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New decompositions of 2-structures

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Abstract

We present a family of decompositions of 2-structures generalizing the modular decomposition, and $O(n^3)$ time algorithms to compute all these decompositions. These results can be applied to non-oriented, oriented and directed graphs. Bi-join decomposition of non-oriented graphs and of tournaments are two special cases of this family of decomposition. Two others special cases are generalisations of the bi-join decomposition on directed graphs.

1 Introduction

The well-known modular decomposition of graph has many applications in graph theory and algorithms. It is unique [8] and can be computed in linear time (i.e. in $O(n + m)$) on non-oriented graphs [11], on directed graphs [10], and in linear time (i.e. in $O(n^2)$) on 2-structures [9]. The bi-join decomposition is a generalisation of the modular decomposition on non-oriented graphs [13, 14] and on tournaments [2]. These two decompositions can be computed in linear time.

We present a family of decompositions of 2-structures which generalize the modular decomposition. We show that these decompositions are unique, and we present an algorithm to compute them in time $O(n^3)$ (for a fixed decomposition in the family). We apply these results to oriented and directed graphs. We give two new different decompositions for directed graphs which generalize the bi-join decomposition of non-oriented graphs and tournaments, and we give a new decomposition for oriented graphs. Bi-join decomposition of non oriented graphs and bi-join decomposition of tournament are also special cases of this family of decompositions.

After some preliminaries in section 2, we introduce in section 3 the G-joins and show that G-joins have the bipartitive property. In section 4 we define the G-join decomposition. For any fixed abelian group with some properties, there is a different G-join decomposition. In section 5 we give some special cases of decompositions on non-oriented, oriented and directed graphs. Finally, we present an $O(n^3)$ algorithm to compute the G-join decomposition in section 6, for any fixed abelian group.

2 Preliminaries

2.1 Graphs and 2-structures

A directed graph $G = (V, A)$ is a pair of a set of vertices $V$ and a set of arcs $A \subseteq V \times V \setminus \{(u, u) : u \in V\}$. A non-oriented graph is a directed graph such that for all $(u, v) \in V^2$, with $u \neq v$, then

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(u, v) ∈ A if and only if (v, u) ∈ A. An oriented graph is a directed graph such that for all (u, v), (u, v) ∈ A ⇒ (v, u) /∈ A. A tournament is a oriented graph such that either (u, v) ∈ A or (v, u) ∈ A.

Let D be a set. A 2-structure on D is a pair (V, e) such that e : V × V → D. In this paper, every 2-structure is finite (i.e., V is finite). A 2-structure is symmetric if e(u, v) = e(v, u) for all u, v ∈ D, u ≠ v. Let σ be an involution on D (i.e., a bijection such that σ(σ(x)) = x for all x ∈ D). A 2-structure (V, e) on D is σ-symmetric if e(u, v) = σ(e(v, u)) for all u, v ∈ D, u ≠ v.

A directed graph can be viewed as a 2-structure on ℤ2, a non oriented graph as a symmetric 2-structure on ℤ2, and a tournament as a σ-symmetric 2-structure on ℤ2 with σ = [0, 1]. There is a way to transform a 2-structure on D into a σ-symmetric 2-structure on D × D: take e′(u, v) = (e(u, v), e(v, u)) and σ((i, j)) = (j, i) for all i, j ∈ D. For example a directed graph can also be viewed as a σ-symmetric 2-structure on ℤ2 × ℤ2, with σ = [[0, 0], [0, 0], [1, 1] [1, 1]].

2.2 Bipartite families

A bipartition of a set V is a partition \{X, Y\} of V such that X ≠ ∅ and Y ≠ ∅. We write sometimes \{X, −\} instead of \{X, V \setminus X\}. Two bipartitions \{X, Y\} and \{X′, Y′\} overlap (or \{X, Y\} overlaps \{X′, Y′\}) if X ∩ X′, X ∩ Y′, Y ∩ X′ and Y ∩ Y′ are non empty. A family \(F\) of bipartitions of V is weakly bipartitive if:

- for all v ∈ V, \{\{v\}, V \setminus \{v\}\} is in \(F\), and
- for all \{X, Y\} and \{X′, Y′\} in \(F\) such that \{X, Y\} overlaps \{X′, Y′\}, then \{X ∩ X′, Y ∪ Y′\}, \{X ∩ Y′, Y ∪ X′\}, \{Y ∩ X′, X ∪ Y′\} and \{Y ∩ Y′, X ∪ X′\} are in \(F\).

Moreover a weakly bipartitive family \(F\) is bipartitive if for all \{X, Y\} and \{X′, Y′\} which overlap in \(F\), \{XΔX′, XΔY′\} is in \(F\) (where AΔB = (A \setminus B) ∪ (B \setminus A)). Bipartite families are close to partitive families \[3, 12\] but deals with bipartitions of V instead of subsets of V.

A member of a bipartite family is strong if it overlaps no other member in the family. A member \{X, Y\} is trivial if \(|X| = 1\) or \(|Y| = 1\). Let T = (V, E) be a tree. We denote by Leaves(T) the leaves of T. For β ∈ V, let \{A_{β}^{1}, ..., A_{β}^{d(β)}\} be the connected components of T − β. Let \(C_{β}^{i} = A_{β}^{i} \cap \text{Leaves}(T)\). For e ∈ E, let \(A_{e}^{1}\) and \(A_{e}^{2}\) be the connected components of T − e, and let \(C_{e}^{1}, C_{e}^{2}\) = \(\{A_{e}^{1} \cap \text{Leaves}(T)\}, A_{e}^{2} \cap \text{Leaves}(T)\}). The following result can be found in \[14\] or in \[6\] using a different formalism. This result can also be easily showed from known results of weakly partitive families \[3, 12\].

