On restrictions of 2-interval graphs

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On restrictions of balanced 2-interval graphs

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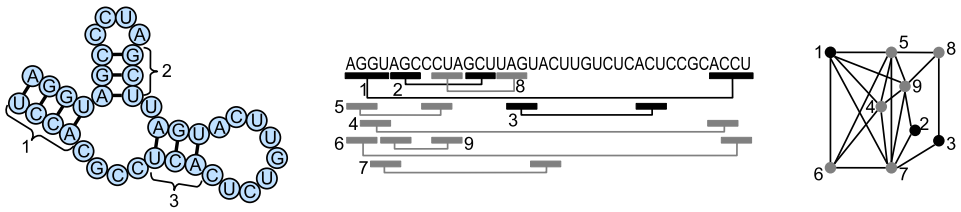
Abstract

The class of 2-interval graphs has been introduced for modelling scheduling and allocation problems, and more recently for specific bioinformatic problems. Some of those applications imply restrictions on the 2-interval graphs, and justify the introduction of a hierarchy of subclasses of 2-interval graphs that generalize line graphs: balanced 2-interval graphs, unit 2-interval graphs, and (x,x)-interval graphs. We provide instances that show that all the inclusions are strict. We extend the NP-completeness proof of recognizing 2-interval graphs to the recognition of balanced 2-interval graphs. Finally we give hints on the complexity of unit 2-interval graphs recognition, by studying relationships with other graph classes: proper circular-arc, quasi-line graphs, $K_{1,5}$-free graphs, ...

Keywords: 2-interval graphs, graph classes, line graphs, quasi-line graphs, claw-free graphs, circular interval graphs, proper circular-arc graphs, bioinformatics, scheduling.

1 2-interval graphs and restrictions

The interval number of a graph, and the classes of k-interval graphs have been introduced as a generalization of the class of interval graphs by McGuigan [McG77] in the context of scheduling and allocation problems. Recently, bioinformatics problems have renewed interest in the class of 2-interval graphs (each vertex is associated to a pair of disjoint intervals and edges denote intersection between two such pairs). Indeed, a pair of intervals can model two associated tasks in scheduling [BYHN+06], but also two similar segments of DNA in the context of DNA comparison [JMT92], or two complementary segments of RNA for RNA secondary structure prediction and comparison [Via04].



(a) (b) (c)

Figure 1: Helices in a RNA secondary structure (a) can be modeled as a set of balanced 2-intervals among all 2-intervals corresponding to complementary and inverted pairs of letter sequences (b), or as an independent subset in the balanced associated 2-interval graph (c).

RNA (ribonucleic acid) are polymers of nucleotides linked in a chain through phosphodiester bonds. Unlike DNA, RNAs are usually single stranded, but many RNA molecules have *secondary structure* in which intramolecular loops are formed by complementary base pairing. RNA secondary structure is generally divided into helices (contiguous base pairs), and various kinds of loops (unpaired nucleotides surrounded by *helices*). The structural stability and function of non-coding RNA (ncRNA) genes are largely determined by the formation of stable secondary structures through complementary bases, and hence ncRNA genes across different species are most similar in the pattern of nucleotide complementary rather than in the genomic sequence. This motivates the use of 2-intervals for modelling RNA secondary structures: each helix of the structure is modeled by a 2-interval. Moreover, the fact that these 2-intervals are usually required to be disjoint in the structure naturally suggests the use of 2-interval graphs. Furthermore, aiming at better modelling RNA secondary structures, it was suggested in [CHLV05] to focus on *balanced 2-interval sets* (each 2-interval is composed of two equally length intervals) and their associated intersection graphs referred as *balanced 2-interval graphs*. Indeed, helices in RNA secondary structures are most of the time composed of equally length contiguous base pairs parts. To the best of our knowledge, nothing is known on the class of balanced 2-interval graphs.

Sharper restrictions have also been introduced in scheduling, where it is possible to consider tasks which all have the same duration, that is 2-interval whose intervals have the same length [BYHN+06, Kar05]. This motivates the study of the classes of unit 2-interval graphs, and (x, x)-interval graphs. In this paper, we consider these subclasses of interval graphs, and in particular we address the problem of recognizing them.

