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# New decompositions of 2-structures

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## Abstract

We present a family of decompositions of 2-structures generalizing the modular decomposition, and  $O(n^3)$  time algorithms to compute all these decompositions. These results can be applied to non-oriented, oriented and directed graphs. Bi-join decomposition of non-oriented graphs and of tournaments are two special cases of this family of decomposition. Two others special cases are generalisations of the bi-join decomposition on directed graphs.

## 1 Introduction

The well-known modular decomposition of graph has many applications in graph theory and algorithms. It is unique [8] and can be computed in linear time (*i.e.* in  $O(n + m)$ ) on non-oriented graphs [11], on directed graphs [10], and in linear time (*i.e.* in  $O(n^2)$ ) on 2-structures [9]. The bi-join decomposition is a generalisation of the modular decomposition on non-oriented graphs [13, 14] and on tournaments [2]. These two decompositions can be computed in linear time.

We present a family of decompositions of 2-structures which generalize the modular decomposition. We show that these decompositions are unique, and we present an algorithm to compute them in time  $O(n^3)$  (for a fixed decomposition in the family). We apply these results to oriented and directed graphs. We give two new different decompositions for directed graphs which generalize the bi-join decomposition of non-oriented graphs and tournaments, and we give a new decomposition for oriented graphs. Bi-join decomposition of non oriented graphs and bi-join decomposition of tournament are also special cases of this family of decompositions.

After some preliminaries in section 2, we introduce in section 3 the G-joins and show that G-joins have the bipartitive property. In section 4 we define the G-join decomposition. For any fixed abelian group with some properties, there is a different G-join decomposition. In section 5 we give some special cases of decompositions on non-oriented, oriented and directed graphs. Finally, we present an  $O(n^3)$  algorithm to compute the G-join decomposition in section 6, for any fixed abelian group.

## 2 Preliminaries

### 2.1 Graphs and 2-structures

A *directed graph*  $G = (V, A)$  is a pair of a set of *vertices*  $V$  and a set of *arcs*  $A \subseteq V \times V \setminus \{(u, u) : u \in V\}$ . A *non-oriented graph* is a directed graph such that for all  $(u, v) \in V^2$ , with  $u \neq v$ , then

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$(u, v) \in A$  if and only if  $(v, u) \in A$ . An *oriented* graph is a directed graph such that for all  $(u, v)$ ,  $(u, v) \in A \Rightarrow (v, u) \notin A$ . A *tournament* is a oriented graph such that either  $(u, v) \in A$  or  $(v, u) \in A$ .

Let  $\mathcal{D}$  be a set. A *2-structure on  $\mathcal{D}$*  is a pair  $(V, e)$  such that  $e : V \times V \rightarrow \mathcal{D}$ . In this paper, every 2-structure is finite (*i.e.*  $V$  is finite). A 2-structure is *symmetric* if  $e(u, v) = e(v, u)$  for all  $u, v \in V$ ,  $u \neq v$ . Let  $\sigma$  be an involution on  $\mathcal{D}$  (*i.e.* a bijection such that  $\sigma(\sigma(x)) = x$  for all  $x \in \mathcal{D}$ ). A 2-structure  $(V, e)$  on  $\mathcal{D}$  is  *$\sigma$ -symmetric* if  $e(u, v) = \sigma(e(v, u))$  for all  $u, v \in \mathcal{D}$ ,  $u \neq v$ .

A directed graph can be viewed as a 2-structure on  $\mathbb{Z}_2$ , a non oriented graph as a symmetric 2-structure on  $\mathbb{Z}_2$ , and a tournament as a  $\sigma$ -symmetric 2-structure on  $\mathbb{Z}_2$  with  $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ . There is a way to transform a 2-structure on  $\mathcal{D}$  into a  $\sigma$ -symmetric 2-structure on  $\mathcal{D} \times \mathcal{D}$ : take  $e'(u, v) = (e(u, v), e(v, u))$  and  $\sigma((i, j)) = (j, i)$  for all  $i, j \in \mathcal{D}$ . For example a directed graph can also be viewed as a  $\sigma$ -symmetric 2-structure on  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , with  $\sigma = \begin{bmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (0,0) & (1,0) & (0,1) & (1,1) \end{bmatrix}$ .

## 2.2 Bipartitive families

A *bipartition* of a set  $V$  is a partition  $\{X, Y\}$  of  $V$  such that  $X \neq \emptyset$  and  $Y \neq \emptyset$ . We write sometimes  $\{X, -\}$  instead of  $\{X, V \setminus X\}$ . Two bipartitions  $\{X, Y\}$  and  $\{X', Y'\}$  *overlap* (or  $\{X, Y\}$  *overlaps*  $\{X', Y'\}$ ) if  $X \cap X'$ ,  $X \cap Y'$ ,  $Y \cap X'$  and  $Y \cap Y'$  are non empty. A family  $\mathcal{F}$  of bipartitions of  $V$  is *weakly bipartitive* if:

- for all  $v \in V$ ,  $\{\{v\}, V \setminus \{v\}\}$  is in  $\mathcal{F}$ , and
- for all  $\{X, Y\}$  and  $\{X', Y'\}$  in  $\mathcal{F}$  such that  $\{X, Y\}$  overlaps  $\{X', Y'\}$ , then  $\{X \cap X', Y \cup Y'\}$ ,  $\{X \cap Y', Y \cup X'\}$ ,  $\{Y \cap X', X \cup Y'\}$  and  $\{Y \cap Y', X \cup X'\}$  are in  $\mathcal{F}$ .

Moreover a weakly bipartitive family  $\mathcal{F}$  is *bipartitive* if for all  $\{X, Y\}$  and  $\{X', Y'\}$  which overlap in  $\mathcal{F}$ ,  $\{X \Delta X', X \Delta Y'\}$  is in  $\mathcal{F}$  (where  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ ). Bipartitive families are close to *partitive families* [3, 12] but deals with bipartitions of  $V$  instead of subsets of  $V$ .

A member of a bipartitive family is *strong* if it overlaps no other member in the family. A member  $\{X, Y\}$  is *trivial* if  $|X| = 1$  or  $|Y| = 1$ . Let  $T = (V, E)$  be a tree. We denote by  $\text{Leaves}(T)$  the leaves of  $T$ . For  $\beta \in V$ , let  $\{A_\beta^1, \dots, A_\beta^{d(\beta)}\}$  be the connected components of  $T - \beta$ . Let  $C_\beta^i = A_\beta^i \cap \text{Leaves}(T)$ . For  $e \in E$ , let  $A_e^1$  and  $A_e^2$  be the connected components of  $T - e$ , and let  $\{C_e^1, C_e^2\} = \{A_e^1 \cap \text{Leaves}(T), A_e^2 \cap \text{Leaves}(T)\}$ . The following result can be found in [14] or in [6] using a different formalism. This result can also be easily showed from known results of weakly partitive families [3, 12].

