{H}(c)\big) E_0\big[1_{(\tau_c < t)} e^{-\frac{\lambda^2}{2}\tau_c}\big].$$

Letting $t \to \infty$ in the expression above and using (2.12) and (1.29) leads to :

$$Q_0^{(\lambda,\tilde{h})}(L_\infty > c) = \big(1 - \widetilde{H}(c)\big) E_0\big[e^{-\frac{\lambda^2}{2}\tau_c}\big] = \big(1 - \widetilde{H}(c)\big) e^{-c\sigma_\lambda}.$$

In order to end the proof of Theorem 1.5 (i.e. points 3. (b) and (c)), we shall use the technique of progressive enlargement of filtrations, with respect to $g = \sup\{t \geq 0 : R_t = 0\}$. Thus, we define $(\mathcal{G}_t, t \geq 0)$ to be the smallest filtration which contains $(\mathcal{F}_t, t \geq 0)$ and which makes g a $(\mathcal{G}_t)_{t\geq 0}$ stopping time. In order to use the enlargement formulae (see for instance [6] or [22]), it is necessary (cf (5.50)) to compute the $\big((\mathcal{F}_t), Q_0^{(\lambda,\tilde{h})}\big)$ supermartingale :

$$Z_t := Q_0^{(\lambda,\tilde{h})}(g > t|\mathcal{F}_t). \tag{5.27}$$

We determine Z_t in the next Lemma 5.1.

Lemma 5.1 *We have :*

$$Z_t = \frac{2^{1-\alpha}}{\Gamma(\alpha)} e^{-\frac{\lambda^2 t}{2}} \Big(1 - \widetilde{H}(L_t)\Big) (\lambda R_t)^\alpha K_\alpha(\lambda R_t) \frac{1}{M_t^{\lambda,\tilde{h}}}, \tag{5.28}$$

$$Q_0^{(\lambda,\tilde{h})}(g < \infty) = 1. \tag{5.29}$$

Proof of Lemma 5.1 1) For any $\Gamma_t \in \mathcal{F}_t$ we compute :

$$E_{Q_0^{(\lambda,\tilde{h})}}\big[1_{\Gamma_t} 1_{\{g>t\}}\big] = E_{Q_0^{\lambda,\tilde{h}}}\big[1_{\Gamma_t} 1_{\{d_t<\infty\}}\big] = E_0\big[1_{\Gamma_t} 1_{\{d_t<\infty\}} M_{d_t}^{\lambda,\tilde{h}}\big],$$

where $d_t = \inf\{s > t; R_s = 0\}$ is the first time of visit of 0 after time t.
Since $M_{d_t} = \big(1 - \widetilde{H}(L_t)\big) e^{-\frac{\lambda^2}{2} d_t}$, then according to Doob's optional stopping theorem we have :

$$E_{Q_0^{(\lambda,\tilde{h})}}\big[1_{\Gamma_t} 1_{\{g>t\}}\big] = E_0\big[1_{\Gamma_t} 1_{\{d_t<\infty\}} \big(1 - \widetilde{H}(L_t)\big) e^{-\frac{\lambda^2}{2} d_t}\big].$$

Applying the Markov property at time t, we get :

$$\begin{aligned} E_{Q_0^{(\lambda,\tilde{h})}}\big[1_{\Gamma_t} 1_{\{g>t\}}\big] &= e^{-\frac{\lambda^2 t}{2}} E_0\Big[1_{\Gamma_t} \big(1 - \widetilde{H}(L_t)\big) E_{R_t}\big[e^{-\frac{\lambda^2}{2} T_0}\big]\Big] \\ &= e^{-\frac{\lambda^2 t}{2}} E_{Q_0^{(\lambda,\tilde{h})}}\Big[1_{\Gamma_t} \big(1 - \widetilde{H}(L_t)\big) E_{R_t}\big[e^{-\frac{\lambda^2}{2} T_0}\big] \frac{1}{M_t^{\lambda,\tilde{h}}}\Big]. \end{aligned} \tag{5.30}$$

Formula (5.28) now follows immediately from (2.15).
2) Taking $\Gamma_t = \Omega$ in (5.30), we have :

$$Q_0^{(\lambda,\tilde{h})}(g > t) = e^{-\frac{\lambda^2 t}{2}} E_0\Big[\big(1 - \widetilde{H}(L_t)\big) E_{R_t}\big[e^{-\frac{\lambda^2}{2} T_0}\big]\Big] \le e^{-\frac{\lambda^2}{2} t}.$$

Thus, $Q_0^{(\lambda,\tilde{h})}(g < \infty) = 1$; and it is clear that $Q_0^{\lambda,\tilde{h}}(g > 0) = 1$, since the probabilities $Q_0^{(\lambda,\tilde{h})}$ and P_0 are equivalent on each σ-algebra $\mathcal{F}_t$.