A graph $G = (V, E)$ of order n is a 2-interval graph if it is the intersection graph of a set of n unions of two disjoint intervals on the real line, that is each vertex corresponds to a union of two disjoint intervals $I^k = I^k_l \cup I^k_r$, $k \in [\![1, n]\!]$ (l for "*left*" and r for "*right*"), and there is an edge between I^j and I^k iff $I^j \cap I^k \neq \emptyset$. Note that for the sake of simplicity we use the same letter to denote a vertex and its corresponding 2-interval. A set of 2-intervals corresponding to a graph G is called a realization of G. The set of all intervals, $\bigcup_{k=1}^{n} \{I^k_l, I^k_r\}$, is called the ground set of G (or the ground set of $\{I^1, \ldots, I^n\}$).

The class of 2-interval graphs is a generalization of interval graphs, and also contains all circular-arc graphs (intersection graphs of arcs of a circle), outerplanar graphs (have a planar embedding with all vertices around one of the faces [KW99]), cubic graphs (maximum degree 3 [GW80]), and line graphs (intersection graphs of edges of a graph).

Unfortunately, most classical graph combinatorial problems turn out to be NP-complete for 2-interval graphs: recognition [WS84], maximum independent set [BNR96, Via01], coloration [Via01], ... Surprisingly enough, the complexity of the maximum clique problem for 2-interval graphs is still open (although it has been recently proven to be NP-complete for 3-interval graphs [BHLR07]).

For practical application, restricted 2-interval graphs are needed. A 2-interval graph is said to be *balanced* if it has a 2-interval realization in which each 2-interval is composed of two intervals of the same length [CHLV05], *unit* if it has a 2-interval realization in which all intervals of the ground set have length 1 [BFV04], and is called a (x, x)-*interval graph* if it has a 2-interval realization in which all intervals of the ground set are open, have integer endpoints, and length x [BYHN+06, Kar05]. In the following sections, we will study those restrictions of 2-interval graphs, and their position in the hierarchy of graph classes illustrated in Figure 2.

Note that all (x, x)-interval graphs are unit 2-interval graphs, and that all unit 2-interval graphs are balanced 2-interval graphs. We can also notice that $(1, 1)$-interval graphs are exactly line graphs: each interval of length 1 of the ground set can be considered as the vertex of a root graph and each 2-interval as an edge in the root graph. This implies for example that the

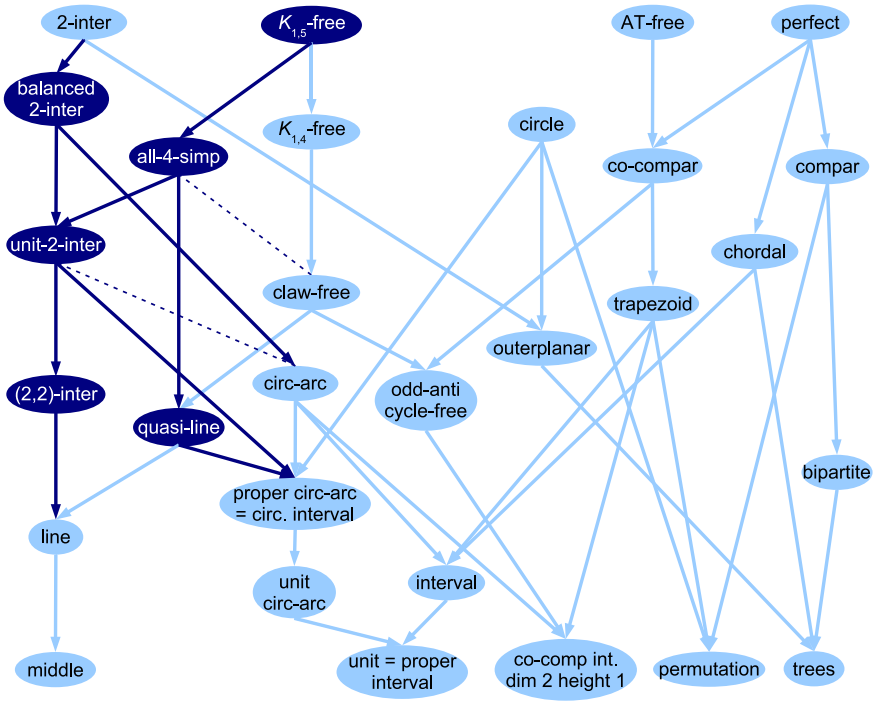
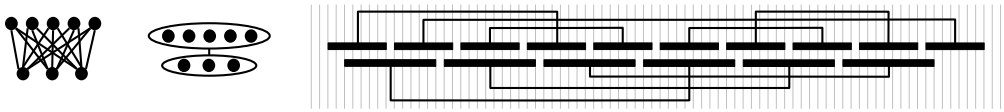


Figure 2: Graph classes related to 2-interval graphs and its restrictions. A class pointing towards another strictly contains it, and the dashed lines mean that there is no inclusion relationship between the two. Dark classes correspond to classes not yet present in the ISGCI Database [BLS+].

coloration problem is also NP-complete for $(2,2)$-interval graphs and wider classes of graphs. It is also known that the complexity of the maximum independent set problem is NP-complete on $(2,2)$-interval graphs [BNR96].