**Theorem 1.** [6, 14] *Let  $\mathcal{F}$  be a weakly bipartitive family  $\mathcal{F}$  on  $V$ . Then there is a unique unrooted tree  $T = (V_T, E_T)$ , call the representative tree, such that  $\text{Leaves}(T) = V$ , and each internal node has at least 3 neighbors and is marked *degenerate*, *linear* or *prime*, such that:*

- For all  $e \in E_T$ ,  $\{C_e^1, C_e^2\}$  is a strong member of  $\mathcal{F}$  and there is no other strong members in  $\mathcal{F}$ .
- Let  $\beta \in V_T$  be an internal node, and let  $k$  be the degree of  $\beta$ .
  - If  $\beta$  is *degenerated*, then for all  $\emptyset \subsetneq I \subsetneq \{1, \dots, k\}$ ,  $\{\cup_{i \in I} C_\beta^i, -\}$  is in  $\mathcal{F}$ .
  - If  $\beta$  is *linear*, there is a ordering  $C_\beta^1, \dots, C_\beta^k$  such that for all  $a, b \in \{1, k\}$  with  $a \leq b$  and  $(a, b) \neq (1, k)$ ,  $\{\cup_{i \in \{a, \dots, b\}} C_\beta^i, -\}$  is in  $\mathcal{F}$ .
- There is no other members in  $\mathcal{F}$ .

Furthermore if  $\mathcal{F}$  is bipartitive, then  $T$  has no *linear node*.

Decompositions based on bipartitive families have been studied in [6] under a formalism called *decomposition frame* with some properties. Some examples of this decomposition frame can be found in [4, 5]. Bipartitive families based decompositions are interesting since the bipartitivity imply a unique decomposition. Furthermore, this imply that a greedy algorithm to decompose the structure will always work: if we can find in polynomial time a decomposable bipartition in the structure, then we can decompose the whole structure in polynomial time.

### 2.3 Modular decomposition and bi-join decomposition

A *module* in a 2-structure  $G = (V, e)$  is a non-empty  $X \subseteq V$  such that for all  $v \notin X$  and  $u, u' \in X$ ,  $e(v, u) = e(v, u')$  and  $e(u, v) = e(u', v)$ . The family of modules of a 2-structure is weakly partitive, and is partitive if the 2-structure is symmetric [7]. If a structure  $G$  has a non-trivial module  $X$ , then it can be decomposed into  $G[X]$  and  $G[V \setminus X \cup \{x\}]$ , where  $x \in X$ . Note that the structure  $G$  can be easily reconstructed from  $G[X]$  and  $G[V \setminus X \cup \{x\}]$ . The *modular decomposition* is defined by recursively decompose the structure by a non-trivial module. It can be represented by a tree, call the *modular decomposition tree*, which is exactly the representative tree of the family of modules.

A *bi-join* in a non-oriented graph  $G = (V, E)$  is a bipartition  $\{X, Y\}$  of  $V$  such that for all  $v, v' \in X$ ,  $\{N(v) \cap Y, Y \setminus N(v)\} = \{N(v') \cap Y, Y \setminus N(v')\}$ . A *bi-join* in a tournament  $G = (V, A)$  is a bipartition  $\{X, Y\}$  of  $V$  such that for all  $v, v' \in X$ ,  $\{N^+(v) \cap Y, Y \setminus N^+(v)\} = \{N^+(v') \cap Y, Y \setminus N^+(v')\}$ .

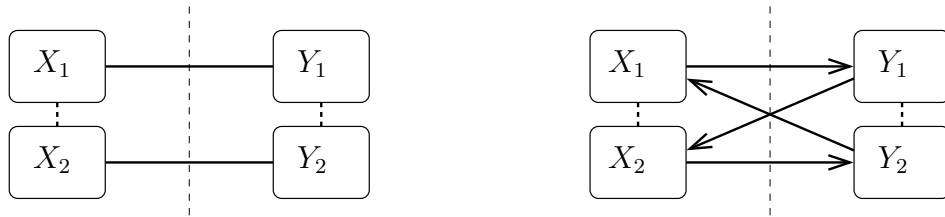


Figure 1: A bi-join in a non-oriented graph and a tournament.  $X = X_1 \cup X_2$  and  $Y = Y_1 \cup Y_2$ .

If  $X \subsetneq V$  is a module of  $G$ , then it is a bi-join of  $G$ . The family of bi-joins of a undirected graph is bipartitive [13], and the family of bi-joins of a tournament is weakly bipartitive [2]. If a graph has a non-trivial bi-join, then it can be decomposed into two graphs, and the *bi-join decomposition* is the recursive decomposition by a strong non-trivial bi-join. Since the family of bi-join is bipartitive, the bi-join decomposition tree is unique (and is isomorphic to the representative tree).

### 2.4 Abelian group

We recall axioms of an abelian group  $(\mathcal{D}, \dot{+})$ .

**Neutral element:** There is an element  $\dot{0}$  in  $\mathcal{D}$  such that for all  $a$  in  $\mathcal{D}$ ,  $\dot{0} \dot{+} a = a \dot{+} \dot{0} = a$ .

**Inverse element:** For each  $a$  in  $\mathcal{D}$  there is an element  $a^{-1}$  in  $\mathcal{D}$  such that  $a \dot{+} a^{-1} = a^{-1} \dot{+} a = \dot{0}$ , where  $\dot{0}$  is the neutral element. (We will write  $\dot{-}a$  for  $a^{-1}$ .)

**Associativity:** For all  $a, b$  and  $c$  in  $\mathcal{D}$ ,  $(a \dot{+} b) \dot{+} c = a \dot{+} (b \dot{+} c)$ .

**Commutativity:** For all  $a$  and  $b$  in  $\mathcal{D}$ ,  $a \dot{+} b = b \dot{+} a$ .

### 3 G-joins

#### 3.1 Definition

Throughout this section, we fix an abelian group  $(\mathcal{D}, \dot{+})$ . Let  $(V, e)$  be a 2-structure on  $\mathcal{D}$ . A pair  $(X, Y)$  with  $X \neq \emptyset$ ,  $Y \neq \emptyset$  and  $V = X \uplus Y$  (i.e.  $X \cup Y = V$  and  $X \cap Y = \emptyset$ ) is a *G-join* if there is pairwise disjoint  $X_i$  and  $Y_i$  (for  $i \in \mathcal{D}$ ) such that  $X = \bigsqcup_{i \in \mathcal{D}} X_i$ ,  $Y = \bigsqcup_{j \in \mathcal{D}} Y_j$ , and for all  $(i, j) \in \mathcal{D}^2$  and  $(u, v) \in (X_i, Y_j)$ ,  $e(u, v) = i \dot{+} j$ . We start with some easy observations.