■

To obtain the laws of $(R_t, t \le g)$ and $(R_{g+t}, t \ge 0)$, the following lemma constitutes a main step.

Lemma 5.2 *There exists a $\big((\mathcal{G}_t, t \ge 0), Q_0^{(\lambda,\tilde{h})}\big)$ Brownian motion $(W_t, t \ge 0)$, starting from 0, such that :*

$$R_t^{2\alpha} = 2\alpha \int_0^t R_s^{2\alpha-1} dW_s + L_t - 2\alpha\lambda \int_0^{t\wedge g} R_s^{2\alpha-1} \frac{K_{\alpha-1}}{K_\alpha}(\lambda R_s) ds + 2\alpha\lambda \int_{t\wedge g}^t R_s^{2\alpha-1} \frac{I_{\alpha-1}}{I_\alpha}(\lambda R_s) ds. \tag{5.31}$$

Proof of Lemma 5.2 We proceed in 4 steps.

i) In order to simplify our notation, we define :

$$\begin{aligned}
&A_1 := \left(\frac{2}{\lambda}\right)^{\alpha}\Gamma(1+\alpha)I_{\alpha-1}(\lambda R_s) && A_2 := \left(\frac{\lambda}{2}\right)^{\alpha}\Gamma(1-\alpha)I_{1-\alpha}(\lambda R_s)\\
&A_3 := \left(\frac{2}{\lambda}\right)^{\alpha}\Gamma(1+\alpha)I_{\alpha}(\lambda R_s) && A_4 := \left(\frac{\lambda}{2}\right)^{\alpha}\Gamma(1-\alpha)I_{-\alpha}(\lambda R_s)\\
&A_5 := \frac{2^{1-\alpha}}{\Gamma(\alpha)}\lambda^{\alpha+1}K_{\alpha-1}(\lambda R_s) && A_6 := \frac{2^{1-\alpha}}{\Gamma(\alpha)}\lambda^{\alpha}K_{\alpha}(\lambda R_s)\\
&\tilde{h} := \tilde{h}(L_s) && \widetilde{H} := \widetilde{H}(L_s).
\end{aligned} \tag{5.32}$$

Hence, with the help of these notation :

$$M_s^{\lambda,\tilde{h}} = e^{-\frac{\lambda^2 s}{2}} R_s^{\alpha}\big(A_3\tilde{h} + A_4(1-\widetilde{H})\big), \tag{5.33}$$

$$Z_s = \frac{A_6(1-\widetilde{H})}{A_3\tilde{h} + A_4(1-\widetilde{H})}. \tag{5.34}$$

ii) With the help of (2.3), we know that there exists a $((\mathcal{F}_t)_{t\geq 0}, P_0)$ Brownian motion $(B_t, t\geq 0)$ such that :

$$R_t^{2\alpha} = 2\alpha\int_0^t R_s^{2\alpha-1}dB_s + L_t. \tag{5.35}$$

This leads us to write the following function $\Psi_1(r)$ as a function of $r^{2\alpha}$:

$$\Psi_1(r) := r^{\alpha}I_{\alpha}(\lambda r) = \overline{\Psi}_1(r^{2\alpha}), \quad r>0, \tag{5.36}$$

where :

$$\overline{\Psi}_1(x) = \sqrt{x}I_{\alpha}(\lambda x^{\frac{1}{2\alpha}}), \quad x>0, \tag{5.37}$$