2 Useful gadgets for 2-interval graphs and restrictions

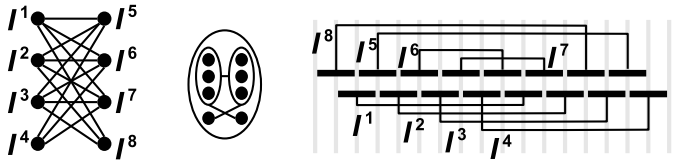
For proving hardness of recognizing 2-interval graphs, West and Shmoys considered in [WS84] the complete bipartite graph $K_{5,3}$ as a useful 2-interval gadget. Indeed, all realizations of this graph are contiguous, that is, for any realization, the union of all intervals in its ground set is an interval. Thus, by putting edges between some vertices of a $K_{5,3}$ and another vertex v, we can force one interval of the 2-interval v (or just one extremity of this interval) to be blocked inside the realization of $K_{5,3}$. It is not difficult to see that $K_{5,3}$ has a balanced 2-interval realization, for example the one in Figure 3.



(a) (b) (c)

Figure 3: The complete bipartite graph $K_{5,3}$ (a,b) has a balanced 2-interval realization (c): vertices of S_5 are associated to balanced 2-intervals of length 7, and vertices of S_3 are associated to balanced 2-intervals of length 11. Any realization of this graph is contiguous, *i.e.*, the union of all 2-intervals is an interval.

However, $K_{5,3}$ is not a unit 2-interval graph. Indeed, each 2-interval $I = I_l \cup I_r$ corresponding to a degree 5 vertex intersect 5 disjoint 2-intervals, and hence one of I_l or I_r intersect at least 3 intervals, which is impossible for unit intervals. Therefore, we introduce the new gadget $K_{4,4} - e$ which is a $(2,2)$-interval graph with only contiguous realizations.



(a) (b) (c)

Figure 4: The graph $K_{4,4} - e$ (a), a nicer representation (b), and a 2-interval realization with open intervals of length 2 (c).

Property 1. *Any 2-interval realization of $K_{4,4} - e$ is contiguous.*

Proof. Write $G = (V, E)$ the graph $K_{4,4} - e$. To study all possible realizations of G, let us study all possible realizations of $G[V - I^8]$.

As 2-intervals I^1, I^2, I^3 and I^4 are disjoints, their ground set $\mathcal{I}_{\text{fixed}} = \{[l_i, r_i],\ 1 \leq i \leq 8,\ r_i < l_{i+1}\}$ is a set of eight disjoint intervals. The ground set $\mathcal{I}_{\text{mobile}}$ of I^5, I^6 and I^7 is a set of six disjoint intervals. Let x_i be the number of intervals of $\mathcal{I}_{\text{mobile}}$ intersecting $i \leq 8$ intervals of $\mathcal{I}_{\text{fixed}}$. We have directly:

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = |\mathcal{I}_{\text{mobile}}| = 6. \tag{1}$$

As there are 12 edges in $G[V \backslash \{v_8\}]$ which is bipartite, we also have:

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 + 7x_7 + 8x_8 = 12. \tag{2}$$

Finally, to build a realization of G from a realization of $G[V \backslash \{v_8\}]$, one must place I^8 so as to intersect three disjoint intervals of $\mathcal{I}_{\text{fixed}}$. Thus one of the intervals of I^8 intersects at least two intervals $]l_k, r_k[$ and $]l_l, r_l[$ $(k < l)$ of $\mathcal{I}_{\text{fixed}}$. So there is "a hole between those two intervals", for example $[r_k, l_{k+1}]$, which is included in one of the intervals of I^8. So we notice that I^8 has to fill one of the seven holes of $\mathcal{I}_{\text{fixed}}$. Thus, the intervals of $\mathcal{I}_{\text{mobile}}$ can not fill more than six holes, and the observation that an interval intersecting i consecutive intervals (for $i \geq 1$) fills $i - 1$ holes, we get:

$$x_2 + 2x_3 + 3x_4 + 4x_5 + 5x_6 + 6x_7 + 7x_8 \leq 6. \tag{3}$$

Equations 1, 2 and 3 are necessary for any valid realization of $G[V \backslash \{v_8\}]$ which gives a valid realization of G.