**Proposition 2.** *If  $(X, Y)$  is a G-join of  $G$  and  $V' \subseteq V$  such that  $V' \cap X \neq \emptyset$  and  $V' \cap Y \neq \emptyset$ , then  $(X \cap V', Y \cap V')$  is a G-join of  $G[V']$ .*

**Proposition 3.** *If  $M$  is a module of  $(V, e)$ , then  $(M, V \setminus M)$  and  $(V \setminus M, M)$  are G-joins of  $(V, e)$ .*

**Proposition 4.** *For every pairwise different  $a, b, c, d \in V$  such that there is a G-join  $\{X, Y\}$  with  $\{a, c\} \subseteq X$  and  $\{b, d\} \subseteq Y$ , then  $e(c, d) = e(c, b) \dot{+} e(a, d) \dot{-} e(a, b)$ .*

**Lemma 5.** *Let  $(V, e)$  be a 2-structure on  $\mathcal{D}$ . Let  $(X, Y)$  and  $(X', Y')$  be two G-joins of  $(V, e)$  such that  $X \cap X' \neq \emptyset$  and  $Y \cap Y' \neq \emptyset$ . Then  $(X \cap X', Y \cup Y')$  is a G-join of  $(V, e)$ .*

*Proof.* Let  $X_i$  and  $Y_i$  (for  $i \in \mathcal{D}$ ) such that  $X = \bigsqcup_{i \in \mathcal{D}} X_i$ ,  $Y = \bigsqcup_{j \in \mathcal{D}} Y_j$ , and for all  $(u, v) \in (X_i, Y_j)$ ,  $e(u, v) = i \dot{+} j$ . Similarly let  $X'_i$  and  $Y'_i$  ( $i \in \mathcal{D}$ ) such that  $X' = \bigsqcup_{i \in \mathcal{D}} X'_i$ ,  $Y' = \bigsqcup_{j \in \mathcal{D}} Y'_j$ , and for all  $(u, v) \in (X'_i, Y'_j)$ ,  $e(u, v) = i \dot{+} j$ .

Since  $Y \cap Y'$  is non-empty, let  $v \in Y \cap Y'$ , and let  $j, j' \in \mathcal{D}$  such that  $v \in Y_j \cap Y'_{j'}$ . Suppose that  $w \in X \cap X'$ , and let  $i, i' \in \mathcal{D}$  such that  $w \in X_i \cap X'_{i'}$ . Then  $e(w, v) = i \dot{+} j = i' \dot{+} j'$ . Thus  $X \cap X' = \bigsqcup_{i \in \mathcal{D}} X_i \cap X'_{i \dot{-} j \dot{-} j'}$ . Moreover, for all  $u \in Y \cap Y'$ ,  $e(w, u) = i \dot{+} k = i' \dot{+} k'$  (with  $u \in Y_k \cap Y'_{k'}$ ), thus  $k' = i \dot{-} i' \dot{+} k = j' \dot{-} j \dot{+} k$ , and  $Y \cap Y' = \bigsqcup_{k \in \mathcal{D}} Y_k \cap Y'_{k \dot{+} j' \dot{-} j}$ .

For all  $k \in \mathcal{D}$ , let  $X''_k = X_k \cap X'_{k \dot{+} j \dot{-} j'}$ , and let  $Y''_k = Y_k \cup Y'_{k \dot{+} j' \dot{-} j - 1}$ .  $X \cap X' = \bigsqcup_{i \in \mathcal{D}} X''_i$  and  $Y \cup Y' = \bigsqcup_{k \in \mathcal{D}} Y''_k$ . For all  $u \in X''_i$  and  $v \in Y''_k$ ,  $e(u, v) = i \dot{+} k$ . Thus  $(X \cap X', Y \cup Y')$  is a G-join.  $\square$

#### 3.2 G-joins in $\sigma$ -symmetric 2-structures

A function  $f$  is a *isomorphism* for  $(\mathcal{D}, \dot{+})$  if  $f(a \dot{+} b) = f(a) \dot{+} f(b)$  for all  $(a, b) \in \mathcal{D}^2$ . From now  $\sigma$  will denote an involution on  $\mathcal{D}$  such that the function  $f : a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$  is an isomorphism for  $(\mathcal{D}, \dot{+})$  (where  $\dot{0}$  is the neutral element).

**Lemma 6.** *Let  $(V, e)$  is a  $\sigma$ -symmetric 2-structure, and let  $X$  and  $Y$  such that  $V = X \uplus Y$ . Then  $(X, Y)$  is a G-join if and only if  $(Y, X)$  is a G-join.*

*Proof.* Let  $X_a$  and  $Y_a$  (for  $a \in \mathcal{D}$ ) such that  $X = \bigsqcup_{a \in \mathcal{D}} X_a$ ,  $Y = \bigsqcup_{a \in \mathcal{D}} Y_a$ , and for all  $(u, v) \in (X_a, Y_b)$ ,  $e(u, v) = a \dot{+} b$ . Let  $X'_a = X_{\sigma(a)}$  and  $Y'_a = Y'_{\sigma(a) \dot{-} \sigma(\dot{0})}$ , for all  $a \in \mathcal{D}$ . Since  $\sigma$  is a bijection,  $X = \bigsqcup_{a \in \mathcal{D}} X'_a$  and  $Y = \bigsqcup_{a \in \mathcal{D}} Y'_a$ . Moreover, for all  $u \in X'_a$  and  $v \in Y'_b$ ,  $e(u, v) = \sigma(a) \dot{+} \sigma(b) \dot{-} \sigma(\dot{0}) = f(a) \dot{+} f(b) \dot{+} \sigma(\dot{0}) = f(a \dot{+} b) \dot{+} \sigma(\dot{0}) = \sigma(a \dot{+} b)$ , and  $e(v, u) = a \dot{+} b$ . Thus  $(Y, X)$  is a G-join.  $\square$

We say that  $\{X, Y\}$  is a G-join of  $(V, e)$  if  $(X, Y)$  is a G-join of  $(V, e)$ . Lemmas 5 and 6 show that if  $\{X, Y\}$  and  $\{X', Y'\}$  are two G-joins such that  $\{X, Y\}$  overlaps  $\{X', Y'\}$ , then  $\{X \cap X', Y \cup Y'\}$  is a G-join. Therefore we have:

**Corollary 7.** *The family of G-joins of a  $\sigma$ -symmetric 2-structure is weakly bipartitive.*

### 3.3 G-joins in symmetric 2-structures

**Lemma 8.** *Let  $(V, e)$  be a symmetric 2-structure. Let  $\{X, Y\}$  and  $\{X', Y'\}$  be two G-joins of  $(V, e)$  such that  $\{X, Y\}$  overlaps  $\{X', Y'\}$ . Then  $\{X\Delta X', X\Delta Y'\}$  is a G-join of  $(V, e)$ .*

*Proof.* Let  $v \in Y \cap Y'$ ,  $w \in X \cap X'$ , and let  $(j, j', l, l') \in \mathcal{D}^4$  such that  $v \in Y_j \cap Y'_{j'}$  and  $w \in X_l \cap X'_{l'}$ . As we show in proof of Lemma 5,  $X \cap X' = \biguplus_{k \in \mathcal{D}} X_k \cap X'_{k \dot{+} j \dot{-} j'}$  and  $Y \cap Y' = \biguplus_{k \in \mathcal{D}} Y_k \cap Y'_{k \dot{+} j \dot{-} j'}$ . Using similar argument,  $Y \cap X' = \biguplus_{k \in \mathcal{D}} Y_k \cap X'_{k \dot{+} l \dot{-} l'}$  and  $X \cap Y' = \biguplus_{k \in \mathcal{D}} X_k \cap Y'_{k \dot{+} l \dot{-} l'}$ .

