Statistical and renewal results for the random sequential absorption model applied to a unidirectional multicracking problem

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Abstract

We work out a stationary process on the real line to represent the positions of the multiple cracks which are observed in some composites materials submitted to a fixed unidirectional stress ε. Our model is the one-dimensional Random Sequential Adsorption. We calculate the intensity of the process and the distribution of the inter-crack distance in the Palm sense. Moreover, the successive crack positions of the one-sided process (denoted by X_i^ε, $i \geq 1$) are described. We prove that the sequence $\{(X_i^\varepsilon, Y_i^\varepsilon), 1 \leq i \leq n\}$ is a "conditional renewal process", where Y_i^ε is the value of the stress at which X_i^ε forms. The approaches "in the Palm sense" and "one-sided process" merge when $n \to +\infty$. The saturation case ($\varepsilon = +\infty$) is also investigated.

Introduction

Let $\{X_i; i \geq 1\}$ be a sequence of independent and uniformly distributed variables on the segment $[0, L]$, $L > 0$. We throw successively $X_1, \cdots, X_N$ on this segment, keeping only some of them according to the following procedure. For $N \geq 2$, we keep X_1 and after that, we erase X_2 if and only if X_2 is in the interval of radius $r > 0$ around X_1. Once we have decided if $X_2, \cdots, X_n$, $2 \leq n \leq (N-1)$ are kept or not, we erase X_{n+1} if it belongs to the union of all the intervals centered on the non-erased points, with length $2r$.

This construction is known as the one-dimensional random sequential adsorption (RSA) [3], [2]. In spite of its simplicity, this model is difficult to deal with: in particular, the law of the number of preserved points is unknown.

In 1958, Rényi [18] worked out a model where the points are placed on the segment up to saturation (i.e. when no more point can be added). He obtained the asymptotic behaviour of the mean number of points in $[0, L]$ when L goes to infinity. This question, known as the car-parking problem, has been largely investigated (see for example [4], [5], [11], [16], [17]).

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In 1966, fixing the number N of thrown variables, Widom [21] demonstrated by heuristic methods that the mean number of points which are separated from their right-neighbor by a fixed length $l > 0$ satisfies a differential equation in l. Moreover, he provided formulas for the empirical distribution function of the inter-point distance when $N, L \to +\infty$, with N/L fixed.

In this paper, we are interested mainly in modelling a unidirectional multicracking phenomenon of brittle coatings. A uniaxial strain is applied to a specimen consisting of a ductile substrate covered with a brittle coating. The applied strain is supposed to result in the coating in a regularly increasing stress denoted by ε, which leads to the formation of cracks parallel and orthogonal to the stress direction [7],[12]. Consequently, the geometrical aspect of the problem reduces practically to the intersections of the cracks with a fixed line parallel with the stress axis. It has been observed [1],[7],[10],[12] that the formation of a crack in the coating results in a relaxation of the stress in the vicinity of this crack so that no new crack can form close to an existing crack because of the smallness of the stress in this area. Consequently, the above-described RSA construction can be considered as a model for the crack positions.

More precisely, we construct through the RSA procedure a one-dimensional stationary point process Λ_ε, that represents the positions of cracks for a fixed value of the applied stress $\varepsilon > 0$. The parameter ε plays a central role in the model. In particular, the limit $\varepsilon \to +\infty$ corresponds to saturation.

The first section of our paper is devoted to the construction of Λ_ε. We start with a two-dimensional Poisson process Φ on $\mathbb{R} \times \mathbb{R}_+$ of intensity measure $\mathbf{1}_{\mathbb{R}_+}(y) f(y) dx dy$. In the physics literature, f is called (see e.g. [13]) rupture probability density of the coating. It is a non-decreasing function and therefore expresses the fact that the number of cracks grows with stress. From a mathematical point of view, there is no loss of generality in assuming that $f = 1$ (see the beginning of Section 1 for details).

By an erasing procedure, we construct a subset Ψ of Φ such that for every point $(x, y) \in \Psi$, the first and second coordinates represent respectively the position of a crack and the exact stress level at which it forms. For every $\varepsilon > 0$,

$$\Lambda_\varepsilon = \{x \in \mathbb{R}; \exists y \in [0, \varepsilon] \mid (x, y) \in \Psi\},$$

is the projection on the x-axis of $\Psi \cap (\mathbb{R} \times [0, \varepsilon])$.

In Section 2, we demonstrate that the process Λ_ε is stationary. In particular, the mean crack number λ_ε and the couple $(D_\varepsilon, L_\varepsilon)$ of the typical inter-crack distance and the stress level in the Palm sense are precisely defined, and a different notion of inter-crack distance I_0^ε is given. The results are expressed through two unknown functions, G and H.

We demonstrate in Section 3 that the function G satisfies an integral equation that can be solved, which allows us to determine the function H.

Let us denote by α the function on $\mathbb{R}_+$ defined by

$$\alpha(s) = \exp\left\{-\int_0^{rs} \frac{1 - e^{-t}}{t} dt\right\}, \quad s \geq 0. \tag{1}$$

Precise formulas for λ_ε and the distribution of $(D_\varepsilon, L_\varepsilon)$ (resp. I_0^ε) can thus be obtained:

Theorem 1 *We have*

(i) $\lambda_\varepsilon = \int_0^\varepsilon \alpha(v)^2 dv$;

(ii) The distribution of D_ε *has a density* $\varphi_{D\varepsilon}$ *on* $[r, +\infty)$ *such that*

$$\varphi_{D_\varepsilon}(x) = \begin{cases} \frac{\varepsilon^2}{\lambda_\varepsilon} \alpha(\varepsilon)^2 e^{-(x-2r)\varepsilon} & \text{if } x > 2r \\ \frac{2}{\lambda_\varepsilon} \int_0^\varepsilon e^{-(x-r)v} \alpha(v)^2 v dv & \text{if } r \leq x \leq 2r. \end{cases}$$

(iii) L_ε *has a density* φ_{L_ε} *such that*

$$\varphi_{L_\varepsilon}(y) = \frac{1}{\lambda_\varepsilon}\alpha(y)^2 \mathbf{1}_{[0,\varepsilon]}(y); \tag{2}$$

(iv) The distribution of D_ε *conditionally on* L_ε *has a density* $\Pi^{D_\varepsilon}(L_\varepsilon;\cdot)$ *such that for every* $y \in [0,\varepsilon]$, $u \geq 0$,

$$\Pi^{D_\varepsilon}(y;u) = \mathbf{1}_{\{u>2r\}}\frac{\varepsilon\alpha(\varepsilon)}{\alpha(y)}e^{-ry}e^{-(u-2r)\varepsilon} \\ +\mathbf{1}_{\{r\leq u\leq 2r\}}\left\{ye^{-(u-r)y} + \frac{e^{-ry}}{\alpha(y)}\int_y^\varepsilon e^{-(u-r)v}\alpha(v)dv\right\}. \tag{3}$$

In order to understand more deeply the process Λ_ε, an alternative point of view is considered in Section 4, i.e. we describe the points X_i^ε of the process on the positive half-line:

$$0 < X_1^\varepsilon < X_2^\varepsilon < \cdots < X_n^\varepsilon, \quad n \geq 1.$$

We denote by Y_n^ε the corresponding stress level of the crack position X_n^ε, $n \geq 1$.

Because of the complexity of the erasing procedure, $(X_n^\varepsilon)_{n\geq 1}$ is not a renewal process. We call it a "conditional renewal process" since we show in Theorem 2 below, that for every $n \geq 1$, $\{X_i^\varepsilon, 1 \leq i \leq n\}$ coincides with the first n points of a renewal process conditioned on some explicit event.

More precisely, let us introduce three probability densities on $\mathbb{R}_+$:

$$\varphi_\eta(x) = \frac{\varepsilon}{r\varepsilon+1}(\mathbf{1}_{[0,r]}(x) + \mathbf{1}_{(r,+\infty)}(x)e^{-(x-r)\varepsilon}), \tag{4}$$

$$\varphi_\rho(x) = \frac{1}{\varepsilon}\mathbf{1}_{[0,\varepsilon]}(x), \tag{5}$$

$$\varphi_{\rho'}(x) = \frac{1}{\int_0^\varepsilon \alpha(u)du}\mathbf{1}_{[0,\varepsilon]}(x)\alpha(x). \tag{6}$$

Theorem 2 *Let* $\{\xi_i; i\geq 1\}$, $\{\eta_i; i \geq 1\}$, $\{\rho_i; i\geq 1\}$ *and* $\{\rho_i'; i \geq 1\}$ *be four mutually independent sequences of i.i.d. variables such that* ξ_1 *is an exponential variable with mean 1 and the distribution of* η_1 *(resp.* ρ_1, ρ_1'*) has the density* φ_η *(resp.* φ_ρ, $\varphi_{\rho'}$*).*

Besides, let us consider the events

$$B_n = \{\xi_n \geq (\eta_n \wedge r)(\rho_n \vee \rho_{n-1}) + r(\rho_n \wedge \rho_{n-1})\},$$
$$B_n' = \{\xi_n \geq (\eta_n \wedge r)(\rho_n' \vee \rho_{n-1}) + r(\rho_n' \wedge \rho_{n-1})\},$$

with the convention $\rho_0 = 0$ *a.s.. Then*
(i) The vector $(X_1^\varepsilon, Y_1^\varepsilon)$ *is distributed as* (η_1, ρ_1') *conditioned on* B_1'*;*
(ii) For every $n \geq 2$*, the vector*

$$(X_1^\varepsilon, Y_1^\varepsilon, X_2^\varepsilon - X_1^\varepsilon, Y_2^\varepsilon, \cdots, X_n^\varepsilon - X_{n-1}^\varepsilon, Y_n^\varepsilon)$$

is distributed as $(\eta_1, \rho_1, r+\eta_2, \rho_2, \cdots, r+\eta_{n-1}, \rho_{n-1}, r+\eta_n, \rho_n')$ *conditioned on the event*

$$C_n = \bigcap_{i=1}^{n-1} B_i \cap B_n'.$$

Theorem 2 provides an algorithm to simulate the successive positions of the cracks (see Remark 20).

We also prove that both $(X_{n+1}^{\varepsilon} - X_n^{\varepsilon}, Y_n^{\varepsilon})_{n\geq 1}$ and $(Y_n^{\varepsilon})_{n\geq 1}$ are Markov chains and the initial distribution, transition probability density and invariant probability measure are determined for each of them. We verify that the process $(X_{n+1}^{\varepsilon} - X_n^{\varepsilon}, Y_n^{\varepsilon})_{n\geq 1}$ converges to its invariant probability measure (i.e. the law of $(D_\varepsilon, L_\varepsilon)$, see Theorem 1).

Section 5 presents the saturation case already considered by Renyi. We define directly the process Λ_∞ associated to saturation as well as its associated quantities (λ_∞, D_∞, L_∞, X_i^∞, Y_i^∞, $i \geq 1$), then we demonstrate that λ_ε tends to λ_∞ as $\varepsilon \to +\infty$ and D_ε (resp. I_0^ε, L_ε) converges in distribution to D_∞ (resp. I_0^∞, L_∞). We also give a complete description of the distribution of $(X_n^\infty, Y_n^\infty)_{n\geq 1}$ and a result of convergence in law of $(X_{n+1}^\infty - X_n^\infty, Y_n^\infty)$ to (D_∞, L_∞).

1 A stationary model with relaxation of stress

In this section, we define a stationary process Λ_ε on $\mathbb{R}$ that represents the crack positions for a given stress ε on the assumption that the stress is relaxed on an interval of radius $r > 0$ around every existing crack.

To this end, we introduce a two-dimensional point process Ψ on $\mathbb{R} \times \mathbb{R}_+$ such that the first and the second coordinates of a point of Ψ represent respectively the position of a crack and the stress level at which the crack forms. Λ_ε is the projection on the x-axis of $\Psi \cap (\mathbb{R} \times [0, \varepsilon])$:

$$\Lambda_\varepsilon = \{x \in \mathbb{R}; \exists y \in [0, \varepsilon] \mid (x, y) \in \Psi\}. \tag{7}$$

Considering a two-dimensional point process is a convenient way to order the crack positions as in the case of the segment $[0, L]$, by associating with any position an “arrival time” of the crack. To define Ψ, we start with the process Φ associated with the cracking phenomenon without stress relaxation.

Φ is a Poisson point process on $\mathbb{R} \times \mathbb{R}_+$, with intensity measure $\nu(dx, dy) = \mathbf{1}_{\mathbb{R}_+}(y)dxdy$. To take into account the physical reality of the cracking process, Φ should be a Poisson point process with intensity measure $f(y)\mathbf{1}_{\mathbb{R}_+}(y)dxdy$ where f is a positive continuous and non-decreasing function on $\mathbb{R}_+$. However, in that case, the random set $\Phi^f = \{(x, F(y)),\ (x, y) \in \Phi\}$ (where $F(y) = \displaystyle\int_0^y f(t)dt,\ y \geq 0$) is a Poisson point process of intensity measure $\mathbf{1}_{\mathbb{R}_+}(y)dxdy$ and the erasing procedure applied to either Φ^f or Φ is the same. Therefore from a mathematical point of view, we can suppose, without any loss of generality, that Φ is a homogeneous Poisson point process.

For any point $(x, y) \in \mathbb{R} \times \mathbb{R}_+$, we define the corresponding domain of relaxation:

$$R(x, y) = [x - r, x + r] \times [y, +\infty) \subset \mathbb{R} \times \mathbb{R}_+. \tag{8}$$

Let us introduce Ψ. This random process is a sub-process of Φ defined by the following recursive algorithm.