Theorem 1. \[6, 14\] Let \(F\) be a weakly bipartitive family \(F\) on V. Then there is a unique unrooted tree T = (V_T, E_T), call the representative tree, such that Leaves(T) = V, and each internal node has at least 3 neighbors and is marked degenerate, linear or prime, such that:

- For all e ∈ E_T, \(\{C_{e}^{1}, C_{e}^{2}\}\) is a strong member of \(F\) and there is no other strong members in \(F\).
- Let β ∈ V_T be an internal node, and let k be the degree of β.
  - If β is degenerated, then for all 0 ⊆ I ⊆ \{1, ..., k\}, \(\cup_{i∈I} C_{β}^{i}, −\) is in \(F\).
  - If β is linear, there is a ordering \(C_{β}^{1}, ..., C_{β}^{k}\) such that for all a, b ∈ \{1, k\} with a ≤ b and (a, b) ≠ (1, k), \(\cup_{i∈\{a,...,b\}} C_{β}^{i}, −\) is in \(F\).
- There is no other members in \(F\).
Furthermore if \( F \) is bipartite, then \( T \) has no linear node.

Decompositions based on bipartite families have been studied in [6] under a formalism called decomposition frame with some properties. Some examples of this decomposition frame can be found in [4, 5]. Bipartite families based decompositions are interesting since the bipartivity imply an unique decomposition. Furthermore, this imply that a greedy algorithm to decompose the structure will always work: if we can find in polynomial time a decomposable bipartition in the structure, then we can decompose the whole structure in polynomial time.

2.3 Modular decomposition and bi-join decomposition

A module in a 2-structure \( G = (V, e) \) is a non-empty \( X \subseteq V \) such that for all \( v \notin X \) and \( u, u' \in X \), \( e(v, u) = e(v, u') \) and \( e(u, v) = e(u', v) \). The family of modules of a 2-structure is weakly partitive, and is partitive if the 2-structure is symmetric [7]. If a structure \( G \) has a non-trivial module \( X \), then it can be decomposed into \( G[X] \) and \( G[V \setminus X \cup \{x\}] \), where \( x \in X \). Note that the structure \( G \) can be easily reconstructed from \( G[X] \) and \( G[V \setminus X \cup \{x\}] \). The modular decomposition is defined by recursively decompose the structure by a non-trivial module. It can be represented by a tree, call the modular decomposition tree, which is exactly the representative tree of the family of modules.

A bi-join in a non-oriented graph \( G = (V, E) \) is a bipartition \( \{X, Y\} \) of \( V \) such that for all \( v, v' \in X \), \( \{N(v) \cap Y, Y \setminus N(v)\} = \{N(v') \cap Y, Y \setminus N(v')\} \). A bi-join in a tournament \( G = (V, A) \) is a bipartition \( \{X, Y\} \) of \( V \) such that for all \( v, v' \in X \), \( \{N^+(v) \cap Y, Y \setminus N^+(v)\} = \{N^+(v') \cap Y, Y \setminus N^+(v')\} \).

![Figure 1: A bi-join in a non-oriented graph and a tournament.](image)

If \( X \subseteq V \) is a module of \( G \), then it is a bi-join of \( G \). The family of bi-joins of a undirected graph is bipartite [13], and the family of bi-joins of a tournament is weakly bipartitive [2]. If a graph has a non-trivial bi-join, then it can be decomposed into two graphs, and the bi-join decomposition is the recursive decomposition by a strong non-trivial bi-join. Since the family of bi-join is bipartitive, the bi-join decomposition tree is unique (and is isomorphic to the representative tree).

2.4 Abelian group

We recall axioms of an abelian group \( (D, +) \).

**Neutral element:** There is an element \( 0 \) in \( D \) such that for all \( a \) in \( D \), \( 0 + a = a + 0 = a \).

**Inverse element:** For each \( a \) in \( D \) there is an element \( a^{-1} \) in \( D \) such that \( a + a^{-1} = a^{-1} + a = 0 \), where \( 0 \) is the neutral element. (We will wrote \( -a \) for \( a^{-1} \).)

**Associativity:** For all \( a, b \) and \( c \) in \( D \), \( (a + b) + c = a + (b + c) \).

**Commutativity:** For all \( a \) and \( b \) in \( D \), \( a + b = b + a \).
3 G-joins

3.1 Definition

Throughout this section, we fix an abelian group \((D, +)\). Let \((V, e)\) be a 2-structure on \(D\). A pair \((X, Y)\) with \(X \neq \emptyset, Y \neq \emptyset\) and \(V = X \cup Y\) (i.e. \(X \cap Y = V\) and \(X \cap Y = \emptyset\)) is a G-join if there is pairwise disjoin \(X_i\) and \(Y_i\) (for \(i \in D\)) such that \(X = \bigcup_{i \in D} X_i, Y = \bigcup_{j \in D} Y_j\), and for all \((i, j) \in D^2\) and \((u, v) \in (X_i, Y_j), e(u, v) = i + j\). We start with some easy observations.

Proposition 2. If \((X, Y)\) is a G-join of \(G\) and \(V' \subseteq V\) such that \(V' \cap X \neq \emptyset\) and \(V' \cap Y \neq \emptyset\), then \((X \cap V', Y \cap V')\) is a G-join of \(G[V']\).

Proposition 3. If \(M\) is a module of \((V, e)\), then \((M, V \setminus M)\) and \((V \setminus M, M)\) are G-joins of \((V, e)\).

Proposition 4. For every pairwise different \(a, b, c, d \in V\) such that there is a G-join \(\{X, Y\}\) with \(\{a, c\} \subseteq X\) and \(\{b, d\} \subseteq Y\), then \(e(c, d) = e(c, b) + e(a, d) - e(a, b)\).