Let's suppose by contradiction that the union of all intervals of the ground set of G is not an interval. Then there is a hole, that is an interval included in the covering interval of $\{I^1, \ldots, I^8\}$, which intersect no I^i. We proceed like for equation 3, with the constraint that another hole cannot be filled by the intervals of $\mathcal{I}_{\text{mobile}}$, so we get instead:

$$x_2 + 2x_3 + 3x_4 + 4x_5 + 5x_6 + 6x_7 + 7x_8 \leq 5. \tag{4}$$

By adding 1 and 4, and subtracting 2, we get $x_0 \leq -1$: impossible! So we have proved that the union of all intervals of the ground set of any realization of G is indeed an interval. □

3 Balanced 2-interval graphs

We show in this section that the class of balanced 2-interval graphs is strictly included in the class of 2-interval graphs, and strictly contains circular-arc graphs. Moreover, we prove that recognizing balanced 2-interval graphs is as hard as recognizing (general) 2-interval graphs.

Property 2. *The class of balanced 2-interval graphs is strictly included in the class of 2-interval graphs.*

Proof. We build a 2-interval graph that has no balanced 2-interval realization. Let's consider a chain of gadgets $K_{5,3}$ (introduced in previous section) to which we add three vertices I^1, I^2, and I^3 as illustrated in Figure 5.

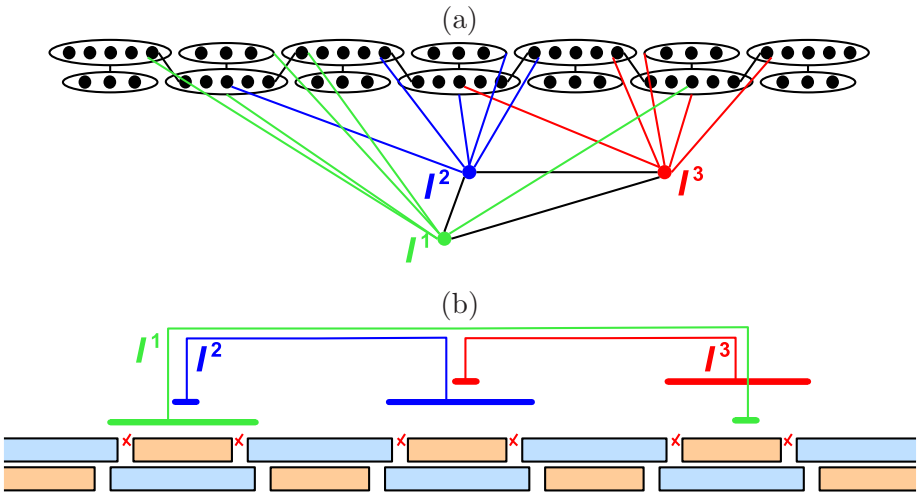


Figure 5: An example of unbalanced 2-interval graph (a) : any realization groups intervals of the seven $K_{5,3}$ in a block, and the chain of seven blocks creates six "holes" between them, which make it impossible to balance the lengths of the three 2-intervals I^1, I^2, and I^3.

In any realization, the presence of holes showed by crosses in the Figure gives the following inequalities for any realization: $l({I_l}^2) < l({I_l}^1)$, $l({I_l}^3) < l({I_r}^2)$, and $l({I_r}^1) < l({I_r}^3)$ (or if the realization of the chain of $K_{5,3}$ appears in the symmetrical order: $l({I_l}^1) < l({I_l}^3)$, $l({I_r}^3) < l({I_l}^2)$, and $l({I_r}^2) < l({I_r}^1)$). If this realization was balanced, then we would have $l({I_l}^1) = l({I_r}^1) < l({I_r}^3) = l({I_l}^3) < l({I_r}^2) = l({I_l}^2)$ (or for the symmetrical case: $l({I_r}^1) = l({I_l}^1) < l({I_l}^3) = l({I_r}^3) < l({I_l}^2) = l({I_r}^2)$) : impossible! So this graph has no balanced 2-interval realization although it has a 2-interval generalization. □

Theorem 1. *Recognizing balanced 2-interval graphs is an NP-complete problem.*

Proof. We just adapt the proof of West and Shmoys [WS84, GW95]. The problem of determining if there is a Hamiltonian cycle in a 3-regular triangle-free graph is proven NP-complete, by reduction from the more general problem without the no triangle restriction. So we reduce the problem of Hamiltonian cycle in a 3-regular triangle-free graph to balanced 2-interval recognition.