Using the first formula of (5.7.9) in ([9], p.110) :

$$\alpha I_{\alpha}(r) + rI'_{\alpha}(r) = rI_{\alpha-1}(r), \tag{5.38}$$

it is easy to prove that :

$$\overline{\Psi}'_1(x) = \frac{\lambda}{2\alpha}x^{\frac{1-\alpha}{2\alpha}}I_{\alpha-1}(\lambda x^{\frac{1}{2\alpha}}), \quad x>0. \tag{5.39}$$

Similarly, introducing :

$$\Psi_2(r) = r^{\alpha}I_{-\alpha}(\lambda r) = \overline{\Psi}_2(r^{2\alpha}), \quad r>0, \tag{5.40}$$

and using the second identity of (5.7.9) in ([9], p.110) :

$$\alpha I_{-\alpha}(r) + rI'_{-\alpha}(r) = rI_{1-\alpha}(r), \tag{5.41}$$

we get :

$$\overline{\Psi}'_2(x) = \frac{\lambda}{2\alpha}x^{\frac{1-\alpha}{2\alpha}}I_{1-\alpha}(\lambda x^{\frac{1}{2\alpha}}), \quad x>0. \tag{5.42}$$

Recall that $(M_t^{\lambda,\tilde{h}})$ is a $((\mathcal{F}_t)_{t\geq 0}, P_0)$ martingale and is given by (5.18). Using the above notation, we have :

$$M_s^{\lambda,\tilde{h}} = e^{-\frac{\lambda^2 s}{2}}\left(\left(\frac{2}{\lambda}\right)^{\alpha}\Gamma(1+\alpha)\tilde{h}\overline{\Psi}_1(R_s^{2\alpha}) + \left(\frac{2}{\lambda}\right)^{-\alpha}\Gamma(1-\alpha)(1-\widetilde{H})\overline{\Psi}_2(R_s^{2\alpha})\right),$$

then Itô's formula and (5.35) imply :

$$\begin{aligned} dM_s^{\lambda,\tilde{h}} &= e^{-\frac{\lambda^2 s}{2}}\Big[\lambda\left(\frac{2}{\lambda}\right)^{\alpha}\Gamma(1+\alpha)\tilde{h}(L_s)R_s^{\alpha}I_{\alpha-1}(\lambda R_s) \\ &\quad +\lambda\left(\frac{2}{\lambda}\right)^{-\alpha}\Gamma(1-\alpha)(1-\widetilde{H}(L_s))R_s^{\alpha}I_{1-\alpha}(\lambda R_s)\Big]dB_s \\ &= \lambda e^{-\frac{\lambda^2 s}{2}}R_s^{\alpha}\left(A_1\tilde{h}+A_2(1-\widetilde{H})\right)dB_s. \end{aligned} \tag{5.43}$$

Next, we consider $\mathcal{N}_s$, the numerator of Z_s, in (5.28), i.e :

$$\mathcal{N}_s := \frac{2^{1-\alpha}}{\Gamma(\alpha)}e^{-\frac{\lambda^2 s}{2}}(1-\widetilde{H}(L_s))\left(\lambda R_s\right)^{\alpha}K_{\alpha}(\lambda R_s) = e^{-\frac{\lambda^2 s}{2}}(1-\widetilde{H})R_s^{\alpha}A_6. \tag{5.44}$$

Since

$$\alpha K_{\alpha}(r)+rK'_{\alpha}(r) = -rK_{\alpha-1}(r), \tag{5.45}$$

reasoning as previously, we can prove that, under P_0, $(\mathcal{N}_s, s\geq 0)$ is a semi-martingale, whose martingale $(\mathbf{M}(\mathcal{N})_s)$ part satisfies :

$$\begin{aligned} d\mathbf{M}(\mathcal{N})_s &= -\frac{2^{1-\alpha}}{\Gamma(\alpha)}\lambda(1-\widetilde{H}(L_s))e^{-\frac{\lambda^2 s}{2}}\left(\lambda R_s\right)^{\alpha}K_{\alpha-1}(\lambda R_s)dB_s \\ &= -(1-\widetilde{H})e^{-\frac{\lambda^2 s}{2}}R_s^{\alpha}A_5 dB_s. \end{aligned} \tag{5.46}$$