Lemma 5. Let \((V, e)\) be a 2-structure on \(D\). Let \((X, Y)\) and \((X', Y')\) be two G-joins of \((V, e)\) such that \(X \cap X' \neq \emptyset\) and \(Y \cap Y' \neq \emptyset\). Then \((X \cap X', Y \cup Y')\) is a G-join of \((V, e)\).

Proof. Let \(X_i\) and \(Y_i\) (for \(i \in D\)) such that \(X = \bigcup_{i \in D} X_i, Y = \bigcup_{j \in D} Y_j\), and for all \((u, v) \in (X_i, Y_j), e(u, v) = i + j\). Similarly let \(X'_i\) and \(Y'_i\) (for \(i \in D\)) such that \(X' = \bigcup_{i \in D} X'_i, Y' = \bigcup_{j \in D} Y'_j\), and for all \((u, v) \in (X'_i, Y'_j), e(u, v) = i + j\).

Since \(Y \cap Y'\) is non-empty, let \(v \in Y \cap Y'\), and let \(j, j' \in D\) such that \(v \in Y_{j'}\cap Y_{j'}\). Suppose that \(w \in X \cap X'\), and let \(i, i' \in D\) such that \(w \in X_i \cap X_{i'}\). Then \(w \in X_i \cap X_{i'}, e(w, v) = i + j = i' + j'.\)

Thus \(X \cap X' = \bigcup_{i \in D} X_i \cap X'_i\). Moreover, for all \(u \in Y \cap Y'\), \(e(w, u) = i + k = i' + k'\) (with \(u \in Y_k \cap Y_{k'}\)), thus \(i' = i - i' + k = i' + j + k\), and \(Y \cap Y' = \bigcup_{k \in D} Y_k \cap Y'_{k + j' - j'}\).

For all \(k \in D\), let \(X''_k = X_k \cap X'_{k + j' - j'}\), and let \(Y''_k = Y_k \cup Y'_{k + j' - j + 1}\). \(X \cap X' = \bigcup_{i \in D} X''_k\) and \(Y \cap Y' = \bigcup_{k \in D} Y''_k\). For all \(u \in X''_k\) and \(v \in Y''_k, e(u, v) = i + k\). Thus \((X \cap X', Y \cup Y')\) is a G-join. \(\square\)

3.2 G-joins in \(\sigma\)-symmetric 2-structures

A function \(f\) is a isomorphism for \((D, +)\) if \(f(a + b) = f(a) + f(b)\) for all \((a, b) \in D^2\). From now \(\sigma\) will denote an involution on \(D\) such that the function \(f : a \rightarrow \sigma(a) - \sigma(0)\) is an isomorphism for \((D, +)\) (where \(0\) is the neutral element).

Lemma 6. Let \((V, e)\) is a \(\sigma\)-symmetric 2-structure, and let \(X\) and \(Y\) such that \(V = X \cup Y\). Then \((X, Y)\) is a G-join if and only if \((Y, X)\) is a G-join.

Proof. Let \(X_a\) and \(Y_a\) (for \(a \in D\)) such that \(X = \bigcup_{a \in D} X_a, Y = \bigcup_{a \in D} Y_a\), and for all \((u, v) \in (X_a, Y_b), e(u, v) = a + b\). Let \(X'_a = X_{\sigma(a)}\) and \(Y'_a = Y'_{\sigma(a) - \sigma(0)}\), for all \(a \in D\). Since \(\sigma\) is a bijection, \(X = \bigcup_{a \in D} X'_a\) and \(Y = \bigcup_{a \in D} Y'_a\). Moreover, for all \(u \in X'_a\) and \(v \in Y'_b, e(u, v) = \sigma(a) + \sigma(b) - \sigma(0) = f(a) + f(b) + \sigma(0) = f(a + b) + \sigma(0) = \sigma(a + b),\), and \(e(v, u) = a + b\). Thus \((Y, X)\) is a G-join. \(\square\)

We say that \(\{X, Y\}\) is a G-join of \((V, e)\) if \((X, Y)\) is a G-join of \((V, e)\). Lemmas 5 and 6 show that if \(\{X, Y\}\) and \(\{X', Y'\}\) are two G-joins such that \(\{X, Y\}\) overlaps \(\{X', Y'\}\), then \(\{X \cap X', Y \cup Y'\}\) is a G-join. Therefore we have:

Corollary 7. The family of G-joins of a \(\sigma\)-symmetric 2-structure is weakly bipartite.
3.3 G-joins in symmetric 2-structures

**Lemma 8.** Let \((V, e)\) be a symmetric 2-structure. Let \(\{X, Y\}\) and \(\{X', Y'\}\) be two G-joins of \((V, e)\) such that \(\{X, Y\}\) overlaps \(\{X', Y'\}\). Then \(\{X \Delta X', X \Delta Y'\}\) is a G-join of \((V, e)\).

**Proof.** Let \(v \in Y \cap Y'\), \(w \in X \cap Y'\), and \((j, j', l, l') \in D^4\) such that \(v \in Y_j \cap Y'_j\) and \(w \in X_l \cap Y'_l\). As we show in proof of Lemma 5, \(X \cap X' = \bigcup_{k \in D} X_k \cap X'_{k+J}\) and \(Y \cap Y' = \bigcup_{k \in D} Y_k \cap Y'_{k+J}\).

Using similar argument, \(Y \cap X' = \bigcup_{k \in D} Y_k \cap X'_{k+J}\) and \(X \cap Y' = \bigcup_{k \in D} X_k \cap Y'_{k+J}\).