Initialization. We start with taking any couple (x, y) in Ψ, such that y is a local minimum, i.e.

$$\Phi \cap ([x - r, x + r] \times [0, y)) = \emptyset.$$

Let us denote by Ψ_1 the set of these points and by Φ_1 the subset of Φ obtained by erasing all the points that are in the domains of relaxation associated to the points of Ψ_1. This means

$$\Phi_1 = \Phi \cap \left(\cup_{(x,y)\in\Psi_1} R(x, y)\right)^c.$$

Iteration. Suppose that for a fixed $n \in \mathbb{N}^*$, the processes $\Phi_1, \cdots, \Phi_n$ and $\Psi_1, \cdots, \Psi_n$ are constructed.
We then take in Ψ_{n+1} the points (x, y) of Φ_n such that y is a local minimum. We define Φ_{n+1} as the set of the points of Φ_n not erased by the domains of relaxation associated to the points of Ψ_{n+1}. In mathematical terms,

$$\begin{cases} \Psi_{n+1} = \{(x, y) \in \Phi_n; \Phi_n \cap [x - r, x + r] \times [0, y) = \emptyset\} \\ \Phi_{n+1} = \Phi_n \cap \left(\cup_{(x,y)\in\Psi_{n+1}} R(x, y)\right)^c \end{cases}$$

We then define

$$\Psi = \cup_{n\geq 1}\Psi_n. \tag{9}$$

From now on the points of Ψ will be named *erasers*, and the points of $\Phi \setminus \Psi$ that are deleted by the domains of relaxation associated to the erasers, will be named *erased points*. So

$$\Phi \setminus \Psi = \{(x, y) \in \Phi; \exists (x', y') \in \Psi \mid (x, y) \in R(x', y')\}.$$

The point process Ψ can also be seen as the complementary set in Φ of the *erasing tree* $\mathcal{A}(\Phi)$, where

$$\mathcal{A}(\Phi) = \cup_{(x,y)\in\Psi}([x - r, x + r] \times (y, +\infty)). \tag{10}$$

We say that a point of $\mathbb{R} \times \mathbb{R}_+$ is *erased* if it is contained in the erasing tree $\mathcal{A}(\Phi)$.

The first properties of Ψ are stated in the following proposition.

Proposition 3 **(i)** *Almost surely the projections of the points of* Ψ *on the* x*-axis are separated by a distance at least equal to* r*.*
(ii) Ψ *is infinite a.s..*
(iii) Ψ *is invariant under horizontal translations.*
(iv) Ψ *is ergodic.*

Proof. (i) Let us consider two points $(x, y), (x', y') \in \Psi$ and suppose that

$$(x, y) \in \Psi_n \text{ and } (x', y') \in \Psi_m, \quad m \geq n.$$

Then $(x', y') \notin R(x, y)$ so $|x' - x| > r$.
(ii) It suffices to show that Ψ_0 is infinite. Let us denote by C_n, $n \in \mathbb{Z}$, the event such that "the minimum of the second coordinates of the points of $\Phi \cap [3nr, 3(n + 1)r] \times \mathbb{R}_+$ is reached at a point of $[(3n + 1)r, (3n + 2)r] \times \mathbb{R}_+$".

Let us remark that

$$C_n \subset \{\Psi_0 \cap [(3n + 1)r, (3n + 2)r] \times \mathbb{R}_+ \neq \emptyset\}, \quad n \in \mathbb{Z}.$$

Since Φ is a Poisson point process, the events C_n are mutually independent and have the same positive probability. So using the Borel-Cantelli lemma leads to

$$\mathbf{P}\{\limsup C_n\} = 1,$$

which proves that Ψ_0 is infinite.
(iii) Let us consider the set $\mathcal{M}_\sigma(\mathbb{R}^2)$ of the locally finite sequences of $\mathbb{R}^2$, endowed with the $\sigma-$field generated by the applications $\phi \longmapsto \#(\phi \cap A)$, $\phi \in \mathcal{M}_\sigma(\mathbb{R}^2)$, where $A \in \mathcal{B}(\mathbb{R}^2)$. We define for every $x \in \mathbb{R}$,

$$T^x : \begin{cases} \mathcal{M}_\sigma(\mathbb{R}^2) & \longrightarrow & \mathcal{M}_\sigma(\mathbb{R}^2) \\ \{(x_i, y_i)\}_{i\geq 1} & \longmapsto & \{(x_i + x, y_i)\}_{i\geq 1} \end{cases}$$

We see immediately that Ψ as well as Φ is invariant in law under the applications T^x, $x \in \mathbb{R}$.
(iv) To prove the ergodicity, let us show that Ψ is strongly mixing for the applications T^x, i.e. every couple $(\mathcal{A}, \mathcal{B})$ of measurable sets of $\mathcal{M}_\sigma(\mathbb{R}^2)$ satisfies

$$\mathbf{P}\{\mathcal{A} \cap T^x(\mathcal{B})\} \underset{|x|\to+\infty}{\longrightarrow} \mathbf{P}\{\mathcal{A}\} \cdot \mathbf{P}\{\mathcal{B}\}. \tag{11}$$

Let us remark (see [9]) that the sets $\{\Psi \cap K = \emptyset\}$, where K runs throughout the compact sets of $\mathbb{R}^2$, generate the σ-field of $\mathcal{M}_\sigma(\mathbb{R}^2)$. Consequently, it suffices to prove the convergence in (11) when $\mathcal{A} = \{\Psi \cap A = \emptyset\}$ and $\mathcal{B} = \{\Psi \cap B = \emptyset\}$, A, B being two compact sets of $\mathbb{R}^2$.

Since Ψ is invariant by $T^{x/2}$, we have

$$\begin{aligned}\mathbf{P}\{(\Psi \cap A = \emptyset) \cap (T^{-x}(\Psi) \cap B = \emptyset)\} &= \mathbf{P}\{(T^{x/2}(\Psi) \cap A = \emptyset) \cap (T^{-x/2}(\Psi) \cap B = \emptyset)\} \\ &= \mathbf{P}\{(\Psi \cap (A - x/2) = \emptyset) \cap (\Psi \cap (B + x/2) = \emptyset)\}. \end{aligned} \tag{12}$$

In order to prove the asymptotic independence of $\{\Psi\cap(A-x/2)) = \emptyset\}$ and $\{\Psi\cap(B+x/2)) = \emptyset\}$, we are going to rewrite these two events with the independent processes Φ^+ and Φ^- defined by

$$\Phi^+ = \Phi \cap (\mathbb{R}_+ \times \mathbb{R}_+) \quad \text{and} \quad \Phi^- = \Phi \cap (\mathbb{R}_- \times \mathbb{R}_+). \tag{13}$$

Let Z_+ (respectively Z_-) be the minimum (respectively the maximum) of the first coordinates of the points of Ψ_0 contained in the domain $[r, +\infty) \times \mathbb{R}_+$ (respectively $(-\infty, -r] \times \mathbb{R}_+$). In other words, Z_+ (respectively Z_-) is the minimal (respectively maximal) first coordinate of the points of $\Phi^+ \cap [r, +\infty) \times \mathbb{R}_+$ (respectively $\Phi^- e \cap (-\infty, -r] \times \mathbb{R}_+$) such that $\Phi^+ \cap [x-r, x+r] \times [0, y) = \emptyset$ (respectively $\Phi^- \cap [x-r, x+r] \times [0, y) = \emptyset$). So Z_+ and Z_- are two independent variables.

Besides, let $\mathcal{A}(\Phi^+)$ (resp. $\mathcal{A}(\Phi^-)$) be the erasing tree of Φ_+ (resp. of Φ_-). We then have

$$\mathcal{A}(\Phi^+) \cap ([Z_+, +\infty) \times \mathbb{R}_+) = \mathcal{A}(\Phi) \cap ([Z_+, +\infty) \times \mathbb{R}_+), \tag{14}$$

and

$$\mathcal{A}(\Phi^-) \cap ((-\infty, Z_-] \times \mathbb{R}_+) = \mathcal{A}(\Phi) \cap ((-\infty, Z_-] \times \mathbb{R}_+). \tag{15}$$

For $n \in \mathbb{N}$, let x be such that $x/2 + \inf p_1(B) > n$ and $-x/2 + \sup p_1(A) < -n$, where $p_1(A)$ denotes the projection on the x-axis of A. Let us consider the events

$$E_n^+ = \{Z_+ \in [0, n]; \Psi \cap (B + x/2) = \emptyset\} \quad \text{and } E_n^- = \{Z_- \in [-n, 0]; \Psi \cap (A - x/2) = \emptyset\}.$$

Then E_n^+ and E_n^- are independent because the equalities (14) and (15) imply that

$$E_n^+ = \{Z_+ \in [0, n]; \Phi^+ \cap [\mathcal{A}(\Phi^+) \cap [Z_+, +\infty) \times \mathbb{R}_+]^c \cap (B + x/2) = \emptyset\},$$

and

$$E_n^- = \{Z_- \in [-n, 0]; \Phi^- \cap [\mathcal{A}(\Phi^-) \cap (-\infty, Z_-] \times \mathbb{R}_+]^c \cap (A - x/2) = \emptyset\}.$$

Consequently, let us fix $\eta > 0$ and choose $n \in \mathbb{N}$ such that

$$\mathbf{P}\{Z_+, Z_- \in [0, n]\} \geq 1 - \eta/3.$$

Then for $x \geq 2 \sup\{(n - \inf p_1(B)), (n + \sup p_1(A))\}$, using the invariance of Ψ under $T^{x/2}$ and the independence of E_n^+ and E_n^-, we have

$$\begin{aligned}&|\mathbf{P}\{(\Psi \cap A = \emptyset) \cap (T^{-x}(\Psi) \cap B = \emptyset)\} - \mathbf{P}\{\Psi \cap A = \emptyset\} \cdot \mathbf{P}\{\Psi \cap B = \emptyset\}| \\ &\leq |\mathbf{P}\{(\Psi \cap (A - x/2) = \emptyset) \cap (\Psi \cap (B + x/2) = \emptyset)\} - \mathbf{P}\{E_n^+ \cap E_n^-\}| \\ &\qquad\qquad + |\mathbf{P}(E_n^+)\mathbf{P}(E_n^-) - \mathbf{P}\{\Psi \cap (A - x/2) = \emptyset\}\mathbf{P}\{\Phi \cap (B + x/2) = \emptyset\}| \\ &\leq \frac{\eta}{3} + 2 \cdot \frac{\eta}{3} = \eta. \end{aligned}$$

So the required convergence (11) is proved.

□

2 The mean crack number and typical inter-crack distance

Let us consider for a fixed $\varepsilon > 0$, the set Λ_ε given by the equality (7). Due to Proposition 3, Λ_ε is stationary and ergodic.

We are interested in two physical quantities, the mean crack number and typical inter-crack distance. First, the mean crack number λ_ε, i.e. the mean number of cracks per unit of length is the intensity of Λ_ε. Secondly, we can define two different characteristic distances:
(i) the typical inter-crack distance D_ε represents the distance in the Palm sense (see (17)) between a point "randomly chosen" in Λ_ε and its successor.
(ii) I_x^ε, $x \in \mathbb{R}$, is the smallest interval whose bounds are in Λ_ε and that contains x (it is unique almost surely). Since Λ_ε is stationary, the distribution of $|I_x^\varepsilon|$ does not depend on x.

We express the distribution of $|I_0^\varepsilon|$ via the law of D_ε and we notice that D_ε and $|I_0^\varepsilon|$ are not identically distributed. Moreover, we determine λ_ε and the distribution of D_ε. In a first step we prove that both λ_ε and the probability distribution function of D_ε can be expressed as an integral of two functions G and H. The calculation of G and H is postponed in Section 3.

Let us start with a precise definition of λ_ε and D_ε. The function that associates to any Borel set $B \subset \mathbb{R}$, the value $\mathbf{E}(\#(\Lambda_\varepsilon \cap B))$ is a positive measure invariant under translations. So it is proportional to the Lebesgue measure on $\mathbb{R}$ denoted by $|\cdot|$. Consequently, we can define the intensity λ_ε of Λ_ε as

$$\lambda_\varepsilon = \frac{1}{|B|}\mathbf{E}(\#(\Lambda_\varepsilon \cap B)), \tag{16}$$

where $B \subset \mathbb{R}$ is a fixed Borel set verifying $0 < |B| < +\infty$.

The law in the Palm sense (see Section 4.4 of [20] or Section 2 of [15] for a complete survey on Palm distributions of stationary point processes on the real line) of the typical inter-crack distance D_ε is defined as follows:

$$\mathbf{E}h(D_\varepsilon) = \frac{1}{\lambda_\varepsilon |B|}\mathbf{E}\left\{\sum_{x\in\Lambda_\varepsilon\cap B} h(v(x,\Lambda_\varepsilon) - x)\right\}, \tag{17}$$

for every measurable function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and every fixed Borel set $B \subset \mathbb{R}$, where

$$v(x,\Gamma) = \inf\{\Gamma \cap (x,+\infty)\} = \inf\{s \in \Gamma; s > x\}, \quad x \in \mathbb{R},\ \Gamma \subset \mathbb{R}, \tag{18}$$

with the conventions $\inf \emptyset = +\infty$ and $\sum_{\emptyset} = 0$. Let us note that the right-hand side does not depend on B.

Let L_ε be the stress level associated with D_ε. We can define the joint distribution of $(D_\varepsilon, L_\varepsilon)$ (in the Palm sense) in the same manner. More precisely, we have

$$\mathbf{E}\left\{h(D_\varepsilon, L_\varepsilon)\right\} = \frac{1}{\lambda_\varepsilon |B|}\mathbf{E}\left\{\sum_{(x,y)\in\Psi\cap(B\times[0,\varepsilon])} h(v(x,\Lambda_\varepsilon) - x, y)\right\}, \tag{19}$$

for every Borel function $h : (\mathbb{R}_+)^2 \longrightarrow \mathbb{R}_+$ and every fixed Borel set $B \subset \mathbb{R}$.

Let us observe that Proposition 3 implies that $D_\varepsilon \geq r$ a.s.. Besides, using the same argument as J. Møller in ([14], page 62), we obtain that D_ε is an integrable r.v. and

$$\mathbf{E}D_\varepsilon = \frac{1}{\lambda_\varepsilon}. \tag{20}$$

We now establish a connection between the distributions of D_ε and $|I_0^\varepsilon|$. Let $\mathcal{I}_L^\varepsilon$, $L > 0$, be the set of intervals I_x^ε included in the segment $[-L, L]$, and N_L^ε the number of such intervals, i.e.

$$N_L^\varepsilon = \#\mathcal{I}_L^\varepsilon = \left(\#\{\Lambda_\varepsilon \cap [-L, L]\} - 1\right)_+.$$

The distributions of D_ε and I_0^ε are connected by the following proposition.