Thus, denoting $(\mathbf{M}(Z)_s)$ the martingale part of the semi-martingale $(Z_s, s\geq 0)$ under P_0, and using (5.33), (5.46), (5.44) and (5.43), we get :

$$\begin{aligned} d\mathbf{M}(Z)_s &= \frac{M_s^{\lambda,\tilde{h}}d\mathbf{M}(\mathcal{N})_s - \mathcal{N}_s dM_s^{\lambda,\tilde{h}}}{\left(M_s^{\lambda,\tilde{h}}\right)^2} \\ &= -(1-\widetilde{H})\,\frac{A_5\big(A_3\tilde{h}+A_4(1-\widetilde{H})\big)+\lambda A_6\big(A_1\tilde{h}+A_2(1-\widetilde{H})\big)}{\big(A_3\tilde{h}+A_4(1-\widetilde{H})\big)^2}\,dB_s. \end{aligned} \tag{5.47}$$

iii) According to Girsanov's Theorem , we deduce from (5.43), (5.33), the existence of a $\big((\mathcal{F}_t)_{t\geq 0}, Q_0^{(\lambda,\tilde{h})}\big)$ Brownian motion $\big(\widetilde{B}_t, t\geq 0\big)$ such that :

$$B_t = \widetilde{B}_t + \lambda\int_0^t \frac{A_1\tilde{h}+A_2(1-\widetilde{H})}{A_3\tilde{h}+A_4(1-\widetilde{H})}\,ds. \tag{5.48}$$

Consequently, from (5.47), under $Q_0^{(\lambda,\tilde{h})}$, the martingale part $\big(\mathbf{M}^Q(Z)_t\big)$ of (Z_t) is given by :

$$d\mathbf{M}^Q(Z)_s = -(1-\widetilde{H})\,\frac{A_5\big(A_3\tilde{h}+A_4(1-\widetilde{H})\big)+\lambda A_6\big(A_1\tilde{h}+A_2(1-\widetilde{H})\big)}{\big(A_3\tilde{h}+A_4(1-\widetilde{H})\big)^2}\,d\widetilde{B}_s. \tag{5.49}$$

iv) In this last step, we now use the technique of progressive enlargement of filtrations (see for instance [6], [7], [11] or [22]) under the probability $Q_0^{(\lambda,\tilde{h})}$.
With respect to $(\mathcal{G}_t)$, the smallest filtration containing $(\mathcal{F}_t)$ and such that g is a $(\mathcal{G}_t)$ stopping time, there exists a $\big((\mathcal{G}_t), Q_0^{(\lambda,\tilde{h})}\big)$ Brownian motion $(W_t, t\geq 0)$ starting from 0, such that :

$$\widetilde{B}_t = W_t + \int_0^{t\wedge g}\frac{1}{Z_u}d<Z,\widetilde{B}>_u - \int_{t\wedge g}^{t}\frac{1}{1-Z_u}d<Z,\widetilde{B}>_u\ . \tag{5.50}$$

It is clear that $< Z, \widetilde{B} > = < \mathbf{M}(Z), \widetilde{B} >$. Consequently ,relations (5.48) and (5.49) imply that :

$$d < Z, \widetilde{B} >_s = -(1-\widetilde{H}) \, \frac{A_5\big(A_3\tilde{h} + A_4(1-\widetilde{H})\big) + \lambda A_6\big(A_1\tilde{h} + A_2(1-\widetilde{H})\big)}{\big(A_3\tilde{h} + A_4(1-\widetilde{H})\big)^2} \, ds. \tag{5.51}$$

Combining (5.35), (5.48) and (5.50), we get :

$$\begin{aligned} R_t^{2\alpha} = L_t \quad &+ \quad 2\alpha \int_0^t R_s^{2\alpha-1} dW_s + 2\alpha\lambda \int_0^t \frac{A_1\tilde{h} + A_2(1-\widetilde{H})}{A_3\tilde{h} + A_4(1-\widetilde{H})} R_s^{2\alpha-1} \, ds \\ &- \quad 2\alpha \int_0^{t\wedge g} \frac{A_5\big(A_3\tilde{h} + A_4(1-\widetilde{H})\big) + \lambda A_6\big(A_1\tilde{h} + A_2(1-\widetilde{H})\big)}{A_6(A_3\tilde{h} + A_4(1-\widetilde{H}))} R_s^{2\alpha-1} \, ds \\ &+ \quad 2\alpha \int_{t\wedge g}^t (1-\widetilde{H}) \, \frac{A_5(A_3\tilde{h} + A_4(1-\widetilde{H})) + \lambda A_6(A_1\tilde{h} + A_2(1-\widetilde{H}))}{\big(A_3\tilde{h} + A_4(1-\widetilde{H})\big)\big(A_3\tilde{h} + A_4(1-\widetilde{H})) - A_6(1-\widetilde{H})\big)} R_s^{2\alpha-1} ds, \end{aligned}$$