Let \(X''_k = \bigcup_{k \in D} X_{k+J} \cup (Y_{k+J} \cap Y'_{k+J})\) and \(Y''_k = \bigcup_{k \in D} Y_{k+J} \cup (X_{k+J} \cap X'_{k+J})\). For all \(u \in X''_k\) and \(v \in Y''_k\), \(e(u, v) = k + l\). Thus \(\bigcup_{k \in D} X''_k, \bigcup_{k \in D} Y''_k\) is a G-join. \(\square\)

With Lemma 5, we obtain:

**Corollary 9.** The family of G-joins of a symmetric 2-structure is bipartite.

4 G-join decomposition

In this section, we fix an abelian group \((D, +)\) and an involution \(\sigma\) such that \(f : a \to \sigma(a) - \sigma(0)\) is an isomorphism for \((D, +)\). For most part, our terminology follows terminology used in [4, 6].

4.1 Simple decomposition

A G-join \(\{X, Y\}\) is **trivial** if \(|X| = 1\) or \(|Y| = 1\). Since every singleton is a module, every bipartition \(\{X, Y\}\) with \(|X| = 1\) or \(|Y| = 1\) is a G-join.

Let \(G = (V, e)\) be a \(\sigma\)-symmetric 2 structure and \(\{X, Y\}\) be a non-trivial G-join. Let \(x \in X\) and \(y \in Y\). A **simple decomposition** of \((V, e)\) by the G-join \((X, Y)\) is the decomposition into \(G_1 = (X \cup \{y\}, e|_{X \cup \{y\}})\) and \(G_2 = (Y \cup \{x\}, e|_{Y \cup \{x\}})\) with an additional **marker triplet** \((x, y, \alpha)\), where \(\alpha = e(x, y) (e|_X\text{ represents the function } e\text{ induced by } X \times X)\). We write \(G \to (G_1, G_2, (x, y, \alpha))\). Note that this decomposition is not unique for a fixed \(\{X, Y\}\).

The **simple composition** of \((V_1, e_1), (V_2, e_2)\) and the marker triplet \((x, y, \alpha)\), with \(V_1 \cap V_2 = \{x, y\}\), is the 2-structure \((V_1 \cup V_2, e)\) where \(e(a, b) = e_1(a, b)\) for all \(a, b \in V_1 \setminus \{y\}\), \(e(a, b) = e_2(a, b)\) for all \(a, b \in V_2 \setminus \{x\}\), and \(e(a, b) = e_1(a, y) - \alpha + e_2(x, b)\) for all \(a \in V_1 \setminus \{y\}\) and \(b \in V_2 \setminus \{x\}\).

By Proposition 4, if \((V_1, e_1), (V_2, e_2), (x, y, \alpha)\) is a simple decomposition of \((V, e)\), then the simple composition of \((V_1, e_1), (V_2, e_2)\) and \((x, y, \alpha)\) is \((V, e)\).

**Lemma 10.** Let \(\{X, Y\}\) be a G-join of \(G\), and \((G_1, G_2, (x, y, \alpha))\) be the simple decomposition of \(G\) by \((X, Y)\). Let \(\{X', Y'\}\) be a bipartition of \(V\) with \(Y' \not\subseteq Y\). Then \(\{X', Y'\}\) is a G-join of \(G\) if and only if \(\{x\} \cup Y \setminus Y', Y'\}) is a G-join of \(G_2\).

**Proof.** If \(\{X', Y'\}\) is a G-join of \(G\) then by Proposition 2 \(\{x\} \cup Y \setminus Y', Y'\}) is a G-join of \(G_2\). Now suppose that \(\{x\} \cup Y \setminus Y', Y'\}) is a G-join of \(G_2\). Let \(X'_a\) and \(Y'_a\) (for \(a \in D\)) such that \(\{x\} \cup Y \setminus Y', Y'\}) = (\bigcup_{a \in D} X'_a, \bigcup_{a \in D} Y'_a)\) and \(e(u, v) = a + b\) for all \(u \in X'_a\) and \(v \in Y'_a\). Since \(\{X, Y\}\) is a G-join of \(G\), let \(X_a\) and \(Y_a\) such that \(\{X, Y\} = (\bigcup_{a \in D} X_a, \bigcup_{a \in D} Y_a)\) and \(e(u, v) = a + b\) for all \(u \in X_a\) and \(v \in Y_a\). Let \(c, d \in D\) such that \(x \in X'_c\) and \(y \in Y_d\). Let \(X''_a = (X'_a \setminus \{x\}) \cup X_{a-d} - c + a\) and \(Y''_a = Y'_d\). Let \(u \in X''_a\) and \(v \in Y''_a\). If \(u \in X'_a\) then \(e(u, v) = a + b\). Otherwise \(u \in X_{a-d} - c + a\) and by definition of simple decomposition, \(e(u, v) = e(u, y) - \alpha + e(x, v) = a + b\) since \(e(u, y) = a - d - c + \alpha + d\) and \(e(x, v) = c + b\). Then \(\bigcup_{a \in D} X''_a, \bigcup_{a \in D} Y''_a\} = \{X', Y'\}) is a G-join of \(G\). \(\square\)
A G-join \( \{X, Y\} \) is **strong** if it is a strong member of the bipartitive family of G-joins of G (i.e. there is no G-join \( \{X', Y'\} \) such that \( \{X, Y\} \) overlaps \( \{X', Y'\} \)). A simple decomposition is strong if it is induced by a strong G-join. The following Corollary follows from previous Lemma.

**Corollary 11.** Let \( \{X, Y\} \) be a G-join of G, and \((G_1, G_2, (x, y, \alpha))\) be the simple decomposition of G by \((X, Y)\). Let \( \{X', Y'\} \) be a bipartition of V with \( Y' \cap Y \). Then \( \{X', Y'\} \) is a strong G-join of G if and only if \( \{x \cup Y \setminus Y', Y'\} \) is a strong G-join of \( G_2 \).