Proposition 4 **(i)** *For any positive measurable function h on $\mathbb{R}_+$,*

$$\mathbf{E}h(D_\varepsilon) = \frac{1}{\mathbf{E}(1/|I_0^\varepsilon|)}\mathbf{E}\left(\frac{h(|I_0^\varepsilon|)}{|I_0^\varepsilon|}\right). \tag{21}$$

(ii) *If there exists $p > 1$ such that $\mathbf{E}(h(D_\varepsilon)^p) < +\infty$, then when L goes to infinity,*

$$\frac{1}{N_L^\varepsilon}\sum_{I\in\mathcal{I}_L^\varepsilon} h(|I|) \longrightarrow \mathbf{E}h(D_\varepsilon) \quad \text{a.s.} \tag{22}$$

Proof. (i) It suffices to combine (20) with the argument used by Møller in ([14], Prop. 3.3.2.), in his study of the typical cell of a Voronoi tessellation on $\mathbb{R}$ generated by a stationary point process.
(ii) Let us define for all $x \in \mathbb{R}$,

$$\widetilde{T}^x : \left\{ \begin{array}{ccc} \mathcal{M}_\sigma(\mathbb{R}) & \longrightarrow & \mathcal{M}_\sigma(\mathbb{R}) \\ \{x_i\}_{i\ge1} & \longmapsto & \{x_i + x\}_{i\ge1}, \end{array} \right.$$

where $\mathcal{M}_\sigma(\mathbb{R})$ is the set of locally finite sequences of $\mathbb{R}$.

According to Wiener's ergodic theorem [22], if $\mathbf{E}(h(|I_0^\varepsilon|)/|I_0^\varepsilon|) < +\infty$, then

$$\frac{1}{2L}\int_{-L}^{L}\frac{h(|I_x^\varepsilon|))}{|I_x^\varepsilon|}dx = \frac{1}{2L}\int_{-L}^{L}\frac{h(|I_0^\varepsilon(\widetilde{T}^{-x}(\Lambda_\varepsilon))|)}{|I_0^\varepsilon(\widetilde{T}^{-x}(\Lambda_\varepsilon))|}dx \longrightarrow \mathbf{E}\left(\frac{h(|I_0^\varepsilon|)}{|I_0^\varepsilon|}\right) \quad \text{a.s.} \tag{23}$$

Moreover, taking $h = 1$, we easily verify that

$$\frac{N_L^\varepsilon}{2L} \longrightarrow \mathbf{E}\left(\frac{1}{|I_0^\varepsilon|}\right), \qquad \text{a.s. when } L \to +\infty.$$

We suppose that h satisfies the condition (ii). Applying the argument used by Goldman (see [8], Lemma 4) in the case of Poissonian tessellations, we demonstrate that

$$\frac{1}{2L}\int_{-L}^{L}\frac{h(|I_x^\varepsilon|)}{|I_x^\varepsilon|}dx - \frac{1}{2L}\sum_{I\in\mathcal{I}_L^\varepsilon} h(|I|) \longrightarrow 0, \qquad \text{when } L \to +\infty.$$

$\square$

Remark 5 It is possible to invert the equality (21), namely

$$\mathbf{E}\{h(|I_0^\varepsilon)\} = \frac{1}{\mathbf{E}(D_\varepsilon)}\mathbf{E}\{D_\varepsilon h(D_\varepsilon)\}, \tag{24}$$

for every positive Borel function h defined on $\mathbb{R}_+$.

From now on, we focus on the calculation of λ_ε and the distribution of D_ε. The following lemma is an essential intermediate result.

Lemma 6 **(i)** $\lambda_\varepsilon = \int_0^\varepsilon \mathbf{P}\{(0,v) \notin \mathcal{A}(\Phi)\}dv.$

(ii) *For every $t \ge 0$,*

$$\mathbf{P}\{D_\varepsilon \ge t\} = \frac{1}{\lambda_\varepsilon}\int_0^\varepsilon \mathbf{P}\{(0,v) \notin \mathcal{A}(\Phi); v(0,\Lambda_{\varepsilon,v}) \ge t\}dv,$$

where $\Lambda_{\varepsilon,v}$ is the cracking process based on $\Phi\cap(\mathbb{R}\times[0,\varepsilon])\cup\{(0,v)\}$ and $v(0,\Lambda_{\varepsilon,v})$ is defined by the equality (18).

Proof. To prove these two equalities, the essential tool is Slivnyak's formula (see for example [14]) satisfied by Φ:

$$\mathbf{E}\left\{\sum_{(x,y)\in\Phi} h((x,y),\Phi)\right\} = \int \mathbf{E}(h((u,v),\Phi\cup\{(u,v)\}))\mathbf{1}_{\mathbb{R}_+}(v)dudv, \tag{25}$$

for every positive measurable function h defined on $\mathbb{R}^2 \times \mathcal{M}_\sigma(\mathbb{R}^2)$.

Using successively (7), (16) and (25), we have for every $L > 0$

$$\begin{aligned} \lambda_\varepsilon &= \frac{1}{L}\mathbf{E}\left[\#(\Lambda_\varepsilon \cap [0,L])\right] \\ &= \frac{1}{L}\mathbf{E}\left[\sum_{(x,y)\in\Phi} \mathbf{1}_{[0,L]}(x)\mathbf{1}_{[0,\varepsilon]}(y)\mathbf{1}_{\mathcal{A}(\Phi)^c}(x,y)\right] \\ &= \frac{1}{L}\int_0^L du \int_0^\varepsilon \mathbf{P}\{(u,v)\notin \mathcal{A}(\Phi\cup\{(u,v)\})\}dv. \end{aligned} \tag{26}$$

Using invariance under horizontal translations of Φ and the equality between the two events $\{(0,v)\notin\mathcal{A}(\Phi\cup\{(0,v)\}\}$ and $\{(0,v)\notin\mathcal{A}(\Phi)\}$, we deduce from (26) that

$$\lambda_\varepsilon = \int_0^\varepsilon \mathbf{P}\{(0,v)\notin\mathcal{A}(\Phi)\}dv.$$

That completes the proof of (i).

Using the equalities (17) and (25), we can prove (ii) in the same manner.

□

Let us consider the continuous function

$$G(x,y) = \mathbf{P}\{\Phi^+ \cap ([0,x]\times[0,y]) \subset \mathcal{A}(\Phi^+)\}, \quad x\in[0,r], y\geq 0, \tag{27}$$

where Φ^+ is defined by (13).

$G(x,y)$ is the probability that either $\Phi\cap[0,x]\times[0,y]=\emptyset$ or the points of $\Phi\cap[0,x]\times[0,y]$ are erased by erasers from the right (i.e. belonging to Φ^+).

More generally, we define the continuous function

$$H(x,y,x',y') = \mathbf{P}\{\Phi^+\cap([0,x]\times[0,y]\setminus[0,x']\times[y',y]))\subset\mathcal{A}(\Phi^+)\},\ 0\leq x'\leq x\leq r,\ 0\leq y'\leq y. \tag{28}$$

$H(x,y,x',y')$ is the probability that either $\Phi\cap([0,x]\times[0,y]\setminus[0,x']\times[y',y])=\emptyset$ or the points of the set $\Phi\cap([0,x]\times[0,y]\setminus[0,x']\times[y',y])$ are erased by erasers from the right.

The following proposition provides the expression of λ_ε and the distribution function of D_ε as integrals of G and H.

Proposition 7 *We have:*

(i) $\lambda_\varepsilon = \int_0^\varepsilon G(r,v)^2dv$;

(ii) *For every* $t\geq 2r$,

$$\mathbf{P}\{D_\varepsilon\geq t\} = \left(\frac{G(r,\varepsilon)}{\lambda_\varepsilon}\int_0^\varepsilon G(r,v)e^{-rv}dv\right)\cdot e^{-(t-2r)\varepsilon}; \tag{29}$$

(iii) *For every $t \in [r, 2r]$,*

$$\mathbf{P}\{D_\varepsilon \geq t\} = \frac{1}{\lambda_\varepsilon} \int_0^\varepsilon G(r,v) H(r, \varepsilon, 2r - t, v) e^{-v(t-r)} dv. \tag{30}$$

Proof. (i) According to the point (i) of Lemma 6, it suffices to prove that for every $v \in [0, \varepsilon]$,

$$\mathbf{P}\{(0,v) \notin \mathcal{A}(\Phi)\} = \mathbf{P}\{(0,v) \text{ not erased}\} = G(r,v)^2. \tag{31}$$

Besides, the point $(0, v)$, $0 \leq v \leq \varepsilon$, is not erased if and only if there is no eraser in $[-r, r] \times [0, v]$, i.e. if the points of $\Phi \cap ([-r, r] \times [0, v])$ have been erased themselves.

In that case, the points of $\Phi \cap ([0, r] \times [0, v])$ (respectively of $\Phi \cap ([-r, 0] \times [0, v])$) could have been erased only by erasers from the right (respectively from the left).

So we have the equivalence

$$(0,v) \notin \mathcal{A}(\Phi) \iff \begin{cases} \Phi^+ \cap ([0,r] \times [0,v]) & \subset \mathcal{A}(\Phi^+) \\ \Phi^- \cap ([-r,0] \times [0,v]) & \subset \mathcal{A}(\Phi^-). \end{cases} \tag{32}$$

Consequently, using the independence of Φ^+ and Φ^-, we obtain

$$\mathbf{P}\{(0,v) \notin \mathcal{A}(\Phi)\} = \mathbf{P}\{\Phi^+ \cap ([0,r] \times [0,v]) \subset \mathcal{A}(\Phi^+)\} \cdot \mathbf{P}\{\Phi^- \cap ([-r,0] \times [0,v]) \subset \mathcal{A}(\Phi^-)\}.$$

It remains to notice that (31) is a direct consequence of the equalities

$$\mathbf{P}\{\Phi^+ \cap ([0,r] \times [0,v]) \subset \mathcal{A}(\Phi^+)\} = \mathbf{P}\{\Phi^- \cap ([-r,0] \times [0,v]) \subset \mathcal{A}(\Phi^-)\} = G(r,v).$$

(ii)(iii) In order to determine the law of D_ε, we deduce from Lemma 6 (ii) that it is sufficient to calculate the expression

$$\mathbf{P}\{(0,v) \notin \mathcal{A}(\Phi); v(0, \Lambda_{\varepsilon,v}) \geq t\}, \quad t \geq r, v \in [0, \varepsilon].$$

We proceed as for (i) and we obtain the equality

$$\{(0,v) \text{ not erased }; v(0, \Lambda_{\varepsilon,v}) \geq t\} = A^- \cap A_t^+, \tag{33}$$

where A^- and A_t^+ are two independent events defined by

$$\begin{aligned} A^- &= \{\Phi \cap ([-r,0] \times [0,v]) = \emptyset \text{ or all points of } \Phi \cap ([-r,0] \times [0,v]) \text{ erased from the left}\}, \\ A_t^+ &= \{\Phi \cap ([0,r] \times [0,v] \cup [r,t] \times [0,\varepsilon]) \text{ is empty or erased from the right}\}. \end{aligned}$$

Let us remark that

$$\mathbf{P}(A^-) = G(r,v). \tag{34}$$

Consequently we obtain the formula

$$\mathbf{P}\{(0,v) \notin \mathcal{A}(\Phi); v(0, \Lambda_{\varepsilon,v}) \geq t\} = G(r,v) \cdot \mathbf{P}(A_t^+). \tag{35}$$

It then remains to determine $\mathbf{P}(A_t^+)$. The computation of this probability depends whether $t \geq 2r$ or $t \in [r, 2r]$.

First case: $t \geq 2r$**.**

Since a point of Φ can be erased only by an eraser located at a distance smaller than r on the x-axis, we can rewrite the event A_t^+ as follows:

$$A_t^+ = \{\Phi \cap ([0,r] \times [0,v] \cup [r, t-r] \times [0,\varepsilon]) = \emptyset\} \cap \{\Phi \cap ([t-r, t] \times [0, \varepsilon]) \text{ erased from the right}\}, \tag{36}$$

the two events of the intersection being independent.

The Poissonian property of Φ provides the equality

$$\mathbf{P}\{\Phi \cap ([0,r] \times [0,v] \cup [r,t-r] \times [0,\varepsilon]) = \emptyset\} = e^{-\nu([0,r]\times[0,v]\cup[r,t-r]\times[0,\varepsilon])} = e^{-(t-2r)\varepsilon}e^{-rv}. \quad (37)$$

Since Φ is invariant under horizontal translations T^{t-r}, we have

$$\mathbf{P}\{\Phi \cap ([t-r,t] \times [0,\varepsilon]) \text{ erased by the right}\} = G(r,\varepsilon). \quad (38)$$

Consequently, we deduce from formulas (36), (37) and (38):

$$\mathbf{P}(A_t^+) = G(r,\varepsilon)e^{-rv}e^{-(t-2r)\varepsilon}.$$

Relation (29) follows immediately.

Second case: $t \in [r, 2r]$**.**

We rewrite the event A_t^+ as the intersection of two independent events:

$$A_t^+ = \{\Phi \cap ([0,t-r] \times [0,v] = \emptyset\} \cap \{\Phi \cap ([t-r,t] \times [0,\varepsilon] \setminus [t-r,r] \times [v,\varepsilon]) \text{ erased from the right}\}.$$

The invariance under horizontal translations of Φ implies that

$$\mathbf{P}\{\Phi \cap ([t-r,t] \times [0,\varepsilon] \setminus [t-r,r] \times [v,\varepsilon]) \text{ erased by the right}\} = H(r,\varepsilon,2r-t,v),$$

and we then have the equality

$$\mathbf{P}(A_t^+) = H(r,\varepsilon,2r-t,v)e^{-v(t-r)}.$$

Using (35), we can conclude as in the first case.

$\square$

3 Explicit formulas for the mean crack number and the distribution function of the typical inter-crack distance

Proposition 7 implies that the mean crack number and the distribution of D_ε (resp. $|I_0^e|$) are known as soon as the functions G and H are determined. We prove in Proposition 8 below that G satisfies an integral equation. Fortunately we can solve it (see Proposition 10) and thereby obtain an explicit formula for both G and H. As for the joint distribution of $(D_\varepsilon, L_\varepsilon)$, we prove that it can be determined via G and H.

3.1 A functional equation satisfied by G

G satisfies the following functional equation.

Proposition 8 *For every* $0 \le x \le r$, $y \ge 0$,

$$G(x,y) = 1 - e^{-xy}\int_0^x \int_0^y G(r-u, y-v)e^{uv}(1+uv)dudv. \quad (39)$$

Proof. Let us first recall that for every fixed $x \in [0,r]$ and $y \in [0,\varepsilon]$, we have:

$$\Phi^+ \cap ([0,x] \times [0,y]) \overset{law}{=} \{(X_i, Y_i); 1 \le i \le N\},$$

where:

(i') $\{(X_i, Y_i); i \geq 1\}$ is a sequence of independent and uniform variables on $[0,x] \times [0,y]$;

(ii') N is a Poisson variable of mean value $\mathbf{E}N = xy$, independent of the preceding sequence.