Let $G = (V, E)$ be a 3-regular triangle-free graph. We build a graph G' which has a 2-interval realization (a special one, very specific, called H-representation and which we prove to be balanced) iff G has a Hamiltonian cycle.

The construction of G', illustrated in Figure 6(a) is almost identical to the one by West and Shmoys, so we just prove that G' has a balanced realization, shown in Figure 6 (b), by

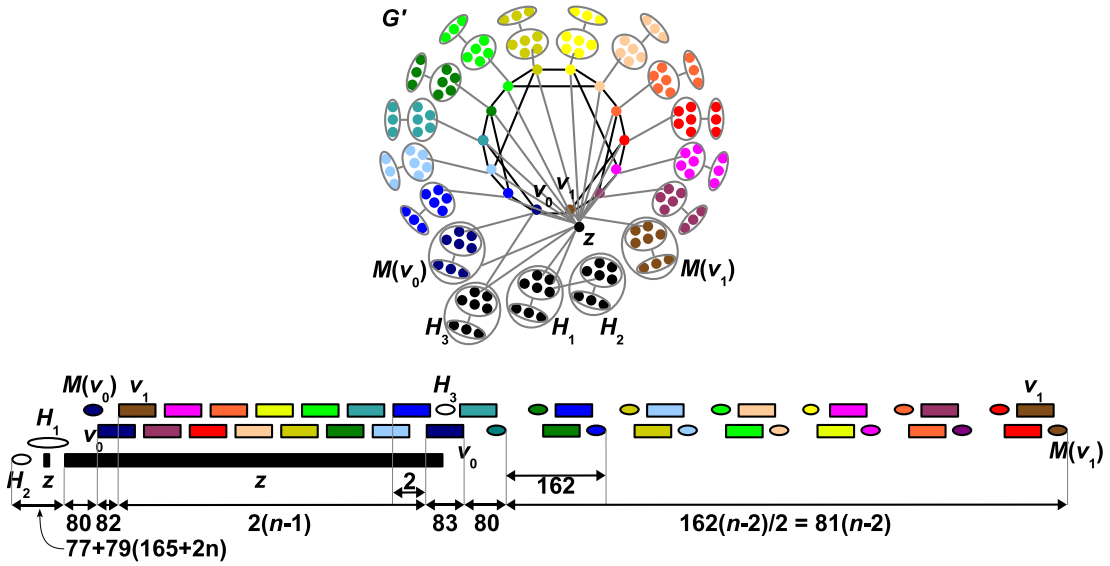


Figure 6: There is a balanced 2-interval of G' (which has been dilated in the drawing to remain readable) iff there is an H-representation (that is a realization where the left intervals of all 2-intervals are grouped together in a contiguous block) for its induced subgraph G iff there is a Hamiltonian cycle in G.

computing lengths for each interval to ensure it. All $K_{5,3}$ have a balanced realization as shown in section 1 of total length 79, in particular H_3. We can thus affect length 83 to the intervals of v_0. The intervals of the other v_i can have length 3, and their $M(v_i)$ length 79, so through the computation illustrated in Figure 6, intervals of z can have length $80 + 82 + 2(n-1) + 3$, that is $163 + 2n$. We dilate H_1 until a hole between two consecutive intervals of its S_3 can contain an interval of z, that is until the hole has length $165 + 2n$: so after this dilating, H_1 has length $79(165 + 2n)$. Finally if G has a Hamiltonian cycle, then we have found a balanced 2-interval realization of G of total length $13,273 + 241n$. □

It is known that circular-arc graphs are 2-interval graphs, they are also balanced 2-interval.