### 4.2 G-join decompositions

A 2-structure is **prime** if all its G-joins are trivial. A 2-structure is **degenerated** if every bipartition is a G-join. A 2-structure \( G = (V, E) \) is **linear** if there is a ordering \( v_1, \ldots, v_n \) of the vertices such that for all \( i, j \in \{1, \ldots, n\} \) with \( i \leq j \) and \((i, j) \neq (1, n), \{v_i, \ldots, v_j\}, -\) is a G-join of G, and G has no others G-join. Every 2-structure with at most 3 vertices is degenerated, linear and prime, and every 2-structure with at least 4 vertices is either prime, degenerated, linear or none of these three cases. The following Lemma comes immediately from the bipartitivity of G-joins.

**Lemma 12.** Let G be a 2-structure. G has no strong non-trivial G-join if and only if G is either prime, degenerated or linear.

**Proof.** If G has no strong non-trivial G-join, then representative tree of G has only one internal node \( \beta \). Then G is prime, degenerated or linear if \( \beta \) is **prime, degenerate or linear**, respectively. \( \square \)

The following Lemma gives a characterisation of degenerated graphs. Its straightforward inductive proof is given in appendix.

**Lemma 13.** Suppose \( \sigma(\emptyset) = \emptyset \). A \( \sigma \)-symmetric 2-structure with at least 4 vertices is degenerated if and only if there is an \( \alpha \in \mathcal{D} \) such that \( \sigma(\alpha) = \alpha \), and a function \( f : V \to \mathcal{D} \) such that for all \( u, v \in V, u \neq v \), \( e(u, v) = \alpha + f(u) + \sigma(f(v)) \).

Let G be a 2-structure. G-join decompositions of G are defined recursively: \( \{\{G\}, \emptyset\} \) is a G-join decomposition of G and if \( (D, M) \) is a G-join decomposition of G, \( H \in \mathcal{D} \), and \( H_1, H_2 \) is a simple decomposition of H with marker triplet \( (u, v, \alpha) \), then \( ((D \setminus \{H\}) \cup \{H_1, H_2\}, M \cup \{(u, v, \alpha)\}) \) is a G-join decomposition of G. In this case we say that \( (D', M') = ((D \setminus \{H\}) \cup \{H_1, H_2\}, M \cup \{(u, v, \alpha)\}) \) is a simple decomposition of \( (D, M) \), and we write \( (D, M) \to (D', M') \).

A G-join decomposition \( (D, M) \) is **minimal** if every 2-structure in \( \mathcal{D} \) is prime. A G-join decomposition \( (D, M) \) is **good** if no \( H \in \mathcal{D} \) has a strong non-trivial G-join. A G-join decomposition \( (D, M) \) is **standard** if it can be obtained from \( \{\{G\}, \emptyset\} \) by a sequence of simple strong decompositions, and no \( H \in \mathcal{D} \) has a strong non-trivial G-join. Note that minimal decompositions and standard decompositions are goods. The proof of the following lemma is similar to the proof given in [6].

**Lemma 14.** Let \( (D, M) \) be a good decomposition of G. If there is no good decomposition \( (D', M') \) such that \( (D', M') \to (D, M) \), then \( (D, M) \) is a standard decomposition of G.

**Proof.** If \( (D, M) \) is not a standard decomposition then there is a simple decomposition in the sequence of decompositions which is not strong. Let \( (D_1, M_1) \to (D_2, M_2) = ((D_1 \setminus \{H\}) \cup \{H_1, H_2\}, M_1 \cup \{m\}) \) be the last non-strong decomposition in the sequence. All the decompositions after \( (D_1, M_1) \to (D_2, M_2) \) are strong and correspond to unique strong G-joins of G. We construct...
the decomposition \((D', M')\) from \((D_1, M_1)\) after simple decompositions of these strong G-joins. 
\((D', M')\) is good since there is a simple decomposition for every strong G-join in \(G\) and \((D', M') \rightarrow (D, M)\) by the simple decomposition of the G-join corresponding to \((D_1, M_1) \rightarrow (D_2, M_2)\). \hfill \Box

The previous Lemma tells us that a standard decomposition can be obtained from a minimal decomposition by a sequence of simple compositions. This will be used in the decomposition algorithm presented in a next section.

A decomposition \((D, M)\) of \(G\) induces a unrooted tree of vertex set \(D\) and \(H_1\) is adjacent to \(H_2\) if there is a \((x, y, \alpha) \in M\) such that \(x\) is a vertex of \(H_1\) and \(y\) is a vertex of \(H_2\). The decomposition tree of a standard decomposition is isomorphic to the representative tree of the weakly bipartite family of G-joins and thus is unique. We call it the standard decomposition tree.

## 5 Special cases of G-join decomposition

### 5.1 Bi-join decomposition

The bi-join decomposition [13, 14] is a special case with \((D, +) = (Z_2, +)\). Lemma 13 says that degenerated graphs are disjoint union of two cliques if \(\alpha = 1\) and complete bipartite graphs if \(\alpha = 0\). This decomposition has no linear node since the structure is symmetric. The bi-join decomposition of tournaments [2] is the decomposition, with \((D, +) = (Z_2, +)\) and \(\sigma = [1, 0]\).

### 5.2 Decomposition of oriented graphs

A directed graph \(G\) can be viewed as an \(\sigma\)-symmetric 2-structure on the set \(\{(0, 0), (1, 0), (0, 1)\}\), with \(\sigma((i, j)) = (j, i)\) for \((i, j) \in \{(0, 0), (1, 0), (0, 1)\}\). There is one abelian group on \(D\) such that \(a \rightarrow \sigma(a) - \sigma(\hat{0})\) is an isomorphism. This abelian group, isomorphic to \((Z_3, +)\), is given in figure 2.