Let us define for all $n \geq 1$,

$$\left(M_1^{(n)}, M_2^{(n)}\right) = \left(\inf_{1\leq i\leq n} X_i, \inf_{1\leq i\leq n} Y_i\right).$$

It is easily verified that the law of the couple $\left(M_1^{(n)}, M_2^{(n)}\right)$ is given by

$$\mathbf{P}\left\{M_1^{(n)} \geq u;\ M_2^{(n)} \geq v\right\} \;=\; \left(1-\frac{u}{x}\right)^n \left(1-\frac{v}{y}\right)^n, \quad u \in [0,x], v \in [0,y]. \tag{40}$$

The key point is the following: let $\left(M_1^{(n)}, \overline{Y}\right)$ (respectively $\left(\widehat{X}, M_2^{(n)}\right)$) be the point of Φ, of first coordinate $M_1^{(n)}$ (resp. of second coordinate $M_2^{(n)}$).

The points of $\Phi^+ \cap ([0,x] \times [0,y])$ cannot be erased by more than one eraser (X,Y). Since (X,Y) has to erase $\left(M_1^{(n)}, \overline{Y}\right)$ (resp. $\left(\widehat{X}, M_2^{(n)}\right)$), then $X \leq M_1^{(n)} + r$ (resp. $Y \leq M_2^{(n)}$).

Consequently, that happens if and only if either $N = 0$ or $N = n$, $n \geq 1$, and there is an eraser in $([x, M_1^{(n)} + r] \times [0, M_2^{(n)}])$.

Combining this argument with the equality (40), we obtain that for every $x \in [0,x]$, $y \in [0,y]$,

$$\begin{aligned}
&G(x,y)\\
&= \mathbf{P}\{\Phi^+ \cap ([0,x] \times [0,y]) \text{ empty or erased from the right}\}\\
&= \mathbf{P}\{N=0\}\\
&\quad + \sum_{n\geq 1} \mathbf{P}\{N=n\}\mathbf{P}\{\Phi^+ \cap ([x, M_1^{(n)} + r] \times [0, M_2^{(n)}]) \text{ not totally erased from the right}\}\\
&= e^{-xy}\left[1 + \sum_{n\geq 1} \frac{(xy)^n}{n!} \int_0^x \int_0^y (1 - G(u+r-x, v))\mathbf{P}(M_1^{(n)} \in du, M_2^{(n)} \in dv)\right]\\
&= 1 - e^{-xy} \sum_{n\geq 1} n^2 \frac{(xy)^{n-1}}{n!} \int_0^x \int_0^y G(u+r-x, v)\left(1-\frac{u}{x}\right)^{n-1}\left(1-\frac{v}{y}\right)^{n-1} dudv\\
&= 1 - e^{-xy} \int_0^x \int_0^y G(u+r-x, v) e^{(x-u)(y-v)} (1 + (x-u)(y-v)) dudv.
\end{aligned}$$

Taking the change of variables (in the integral) $u' = x - u$, $v' = y - v$, we deduce (39) from the preceding equality.

□

Let us consider the bounded operator L on the space of continuous functions $C([0,r] \times \mathbb{R}_+)$ (endowed with the topology of uniform convergence on every compact set) defined by

$$L(Q) : (x,y) \longmapsto e^{-xy} \int_0^x \int_0^y Q(r-u, y-v) e^{uv} (1+uv) dudv, \quad x \in [0,r], y \geq 0, \tag{41}$$

where $Q \in C([0,r] \times \mathbb{R}_+)$.

The following theorem provides the uniqueness of the solution of the functional equation (39) in the space $C([0,r] \times \mathbb{R}_+)$.

Proposition 9 *We have*

$$G = \sum_{n\geq 0}(-1)^n L^n(1), \tag{42}$$

the convergence of the series being uniform on $[0,r]\times[0,k]$*, for any* $k>0$.

Proof. Equation (39) can be rewritten as

$$G + L(G) = 1, \quad G \in C([0,r]\times\mathbb{R}_+). \tag{43}$$

Let us remark that

$$L(1)(x,y) = 1 - e^{-xy}, \quad x\in[0,r], y\in\mathbb{R}_+. \tag{44}$$

Let $k>0$ be fixed. We suppose that the set $C([0,r]\times[0.k])$ of continuous fonctions defined on $[0,r]\times[0,k]$ is equipped with the uniform norm.

We deduce easily from (44) that the restriction of L to $C([0,r]\times[0,k])$ has a finite norm equal to $(1-e^{rk})$. Consequently, the series

$$\sum_{n\geq 0}(-1)^n L^n$$

converges to the inverse of $(I+L)$. Therefore (42) is a direct consequence of (43).

□

The functional equation (39) is the key point to calculate the expressions of the functions G and H. We will then deduce from Proposition 7 the mean crack number and distribution function of the typical inter-crack distance.

3.2 Explicit formulas for λ_ε and $\mathbf{P}\{D_\varepsilon \geq t\}$, $t\geq r$

Let us remark that the function α defined in (1) satisfies the two following identities:

$$(\alpha(t)t)' = \alpha(t)e^{-rt}, \tag{45}$$

$$\alpha(t) = \frac{e^{-\gamma}}{rt}\exp\{-\mathrm{Ei}(1,rt)\}, \quad t>0, \tag{46}$$

where γ is Euler's constant and $\mathrm{Ei}(n,x) = \displaystyle\int_1^{+\infty}\frac{e^{-xs}}{s^n}ds$.

Proposition 10 *For every* $0\leq x\leq r$, $y\geq 0$,

$$G(x,y) = 1 - \int_0^y \alpha(s)\frac{1-e^{-sx}}{s}ds = 1 - \int_0^y \exp\left\{-\int_0^{rs}\frac{1-e^{-t}}{t}dt\right\}\frac{1-e^{-sx}}{s}ds.$$

In particular,

$$G(r,y) = \alpha(y) = \exp\left\{-\int_0^{ry}\frac{1-e^{-v}}{v}dv\right\}. \tag{47}$$

Proof. Let us recall that Proposition 9 provides the uniqueness of the solution of the integral equation (39) in the space $C([0,r]\times\mathbb{R}_+)$. Consequently, it suffices to verify that the continuous function

$$U(x,y) = 1 - V(x,y) = 1 - \int_0^y \alpha(s)\frac{1-e^{-sx}}{s}ds, \quad x\in[0,r], y\geq 0,$$

satisfies the identity $U + L(U) = 1$. Using (44), we obtain that it is equivalent to

$$L(V)(x,y) = U(x,y) - e^{-xy}. \tag{48}$$

We need to calculate $L(V)$ where L is the operator defined by (41).

For $x \in [0,r]$ and $y \geq 0$ fixed, we have

$$\begin{aligned}
L(V)(x,y) &= e^{-xy}\int_0^x\int_0^y\int_0^{y-v}\alpha(s)\frac{1-e^{-s(r-u)}}{s}e^{uv}(1+uv)dudvds \\
&= e^{-xy}\int_0^x\int_0^y\alpha(s)\frac{1-e^{-s(r-u)}}{s}\left[\int_0^{y-s}e^{uv}(1+uv)dv\right]dsdu \\
&= e^{-xy}\int_0^y\frac{\alpha(s)}{s}\left[\int_0^x(y-s)(1-e^{-s(r-u)})e^{u(y-s)}du\right]ds \\
&= \int_0^y\frac{\alpha(s)}{s}\left\{e^{-xs}-e^{-xy}-(y-s)e^{-rs}\frac{1-e^{-xy}}{y}\right\}ds \\
&= -\int_0^y\frac{\alpha(s)}{s}(1-e^{-xs})ds - \frac{1-e^{-xy}}{y}\left(\int_0^y(y-s)\alpha(s)\frac{e^{-sr}-1}{s}ds - \int_0^y\alpha(s)ds\right) \\
&= U(x,y) - 1 - \frac{1-e^{-xy}}{y}\left(\int_0^y(y-s)\alpha'(s)ds - \int_0^y\alpha(s)ds\right) \\
&= U(x,y) - 1 + (1-e^{-xy}) = U(x,y) - e^{-xy}.
\end{aligned}$$

We then obtain (48). This implies Proposition 10.

□

Remark 11 Let us briefly explain how the right function G was determined.

Let us fix $x \in (0,r)$ and a continuously derivable function h defined on $\mathbb{R}_+$ and

$$H(y) = \int_0^y h(y-v)e^{xv}(1+xv)dv, \ y \geq 0.$$

It is easy to check that H solves the following linear ordinary equation:

$$H''(y) - 2xH'(y) + x^2H(y) = h'(y)$$

with the boundary conditions $H(0) = 0$ and $H'(0) = h(0)$.

Let G be a solution of $G = 1 - LG$. We introduce $G_1(x,y) = \frac{\partial G}{\partial y}(x,y)$. The previous step implies that G_1 solves:

$$2G_1(x,y) + y\frac{\partial G_1}{\partial y}(x,y) + \frac{\partial^2 G_1}{\partial x\partial y}(x,y) = -e^{-xy}G_1(r-x,y), \ 0 \leq x \leq r, y \geq 0.$$

We notice that $(x,y) \longmapsto xe^{-xy}$ and $(x,y) \longmapsto 1/y^2$ are two particular solutions of:

$$2A(x,y) + y\frac{\partial A}{\partial y}(x,y) + \frac{\partial^2 A}{\partial x\partial y}(x,y) = 0.$$

It is then possible (after tedious calculations) to deduce $G_1(x,y)$.

Let us define the cracking process Λ_ε^+ on the positive half-line. Let Ψ^+ be the set obtained by the erasing procedure (developed in Section 1) applied to the intersection Φ^+ of Φ and $(\mathbb{R}_+)^2$. Λ_ε^+ is defined as the projection on the first axis of $\Psi^+ \cap (\mathbb{R} \times [0,\varepsilon])$, namely:

$$\Lambda_\varepsilon^+ = \{x \in \mathbb{R}_+; \exists y \in [0,\varepsilon] \mid (x,y) \in \Psi^+\}. \tag{49}$$

Λ_ε^+ represents the position of the cracks when the stress is less than ε.

Let us consider the first positive crack position:

$$X_1^\varepsilon = \inf \Lambda_\varepsilon^+. \tag{50}$$

The calculation of the law of X_1^ε, $\varepsilon \geq 0$, is essential to obtain an explicit formula of the function $H(r,\cdot,\cdot,\cdot)$ defined in (28). The following theorem provides the exact distribution of X_1^ε.

Theorem 12 *The law of X_1^ε has a density $\varphi_{X_1^\varepsilon}$ such that*

$$\varphi_{X_1^\varepsilon}(x) = \begin{cases} \varepsilon\alpha(\varepsilon)e^{-\varepsilon(x-r)} & \text{if } x \geq r \\ \int_0^\varepsilon \alpha(v)e^{-xv}dv & \text{if } x \in [0,r]. \end{cases} \tag{51}$$

Proof. Using Proposition 10, we only have to verify that

$$\mathbf{P}\{X_1^\varepsilon \geq x\} = \begin{cases} G(x,\varepsilon) & \text{if } x \in [0,r] \\ e^{-(x-r)\varepsilon}\alpha(\varepsilon) & \text{if } x \geq r. \end{cases} \tag{52}$$

Let us notice the equality of events

$$\{\Phi^+ \cap ([0,x] \times [0,\varepsilon]) \text{ erased from the right}\} = \{X_1^\varepsilon \geq x\}. \tag{53}$$

The equality (52), with $x \in [0,r]$, follows directly from (53) and (27).

When $x \geq r$, using (53) and the invariance under every horizontal translation of Φ^+, we have

$$\begin{aligned}
\mathbf{P}\{X_1^\varepsilon \geq x\} &= \mathbf{P}\{\Phi^+ \cap ([0,x] \times [0,\varepsilon]) \text{ empty or erased from the right}\} \\
&= \mathbf{P}\{\Phi^+ \cap ([0,x-r] \times [0,\varepsilon]) = \emptyset; \\
&\qquad \Phi^+ \cap ([x-r,x] \times [0,\varepsilon]) \text{ empty or erased from the right})\} \\
&= \mathbf{P}\{\Phi^+ \cap ([0,x-r] \times [0,\varepsilon]) = \emptyset\} \\
&\qquad \cdot\mathbf{P}\{\Phi^+ \cap ([0,r] \times [0,\varepsilon]) \text{ empty or erased from the right})\} \\
&= e^{-(x-r)\varepsilon}G(r,\varepsilon).
\end{aligned}$$

This proves the second part of (52). In particular,

$$\mathbf{P}\{\Phi^+ \cap ([0,r] \times [0,\varepsilon]) \text{ erased from the right}\} = \mathbf{P}\{X_1^\varepsilon \geq r\} = \alpha(\varepsilon). \tag{54}$$

□

Proposition 13 *For every $0 \leq x \leq r$, $0 \leq y \leq \varepsilon$,*

$$H(r,\varepsilon,x,y) = \alpha(y) - e^{-xy}\int_y^\varepsilon \alpha(s)\frac{1-e^{-(r-x)s}}{s}ds.$$

Proof. We observe that

$$\{\Phi^+ \cap ([0,r] \times [0,\varepsilon] \setminus [0,x] \times [y,\varepsilon]) \text{ erased from the right}\} = A_1 \cup A_2,$$

where

$$\begin{aligned}
A_1 &= \{X_1^y \in [r, x+r]\}, \\
A_2 &= \{\Phi^+ \cap ([0,x] \times [0,y]) = \emptyset; \Phi^+ \cap ([x,r] \times [0,\varepsilon]) \text{ erased from the right}\}.
\end{aligned}$$

We then obtain the following formula which is the key point of the proof of Proposition 13:

$$H(r,\varepsilon,x,y) = \mathbf{P}(A_1) + \mathbf{P}(A_2) - \mathbf{P}(A_1 \cap A_2). \tag{55}$$

Using Theorem 12, we have

$$\mathbf{P}(A_1) = \int_r^{x+r} y\alpha(y)e^{-y(u-r)}du = \alpha(y)(1-e^{-xy}). \tag{56}$$

The invariance of Φ^+ under every positive translation combined with Theorem 10 implies:

$$\begin{aligned}
\mathbf{P}(A_2) &= \mathbf{P}\{\Phi^+ \cap ([0,x]\times[0,y]) = \emptyset\} \cdot \mathbf{P}\{\Phi^+ \cap ([0,r-x]\times[0,\varepsilon]) \text{ erased by the right}\} \\
&= e^{-xy} G(r-x,\varepsilon) \\
&= e^{-xy}\left(1 - \int_0^\varepsilon \alpha(s)\frac{1-e^{-s(r-x)}}{s}ds\right).
\end{aligned} \tag{57}$$

It remains to determine $\mathbf{P}(A_1 \cap A_2)$. To this end, we remark that the law of the process Φ^+ conditioned on the event $\{\Phi^+ \cap ([0,x]\times[0,y]) = \emptyset\}$ is the same as $T^x(\Phi^+)$.