Property 3. *The class of circular-arc graphs is strictly included in the class of balanced 2-interval graphs.*

Proof. The transformation is simple: if we have a circular-arc representation of a graph $G = (V, E)$, then we choose some point P of the circle. We partition V in $V_1 \cup V_2$, where P intersects all the arcs corresponding to vertices of V_1 and none of the arcs of the vertices of V_2. Then we cut the circle at point P to map it to a line segment: every arc of V_2 becomes an interval, and every arc of V_1 becomes a 2-interval. To obtain a balanced realization we just cut in half the intervals of V_2 to obtain two intervals of equal length for each. And for each 2-interval $[g(I_l), d(I_l)] \cup [g(I_r), d(I_r)]$ of V_1, as both intervals are located on one of the extremities of the realization, we can increase the length of the shortest so that it reaches the length of the longest without changing intersections with the other intervals. The inclusion is strict because $K_{2,3}$ is a balanced 2-interval graph (as a subgraph of $K_{5,3}$ for example) but is not a circular-arc graph (we can find two C_4 in $K_{2,3}$, and only one can be realized with a circular-arc representation). □

4 Unit 2-interval and (x,x)-interval graphs

Property 4. *Let $x \in \mathbb{N}, x \geq 2$. The class of (x,x)-interval graphs is strictly included in the class of $(x+1,x+1)$-interval graphs.*

Proof. We first prove that an interval graph with a representation where all intervals have length k (and integer open bounds) has a representation where all intervals have length $k+1$.

We use the following algorithm. Let S be initialized as the set of all intervals of length k, and let T be initially the empty set. As long as S is not empty, let $I = [a,b]$ be the left-most interval of S, remove from S each interval $[\alpha,\beta]$ such that $\alpha < b$ (including I), add $[\alpha,\beta+1]$ to T, and translate by $+1$ all the remaining intervals in S.

When S is empty, the intersection graph of T, where all intervals have length $k+1$ is the same as the intersection graph for the original S. We also build for each $x \geq 2$ a $(x+1,x+1)$-interval graph which is not a (x,x)-interval graph. We consider the bipartite graph K_{2x} and a perfect matching $\{(v_i, v'_i), i \in [\![1,x]\!]\}$. We call K'_x the graph obtained from K_{2x} with the following transformations, illustrated in Figure 7(a): remove edges (v_i, v'_i) of the perfect matching, add four graphs $K_{4,4} - e$ called X_1, X_2, X_3, X_4 (for each X_i, we call v_l^i and v_r^i the vertices of degree 3), link v_r^2 and v_l^3, link all v_i to v_r^1 and v_l^4, link all v'_i to v_l^2 and v_r^3, and finally add a vertex a (resp. b) linked to all v_i, v'_i, and to two adjacent vertices of X_1 (resp. X_4) of degree 4. We illustrate in Figure 7(b) that K'_x has a realization with intervals of length $x+1$. We can prove by induction on x that K'_x has no realization with intervals of length x: it is rather technical, so we just give the idea. Any realization of K'_x forces the block X_2 to share an extremity with the block X_3, so each 2-interval v'_i has one interval intersecting the other extremity of X_2, and the other intersecting the other extremity of X_3. Then constraints on the position of vertices v_i force their intervals to appear as two "stairways" as shown in Figure 7(b). So v_r^1 must contain x extremities of intervals which have to be different, so it must have length $x+1$. □