### 5.3 Decompositions of directed graphs

A directed graph \(G\) is a 2-structure on \(Z_2\), and can be viewed as a \(\sigma\)-symmetric 2-structure \((V, e)\) on \(Z_2 \times Z_2\), with \(\sigma((i, j)) = (j, i)\). There are two abelian groups such that \(a \rightarrow \sigma(a) - \sigma(\hat{0})\) is an isomorphism. The first one (isomorphic to \((Z_2^2, +)\)) is given in figure 3, and the second (isomorphic to \((Z_4, +)\)) is given in figure 4. These two decompositions are generalizations of the bi-join decomposition on both non-oriented graphs and on tournament. They are mutually exclusive, that is there is a graph prime for the first one and completely decomposable for the other one, and vice versa.

## 6 Decomposition algorithm

From now, we fix an abelian group \((D, +)\) and an involution \(\sigma\) such that \(f: a \rightarrow \sigma(a) - \sigma(\hat{0})\) is an isomorphism for \((D, +)\).

### 6.1 Find a non trivial G-join

We give in this section a \(O(n^2)\) algorithm for the following problem: given a 2-structure \(G = (V, e)\) and \(u, v \in V\), output a non trivial G-join \(\{X, Y\}\) such that \(u \in X\) and \(v \in Y\), or output “No” if there is no such partition.
Figure 2: Decomposition for oriented graphs. (Dashed edge signify that two vertex can be adjacent or not.)

Figure 3: Decomposition for directed graphs (first).

Figure 4: Decomposition for directed graphs (second).
A directed graph $G = (V, A)$ is **strongly connected** if for every $u, v \in V$ there is a path from $u$ to $v$ (i.e. there is a sequence $u_0 = u, u_1, \ldots, u_k = v$ such that for all $i \in \{0, \ldots, k - 1\}$, $(u_i, u_{i+1}) \in A$). A **strongly connected component** is a maximal subset $W \subseteq V$ such that $G[W]$ is strongly connected. The strongly connected components form a partition of the vertices of $G$, and can be found in linear time [1]. Moreover, there is always a strongly connected component $W$ such that there is no arcs from $W$ to $V \setminus W$, since the incidence graph of strongly connected components is acyclic.

We transform our problem into a 2-SAT problem. We suppose w.l.o.g. that $u \in X_0$ and thus $v \in Y_{e(u,v)}$. If a vertex $w \not\in \{u, v\}$ is in $X$ then it is in $X_{e(w,v)-e(u,v)}$; and if $w$ is in $Y$, it is in $Y_{e(u,v)}$. Let the 2-SAT problem with variable set $V \setminus \{u, v\}$, and $w \Rightarrow t$ if $e(w,v) - e(u,v) + e(u,t) \neq e(w,t)$. A variable $w$ is true means that $w \in X$. Then there is a non trivial G-join if and only is there is a non trivial solution for the 2-SAT problem. Let $G_f = (V \setminus \{u, v\}, E_f)$ with $E_f = \{(w,t) : w, t \in V \setminus \{u, v\} \text{ and } e(w,v) - e(u,v) + e(u,t) \neq e(w,t)\}$. The 2-SAT problem has a non trivial solution if and only if the graph $G_f$ is not strongly connected. In this case $\{X \cup \{u\}, V \setminus (X \cup \{u\})\}$ is a non trivial G-join of $G$. All these operations can be done in time $O(n^2)$. (Algorithms in pseudo-code are given in appendix.)

### 6.2 Compute a minimal G-join decomposition

If a 2-structure is not prime, then a G-join can be found in $O(n^3)$ time using the previous algorithm for a fixed $u \in V$ and for all $v \neq u$. So a naive algorithm to compute a minimal decomposition take $O(n^4)$ time. We can reach $O(n^3)$ by the following way. We remember the set $P$ of subsets of $V$ such that there is no non-trivial G-join which overlaps $U$ for all $U \in P$. $P$ is a partition of $V$, and at each call of the sub-routine, either it succeed and we decompose the 2-structure, either it fails and we merge two sets in $P$. So a minimal decomposition can be obtained with $O(n)$ call to the algorithm of section 6.1, and can computed in $O(n^3)$.

### 6.3 Compute a standard G-join decomposition

Lemma 14 says that a standard decomposition of $G$ can be computed from a minimal decomposition, after some re-compositions. We show that we can test in time $O(n^2)$ if a composition of two 2-structures degenerate or linear is degenerate or linear.

Let $G_1 = (V_1, e_1)$ and $G_2 = (V_2, e_2)$ and a marker triplet $(x, y, \alpha)$, such that $G_1$ and $G_2$ have no strong non-trivial G-join. If $G$ has no strong non-trivial G-join, then by Lemma 12, $G$ is either degenerate or linear (since it cannot be prime). If $G_1$ or $G_2$ is not degenerate, then $G$ must be linear. Moreover if $G_1$ and $G_2$ are linear, let $v_1, \ldots, v_k$ be a linear ordering of the vertex of $G_1$, and let $v'_1, \ldots, v'_k$ be a linear ordering of $G_2$. W.l.o.g. $v_1 = y$ and $v'_1 = x$. Then if $G$ is linear, $v_2, \ldots, v_k, v'_2, \ldots, v'_k$ or $v_2, \ldots, v_k, v'_k, \ldots, v'_2$ must be a linear ordering of $G$, and so either $\{v_2, v'_2\}, -$ or $\{v_2, v'_k\}, -$ must be a G-join of $G$.