Consequently, we have

$$\begin{aligned}
\mathbf{P}(A_1 \cap A_2) &= \mathbf{P}\{X_1^y \in [r,x+r] | \Phi^+ \cap ([0,x]\times[0,y]) = \emptyset\} \cdot \mathbf{P}\{\Phi^+ \cap ([0,x]\times[0,y]) = \emptyset\} \\
&= \mathbf{P}\{X_1^y \in [r-x,r]\} \cdot e^{-xy} \\
&= e^{-xy}(\mathbf{P}\{X_1^y \geq r-x\} - \mathbf{P}\{X_1^y > r\}) \\
&= e^{-xy}\int_0^y \alpha(v)\frac{e^{-(r-x)v} - e^{-rv}}{v}dv.
\end{aligned} \tag{58}$$

Inserting the formulas (56), (57) and (58) in (55), we get

$$\begin{aligned}
H(r,\varepsilon,x,y) &= \alpha(y)(1-e^{-xy}) \\
&\quad + e^{-xy}\left\{1 - \int_0^\varepsilon \alpha(s)\frac{1-e^{-s(r-x)}}{s}ds - \int_0^y \alpha(s)\frac{e^{-s(r-x)} - e^{-rs}}{s}ds\right\}.
\end{aligned} \tag{59}$$

We calculate the last integral in the following way:

$$\begin{aligned}
\int_0^y \alpha(s)\frac{e^{-s(r-x)} - e^{-rs}}{s}ds &= \int_0^y \alpha(s)\frac{e^{-s(r-x)} - 1}{s}ds + \int_0^y \alpha(s)\frac{1-e^{-rs}}{s}ds \\
&= \int_0^\varepsilon \alpha(s)\frac{e^{-s(r-x)} - 1}{s}ds - \int_y^\varepsilon \alpha(s)\frac{e^{-s(r-x)} - 1}{s}ds \\
&\qquad - [\alpha(s)]_0^y \\
&= \int_y^\varepsilon \alpha(s)\frac{1-e^{-s(r-x)}}{s}ds - \int_0^\varepsilon \alpha(s)\frac{1-e^{-s(r-x)}}{s}ds \\
&\qquad + 1 - \alpha(y).
\end{aligned} \tag{60}$$

Combining the equalities (59) et (60), we obtain Proposition 13.

□

Proof of Theorem 1. The point (i) follows immediately from Propositions 7 and 10.
To prove the point (ii), it suffices to demonstrate that

$$\mathbf{P}\{D_\varepsilon \geq t\} = \begin{cases} \frac{\varepsilon}{\lambda_\varepsilon}\alpha(\varepsilon)^2 e^{-(t-2r)\varepsilon}; & \text{if } t \geq 2r \\ \frac{2}{\lambda_\varepsilon}\int_0^\varepsilon \alpha(v)^2 e^{-(t-r)v}dv - 1 & \text{if } t \in [r,2r]. \end{cases} \tag{61}$$

Using Proposition 7(ii) and (45), we have when $t > 2r$,

$$\begin{aligned}
\mathbf{P}\{D_\varepsilon \geq t\} &= \frac{\alpha(\varepsilon)}{\lambda_\varepsilon} \int_0^\varepsilon \alpha(v) e^{-rv} dv \cdot e^{-(t-2r)\varepsilon} \\
&= \frac{\alpha(\varepsilon)}{\lambda_\varepsilon} [v\alpha(v)]_0^\varepsilon \cdot e^{-(t-2r)\varepsilon} \\
&= \frac{\varepsilon}{\lambda_\varepsilon} \alpha(\varepsilon)^2 e^{-(t-2r)\varepsilon}.
\end{aligned}$$

Ir remains to calculate $\mathbf{P}\{D_\varepsilon \geq t\}$ when $t \in [r, 2r]$. Using Propositions 7(iii) and 13, we obtain:

$$\begin{aligned}
\mathbf{P}\{D_\varepsilon \geq t\} &= \frac{1}{\lambda_\varepsilon} \int_0^\varepsilon \alpha(v) e^{-(t-r)v} \left\{ \alpha(v) - e^{-(2r-t)v} \int_v^\varepsilon \alpha(s) \frac{1 - e^{-(t-r)s}}{s} ds \right\} dv \\
&= \frac{1}{\lambda_\varepsilon} \int_0^\varepsilon \alpha(v)^2 e^{-(t-r)v} dv - \frac{1}{\lambda_\varepsilon} \int_0^\varepsilon \alpha(s) \frac{1 - e^{-(t-r)s}}{s} \left[\int_0^s e^{-rv} \alpha(v) dv \right] ds \\
&= \frac{1}{\lambda_\varepsilon} \int_0^\varepsilon \alpha(v)^2 e^{-(t-r)v} dv - \frac{1}{\lambda_\varepsilon} \int_0^\varepsilon \alpha(s) \frac{1 - e^{-(t-r)s}}{s} s\alpha(s) ds \\
&= \frac{2}{\lambda_\varepsilon} \int_0^\varepsilon \alpha(v)^2 e^{-(t-r)v} dv - 1.
\end{aligned}$$

This completes the proof of the equality (61).

We now generalize (ii) in determining the joint density of the couple $(D_\varepsilon, L_\varepsilon)$. Fixing $t \geq r, s \in [0, \varepsilon]$, we use the equality (19) of $(D_\varepsilon, L_\varepsilon)$ and apply Slivnyak's formula (25) as in the proof of Lemma 6 to obtain that

$$\mathbf{P}\{D_\varepsilon \geq t; L_\varepsilon \leq s\} = \frac{1}{\lambda_\varepsilon} \int_0^s \mathbf{P}\{(0, v) \notin \mathcal{A}(\Phi); v(0, \Lambda_{\varepsilon,v}) \geq t\} dv.$$

Consequently, we get as in Proposition 7 that

$$\mathbf{P}\{D_\varepsilon \geq t; L_\varepsilon \leq s\} = \begin{cases} \left(\frac{G(r,\varepsilon)}{\lambda_\varepsilon} \int_0^s G(r, v) e^{-rv} dv \right) \cdot e^{-(t-2r)\varepsilon} & \text{if } t \geq 2r \\ \frac{1}{\lambda_\varepsilon} \int_0^s G(r, v) H(r, \varepsilon, 2r - t, v) e^{-v(t-r)} dv & \text{if } t \in [r, 2r]. \end{cases} \tag{62}$$

It then suffices to insert the expressions of G and H (see Propositions 10 and 13) into (62) to deduce that

$$\mathbf{P}\{D_\varepsilon \geq t; L_\varepsilon \leq s\} \tag{63}$$

$$= \begin{cases} \frac{\alpha(\varepsilon)}{\lambda_\varepsilon} s\alpha(s) e^{-(t-2r)\varepsilon} & \text{if } t \geq 2r \\ \frac{1}{\lambda_\varepsilon} \int_0^s \alpha(v)^2 (2e^{-(t-r)v} - 1) dv - \frac{s\alpha(s)}{\lambda_\varepsilon} \int_s^\varepsilon \alpha(v) \frac{1 - e^{-(t-r)v}}{v} dv & \text{if } t \in [r, 2r]. \end{cases} \tag{64}$$

Points (iii) and (iv) of Theorem 1 are easy consequences of this last equality.

□

Remark 14 (i) A similar formulation of points (i)-(ii) of Theorem 1 has been obtained by Widom (see [21], page 3893, results (37)-(39)) through heuristic methods. Besides, Coffman et al. [5], [4] constructed a point process on a finite interval, by the same erasing procedure as ours, and deduced analogous results by taking the limit when the length of the interval goes to infinity. Their work mostly used analytic tools such as Fourier transform and analytic functions.
(ii) Let us remark that the distribution of D_ε has a decreasing density on $[r, +\infty)$, with a

transition at $2r$. Since the distribution is of exponential type in the interval $[2r,+\infty)$, D_ε has finite moments of any order. Applying Proposition 4, we obtain

$$\frac{1}{N_L}\sum_{I\in\mathcal{I}_L}|I|^n \longrightarrow \mathbf{E}(D_\varepsilon^n), \qquad \textit{when } L\to+\infty,\ n\geq 1.$$

Besides, it is easy to verify that the first moment of D_ε satisfies (20).

Using (24), we can prove easily that $|I_0^\varepsilon|$ has an explicit density.

Proposition 15 *The law of $|I_0^\varepsilon|$ has a density $\varphi_{|I_0^\varepsilon|}$ on $[r,+\infty)$ such that*

$$\varphi_{|I_0^\varepsilon|}(x)=\begin{cases}\varepsilon^2\alpha(\varepsilon)^2xe^{-(x-2r)\varepsilon} & \textit{if } x>2r\\ 2x\int_0^\varepsilon e^{-(x-r)v}\alpha(v)^2vdv & \textit{if } r\leq x\leq 2r\end{cases}$$

4 The law of the successive cracks on the positive half-line

Using stationarity of Λ_ε, we have determined some of its statistical characteristics such as the distribution of the inter-crack distance. We would like to give an enumerative description of the points in Λ_ε. It is actually more convenient to fix an origin, namely 0, and to replace Λ_ε by Λ_ε^+ (defined by (49)). We have already considered the first crack position X_1^ε and determined its distribution in Theorem 12. Here we plan to go further, enumerating the point of Λ_ε^+ as follows:

$$\Lambda_\varepsilon^+=\{X_n^\varepsilon; n\in\mathbb{N}^*\},$$

where $0<X_1^\varepsilon<X_2^\varepsilon<\cdots<X_n^\varepsilon<\cdots$, $n\geq 1$.
Let Y_n^ε, $n\geq 1$, be the positive real number such that

$$(X_n^\varepsilon,Y_n^\varepsilon)\in\Psi^+\cap(\mathbb{R}_+\times[0,\varepsilon]).$$

The aim of this section is the description of the distribution of $\{(X_i^\varepsilon,Y_i^\varepsilon);\ 1\leq i\leq n\}$ for any $n\geq 1$. A first answer is given by a recursive algorithm (see Theorems 16 and 17): we compute the distribution of $(X_1^\varepsilon,Y_1^\varepsilon)$ and the distribution of $(X_{i+1}^\varepsilon,Y_{i+1}^\varepsilon)$ conditionally on $(X_1^\varepsilon,Y_1^\varepsilon,\cdots,X_i^\varepsilon,Y_i^\varepsilon)$, $1\leq i\leq n-1$. We interpret this result by using a Markov chain model (see Theorem 18) and we prove the convergence in law of the couple $(X_{n+1}^\varepsilon-X_n^\varepsilon,Y_n^\varepsilon)$ to $(D_\varepsilon,L_\varepsilon)$ (see Theorem 19).

We observe in particular that $\{X_n^\varepsilon;n\geq 1\}$ is not a renewal sequence, for instance $(X_2^\varepsilon-X_1^\varepsilon)$ is not independent of X_1^ε. However we prove (see Theorem 2) that $\{X_n^\varepsilon;n\geq 1\}$ is a "conditional renewal process" (see Theorem 2 for a detailed explanation of this expression).

Let us start with the density of $(X_1^\varepsilon,Y_1^\varepsilon)$.

Theorem 16 *The law of the couple $(X_1^\varepsilon,Y_1^\varepsilon)$ has a density $\varphi_{(X_1^\varepsilon,Y_1^\varepsilon)}$ such that for every $u,v\in\mathbb{R}$,*

$$\varphi_{(X_1^\varepsilon,Y_1^\varepsilon)}(u,v)=\Big(\mathbf{1}_{\{u>r\}}e^{-(u-r)\varepsilon}e^{-rv}+\mathbf{1}_{\{0\leq u\leq r\}}e^{-uv}\Big)\alpha(v)\mathbf{1}_{\{0\leq v\leq\varepsilon\}}. \tag{65}$$

Proof. It suffices to prove that for every $x\geq 0$ and $0\leq y\leq\varepsilon$:

$$\mathbf{P}\{X_1^\varepsilon\geq x;Y_1^\varepsilon\leq y\}=\begin{cases}\frac{y}{\varepsilon}\alpha(y)e^{-(x-r)\varepsilon} & \text{if } x>r\\ 1-\alpha(y)-\int_0^y\frac{\alpha(v)}{v}(1-e^{-xv})dv+\frac{y}{\varepsilon}\alpha(y) & \text{otherwise.}\end{cases} \tag{66}$$

We notice that

$$\begin{aligned}
\mathbf{P}\{X_1^\varepsilon \geq x; Y_1^\varepsilon \leq y\} &= \mathbf{P}\{X_1^\varepsilon \geq x; X_1^\varepsilon = X_1^y\} \\
&= \mathbf{P}\{X_1^y \geq x; X_1^\varepsilon = X_1^y\} \\
&= \int_x^{+\infty} \mathbf{P}\{X_1^\varepsilon = X_1^y | X_1^y = u\} \mathbf{P}\{X_1^y \in du\}.
\end{aligned} \tag{67}$$

Moreover, $(X_1^\varepsilon = X_1^y)$ if and only if there is no positive eraser in $([0, X_1^y] \times [y, \varepsilon])$, which means that for every $u \geq 0$,

$$\mathbf{P}\{X_1^\varepsilon = X_1^y | X_1^y = u\} = \begin{cases} 1 & \text{if } u \leq r \\ \mathbf{P}\{\Phi \cap [0, u-r] \times [y, \varepsilon] = \emptyset\} & \text{otherwise.} \end{cases}$$

Since Φ is a Poisson point process,

$$\mathbf{P}\{X_1^\varepsilon = X_1^y | X_1^y = u\} = \begin{cases} 1 & \text{if } u \leq r \\ e^{-(u-r)(\varepsilon - y)} & \text{otherwise.} \end{cases} \tag{68}$$

Inserting equalities (51) and (68) in (67), we get the result (66), via (60).