Let  $X''_k = (X_k \cap X'_{k \dot{+} j \dot{-} j'}) \cup (Y_{k \dot{+} j \dot{-} j' \dot{+} l \dot{-} l'} \cap Y'_{k \dot{+} l \dot{-} l'})$  and  $Y''_k = (Y_k \cap X'_{k \dot{+} l \dot{-} l'}) \cup (X_{k \dot{+} j \dot{-} j \dot{+} l \dot{-} l'} \cap Y'_{k \dot{+} j \dot{-} j'})$ . For all  $u \in X''_k$  and  $v \in Y''_l$ ,  $e(u, v) = k \dot{+} l$ . Thus  $\{\biguplus_{k \in \mathcal{D}} X''_k, \biguplus_{k \in \mathcal{D}} Y''_k\}$  is a G-join.  $\square$

With Lemma 5, we obtain:

**Corollary 9.** *The family of G-joins of a symmetric 2-structure is bipartitive.*

## 4 G-join decomposition

In this section, we fix an abelian group  $(\mathcal{D}, \dot{+})$  and an involution  $\sigma$  such that  $f : a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$  is an isomorphism for  $(\mathcal{D}, \dot{+})$ . For most part, our terminology follows terminology used in [4, 6].

### 4.1 Simple decomposition

A G-join  $\{X, Y\}$  is *trivial* if  $|X| = 1$  or  $|Y| = 1$ . Since every singleton is a module, every bipartition  $\{X, Y\}$  with  $|X| = 1$  or  $|Y| = 1$  is a G-join.

Let  $G = (V, e)$  be a  $\sigma$ -symmetric 2 structure and  $\{X, Y\}$  be a non-trivial G-join. Let  $x \in X$  and  $y \in Y$ . A *simple decomposition* of  $(V, e)$  by the G-join  $(X, Y)$  is the decomposition into  $G_1 = (X \cup \{y\}, e|_{X \cup \{y\}})$  and  $G_2 = (Y \cup \{x\}, e|_{Y \cup \{x\}})$  with an additional *marker triplet*  $(x, y, \alpha)$ , where  $\alpha = e(x, y)$  ( $e|_X$  represents the function  $e$  induced by  $X \times X$ ). We write  $G \rightarrow (G_1, G_2, (x, y, \alpha))$ . Note that this decomposition is not unique for a fixed  $\{X, Y\}$ .

The *simple composition* of  $(V_1, e_1)$ ,  $(V_2, e_2)$  and the marker triplet  $(x, y, \alpha)$ , with  $V_1 \cap V_2 = \{x, y\}$ , is the 2-structure  $(V_1 \cup V_2, e)$  where  $e(a, b) = e_1(a, b)$  for all  $a, b \in V_1 \setminus \{y\}$ ,  $e(a, b) = e_2(a, b)$  for all  $a, b \in V_2 \setminus \{x\}$ , and  $e(a, b) = e_1(a, y) \dot{-} \alpha \dot{+} e_2(x, b)$  for all  $a \in V_1 \setminus \{y\}$  and  $b \in V_2 \setminus \{x\}$ . By Proposition 4, if  $((V_1, e_1), (V_2, e_2), (x, y, \alpha))$  is a simple decomposition of  $(V, e)$ , then the simple composition of  $(V_1, e_1)$ ,  $(V_2, e_2)$  and  $(x, y, \alpha)$  is  $(V, e)$ .

**Lemma 10.** *Let  $\{X, Y\}$  be a G-join of  $G$ , and  $(G_1, G_2, (x, y, \alpha))$  be the simple decomposition of  $G$  by  $(X, Y)$ . Let  $\{X', Y'\}$  be a bipartition of  $V$  with  $Y' \subsetneq Y$ . Then  $\{X', Y'\}$  is a G-join of  $G$  if and only if  $\{\{x\} \cup Y \setminus Y', Y'\}$  is a G-join of  $G_2$ .*

*Proof.* If  $\{X', Y'\}$  is a G-join of  $G$  then by Proposition 2  $\{\{x\} \cup Y \setminus Y', Y'\}$  is a G-join of  $G_2$ . Now suppose that  $\{\{x\} \cup Y \setminus Y', Y'\}$  is a G-join of  $G_2$ . Let  $X'_a$  and  $Y'_a$  (for  $a \in \mathcal{D}$ ) such that  $(\{x\} \cup Y \setminus Y', Y') = (\biguplus_{a \in \mathcal{D}} X'_a, \biguplus_{a \in \mathcal{D}} Y'_a)$  and  $e(u, v) = a \dot{+} b$  for all  $u \in X'_a$  and  $v \in Y'_b$ . Since  $\{X, Y\}$  is a G-join of  $G$ , let  $X_a$  and  $Y_a$  such that  $(X, Y) = (\biguplus_{a \in \mathcal{D}} X_a, \biguplus_{a \in \mathcal{D}} Y_a)$  and  $e(u, v) = a \dot{+} b$  for all  $u \in X_a$  and  $v \in Y_b$ . Let  $c, d \in \mathcal{D}$  such that  $x \in X'_c$  and  $y \in Y_d$ . Let  $X''_a = (X'_a \setminus \{x\}) \cup X_{a \dot{-} d \dot{-} c \dot{+} \alpha}$  and  $Y''_a = Y'_a$ . Let  $u \in X''_a$  and  $v \in Y''_b$ . If  $u \in X'_a$  then  $e(u, v) = a \dot{+} b$ . Otherwise  $u \in X_{a \dot{-} d \dot{-} c \dot{+} \alpha}$  and by definition of simple decomposition,  $e(u, v) = e(u, y) \dot{-} \alpha \dot{+} e(x, v) = a \dot{+} b$  since  $e(u, y) = a \dot{-} d \dot{-} c \dot{+} \alpha \dot{+} d$  and  $e(x, v) = c \dot{+} b$ . Then  $\{\biguplus_{a \in \mathcal{D}} X''_a, \biguplus_{a \in \mathcal{D}} Y''_a\} = \{X', Y'\}$  is a G-join of  $G$ .  $\square$

A G-join  $\{X, Y\}$  is *strong* if it is a strong member of the bipartitive family of G-joins of  $G$  (i.e. there is no G-join  $\{X', Y'\}$  such that  $\{X, Y\}$  overlaps  $\{X', Y'\}$ ). A simple decomposition is strong if it is induced by a strong G-join. The following Corollary follows from previous Lemma.