where we have used (5.34) and :

$$\frac{1}{1-Z_s} = \frac{A_3\tilde{h} + A_4(1-\widetilde{H})}{A_3\tilde{h} + A_4(1-\widetilde{H}) - A_6(1-\widetilde{H})}.$$

Hence :

$$R_t^{2\alpha} = L_t + 2\alpha \int_0^t R_s^{2\alpha-1} dW_s + 2\alpha \int_0^{t\wedge g} R_s^{2\alpha-1} \varphi_1(s) ds) + 2\alpha \int_{t\wedge g}^t R_s^{2\alpha-1} \varphi_2(s) ds, \tag{5.52}$$

with :

$$\begin{aligned} \varphi_1(s) \quad &= \quad \lambda \frac{A_1\tilde{h} + A_2(1-\widetilde{H})}{A_3\tilde{h} + A_4(1-\widetilde{H})} - \frac{A_5\big(A_3\tilde{h} + A_4(1-\widetilde{H})\big) + \lambda A_6\big(A_1\tilde{h} + A_2(1-\widetilde{H})\big)}{A_6\big(A_3\tilde{h} + A_4(1-\widetilde{H})\big)} \\ &= \quad -\frac{A_5}{A_6} = -\lambda \, \frac{K_{\alpha-1}(\lambda R_s)}{K_\alpha(\lambda R_s)}, \end{aligned} \tag{5.53}$$

and :

$$\begin{aligned} \varphi_2(s) \quad &= \quad \lambda \frac{A_1\tilde{h} + A_2(1-\widetilde{H})}{A_3\tilde{h} + A_4(1-\widetilde{H})} + (1-\widetilde{H}) \, \frac{A_5(A_3\tilde{h} + A_4(1-\widetilde{H})) + \lambda A_6(A_1\tilde{h} + A_2(1-\widetilde{H}))}{(A_3\tilde{h} + A_4(1-\widetilde{H}))(A_3\tilde{h} + A_4(1-\widetilde{H})) - A_6(1-\widetilde{H})} \\ &= \quad \frac{\lambda A_1\tilde{h} + (1-\widetilde{H})(\lambda A_2 + A_5)}{A_3\tilde{h} + (A_4 - A_6)(1-\widetilde{H})}. \end{aligned} \tag{5.54}$$

Using (5.15) we have :

$$A_4 - A_6 = \left(\frac{\lambda}{2}\right)^\alpha \Gamma(1-\alpha) I_{-\alpha}(\lambda R_s) - \frac{2^{1-\alpha}}{\Gamma(\alpha)} \lambda^\alpha K_\alpha(\lambda R_s) = \left(\frac{\lambda}{2}\right)^\alpha \Gamma(1-\alpha) I_\alpha(\lambda R_s).$$

Similarly :

$$\begin{aligned} \lambda A_2 + A_5 \quad &= \quad \lambda \left(\frac{\lambda}{2}\right)^\alpha \Gamma(1-\alpha) I_{1-\alpha}(\lambda R_s) + \frac{2^{1-\alpha}}{\Gamma(\alpha)} \lambda^{\alpha+1} K_{\alpha-1}(\lambda R_s) \\ &= \quad \lambda \left(\frac{\lambda}{2}\right)^\alpha \Gamma(1-\alpha) I_{\alpha-1}(\lambda R_s) = \lambda (A_4 - A_6) \frac{A_1}{A_3}. \end{aligned}$$

Hence :

$$\varphi_2(s) = \frac{\lambda A_1\tilde{h} + (1-\widetilde{H})\lambda(A_4 - A_6)\frac{A_1}{A_3}}{A_3\tilde{h} + (A_4 - A_6)(1-\widetilde{H})} = \frac{\lambda A_1}{A_3} = \lambda \frac{I_{\alpha-1}(\lambda R_s)}{I_\alpha(\lambda R_s)}. \tag{5.55}$$

It is clear that plugging (5.53) and (5.55) in (5.52) proves Lemma 5.2.