□

The following theorem provides the law of the couple $(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_{n+1}^\varepsilon)$ conditionally on $(X_1^\varepsilon, Y_1^\varepsilon, \cdots, X_n^\varepsilon, Y_n^\varepsilon)$.

Proposition 17 *For every $n \geq 1$, the distribution of the couple $(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_{n+1}^\varepsilon)$ conditionally on $(X_1^\varepsilon, Y_1^\varepsilon, \cdots, X_n^\varepsilon, Y_n^\varepsilon)$ has a density $\theta^\varepsilon(Y_n^\varepsilon; \cdot)$:*

$$\begin{aligned}
\theta^\varepsilon(y; u, v) = \Big[&\mathbf{1}_{\{u > 2r\}} \mathbf{1}_{\{0 \leq v \leq \varepsilon\}} e^{-r(y+v)} e^{-(u-2r)\varepsilon} \\
&+ \mathbf{1}_{\{r \leq u \leq 2r\}} \left\{ \mathbf{1}_{\{0 \leq v \leq y\}} e^{-(u-r)y} e^{-rv} + \mathbf{1}_{\{y < v \leq \varepsilon\}} e^{-ry} e^{-(u-r)v} \right\} \Big] \frac{\alpha(v)}{\alpha(y)},
\end{aligned} \tag{69}$$

where $y, v \in [0, \varepsilon]$ and $u \geq 0$.

Proof. Let $Z_n = (X_1^\varepsilon, Y_1^\varepsilon, \cdots, X_n^\varepsilon, Y_n^\varepsilon)$ and $z = (x_1, \cdots, x_{n-1}, x, y_1, \cdots, y_{n-1}, y)$ where $0 < x_1 < \cdots < x_{n-1} < x$ and $y_1, \cdots, y_{n-1}, y \in [0, \varepsilon]$. It suffices to demonstrate

$$\begin{aligned}
&\mathbf{P}\{X_{n+1}^\varepsilon - X_n^\varepsilon \geq u; Y_{n+1}^\varepsilon \leq v | Z_n = z\} \\
&= \begin{cases}
\frac{e^{-ry}}{\alpha(y)} \frac{v}{\varepsilon} \alpha(v) e^{-(u-2r)\varepsilon} & \text{if } u > 2r \\
(e^{-(u-r)y} - e^{-ry}) \frac{\alpha(v) v}{\alpha(y) y} + e^{-ry} \frac{\alpha(v) v}{\alpha(y)\varepsilon} & \text{if } r \leq u \leq 2r \\
 & \text{and } v \leq y, \\
e^{-(u-r)y} - \frac{e^{-ry}}{\alpha(y)} \left\{ \int_y^v (1 - e^{-(u-r)s}) \frac{\alpha(s)}{s} ds + \alpha(v) \left(1 - \frac{v}{\varepsilon}\right) \right\} & \text{if } r \leq u \leq 2r \\
 & \text{and } v > y.
\end{cases}
\end{aligned} \tag{70}$$

Our approach is based on the following properties:
(i) the distribution of $\widehat{\Phi} = T^{-X_n^\varepsilon}(\Phi_+) \cap (\mathbb{R}_+)^2$ conditionally on Z_n is the same as the distribution of Φ_+ conditionally on $\{\Phi_+ \cap ([0, r] \times [0, Y_n^\varepsilon])$ empty or erased from the right$\}$.
(ii) $(X_{n+1}^\varepsilon - X_n^\varepsilon - r, Y_{n+1}^\varepsilon)$ is the first point on the right of the point process $T^{-r}(\widehat{\Phi}) \cap (\mathbb{R}_+)^2$.
Using points (i) and (ii) above, (47) and (54), we get

$$\begin{aligned}
&\mathbf{P}\{X_{n+1}^\varepsilon - X_n^\varepsilon \geq u; Y_{n+1}^\varepsilon \leq v \mid Z_n = z\} \\
&= \frac{\mathbf{P}\{\Phi_+ \cap ([0, r] \times [0, y]) \text{ empty or erased from the right}; X_1^\varepsilon(T^{-r}(\Phi_+)) \geq u - r; Y_1^\varepsilon(T^{-r}(\Phi_+)) \leq v\}}{\mathbf{P}\{\Phi_+ \cap ([0, r] \times [0, y]) \text{ empty or erased from the right}\}} \\
&= \frac{1}{\alpha(y)} \mathbf{P}\{\Phi_+ \cap ([0, r] \times [0, y]) \text{ empty or erased from the right}; X_1^\varepsilon(T^{-r}(\Phi_+)) \geq u - r; Y_1^\varepsilon(T^{-r}(\Phi_+)) \leq v\}
\end{aligned} \tag{71}$$

where $(X_1^\varepsilon(T^{-r}(\Phi_+)), Y_1^\varepsilon(T^{-r}(\Phi_+)))$ is the first point on the right of the process $T^{-r}(\Phi_+) \cap (\mathbb{R}_+)^2$. In particular, $(X_1^\varepsilon(T^{-r}(\Phi_+)), Y_1^\varepsilon(T^{-r}(\Phi_+)))$ is distributed as $(X_1^\varepsilon, Y_1^\varepsilon)$.

First case: $u > 2r$**.** We have that

$$\begin{aligned}
&\{\Phi_+ \cap ([0,r] \times [0,y]) \text{ empty or erased from the right;} \\
&\qquad\qquad\qquad\qquad X_1^\varepsilon(T^{-r}(\Phi_+)) \geq u - r; Y_1^\varepsilon(T^{-r}(\Phi_+)) \leq v\} \\
&\quad = \{\Phi_+ \cap ([0,r] \times [0,y]) = \emptyset\} \cap \{X_1^\varepsilon(T^{-r}(\Phi_+)) \geq u - r; Y_1^\varepsilon(T^{-r}(\Phi_+)) \leq v\},
\end{aligned}$$

the two events of the intersection being independent.

Using this remark, (71) and (66), we obtain

$$\begin{aligned}
\mathbf{P}\{&X_{n+1}^\varepsilon - X_n^\varepsilon \geq u; Y_{n+1}^\varepsilon \leq v | Z_n = z\} \\
&= \frac{1}{\alpha(y)} \mathbf{P}\{\Phi_+ \cap ([0,r] \times [0,y]) = \emptyset\} \cdot \mathbf{P}\{X_1^\varepsilon \geq u - r; Y_1^\varepsilon \leq v\} \\
&= \frac{e^{-ry}}{\alpha(y)} \frac{v}{\varepsilon} \alpha(v) e^{-(u-2r)\varepsilon}. \qquad (72)
\end{aligned}$$

Second case: $r \leq u \leq 2r$**.**

The independence property is not satisfied, but $(X_1^\varepsilon(T^{-r}(\Phi_+)), Y_1^\varepsilon(T^{-r}(\Phi_+)))$ is still distributed with density $\varphi_{(X_1^\varepsilon, Y_1^\varepsilon)}$ given by (65). Then going back to (71), we get

$$\mathbf{P}\{X_{n+1}^\varepsilon - X_n^\varepsilon \geq u; Y_{n+1}^\varepsilon \leq v | Y_n = y\} = \frac{1}{\alpha(y)} \int_{u-r}^{+\infty} dw \int_0^v A(w,t,y) \varphi_{(X_1^\varepsilon, Y_1^\varepsilon)}(w,t) dt, \qquad (73)$$

where for every $w \geq 0, 0 \leq y, t \leq \varepsilon$,

$$A(w,t,y) = \mathbf{P}\{\Phi_+ \cap ([0,r] \times [0,y]) \text{ erased by the right } | X_1^\varepsilon(T^{-r}(\Phi_+)) = w, Y_1^\varepsilon(T^{-r}(\Phi_+)) = t\}.$$

It remains to determine the function A. To this end, let us notice that $([0,r] \times [0,y])$ has a non-empty intersection with the domain of relaxation $R(w+r,t)$ if and only if $w \leq r$ and $t \leq y$. Consequently, we obtain

$$A(w,t,y) = \begin{cases} e^{-wy-(r-w)t} & \text{if } w \leq r \text{ and } t \leq y \\ e^{-ry} & \text{otherwise.} \end{cases} \qquad (74)$$

Inserting formulas (74) and (65) in (73), we deduce the result (70), via (45), which completes the proof of Theorem 17.

□

We explicit the distribution of $\{(X_i^\varepsilon, Y_i^\varepsilon); 1 \leq i \leq n\}$ starting with the law of $\{Y_i^\varepsilon; i \geq 1\}$.

Theorem 18 *$(Y_n^\varepsilon)_{n \geq 1}$ is a homogeneous Markov chain such that:*

(i) Y_1^ε has a density $\varphi_{Y_1^\varepsilon}$ such that

$$\varphi_{Y_1^\varepsilon}(y) = \left(\frac{e^{-ry}}{\varepsilon} + \frac{1 - e^{-ry}}{y}\right) \alpha(y) \mathbf{1}_{[0,\varepsilon]}(y);$$

(ii) the transition kernel of $\{Y_n^\varepsilon; n \geq 1\}$ admits a transition probability density $\Pi^{Y,\varepsilon}(y;\cdot)$ such that for every $y, v \in [0,\varepsilon]$,

$$\Pi^{Y,\varepsilon}(y;v) = \Bigg(\mathbf{1}_{\{0\leq v\leq\varepsilon\}} \frac{e^{-r(y+v)}}{\varepsilon} + \mathbf{1}_{\{0\leq v\leq y\}} e^{-rv} \frac{1-e^{-ry}}{y} + \mathbf{1}_{\{y<v\leq\varepsilon\}} e^{-ry} \frac{1-e^{-rv}}{v} \Bigg) \frac{\alpha(v)}{\alpha(y)}; \tag{75}$$

(iii) the stationary law of $(Y_n^\varepsilon)_{n\geq1}$ is the distribution of L_ε (cf (2));

(iv) conditionally on $(X_1^\varepsilon, Y_1^\varepsilon, \cdots, X_n^\varepsilon, Y_n^\varepsilon)$ the r.v. $(X_{n+1}^\varepsilon - X_n^\varepsilon)$ has a density which depends only on Y_n^ε and is equal to $\Pi^{D_\varepsilon}(Y_n^\varepsilon;\cdot)$ (where $\Pi^{D_\varepsilon}(y;\cdot)$ is defined by (3)).

Proof. The points (i), (ii) and (iv) follow easily from Theorems 16 and 17. In order to obtain (iii), it suffices to prove that for every $v \in [0,\varepsilon]$,

$$\mathbf{P}\{L_\varepsilon \leq v\} = \int_0^\varepsilon \Pi^{Y,\varepsilon}(y;v)\mathbf{P}\{L_\varepsilon \in dy\}. \tag{76}$$

Using (75) and (45), we have for any $v \in [0,\varepsilon]$,

$$\begin{aligned}
&\int_0^\varepsilon \Pi^{Y,\varepsilon}(y;v)\alpha(y)^2 dy \\
&= \alpha(v)\left[\frac{e^{-rv}}{\varepsilon}\int_0^\varepsilon e^{-ry}\alpha(y)dy + e^{-rv}\int_v^\varepsilon \frac{1-e^{-ry}}{y}\alpha(y)dy + \frac{1-e^{-rv}}{v}\int_0^v e^{-ry}\alpha(y)dy\right] \\
&= \alpha(v)\left[\frac{e^{-rv}}{\varepsilon}\varepsilon\alpha(\varepsilon) + e^{-rv}(\alpha(v)-\alpha(\varepsilon)) + \frac{1-e^{-rv}}{v}v\alpha(v)\right] \\
&= \alpha(v)^2.
\end{aligned} \tag{77}$$

Combining (77) with (2), we obtain the equality (76) which completes the proof of Theorem 18.

□

Proposition 17 implies that $(X_n^\varepsilon, Y_n^\varepsilon)_{n\geq1}$ is a Markov chain. It seems natural to investigate its limit distribution. More precisely, we have the following result.

Theorem 19 *The couple $(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_n^\varepsilon)$ converges in law when $n \to +\infty$, and the limit distribution coincides with the law of $(D_\varepsilon, L_\varepsilon)$ (see Theorem 1).*

Proof. Let us begin with proving the convergence of the Markov chain $(Y_n^\varepsilon)_{n\geq1}$ to its stationary distribution μ, i.e. the distribution of L_ε.

The transition probability of $(Y_n^\varepsilon)_{n\geq1}$ has a density $\Pi^{Y,\varepsilon}(y;\cdot)$, $y \in [0,\varepsilon]$ (see Theorem 18 (ii)) such that the function $(y,v) \longmapsto \Pi^{Y,\varepsilon}(y;v)$ is continuous and everywhere positive on $(0,\varepsilon]^2$. Consequently, following ([6], example 6.2. of Section 5), we deduce that $(Y_n^\varepsilon)_{n\geq1}$ is a Harris chain. Moreover, since we have proved the existence of a stationary distribution (see Theorem 18 (iii)), it is also recurrent (see [6], exercise 6.11. of Section 5). An application of the beginning of Section 5.6.c of [6] shows that $(Y_n^\varepsilon)_{n\geq1}$ is an aperiodic recurrent Harris chain. Consequently, applying the convergence theorem for Harris chains (see [6], Theorem (6.8)), we deduce that $(Y_n^\varepsilon)_{n\geq1}$ converges to μ in the sense of the total variation distance $||\cdot||$ (let us notice that the note following Durrett's theorem guarantees that the starting law of $(Y_n^\varepsilon)_{n\geq1}$ given by Theorem 18

(i) satisfies the required hypothesis for the convergence). We recall that the variation distance between two probability measures μ_1, μ_2 with support in $[0,\varepsilon]$ is:

$$||\mu_1 - \mu_2|| = \sup_f \left| \int f d\mu_1 - \int f d\mu_2 \right|,$$

where f belongs to the set of measurable functions defined on $[0,\varepsilon]$, with values in $[0,1]$. In particular, $(Y_n^\varepsilon)_{n\geq 1}$ converges in distribution to μ.