**Corollary 11.** *Let  $\{X, Y\}$  be a G-join of  $G$ , and  $(G_1, G_2, (x, y, \alpha))$  be the simple decomposition of  $G$  by  $(X, Y)$ . Let  $\{X', Y'\}$  be a bipartition of  $V$  with  $Y' \cap Y$ . Then  $\{X', Y'\}$  is a strong G-join of  $G$  if and only if  $\{\{x\} \cup Y \setminus Y', Y'\}$  is a strong G-join of  $G_2$ .*

## 4.2 G-join decompositions

A 2-structure is *prime* if all its G-joins are trivial. A 2-structure is *degenerated* if every bipartition is a G-join. A 2-structure  $G = (V, E)$  is *linear* if there is a ordering  $v_1, \dots, v_n$  of the vertices such that for all  $i, j \in \{1, \dots, n\}$  with  $i \leq j$  and  $(i, j) \neq (1, n)$ ,  $\{\{v_i, \dots, v_j\}, -\}$  is a G-join of  $G$ , and  $G$  has no others G-join. Every 2-structure with at most 3 vertices is degenerated, linear and prime, and every 2-structure with at least 4 vertices is either prime, degenerated, linear or none of these three cases. The following Lemma comes immediately from the bipartitivity of G-joins.

**Lemma 12.** *Let  $G$  be a 2-structure.  $G$  has no strong non-trivial G-join if and only if  $G$  is either prime, degenerated or linear*

*Proof.* If  $G$  has no strong non-trivial G-join, then representative tree of  $G$  has only one internal node  $\beta$ . Then  $G$  is prime, degenerated or linear if  $\beta$  is **prime**, **degenerate** or **linear**, respectively.  $\square$

The following Lemma gives a characterisation of degenerated graphs. Its straightforward inductive proof is given in appendix.

**Lemma 13.** *Suppose  $\sigma(\dot{0}) = \dot{0}$ . A  $\sigma$ -symmetric 2-structure with at least 4 vertices is degenerated if and only if there is an  $\alpha \in \mathcal{D}$  such that  $\sigma(\alpha) = \alpha$ , and a function  $f : V \rightarrow \mathcal{D}$  such that for all  $u, v \in V$ ,  $u \neq v$ ,  $e(u, v) = \alpha \dot{+} f(u) \dot{+} \sigma(f(v))$ .*

Let  $G$  be a 2-structure. *G-join decompositions* of  $G$  are defined recursively:  $(\{G\}, \emptyset)$  is a G-join decomposition of  $G$  and if  $(\mathcal{D}, M)$  is a G-join decomposition of  $G$ ,  $H \in \mathcal{D}$ , and  $H_1, H_2$  is a simple decomposition of  $H$  with marker triplet  $(u, v, \alpha)$ , then  $((\mathcal{D} \setminus \{H\}) \cup \{H_1, H_2\}, M \cup \{(u, v, \alpha)\})$  is a G-join decomposition of  $G$ . In this case we say that  $(\mathcal{D}', M') = ((\mathcal{D} \setminus \{H\}) \cup \{H_1, H_2\}, M \cup \{(u, v, \alpha)\})$  is a simple decomposition of  $(\mathcal{D}, M)$ , and we write  $(\mathcal{D}, M) \rightarrow (\mathcal{D}', M')$ .

A G-join decomposition  $(\mathcal{D}, M)$  is *minimal* if every 2-structure in  $\mathcal{D}$  is prime. A G-join decomposition  $(\mathcal{D}, M)$  is *good* if no  $H \in \mathcal{D}$  has a strong non-trivial G-join. A G-join decomposition  $(\mathcal{D}, M)$  is *standard* if it can be obtained from  $(\{G\}, \emptyset)$  by a sequence of simple strong decompositions, and no  $H \in \mathcal{D}$  has a strong non-trivial G-join. Note that minimal decompositions and standard decompositions are goods. The proof of the following lemma is similar to the proof given in [6].

**Lemma 14.** *Let  $(\mathcal{D}, M)$  be a good decomposition of  $G$ . If there is no good decomposition  $(\mathcal{D}', M')$  such that  $(\mathcal{D}', M') \rightarrow (\mathcal{D}, M)$ , then  $(\mathcal{D}, M)$  is a standard decomposition of  $G$ .*

*Proof.* If  $(\mathcal{D}, M)$  is not a standard decomposition then there is a simple decomposition in the sequence of decompositions which is not strong. Let  $(\mathcal{D}_1, M_1) \rightarrow (\mathcal{D}_2, M_2) = ((\mathcal{D}_1 \setminus \{H\}) \cup \{H_1, H_2\}, M_1 \cup \{m\})$  be the last non-strong decomposition in the sequence. All the decompositions after  $(\mathcal{D}_1, M_1) \rightarrow (\mathcal{D}_2, M_2)$  are strong and correspond to unique strong G-joins of  $G$ . We construct

the decomposition  $(\mathcal{D}', M')$  from  $(\mathcal{D}_1, M_1)$  after simple decompositions of these strong G-joins.  $(\mathcal{D}', M')$  is good since there is a simple decomposition for every strong G-join in  $G$  and  $(\mathcal{D}', M') \rightarrow (\mathcal{D}, M)$  by the simple decomposition of the G-join corresponding to  $(\mathcal{D}_1, M_1) \rightarrow (\mathcal{D}_2, M_2)$ .  $\square$

The previous Lemma tells us that a standard decomposition can be obtained from a minimal decomposition by a sequence of simple compositions. This will be used in the decomposition algorithm presented in a next section.

A decomposition  $(\mathcal{D}, M)$  of  $G$  induces a unrooted tree of vertex set  $\mathcal{D}$  and  $H_1$  is adjacent to  $H_2$  if there is a  $(x, y, \alpha) \in M$  such that  $x$  is a vertex of  $H_1$  and  $y$  is a vertex of  $H_2$ . The decomposition tree of a standard decomposition is isomorphic to the representative tree of the weakly bipartitive family of G-joins and thus is unique. We call it the *standard decomposition tree*.

## 5 Specials cases of G-join decomposition

### 5.1 Bi-join decomposition

The bi-join decomposition [13, 14] is a special case with  $(\mathcal{D}, \dagger) = (\mathbb{Z}_2, +)$ . Lemma 13 says that degenerated graphs are disjoint union of two cliques if  $\alpha = 1$  and complete bipartite graphs if  $\alpha = 0$ . This decomposition has no linear node since the structure is symmetric. The bi-join decomposition of tournaments [2] is the decomposition, with  $(\mathcal{D}, \dagger) = (\mathbb{Z}_2, +)$  and  $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

### 5.2 Decomposition of oriented graphs

A directed graph  $G$  can be viewed as an  $\sigma$ -symmetric 2-structure on the set  $\{(0, 0), (1, 0), (0, 1)\}$ , with  $\sigma((i, j)) = (j, i)$  for  $(i, j) \in \{(0, 0), (1, 0), (0, 1)\}$ . There is one abelian group on  $\mathcal{D}$  such that  $a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$  is an isomorphism. This abelian group, isomorphic to  $(\mathbb{Z}_3, +)$ , is given in figure 2.