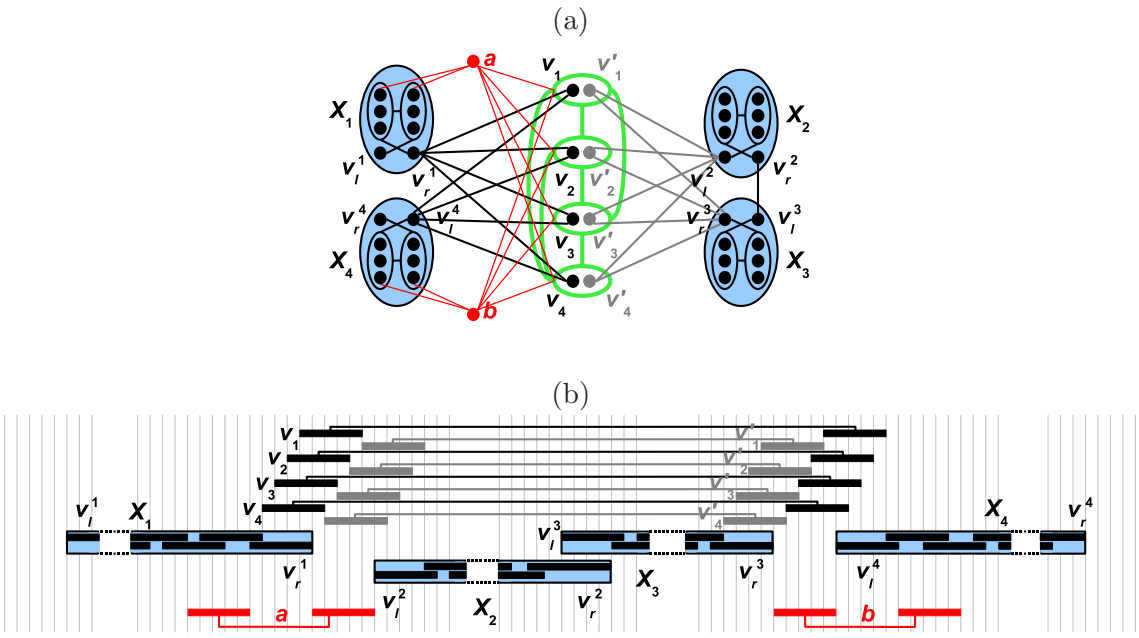


Figure 7: The graph K'_4 (a) is (5,5)-interval but not (4,4)-interval.

The complexity of recognizing unit 2-interval graphs and (x,x)-interval graphs remains open, however the following shows a relationship between those complexities.

Lemma 1. $\{$*unit 2-interval graphs*$\} = \bigcup_{x \in \mathbb{N}^*} \{(x,x)$*-interval graphs*$\}$.

Proof. The $\supset$ part is trivial. To prove $\subset$, let $G = (V, E)$ be a unit 2-interval graph. Then it has a realization with $|V| = n$ 2-intervals, that is $2n$ 2-intervals of the ground set. So we consider the interval graph of the ground set, which is a unit interval graph. There is a linear time algorithm based on breadth-first search to compute a realization of such a graph where interval endpoints are rational, with denominator $2n$ [CKN+95]. So by dilating by a factor $2n$ such a realization, we obtain a realization of G where intervals of the ground set have length $2n$. □

Theorem 2. *If recognizing (x,x)-interval graphs is polynomial for any integer x then recognizing unit 2-interval graphs is polynomial.*

5 Investigating the complexity of unit 2-interval graphs

In this section we show that all proper circular-arc graphs (circular-arc graphs such that no arc is included in another in the representation) are unit 2-interval graphs, and we study a class of graphs which generalizes quasi-line graphs and contains unit 2-interval graphs.

Recall that, according to Property 3, circular-arc graphs are balanced 2-interval graphs. However, circular-arc graphs are not necessarily unit 2-interval graphs.

Property 5. *The class of proper circular-arc graphs is strictly included in the class of unit 2-interval graphs.*

Proof. As in the proof of Property 3, we choose a point P on the circle of the representation of a proper circular-arc graph G, and maps the cut circle into a line segment. We extend the outer extremities of intervals that have been cut so that no interval contains another. Thus we obtain a set of 2-intervals for arcs containing P, and a set I of intervals for arcs not containing P. For each interval of I, we add a new interval disjoint of any other to get a 2-interval. If we consider the intersection graph of the ground set of such a representation, it is a proper interval graph. So it is also a unit interval graph [Rob69], which provides a unit 2-interval representation of G.

To complete the proof, we notice that the domino (two cycles C_4 having an edge in common) is a unit 2-interval graph but not a circular-arc graph. □

Quasi-line graphs are those graphs whose vertices are bisimplicial, *i.e.*, the closed neighborhood of each vertex is the union of two cliques. This graph class has been introduced as a generalization of line graphs and a useful subclass of claw-free graphs [Ben81, FFR97, CS05, KR07]. Following the example of quasi-line graphs that generalize line graphs, we introduce here a new class of graphs for generalizing unit 2-interval graphs. Let $k \in \mathbb{N}^*$. A graph $G = (V, E)$ is *all-k-simplicial* if the neighborhood of each vertex $v \in V$ can be partitioned into at most k cliques. The class of quasi-line graphs is thus exactly the class of all-2-simplicial graphs. Notice that this definition is equivalent to the following: in the complement graph of G, for each vertex u, the vertices that are not in the neighborhood of u are k-colorable.