We now prove the convergence in distribution of $(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_n^\varepsilon)$. Let us consider a continuous bounded measurable function $h : \mathbb{R}_+ \times [0,\varepsilon] \longrightarrow \mathbb{R}$. Using Theorem 18 (iv), we get for every $n \geq 1$,

$$\mathbf{E}\{h(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_n^\varepsilon)\} = \int_0^\varepsilon \left[\int_0^{+\infty} h(x,y)\Pi^{D_\varepsilon}(y;x)dx \right] \mathbf{P}\{Y_n^\varepsilon \in dy\} \tag{78}$$

But $y \longmapsto \int_0^{+\infty} h(x,y)\Pi^{D_\varepsilon}(y;x)dx$ is a bounded and continuous function, therefore

$$\lim_{n\to+\infty} \mathbf{E}\{h(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_n^\varepsilon)\} = \int_0^\varepsilon \left[\int_0^{+\infty} h(x,y)\Pi^{D_\varepsilon}(y;x)dx \right] \mathbf{P}\{L_\varepsilon \in dy\}. \tag{79}$$

Using (iii) and (iv) of Theorem 1, we obtain that the right hand-side of (79) is equal to $\mathbf{E}\{h(D_\varepsilon, L_\varepsilon)\}$.

□

Thanks to Proposition 17, we are able to prove Theorem 2.

Proof of Theorem 2. (i) Using Theorem 16, it suffices to have

$$\mathbf{E}\{h(\eta_1, \rho_1')\mathbf{1}_{B_1'}\} = \mathbf{P}(B_1') \int h(u,v)\varphi_{(X_1^\varepsilon, Y_1^\varepsilon)}(u,v)dudv, \tag{80}$$

for every measurable and bounded function h on $\mathbb{R}^2$. Taking the conditional expectation of $h(\eta_1, \rho_1')\mathbf{1}_{B_1'}$ with respect to η_1, ρ_1', we have

$$\begin{aligned}
\mathbf{E}\{h(\eta_1, \rho_1')\mathbf{1}_{B_1'}\} &= \mathbf{E}\{h(\eta_1, \rho_1')e^{-(\eta_1\wedge r)\rho_1'}\} \\
&= C\int_0^{+\infty} du \int_0^\varepsilon h(u,v) \big(\mathbf{1}_{[0,r]}(u)e^{-uv} \\
&\qquad\qquad + \mathbf{1}_{(r,+\infty)}(u)e^{-(u-r)\varepsilon}e^{-rv}\big)\, \alpha(v)dv \\
&= C\int\int h(u,v)\varphi_{(X_1^\varepsilon, Y_1^\varepsilon)}(u,v)dudv,
\end{aligned}$$

where C is a positive constant. Consequently, we get (80).

(ii) - Let us first prove that $(X_1^\varepsilon, Y_1^\varepsilon, X_2^\varepsilon - X_1^\varepsilon, Y_2^\varepsilon)$ is distributed as the vector $(\eta_1, \rho_1, r+\eta_2, \rho_2')$ conditioned on the event $B_1 \cap B_2'$, i.e.

$$\mathbf{E}\left\{g(\eta_1,\rho_1)h(r+\eta_2,\rho_2')\mathbf{1}_{B_1}\mathbf{1}_{B_2'}\right\} = K\mathbf{E}\left\{g(X_1^\varepsilon, Y_1^\varepsilon)h(X_2^\varepsilon - X_1^\varepsilon, Y_2^\varepsilon)\right\} \tag{81}$$

where g and h are two bounded Borel functions defined on $\mathbb{R}^2$ and K is a positive constant, independent from g and h. It is clear that (5) and (6) imply:

$$\varphi_{\rho'}(x) = \frac{\varepsilon}{\int_0^\varepsilon \alpha(v)dv}\alpha(x)\varphi_\rho(x). \tag{82}$$

Consequently, the left-hand side of (81) is equal (up to a multiplicative constant) to

$$\mathbf{E}\left\{g(\eta_1,\rho_1')h(r+\eta_2,\rho_2)\frac{\alpha(\rho_2)}{\alpha(\rho_1')}\mathbf{1}_{B_1'}\mathbf{1}_{\widetilde{B_2}}\right\},$$

where $\widetilde{B}_2=\{\xi_2\geq(\eta_2\wedge r)(\rho_2\vee\rho_1')+r(\rho_2\wedge\rho_1')$.

We now take the conditional expectation with respect to $\xi_1,\eta_1,\rho_1',\eta_2,\rho_2$ in the previous expectation and we obtain:

$$\begin{aligned}&\mathbf{E}\left\{g(\eta_1,\rho_1)h(r+\eta_2,\rho_2')\mathbf{1}_{B_1}\mathbf{1}_{B_2'}\right\}\\&\qquad=K_1\mathbf{E}\left\{g(\eta_1,\rho_1')h(r+\eta_2,\rho_2)\frac{\alpha(\rho_2)}{\alpha(\rho_1')}\mathbf{1}_{B_1'}e^{-(\eta_2\wedge r)(\rho_2\vee\rho_1')-r(\rho_2\wedge\rho_1')}\right\}.\end{aligned}\tag{83}$$

Inserting the densities (4) and (5) of η_2 and ρ_2 in (83), we thus get

$$\begin{aligned}&\mathbf{E}\left\{g(\eta_1,\rho_1)h(r+\eta_2,\rho_2')\mathbf{1}_{B_1}\mathbf{1}_{B_2'}\right\}\\&=K_1\mathbf{E}\left\{g(\eta_1,\rho_1')\mathbf{1}_{B_1'}\int_r^{+\infty}du\int_0^{\varepsilon}h(u,v)e^{-((u-r)\wedge r)(v\vee\rho_1')-r(v\wedge\rho_1')}\varphi_\eta(u-r)\varphi_\rho(v)\frac{\alpha(v)}{\alpha(\rho_1')}dv\right\}\\&=K_2\mathbf{E}\left\{g(\eta_1,\rho_1')\mathbf{1}_{B_1'}\int_{\mathbb{R}_+^2}h(u,v)\left[\mathbf{1}_{\{u>2r\}}\mathbf{1}_{\{0\leq v\leq\varepsilon\}}e^{-r(v+\rho_1')}e^{-(u-2r)\varepsilon}+\mathbf{1}_{\{r\leq u\leq 2r\}}\cdot\right.\right.\\&\qquad\left.\left.\cdot\left(\mathbf{1}_{\{0\leq v\leq\rho_1'\}}e^{-(u-r)\rho_1'}e^{-rv}+\mathbf{1}_{\{\rho_1'<v\leq\varepsilon\}}e^{-(u-r)v}e^{-r\rho_1'}\right)\right]\frac{\alpha(v)}{\alpha(\rho_1')}dudv\right\}.\end{aligned}\tag{84}$$

Moreover, using (69), we obtain that

$$\mathbf{E}\left\{g(\eta_1,\rho_1)h(r+\eta_2,\rho_2')\mathbf{1}_{B_1}\mathbf{1}_{B_2'}\right\}=K_2\mathbf{E}\left\{g(\eta_1,\rho_1')\mathbf{1}_{B_1'}\int\int h(u,v)\theta^\varepsilon(\rho_1',u,v)dudv\right\}.\tag{85}$$

Let us recall that Point (i) shows that, up to a multiplicative constant, the right-hand side of (85) is equal to

$$\mathbf{E}\left\{g(X_1^\varepsilon,Y_1^\varepsilon)\int\int h(u,v)\theta^\varepsilon(X_1^\varepsilon,u,v)dudv\right\}.$$

Using Proposition 17, we obtain that the expectation above is equal to

$$\mathbf{E}\left\{g(X_1^\varepsilon,Y_1^\varepsilon)h(X_2^\varepsilon-X_1^\varepsilon,Y_2^\varepsilon)\right\},$$

which completes the proof of (81).

‒ Our next objective is to prove Point (ii) in the general case, i.e. that for any $n\geq 2$, the distribution of $(X_1^\varepsilon,Y_1^\varepsilon,\cdots,X_{n-1}^\varepsilon-X_{n-2}^\varepsilon,Y_{n-1}^\varepsilon,X_n^\varepsilon-X_{n-1}^\varepsilon,Y_n^\varepsilon)$ is the distribution of $(\eta_1,\rho_1,\cdots,r+\eta_{n-1},\rho_{n-1},r+\eta_n,\rho_n')$ conditioned on the event $B_1\cap\cdots B_{n-1}\cap B_n'$. As in the case $n=2$, it is equivalent to show that

$$\begin{aligned}&\mathbf{E}\left\{g(\eta_1,\rho_1,r+\eta_2,\rho_2,\cdots,r+\eta_{n-1},\rho_{n-1})h(r+\eta_n,\rho_n')\mathbf{1}_{\cap_{i=1}^{n-1}B_i}\mathbf{1}_{B_n'}\right\}\\&\qquad=K\mathbf{E}\left\{g(X_1^\varepsilon,Y_1^\varepsilon,\cdots,X_{n-1}^\varepsilon-X_{n-2}^\varepsilon,Y_{n-1}^\varepsilon)h(X_n^\varepsilon-X_{n-1}^\varepsilon,Y_n^\varepsilon)\right\},\end{aligned}\tag{86}$$

where g (resp. h) is a bounded Borel function on $\mathbb{R}^{2n}$ (resp. $\mathbb{R}^2$) and K is a constant independent from g and h.

In much the same way as (85), we can obtain that for every $n \geq 2$,

$$\mathbf{E}\left\{g(\eta_1,\rho_1,r+\eta_2,\rho_2,\cdots,r+\eta_{n-1},\rho_{n-1})h(r+\eta_n,\rho'_n)\mathbf{1}_{\bigcap_{i=1}^{n-1}B_i}\mathbf{1}_{B'_n}\right\}$$
$$= K\mathbf{E}\left\{g(\eta_1,\rho_1,r+\eta_2,\rho_2,\cdots,r+\eta_{n-1},\rho'_{n-1})\mathbf{1}_{\bigcap_{i=1}^{n-2}B_i}\mathbf{1}_{B'_{n-1}}\int_{\mathbb{R}_+^2}h(u,v)\theta^{\varepsilon}(\rho'_{n-1},u,v)dudv\right\} \tag{87}$$

It remains to use a reasoning by induction to deduce (86) from (87) and Proposition 17. This completes the proof of Theorem 2.

□

Remark 20 1. We emphasize that Theorem 2 leads us to simulate to $2n$-vector

$$Z_n = (X_1^{\varepsilon}, Y_1^{\varepsilon}, X_2^{\varepsilon} - X_1^{\varepsilon}, Y_2^{\varepsilon}, \cdots, X_n^{\varepsilon} - X_{n-1}^{\varepsilon}, Y_n^{\varepsilon}),\ n \geq 1.$$

Let us consider

$$Z'_n = (\eta_1, \rho_1, r+\eta_2, \cdots, r+\eta_{n-1}, \rho_{n-1}, r+\eta_n, \rho'_n),\ n \geq 1.$$

We will denote by φ_{Z_n} (resp. $\varphi_{Z'_n}$) the density of Z_n (resp. Z'_n).

Since $\xi_1, \cdots, \xi_n$ are independent exponential variables with mean 1, we have

$$\mathbf{P}\left\{\cap_{i=1}^{n-1}B_i \cap B'_n | \eta_1,\cdots,\eta_n,\rho_1,\cdots,\rho_{n-1},\rho'_n\right\}$$
$$= \exp\left(-\sum_{k=1}^{n-1}\left[(\eta_k\wedge r)(\rho_k\vee\rho_{k-1})+r(\rho_k\wedge\rho_{k-1})\right]-\left[(\rho_n\wedge r)(\rho'_n\vee\rho_{n-1})+r(\rho'_n\wedge\rho_{n-1})\right]\right) \tag{88}$$

Consequently, Theorem 2 (ii) implies that

$$\varphi_{Z_n} = \frac{1}{\mathbf{P}(C_n)}e^{-\Gamma_n}\varphi_{Z'_n}, \tag{89}$$

where Γ_n is the positive function defined on $(\mathbb{R}_+)^{2n}$ by:

$$\Gamma_n(x_1,y_1,\cdots,x_n,y_n) = \mathcal{A}(x_1,y_1,0) + \sum_{i=2}^{n}\mathcal{A}(x_i - r, y_i, y_{i-1}), \tag{90}$$

with

$$\mathcal{A}(x,y,y') = (x\wedge r)(y\vee y') + r(y\wedge y').$$

In particular, combining (88) with (90), we obtain

$$\Gamma_n(\eta_1,\rho_1,r+\eta_2,\rho_2,\cdots,r+\eta_{n-1},\rho_{n-1},r+\eta_n,\rho'_n)$$
$$= -\ln\left(\mathbf{P}\left\{\cap_{i=1}^{n-1}B_i\cap B'_n|\eta_1,\cdots,\eta_n,\rho_1,\cdots,\rho_{n-1},\rho'_n\right\}\right) \quad a.s.$$

We may apply the Hit or Miss Monte-Carlo Method (see [19], Chapter 4). More precisely, we first simulate $Z'_n = \omega$ and keep it with probability $p = e^{-\Gamma_n(\omega)}$. Otherwise we simulate a new independent copy of Z'_n and so on.

Let us remark that this procedure is not on-line in the following sense: if the algorithm has been applied to construct the first n points $(X_i^{\varepsilon}, Y_i^{\varepsilon})$, $1 \leq i \leq n$, then $(X_{n+1}^{\varepsilon}, Y_{n+1}^{\varepsilon})$ cannot be obtained directly. In fact, the whole procedure has to be applied once more to provide the $(n+1)$ points $(X_i^{\varepsilon}, Y_i^{\varepsilon})$, $1 \leq i \leq (n+1)$.