### 5.3 Decompositions of directed graphs

A directed graph  $G$  is a 2-structure on  $\mathbb{Z}_2$ , and can be viewed as a  $\sigma$ -symmetric 2-structure  $(V, e)$  on  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , with  $\sigma((i, j)) = (j, i)$ . There are two abelian groups such that  $a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$  is an isomorphism. The first one (isomorphic to  $(\mathbb{Z}_2^2, +)$ ) is given in figure 3, and the second (isomorphic to  $(\mathbb{Z}_4, +)$ ) is given in figure 4. These two decompositions are generalizations of the bi-join decomposition on both non-oriented graphs and on tournament. They are mutually exclusive, that is there is a graph prime for the first one and completely decomposable for the other one, and *vice versa*.

## 6 Decomposition algorithm

From now, we fix an abelian group  $(\mathcal{D}, \dagger)$  and an involution  $\sigma$  such that  $f : a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$  is an isomorphism for  $(\mathcal{D}, \dagger)$ .

### 6.1 Find a non trivial G-join

We give in this section a  $O(n^2)$  algorithm for the following problem: given a 2-structure  $G = (V, e)$  and  $u, v \in V$ , output a non trivial G-join  $\{X, Y\}$  such that  $u \in X$  and  $v \in Y$ , or output “No” if there is no such partition.



$\dagger$	(0,0)	(1,0)	(0,1)
(0,0)	(0,0)	(1,0)	(0,1)
(1,0)	(1,0)	(0,1)	(0,0)
(0,1)	(0,1)	(0,0)	(1,0)

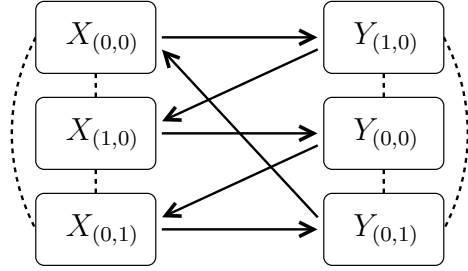


Figure 2: Decomposition for oriented graphs. (Dashed edge signify that two vertex can be adjacent or not.)

$\dagger$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

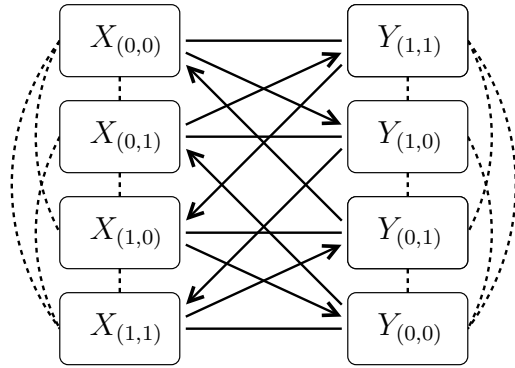


Figure 3: Decomposition for directed graphs (first).

$\dagger$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

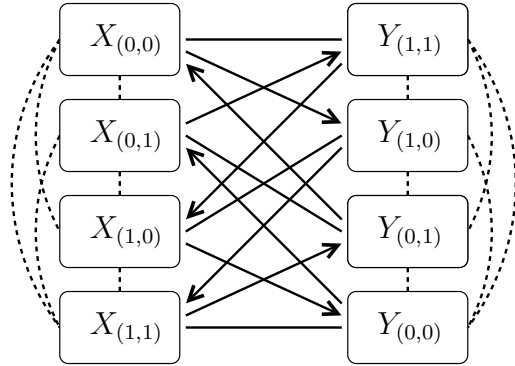


Figure 4: Decomposition for directed graphs (second).

A directed graph  $G = (V, A)$  is *strongly connected* if for every  $u, v \in V$  there is a path from  $u$  to  $v$  (i.e. there is a sequence  $u_0 = u, u_1, \dots, u_k = v$  such that for all  $i \in \{0, \dots, k-1\}$ ,  $(u_i, u_{i+1}) \in A$ ). A *strongly connected component* is a maximal subset  $W \subseteq V$  such that  $G[W]$  is strongly connected. The strongly connected components form a partition of the vertices of  $G$ , and can be found in linear time [1]. Moreover, there is always a strongly connected component  $W$  such that there is no arcs from  $W$  to  $V \setminus W$ , since the incidence graph of strongly connected components is acyclic.

We transform our problem into a 2-SAT problem. We suppose w.l.o.g. that  $u \in X_0$  and thus  $v \in Y_{e(u,v)}$ . If a vertex  $w \notin \{u, v\}$  is in  $X$  then it is in  $X_{e(w,v) \dot{-} e(u,v)}$ , and if  $w$  is in  $Y$ , it is in  $Y_{e(u,v)}$ . Let the 2-SAT problem with variable set  $V \setminus \{u, v\}$ , and  $w \Rightarrow t$  if  $e(w, v) \dot{-} e(u, v) \dot{+} e(u, t) \neq e(w, t)$ . A variable  $w$  is true means that  $w \in X$ . Then there is a non trivial G-join if and only if there is a non trivial solution for the 2-SAT problem. Let  $G_f = (V \setminus \{u, v\}, E_f)$  with  $E_f = \{(w, t) : w, t \in V \setminus \{u, v\} \text{ and } e(w, v) \dot{-} e(u, v) \dot{+} e(u, t) \neq e(w, t)\}$ . The 2-SAT problem has a non trivial solution if and only if the graph  $G_f$  is not strongly connected. In this case  $\{X \cup \{u\}, V \setminus (X \cup \{u\})\}$  is a non trivial G-join of  $G$ . All these operations can be done in time  $O(n^2)$ . (Algorithms in pseudo-code are given in appendix.)

## 6.2 Compute a minimal G-join decomposition

If a 2-structure is not prime, then a G-join can be found in  $O(n^3)$  time using the previous algorithm for a fixed  $u \in V$  and for all  $v \neq u$ . So a naive algorithm to compute a minimal decomposition take  $O(n^4)$  time. We can reach  $O(n^3)$  by the following way. We remember the set  $\mathcal{P}$  of subsets of  $V$  such that there is no non-trivial G-join which overlaps  $U$  for all  $U \in \mathcal{P}$ .  $\mathcal{P}$  is a partition of  $V$ , and at each call of the sub-routine, either it succeed and we decompose the 2-structure, either it fails and we merge two sets in  $\mathcal{P}$ . So a minimal decomposition can be obtained with  $O(n)$  call to the algorithm of section 6.1, and can computed in  $O(n^3)$ .

## 6.3 Compute a standard G-join decomposition

Lemma 14 says that a standard decomposition of  $G$  can be computed from a minimal decomposition, after some re-compositions. We show that we can test in time  $O(n^2)$  if a composition of two 2-structures degenerated or linear is degenerated or linear.