Let $G_1$ and $G_2$ be two 2-structures without strong non-trivial G-join. We want to known if the composition $G$ of $G_1$ and $G_2$ with the marker triplet $(x, y, \alpha)$ is degenerate or linear (and to know a ordering of $G$ if it is linear). Case 1: $|V_1| = |V_2| = 3$. All bipartitions of $G$ can be tested in constant time, so the type of $G$ (and an ordering if $G$ is linear) can be computed in $O(1)$. Case 2: $|V_1| \neq 3$ or $|V_2| \neq 3$. Then $G_1$ or $G_2$ is non degenerate, or non linear. If $G_1$ and $G_2$ are degenerate, then $G$ is degenerate if and only if $\{\{x, y\}, -\}$ is a G-join of $G$. If $G_1$ and $G_2$ are linear, with ordering $\{v_1 = y, \ldots, v_k\}$ and $\{v'_1 = x, \ldots, v'_k\}$, then $G$ is linear if $\{v_2, v'_2\}, -$ or $\{v_2, v'_k\}, -$ is a G-join of $G$. In this case, $v_2, \ldots, v_k, v'_2, \ldots, v'_k$ or $v_2, \ldots, v_k, v'_k, \ldots, v'_2$ is ordering of $G$. In others cases, $G$
is neither degenerate nor linear. Moreover, to test if a bipartition is a $G$-join of a 2-structure can be done in $O(n^2)$.

There is at most $O(n)$ re-compositions (at most one for each edge in the decomposition tree). To summarize, we obtain:

**Theorem 15.** A standard $G$-join decomposition can be computed in time $O(n^3)$.

**References**


7 Appendix

7.1 Proof of lemma 13

Let \((V, e)\) be a 2-structure such that there is a \(f : V \rightarrow \mathcal{D}\) and a \(\alpha\) with \(\sigma(\alpha) = \alpha\) and \(e(u, v) = \alpha + f(u) + \sigma(f(v))\) for all \(u, v \in V, u \neq v\). It is easy to see that \((V, e)\) is \(\sigma\)-symmetric since \(\sigma(0) = 0\) and thus \(\sigma\) is an isomorphism for \((\mathcal{D}, \pm)\). Let \(\{X, Y\}\) be a bipartition of \(V\). Let \(X_a = \{v \in X : a = f(v)\}\) and \(Y_a = \{v \in Y : a = \alpha + \sigma(f(v))\}\). For all \(u \in X_a\) and \(b \in Y_b\), \(a + b = f(u) + \alpha + \sigma(f(v)) = e(u, v)\), thus \(\{X, Y\}\) is a G-join.

On the other hand, let \((V, e)\) be a degenerated 2-structure such that \(|V| \geq 4\).

Claim 1. For every pairwise different \(a, b, c, d \in V\), \(e(c, d) = e(c, b) + e(a, d) - e(a, b)\).

Proof. Since \((V, e)\) is degenerated, \(\{\{a, c\}, -\}\) is a G-join. By Proposition 4 we have the equality. \(\square\)

Claim 2. For every pairwise different \(a, b, c \in V\):

\[
e(a, b) + e(b, c) + e(c, a) = e(b, a) + e(c, b) + e(a, c).
\]

Proof. Let \(d \in V \setminus \{a, b, c\}\). Applying Claim 1, we get:

\[
e(d, a) - e(d, b) = e(c, a) - e(c, b)
\]
\[
e(d, b) - e(d, c) = e(a, b) - e(a, c)
\]
\[
e(d, c) - e(d, a) = e(b, c) - e(b, a).
\]

Thus

\[
e(a, b) + e(b, c) + e(c, a) = e(b, a) + e(c, b) + e(a, c)
= \sigma(e(a, b)) + \sigma(e(b, c)) + \sigma(e(c, a)).
\]

\(\square\)

Case 1: \(|V| = 4\). W.l.o.g \(V = \{a, b, c, d\}\). Let:

\[
\alpha = e(a, b) + e(b, c) + e(c, a)
\]
\[
f(a) = -e(b, c)
\]
\[
f(b) = -e(a, c)
\]
\[
f(c) = -e(b, a) - e(a, c) + e(c, a)
\]
\[
f(d) = e(d, a) + e(c, b) - \alpha
\]
From Claim 2, $\sigma(\alpha) = \alpha$. We get:

\[
\begin{align*}
    f(a) + \sigma(f(b)) + \alpha &= -e(b, c) - e(c, a) + e(a, b) + e(b, c) + e(c, a) \\
    &= e(a, b) \\
    f(a) + \sigma(f(c)) + \alpha &= -e(b, c) - e(a, b) - e(c, a) + e(a, c) + e(a, b) + e(b, c) + e(c, a) \\
    &= e(a, c) \\
    f(b) + \sigma(f(c)) + \alpha &= -e(a, c) - e(a, b) - e(c, a) + e(a, c) + e(a, b) + e(b, c) + e(c, a) \\
    &= e(b, c) \\
    f(a) + \sigma(f(d)) + \alpha &= -e(b, c) + e(a, d) + e(b, c) - \alpha + \alpha \\
    &= e(a, d) \\
    f(b) + \sigma(f(d)) + \alpha &= -e(a, c) + e(a, d) + e(b, c) - \alpha + \alpha \\
    &= e(b, d) \quad \text{(by Claim 1)} \\
    f(c) + \sigma(f(d)) + \alpha &= -e(b, a) - e(a, c) + e(c, a) + e(a, d) + e(b, c) - \alpha + \alpha \\
    &= e(a, d) + e(c, b) - e(a, b) \quad \text{(by Claim 2)} \\
    &= e(c, d) \quad \text{(by Claim 1.)}
\end{align*}
\]

Thus $f$ and $\alpha$ have the required property.