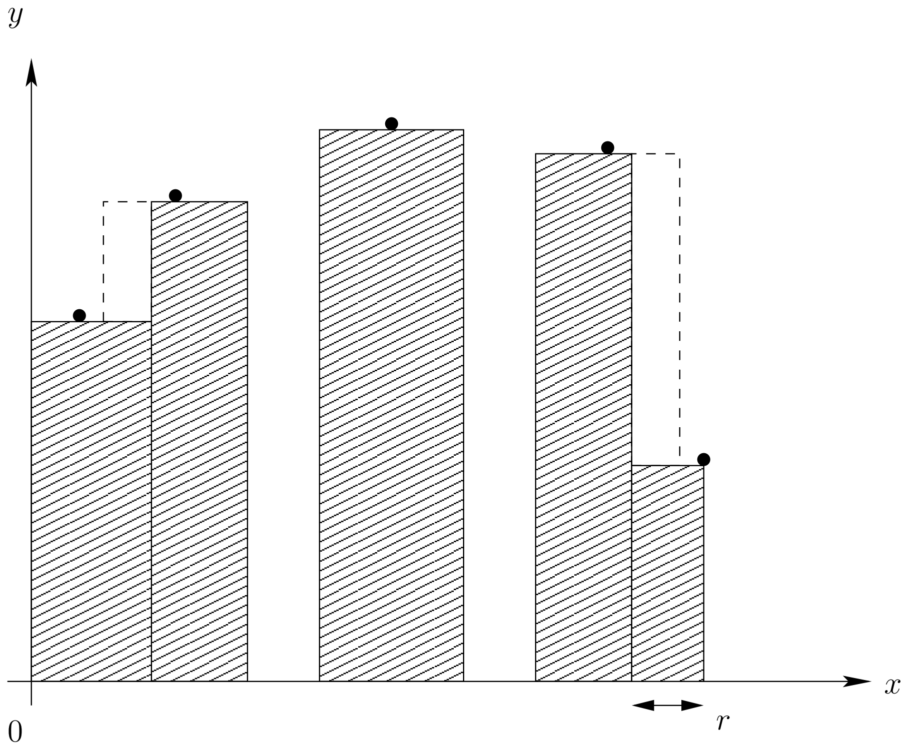

Figure 1: Hachured domain of “potential erasers” of $\{(\widehat{x}_i, y_i),\ 1 \le i \le 5\}$ (black points)

2. Besides, the function Γ_n has a geometrical interpretation. $\mathcal{A}(x, y, y')$ is the area of the union of two rectangles, i.e.

$$\mathcal{A}(x,y,y') = \begin{cases} \nu\left[([0,(x\wedge r)] \times [0,y']) \cup ([x, x+r] \times [0,y])\right] & \text{if } y \le y' \\ \nu\left[([0,r] \times [0,y']) \cup ([x+r-(x\wedge r), x+r] \times [0,y])\right] & \text{else,} \end{cases} \tag{91}$$

where ν is the Lebesgue measure on $\mathbb{R} \times \mathbb{R}_+$.

We deduce easily from (91) that

$$\Gamma_n(x_1, y_1, \cdots, x_n, y_n) = \nu(\cup_{i=1}^n \mathcal{D}(x_i, y_i)),$$

where

$$\begin{cases} \mathcal{D}(x_1,y_1) = ([(\widehat{x_1}-r)\vee 0, \widehat{x_1}+r] \times [0,y_1]) \setminus R(\widehat{x_2}, y_2) \\ \mathcal{D}(x_i,y_i) = ([\widehat{x}_i - r, \widehat{x}_i + r] \times [0,y_i]) \setminus [R((\widehat{x_{i-1}}, y_{i-1}) \cup R(\widehat{x_{i+1}}, y_{i+1})] \ \text{ if } 2 \le i \le (n-1) \\ \mathcal{D}(x_n,y_n) = ([\widehat{x_n}-r, \widehat{x_n}] \times [0,y_n]) \setminus R(\widehat{x_{n-1}}, y_{n-1}), \end{cases}$$

$R(x, y)$, $(x, y) \in \mathbb{R} \times \mathbb{R}_+$, is the set defined in (8) and with the convention $\widehat{x}_i = \sum_{k=1}^i x_k$, $1 \le i \le n$.

By definition of the erasing procedure, the set $\displaystyle\bigcup_{i=1}^n \mathcal{D}(x_i, y_i)$ is exactly the set of points of $[0, \widehat{x}_n] \times \mathbb{R}_+$ which erase at least one of the points $(\widehat{x}_i, y_i)$, $1 \le i \le n$ and are not erased by any of these n points. In other words, the set $\displaystyle\bigcup_{i=1}^n \mathcal{D}(x_i, y_i)$ (denoted by $\mathcal{PE}(x_1, y_1, \cdots, x_n, y_n)$) is the domain of “potential erasers” of the points $(\widehat{x}_i, y_i)$, $1 \le i \le n$ (see Figure 1). This geometrical interpretation provides us a more intuitive way to rewrite Theorem 2: let us consider the sequences $(\eta_i)_{i\geq 1}$, $(\rho_i)_{i\geq 1}$ and $(\rho'_i)_{\imath\geq 1}$ defined as in Theorem 2 and a Poisson point process Φ_+ on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $\mathbf{1}_{(\mathbb{R}_+)^2}(x,y)dxdy$ independent from the three sequences above. Then for every $n \geq 1$,

$$(X_1^\varepsilon, Y_1^\varepsilon, (X_2^\varepsilon - X_1^\varepsilon), Y_2^\varepsilon, \cdots, (X_n^\varepsilon - X_{n-1}^\varepsilon), Y_n^\varepsilon)$$

is distributed as $(\eta_1, \rho_1, r+\eta_2, \rho_2, \cdots, r+\eta_{n-1}, \rho_{n-1}, r+\eta_n, \rho'_n)$ conditioned on the event

$$\{\Phi_+ \cap \mathcal{PE}(\eta_1, \rho_1, r+\eta_2, \rho_2, \cdots, r+\eta_n, \rho'_n) = \emptyset\}.$$

5 The saturation case

In this section, the saturation model Λ_∞ is directly defined and we prove that Λ_∞ is the limiting process of Λ_ε when $\varepsilon \to +\infty$. As in Sections 2 and 4, the following quantities related to Λ_∞ are introduced: λ_∞, D_∞, L_∞, $|I_0^\infty|$ and (X_i^∞, Y_i^∞), $i \geq 1$. Most of the calculations done in the unsaturated case (i.e. Λ_ε) are valid in the saturated case (i.e. Λ_∞) as well. Moreover, we obtain the convergence of λ_ε, D_ε, L_ε, X_1^ε,..., to their analogues for Λ_∞.

More precisely, let us consider the process

$$\Lambda_\infty = \{x \in \mathbb{R}; \exists y \geq 0 | (x,y) \in \Psi\},$$

where Ψ is the point process on $\mathbb{R} \times \mathbb{R}_+$ defined in the first section. Λ_∞ is a saturation model in the sense that no new crack can be added. Consequently, two successive points of Λ_∞ are separated by at least a distance r and at most $2r$.

As in the unsaturated case, λ_∞, D_∞, L_∞ and $|I_0^\infty|$ are respectively defined as the mean number of points of Λ_∞ per unit length, the typical inter-crack distance of Λ_∞ in the Palm sense, the typical stress level of a point of Λ_∞ and the length of the smallest interval containing the origin and bounded by two points of Λ_∞. The calculations of Sections 2 and 3 still hold when ε is replaced by $+\infty$. Consequently, the analogues of Theorem 1 and Proposition 15 can be obtained:

Theorem 21 *We have :*

(i) $\lambda_\infty = \int_0^{+\infty} \alpha(v)^2 dv$;

(ii) a) The distribution of D_∞ has a density φ_{D_∞} given by

$$\varphi_{D_\infty}(x) = \frac{2}{\lambda_\infty} \mathbf{1}_{[r,2r]}(x) \int_0^{+\infty} e^{-(x-r)v} v\alpha(v)^2 dv; \tag{92}$$

b) The two-dimensional r.v. (D_∞, L_∞) has the following density:

$$\varphi_{(D_\infty, L_\infty)}(y,u) = \varphi_{L_\infty}(y) \Pi^{D_\infty}(y;u), \tag{93}$$

where φ_{L_∞} (resp. $\Pi^{D_\infty}(y;\cdot)$) is the density of L_∞ (resp. the density of D_∞ conditionally on $L_\infty = y$) and

$$\varphi_{L_\infty}(y) = \frac{1}{\lambda_\infty} \alpha(y)^2 \mathbf{1}_{\mathbb{R}_+}(y), \tag{94}$$

$$\Pi^{D_\infty}(y;u) = \mathbf{1}_{\{r\leq u\leq 2r\}} \left\{ y e^{-(u-r)y} + \frac{e^{-ry}}{\alpha(y)} \int_y^{+\infty} e^{-(u-r)v} \alpha(v) dv \right\}; \tag{95}$$

(iii) The distribution of $|I_0^\infty|$ has a density $\varphi_{|I_0^\infty|}$ given by

$$\varphi_{|I_0^\infty|}(x) = 2x \int_0^{+\infty} e^{-(x-r)v} v\alpha(v)^2 dv \cdot \mathbf{1}_{[r,2r]}(x). \tag{96}$$

Remark 22 (i) In a different theoretical context, Rényi gave an equivalent formulation of the point (i) in [18] (see result (0.10)) and he estimated that the mean crack number at saturation is approximately 0.748 (for $r = 1$). To our knowledge, the other results of Theorem 23 are new.
(ii) Let us notice that the distribution of D_∞ has a decreasing density on $[r, 2r]$ and as for (20), by an easy calculation, we have:

$$\mathbf{E}D_\infty = \frac{1}{\lambda_\infty}.$$

We see at once that Λ_ε converges in law [15] to Λ_∞ and λ_ε tends to λ_∞ when $\varepsilon \to +\infty$. Moreover, we have the following:

Theorem 23 *When ε goes to infinity (saturation), $(D_\varepsilon, L_\varepsilon)$ (resp. $|I_0^\varepsilon|$) converges in distribution to (D_∞, L_∞). (resp. $|I_0^\infty|$).*

Proof. Let us investigate the convergence of $\mathbf{P}\{D_\varepsilon \geq t; L_\varepsilon \leq s\}$, $t \geq r$, $s \geq 0$, when $\varepsilon \to +\infty$. We notice that

$$\alpha(t) \underset{t\to+\infty}{\sim} \frac{\alpha_0}{t}, \tag{97}$$

where $\alpha_0 = (1/r)\exp\left\{-\int_0^1 \frac{1-e^{-s}}{s} ds + \int_1^{+\infty} \frac{e^{-s}}{s} ds\right\}$.

Combining Theorem 1 (ii) with (97), we obtain that

$$\mathbf{P}\{D_\varepsilon \geq 2r\} = \frac{\varepsilon}{\lambda_\varepsilon}\alpha(\varepsilon)^2 \underset{\varepsilon\to+\infty}{\longrightarrow} 0.$$

It remains to prove that for any $t \in [r, 2r]$ and $s \geq 0$, $\mathbf{P}\{D_\varepsilon \geq t; L_\varepsilon \leq s\}$ converges to $\mathbf{P}\{D_\infty \geq t; L_\infty \leq s\}$ which is clear from (63) and (93).

The same method holds for the convergence of $|I_0^\varepsilon|$.

□

Let us define in the same way as Λ_∞ the one-sided process Λ_∞^+ of the cracks on the positive half-line. $X_1^\infty, \cdots, X_n^\infty, \cdots$ denote the successive crack positions of this process and $Y_1^\infty, \cdots, Y_n^\infty, \cdots$ their corresponding stress levels. Similarly to Section 4, we are able to determine the joint distribution of

$$\{(X_i^\infty, Y_i^\infty); 1 \leq i \leq n\}, \quad n \geq 1.$$

Theorem 24 *(i) (X_1^∞, Y_1^∞) is a two-dimensional r.v. with density:*

$$\varphi_{(X_1^\infty, Y_1^\infty)}(u, v) = e^{-uv}\alpha(v)\mathbf{1}_{\{0\leq u\leq r\}}\mathbf{1}_{\{v\geq 0\}};$$

(ii) Conditionally on $(X_1^\infty, \cdots, X_n^\infty, Y_1^\infty, \cdots, Y_n^\infty)$, the couple $(X_{n+1}^\infty - X_n^\infty, Y_{n+1}^\infty)$ has a density $\theta^\infty(Y_n^\infty; \cdot)$ such that for every $y, u, v \geq 0$:

$$\theta^\infty(y; u, v) = \left(e^{-(u-r)y}e^{-rv}\mathbf{1}_{\{0\leq v\leq y\}} + e^{-ry}e^{-(u-r)v}\mathbf{1}_{\{v>y\}}\right)\frac{\alpha(v)}{\alpha(y)}\mathbf{1}_{\{r\leq u\leq 2r\}}.$$

Remark 25 (i) Since the process Λ_∞^+ is a saturation model, we obviously have

$$X_1^\infty \leq r \text{ and } (X_{n+1}^\infty - X_n^\infty) \in [r, 2r], \quad \forall n \geq 1 \ a.s..$$

However we are not able to prove, as we did in Theorem 2, that $(X_n^\infty)_{n\geq 1}$ is a “conditional renewal process”.
(ii) As in the non-saturated case, it suffices to have the law of (X_1^∞, Y_1^∞) on the one hand and the law of $(Y_n^\infty)_{n\geq 1}$ on the other to determine the positions $(X_n^\infty)_{n\geq 1}$.
(iii) It is immediate that the sequence $(X_i^\varepsilon, Y_i^\varepsilon)_{i\geq 1}$ converges in distribution to $(X_i^\infty, Y_i^\infty)_{i\geq 1}$.

Theorem 18 may be easily generalized to the saturated case in the following way.

Theorem 26 *$(Y_n^\infty)_{n\geq 1}$ is a homogeneous Markov chain such that:*

(i) Y_1^∞ has a density $\varphi_{Y_1^\infty}$ such that

$$\varphi_{Y_1^\infty}(y) = \frac{1 - e^{-ry}}{y} \alpha(y) \mathbf{1}_{\mathbb{R}_+}(y);$$

(ii) the law of Y_{n+1}^∞ conditionally on $\{Y_n^\infty = y\}$, $y \geq 0$, is independent from n and has a density $\Pi^{Y,\infty}(y; \cdot)$ such that for every $v \geq 0$,

$$\Pi^{Y,\infty}(y; v) = \left(\mathbf{1}_{\{0\leq v\leq y\}} (1 - e^{-ry}) \frac{e^{-rv}}{y} + \mathbf{1}_{\{y<v\}} e^{-ry} \frac{1 - e^{-rv}}{v} \right) \frac{\alpha(v)}{\alpha(y)};$$

(iii) the stationary law of $(Y_n^\infty)_{n\geq 1}$ is the distribution of L_∞, with density given by (94);

(iv) conditionally on $(X_1^\infty, Y_1^\infty, \cdots, X_n^\infty, Y_n^\infty)$, the distribution of the r.v. $(X_{n+1}^\infty - X_n^\infty)$ has a density which only depends on Y_n^∞ and is equal to $\Pi^{D_\infty}(Y_n^\infty; \cdot)$ (where $\Pi^{D_\infty}(y; \cdot)$ is given by (95)).

Finally, Theorem 19 is generalized to the saturated case.

Theorem 27 *The couple $(X_{n+1}^\infty - X_n^\infty, Y_n^\infty)$ converges in law when $n \to +\infty$, and the limit distribution is the law of (D_∞, L_∞) (provided by (93)).*

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