Let  $G_1 = (V_1, e_1)$  and  $G_2 = (V_2, e_2)$  and a marker triplet  $(x, y, \alpha)$ , such that  $G_1$  and  $G_2$  have no strong non-trivial G-join. If  $G$  has no strong non-trivial G-join, then by Lemma 12,  $G$  is either degenerated or linear (since it cannot be prime). If  $G_1$  or  $G_2$  is not degenerated, then  $G$  must be linear. Moreover if  $G_1$  and  $G_2$  are linear, let  $v_1, \dots, v_k$  be a linear ordering of the vertex of  $G_1$ , and let  $v'_1, \dots, v'_{k'}$  be a linear ordering of  $G_2$ . W.l.o.g.  $v_1 = y$  and  $v'_1 = x$ . Then if  $G$  is linear,  $v_2, \dots, v_k, v'_2, \dots, v'_{k'}$  or  $v_2, \dots, v_k, v'_{k'}, \dots, v'_2$  must be a linear ordering of  $G$ , and so either  $\{\{v_2, v'_2\}, -\}$  or  $\{v_2, v'_{k'}\}, -\}$  must be a G-join of  $G$ .

Let  $G_1$  and  $G_2$  be two 2-structures without strong non-trivial G-join. We want to know if the composition  $G$  of  $G_1$  and  $G_2$  with the marker triplet  $(x, y, \alpha)$  is degenerated or linear (and to know a ordering of  $G$  if it is linear). Case 1:  $|V_1| = |V_2| = 3$ . All bipartitions of  $G$  can be tested in constant time, so the type of  $G$  (and a ordering if  $G$  is linear) can be computed in  $O(1)$ . Case 2:  $|V_1| \neq 3$  or  $|V_2| \neq 3$ . Then  $G_1$  or  $G_2$  is non degenerated, or non linear. If  $G_1$  and  $G_2$  are degenerated, then  $G$  is degenerated if and only if  $\{\{x, y\}, -\}$  is a G-join of  $G$ . If  $G_1$  and  $G_2$  are linear, with ordering  $\{v_1 = y, \dots, v_k\}$  and  $\{v'_1 = x, \dots, v'_{k'}\}$ , then  $G$  is linear if  $\{\{v_2, v'_2\}, -\}$  or  $\{\{v_2, v'_{k'}\}, -\}$  is a G-join of  $G$ . In this case,  $v_2, \dots, v_k, v'_2, \dots, v'_{k'}$  or  $v_2, \dots, v_k, v'_{k'}, \dots, v'_2$  is ordering of  $G$ . In others cases,  $G$

is neither degenerate nor linear. Moreover, to test if a bipartition is a G-join of a 2-structure can be done in  $O(n^2)$ .

There is at most  $O(n)$  re-compositions (at most one for each edge in the decomposition tree). To summarize, we obtain:

**Theorem 15.** *A standard G-join decomposition can be computed in time  $O(n^3)$ .*

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## 7 Appendix

### 7.1 Proof of lemma 13

Let  $(V, e)$  be a 2-structure such that there is a  $f : V \rightarrow \mathcal{D}$  and a  $\alpha$  with  $\sigma(\alpha) = \alpha$  and  $e(u, v) = \alpha \dot{+} f(u) \dot{+} \sigma(f(v))$  for all  $u, v \in V$ ,  $u \neq v$ . It is easy to see that  $(V, e)$  is  $\sigma$ -symmetric since  $\sigma(\dot{0}) = \dot{0}$  and thus  $\sigma$  is an isomorphism for  $(\mathcal{D}, \dot{+})$ . Let  $\{X, Y\}$  be a bipartition of  $V$ . Let  $X_a = \{v \in X : a = f(v)\}$  and  $Y_a = \{v \in Y : a = \alpha \dot{+} \sigma(f(v))\}$ . For all  $u \in X_a$  and  $b \in Y_b$ ,  $a \dot{+} b = f(u) \dot{+} \alpha \dot{+} \sigma(f(v)) = e(u, v)$ , thus  $\{X, Y\}$  is a G-join.

On the other hand, let  $(V, e)$  be a degenerated 2-structure such that  $|V| \geq 4$ .

*Claim 1.* For every pairwise different  $a, b, c, d \in V$ ,  $e(c, d) = e(c, b) \dot{+} e(a, d) \dot{-} e(a, b)$ .

*Proof.* Since  $(V, e)$  is degenerated,  $\{\{a, c\}, -\}$  is a G-join. By Proposition 4 we have the equality.  $\square$

*Claim 2.* For every pairwise different  $a, b, c \in V$ :

$$e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) = e(b, a) \dot{+} e(c, b) \dot{+} e(a, c).$$

*Proof.* Let  $d \in V \setminus \{a, b, c\}$ . Applying Claim 1, we get:

$$\begin{aligned} e(d, a) \dot{-} e(d, b) &= e(c, a) \dot{-} e(c, b) \\ e(d, b) \dot{-} e(d, c) &= e(a, b) \dot{-} e(a, c) \\ e(d, c) \dot{-} e(d, a) &= e(b, c) \dot{-} e(b, a). \end{aligned}$$

Thus

$$\begin{aligned} e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) &= e(b, a) \dot{+} e(c, b) \dot{+} e(a, c) \\ &= \sigma(e(a, b)) \dot{+} \sigma(e(b, c)) \dot{+} \sigma(e(c, a)). \end{aligned}$$

$\square$

Case 1:  $|V| = 4$ . W.l.o.g  $V = \{a, b, c, d\}$ . Let:

$$\begin{aligned} \alpha &= e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) \\ f(a) &= \dot{-} e(b, c) \\ f(b) &= \dot{-} e(a, c) \\ f(c) &= \dot{-} e(b, a) \dot{-} e(a, c) \dot{+} e(c, a) \\ f(d) &= e(d, a) \dot{+} e(c, b) \dot{-} \alpha \end{aligned}$$

From Claim 2,  $\sigma(\alpha) = \alpha$ . We get:

$$\begin{aligned}
f(a) \dot{+} \sigma(f(b)) \dot{+} \alpha &= \dot{-}e(b, c) \dot{-}e(c, a) \dot{+} e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) \\
&= e(a, b) \\
f(a) \dot{+} \sigma(f(c)) \dot{+} \alpha &= \dot{-}e(b, c) \dot{-}e(a, b) \dot{-}e(c, a) \dot{+} e(a, c) \dot{+} e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) \\
&= e(a, c) \\
f(b) \dot{+} \sigma(f(c)) \dot{+} \alpha &= \dot{-}e(a, c) \dot{-}e(a, b) \dot{-}e(c, a) \dot{+} e(a, c) \dot{+} e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) \\
&= e(b, c) \\
f(a) \dot{+} \sigma(f(d)) \dot{+} \alpha &= \dot{-}e(b, c) \dot{+} e(a, d) \dot{+} e(b, c) \dot{-} \alpha \dot{+} \alpha \\
&= e(a, d) \\
f(b) \dot{+} \sigma(f(d)) \dot{+} \alpha &= \dot{-}e(a, c) \dot{+} e(a, d) \dot{+} e(b, c) \dot{-} \alpha \dot{+} \alpha \\
&= e(b, d) \quad (\text{by Claim 1}) \\
f(c) \dot{+} \sigma(f(d)) \dot{+} \alpha &= \dot{-}e(b, a) \dot{-}e(a, c) \dot{+} e(c, a) \dot{+} e(a, d) \dot{+} e(b, c) \dot{-} \alpha \dot{+} \alpha \\
&= e(a, d) \dot{+} e(c, b) \dot{-}e(a, b) \quad (\text{by Claim 2}) \\
&= e(c, d) \quad (\text{by Claim 1.})
\end{aligned}$$

Thus  $f$  and  $\alpha$  have the required property.