Case 2: $|V| > 4$. Let $v \in V$. $(V \setminus \{v\}, e)$ is degenerated and thus there is a $f' : V \setminus \{v\} \rightarrow \mathcal{D}$ and an $\alpha \in \mathcal{D}$ such that for all $u, v \in V \setminus \{v\}$, $e(u, v) = f'(u) + \sigma(f'(v)) + \alpha$. Let $u \neq v$, and let $f$ such that $f(w) = f'(w)$ if $x \in V \setminus \{v\}$ and $f(v) = e(v, u) - \sigma(f'(u)) - \alpha$.

\[
\begin{align*}
    f(u) + \sigma(f(v)) + \alpha &= f'(u) + e(u, v) - \sigma(\sigma(f'(u))) - \alpha + \alpha \\
    &= e(u, v)
\end{align*}
\]

Let $w \in V \setminus \{u, v\}$ and $x \in V \setminus \{u, v, w\}$.

\[
\begin{align*}
    f(w) + \sigma(f(v)) + \alpha &= f'(w) + e(u, v) - \sigma(\sigma(f'(u))) - \alpha + \alpha \\
    &= f'(w) + f'(u) + e(u, x) + e(w, v) - e(w, x) \quad \text{(by Claim 1)} \\
    &= e(w, v) + f'(w) - f'(u) + f'(u) + \sigma(f'(x)) + \alpha - f'(w) - \sigma(f'(x)) - \alpha \\
    &= e(w, v).
\end{align*}
\]

Thus $f$ and $\alpha$ have the required property.
7.2 Algorithm to find a non trivial G-join

Function \textsc{FindGJoin}(G = (V, e), u, v)
Input: a 2-structure \( G = (V, e) \) and \( u, v \in V, u \neq v \)
Output: a non trivial G-join \{X, Y\} of \( G \) such that \( u \in X \) and \( v \in Y \),
or “No” is there is no such G-join

begin
\( f_1(u) := 0 \)
\( f_2(v) := e(u, v) \)
For every \( w \in V \setminus \{u, v\} \)
\( f_1(w) := e(w, v) - e(u, v) \)
\( f_2(w) := e(u, w) \)
\( E_f := \{(w, t) : w, t \in V \setminus \{u, v\} \text{ and } f_1(w) + f_2(t) \neq e(w, t)\} \)
\( G_f := (V \setminus \{u, v\}, E_f) \)
if \( G_f \) is strongly connected
output “No”
Otherwise
Let \( W \) be a strongly connected component of \( G_f \)
such that there is no arc in \( G_f \) from \( W \) to \( V \setminus W \setminus \{u, v\} \)
output \( \{\{u\} \cup W, V \setminus \{u\} \setminus W\} \)
end \{\textsc{FindGJoin}\}

7.3 Algorithm to compute a minimal G-join decomposition

Function \textsc{DecomposeP}(G, \mathcal{P})
Input: a 2-structure \( G = (V, e) \) and a partition \( \mathcal{P} \) of \( V \)
Output: a minimal G-join decomposition \( G \)
begin
If \( |\mathcal{P}| = 1 \) then
return \( (\{G\}, \emptyset) \)
Let \( A, B \in \mathcal{P}, a \in A \) and \( b \in B \)
If \textsc{FindGJoin}(G, a, b) returns “no” then
\( P := \{A \cup B\} \cup (\mathcal{P} \setminus \{A, B\}) \)
return \textsc{DecomposeP}(G, \mathcal{P})
Let \( \{X, Y\} \) be the G-join returned by \textsc{FindGJoin}
Decompose \( G \) into \( G_1 \) and \( G_2 \) by the G-join \( \{X, Y\} \) with marker triplet \((x, y, \alpha)\)
\( \mathcal{P}_1 := \{P \in \mathcal{P} : P \subseteq X\} \)
\( \mathcal{P}_2 := \{P \in \mathcal{P} : P \subseteq Y\} \)
\( (\mathcal{D}_1, M_1) := \textsc{DecomposeP}(G_1, \mathcal{P}_1) \)
\( (\mathcal{D}_2, M_2) := \textsc{DecomposeP}(G_2, \mathcal{P}_2) \)
return \( (\mathcal{D}_1 \cup \mathcal{D}_2, M_1 \cup M_2 \cup (x, y, \alpha)) \)
end \{\textsc{DecomposeP}\}
Function \textsc{Decompose}(G)
\begin{itemize}
\item \textbf{Input:} a 2-structure $G = (V, e)$
\item \textbf{Output:} a minimal G-join decomposition $G$
\end{itemize}
begin
\begin{itemize}
\item $P := \{\{v\} : v \in V\}$
\item return \textsc{DecomposeP}(G, P)
\end{itemize}
end \{\textsc{Decompose}\}

7.4 Algorithm to compute a standard G-join decomposition

Function \textsc{DecomposeStandard}((D, M))
\begin{itemize}
\item \textbf{Input:} a minimal G-join decomposition
\item \textbf{Output:} a standard G-join decomposition
\end{itemize}
begin
\begin{itemize}
\item for all $H \in D$
  \begin{itemize}
  \item if $H$ has exactly 3 vertices then
  \begin{itemize}
    \item mark $H$ degenerated and linear, and set an arbitrary linear ordering for $H$
  \end{itemize}
  \end{itemize}
\item for all $(x, y, \alpha) \in M$
  \begin{itemize}
    \item let $H_1 \in D$ having vertex $x$, and let $H_2 \in D$ having vertex $y$
    \item compute the composition $H$ of $H_1$ and $H_2$
    \item if $H$ is degenerated or linear then
      \begin{itemize}
        \item $(D, M) := (D \setminus \{H_1, H_2\} \cup \{H\}, M \setminus \{(x, y, \alpha)\})$
      \end{itemize}
    \end{itemize}
\item return $(D, M)$
\end{itemize}
end \{\textsc{DecomposeStandard}\}