■

4) Proof of point 3. (c) of Theorem 1.5.

4. a) First, we study $(R_t, t \leq g)$ under $Q_0^{(\lambda,h)}$.

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function of class C^2, whose support does not contain 0. We apply Itô's formula to (5.31) and with the function $g(x) = f(x^{1/2\alpha})$.

We compute the two first derivatives of g in terms of those of f :

$$\begin{aligned} g'(x) &= f'(x^{1/2\alpha})\frac{1}{2\alpha}x^{\frac{1-2\alpha}{2\alpha}} \\ g''(x) &= f''(x^{1/2\alpha})\frac{1}{4\alpha^2}x^{\frac{2(1-2\alpha)}{2\alpha}} + \frac{1-2\alpha}{4\alpha^2}f'(x^{1/2\alpha})x^{\frac{1-4\alpha}{2\alpha}}. \end{aligned}$$

Consequently, we get, for $t < g$:

$$\begin{aligned} f(R_t) &= g(R_t^{2\alpha}) = \int_0^t f'(R_s)\frac{1}{2\alpha}R_s^{1-2\alpha}\Big[2\alpha R_s^{2\alpha-1}\big(dW_s - \frac{\lambda K_{\alpha-1}}{K_\alpha}(\lambda R_s)ds\big)\Big] \\ &\quad + \frac{1}{2}\int_0^t \Big[f''(R_s)\frac{1}{4\alpha^2}R_s^{2(1-2\alpha)} + \frac{1-2\alpha}{4\alpha^2}f'(R_s)R_s^{1-4\alpha}\Big]4\alpha^2 R_s^{2(2\alpha-1)}ds \\ &= \int_0^t f'(R_s)dW_s + \int_0^t \Big[\frac{1}{2}f''(R_s) + f'(R_s)\Big(\frac{1-2\alpha}{2R_s} - \frac{\lambda K_{\alpha-1}}{K_\alpha}(\lambda R_s)\Big)\Big]ds. \end{aligned}$$

This proves that the process $(R_t, t \leq g)$ admits the infinitesimal generator :

$$\mathcal{L}^{\downarrow} f(r) = \frac{1}{2}f''(r) + \Big(\frac{1-2\alpha}{2r} - \frac{\lambda K_{\alpha-1}}{K_\alpha}(\lambda r)\Big)f'(r).$$

4. b) In this last step we focus on $(R_{g+t}, t \geq 0)$.

It can be proved analogously, that the infinitesimal generator of this process is $\mathcal{L}^{\uparrow}$ (this operator is defined by (1.37)). The independence of the processes $(R_t, t \leq g)$ and $(R_{g+t}, t \geq 0)$ follows from the fact that the stochastic differential equation :

$$\widetilde{R}_t = W_t + \int_0^t \left[\frac{1-2\alpha}{2\widetilde{R}_s} + \frac{\lambda I_{\alpha-1}}{I_\alpha}(\lambda \widetilde{R}_s)\right] ds$$

admits a unique strong solution.

Note that from (5.16) :

$$\frac{1-2\alpha}{2r} + \lambda\frac{I_{\alpha-1}}{I_\alpha}(\lambda r) \underset{r\to 0}{\sim} \frac{1}{2r}(1-2\alpha+4\alpha) = \frac{1+2\alpha}{2r}.$$

Then, the process $(R_{g+t}, t \geq 0)$ behaves, near 0, as a Bessel process with dimension : $\delta = 2(1+\alpha) = 4-d$. In particular, starting from the origin, it immediately leaves 0, and never comes back to it (since $\alpha > 0$, hence $\delta > 2$).

Remark 5.3 *We note that, most likely, the description we have just given of the $Q_0^{(\lambda,h)}$-process may be reproduced for a "general" diffusion. The role of the functions $r^\alpha I_\alpha(\lambda r)$ and $r^\alpha I_{-\alpha}(\lambda r)$ being then played by two linearly independent eigenfunctions of the infinitesimal generator. (See, e.g, [14] for a general framework).*

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