Case 2:  $|V| > 4$ . Let  $v \in V$ .  $(V \setminus \{v\}, e)$  is degenerated and thus there is a  $f' : V \setminus \{v\} \rightarrow \mathcal{D}$  and an  $\alpha \in \mathcal{D}$  such that for all  $u, v \in V \setminus \{v\}$ ,  $e(u, v) = f'(u) \dot{+} \sigma(f'(v)) \dot{+} \alpha$ . Let  $u \neq v$ , and let  $f$  such that  $f(w) = f'(w)$  if  $x \in V \setminus \{v\}$  and  $f(v) = e(v, u) \dot{-} \sigma(f'(u)) \dot{-} \alpha$ .

$$\begin{aligned}
f(u) \dot{+} \sigma(f(v)) \dot{+} \alpha &= f'(u) \dot{+} e(u, v) \dot{-} \sigma(\sigma(f'(u))) \dot{-} \alpha \dot{+} \alpha \\
&= e(u, v)
\end{aligned}$$

Let  $w \in V \setminus \{u, v\}$  and  $x \in V \setminus \{u, v, w\}$ .

$$\begin{aligned}
f(w) \dot{+} \sigma(f(v)) \dot{+} \alpha &= f'(w) \dot{+} e(u, v) \dot{-} \sigma(\sigma(f'(u))) \dot{-} \alpha \dot{+} \alpha \\
&= f'(w) \dot{-} f'(u) \dot{+} e(u, x) \dot{+} e(w, v) \dot{-} e(w, x) \quad (\text{by Claim 1}) \\
&= e(w, v) \dot{+} f'(w) \dot{-} f'(u) \dot{+} f'(u) \dot{+} \sigma(f'(x)) \dot{+} \alpha \dot{-} f'(w) \dot{-} \sigma(f'(x)) \dot{-} \alpha \\
&= e(w, v).
\end{aligned}$$

Thus  $f$  and  $\alpha$  have the required property.

## 7.2 Algorithm to find a non trivial G-join

**Function** FINDGJOIN( $G = (V, e), u, v$ )

**Input:** a 2-structure  $G = (V, e)$  and  $u, v \in V, u \neq v$

**Output:** a non trivial G-join  $\{X, Y\}$  of  $G$  such that  $u \in X$  and  $v \in Y$ ,  
or “No” is there is no such G-join

**begin**

$f_1(u) := \dot{0}$

$f_2(v) := e(u, v)$

**For every**  $w \in V \setminus \{u, v\}$

$f_1(w) := e(w, v) \dot{-} e(u, v)$

$f_2(w) := e(u, w)$

$E_f := \{(w, t) : w, t \in V \setminus \{u, v\} \text{ and } f_1(w) \dot{+} f_2(t) \neq e(w, t)\}$

$G_f := (V \setminus \{u, v\}, E_f)$

**if**  $G_f$  is strongly connected

**output** “No”

**Otherwise**

**Let**  $W$  be a strongly connected component of  $G_f$

such that there is no arc in  $G_f$  from  $W$  to  $V \setminus W \setminus \{u, v\}$

**output**  $\{\{u\} \cup W, V \setminus \{u\} \setminus W\}$

**end** {FINDGJOIN}

## 7.3 Algorithm to compute a minimal G-join decomposition

**Function** DECOMPOSEP( $G, \mathcal{P}$ )

**Input:** a 2-structure  $G = (V, e)$  and a partition  $\mathcal{P}$  of  $V$

**Output:** a minimal G-join decomposition  $G$

**begin**

**If**  $|\mathcal{P}| = 1$  **then**

**return**  $(\{G\}, \emptyset)$

**Let**  $A, B \in \mathcal{P}, a \in A$  and  $b \in B$

**If** FINDGJOIN( $G, a, b$ ) returns “no” **then**

$P := \{A \cup B\} \cup (\mathcal{P} \setminus \{A, B\})$

**return** DECOMPOSEP( $G, \mathcal{P}$ )

**Let**  $\{X, Y\}$  be the G-join returned by FINDGJOIN

Decompose  $G$  into  $G_1$  and  $G_2$  by the G-join  $\{X, Y\}$  with marker triplet  $(x, y, \alpha)$

$\mathcal{P}_1 := \{P \in \mathcal{P} : P \subseteq X\}$

$\mathcal{P}_2 := \{P \in \mathcal{P} : P \subseteq Y\}$

$(\mathcal{D}_1, M_1) := \text{DECOMPOSEP}(G_1, \mathcal{P}_1)$

$(\mathcal{D}_2, M_2) := \text{DECOMPOSEP}(G_2, \mathcal{P}_2)$

**return**  $(\mathcal{D}_1 \cup \mathcal{D}_2, M_1 \cup M_2 \cup (x, y, \alpha))$

**end** {DECOMPOSEP}

**Function** DECOMPOSE( $G$ )  
**Input:** a 2-structure  $G = (V, e)$   
**Output:** a minimal G-join decomposition  $G$   
**begin**  
 $\mathcal{P} := \{\{v\} : v \in V\}$   
**return** DECOMPOSEP( $G, \mathcal{P}$ )  
**end** {DECOMPOSE}

#### 7.4 Algorithm to compute a standard G-join decomposition

**Function** DECOMPOSESTANDARD( $(\mathcal{D}, M)$ )  
**Input:** a minimal G-join decomposition  
**Output:** a standard G-join decomposition  
**begin**  
**for all**  $H \in \mathcal{D}$   
  **if**  $H$  has exactly 3 vertices **then**  
    mark  $H$  **degenerated** and **linear**, and set an arbitrary linear ordering for  $H$   
  **for all**  $(x, y, \alpha) \in M$   
    **let**  $H_1 \in \mathcal{D}$  having vertex  $x$ , and **let**  $H_2 \in \mathcal{D}$  having vertex  $y$   
    compute the composition  $H$  of  $H_1$  and  $H_2$   
    **if**  $H$  is degenerated or linear **then**  
       $(\mathcal{D}, M) := (\mathcal{D} \setminus \{H_1, H_2\} \cup \{H\}, M \setminus \{(x, y, \alpha)\})$   
      mark  $H$  **degenerated** or **linear**, and set the linear ordering of  $H$  if  $H$  is linear  
  **return**  $(\mathcal{D}, M)$   
**end** {DECOMPOSESTANDARD}