Property 6. *The class of unit 2-interval graphs is strictly included in the class of all-4-simplicial graphs.*

Proof. The inclusion is trivial. What is left is to show that the inclusion is strict. Consider the following graph which is all-4-simplicial but not unit 2-interval: start with the cycle C_4, call its vertices v_i, $i \in [\![1,4]\!]$, add add four $K_{4,4} - e$ gadgets called X_i, and for each i we connect the vertex v_i to two connected vertices of degree 4 in X_i. This graph is certainly all-4-simplicial. But if we try to build a 2-interval realization of this graph, then each of the 2-intervals v_k has

an interval trapped into the block X_k. So each 2-interval v_k has only one interval to realize the intersections with the other v_i: this is impossible as we have to realize a C_4 which has no interval representation. □

Property 7. *The class of claw-free graphs is not included in the class of all-4-simplicial graphs.*

Proof. The Kneser Graph $KG(7,2)$ is triangle-free, but not 4-colorable [Lov78]. We consider the graph obtained by adding an isolated vertex v and then taking the complement graph, *i.e.*, $\overline{KG(7,2) \uplus \{v\}}$. It is claw-free as $KG(7,2)$ is triangle-free. And if it was all-4-simplicial, then the neighborhood of v in $\overline{KG(7,2) \uplus \{v\}}$, that is $\overline{KG(7,2)}$, would be a union of at most four cliques, so $KG(7,2)$ would be 4-colorable: impossible so this graph is claw-free but not all-4-simplicial. □

Property 8. *The class of all-k-simplicial graphs is strictly included in the class of $K_{1,k+1}$-free graphs.*

Proof. If a graph G contains $K_{1,k+1}$, then it has a vertex with $k+1$ independent neighbors, and hence G is not all-k-simplicial. The wheel W_{2k+1} is a simple example of a graph which is $K_{1,k+1}$-free but in which the center can not have its neighborhood (a C_{2k+1}) partitioned into k cliques or less. □

Unfortunately, all-k-simplicial graphs do not have a nice structure which could help unit 2-interval graph recognition.

Theorem 3. *Recognizing all-k-simplicial graphs is NP-complete for $k \geq 3$.*

Proof. We reduce from the Graph k-colorability problem, which is known to be NP-complete for $k \geq 3$ [Kar72]. Let $G = (V, E)$ be a graph, and let G' be the complement graph of G to which we add a universal vertex v. We claim that G is k-colorable iff G' is all-k-simplicial.

If G is k-colorable, then the non-neighborhood of any vertex in G is k-colorable, so the neighborhood of any vertex in $\overline{G}$ is a union of at most k cliques. And the neighborhood of v is also a union of at most k cliques, so G' is all-k-simplicial.

Conversely, if G' is all-k-simplicial, then in particular the neighborhood of v is a union of at most k cliques. Let's partition it into k vertex-disjoint cliques $X_1, \ldots, X_k$. Then, coloring G such that two vertices have the same color iff they are in the same X_i leads to a valid k-coloring of G. □

6 Conclusion

Motivated by practical applications in scheduling and computational biology, we focused in this paper on balanced 2-interval graphs and unit 2-intervals graphs. Also, we introduced two natural new classes: (x,x)-interval graphs and all-k-simplicial graphs.

We mention here some directions for future works. First, the complexity of recognizing unit 2-interval graphs and (x,x)-interval graphs remains open. Second, the relationships between quasi-line graphs and subclasses of balanced 2-intervals graphs still have to be investigated. Last, since most problems remains NP-hard for balanced 2-interval graphs, there is thus a natural interest in investigating the complexity and approximation of classical optimization problems on unit 2-interval graphs and (x,x)-interval graphs.

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7 Appendix

We give the detailed proofs of Theorem 1 and Property 4.

Proof of Theorem 1. Let $G = (V, E)$ be a 3-regular triangle-free graph. We build a graph G' which has a 2-interval realization (a special one, very specific, and which we prove to be balanced) iff G has a Hamiltonian cycle.

First we will detail how we build G' starting from the graph G, and adding some vertices, in particular $K_{5,3}$ gadgets. The idea is that the edges of G will partition into a Hamiltonian cycle and a perfect matching iff all 2-intervals of the realization of G' can have their left interval realizing the Hamiltonian cycle, and their right interval realizing the perfect matching. A realization with such a placement of the intervals is called an "H-representation" of G.

We proceed as illustrated in Figure 6. We choose some vertex of G that we call v_0 (which will be the "origin" of the Hamiltonian cycle), and the other are called $v_1, \ldots, v_n$. For each vertex v_i of G we link it to a vertex of the S_5 of a $K_{5,3}$ called $M(v_i)$ (which will block one of the four extremities of the 2-interval v_i). We link all vertices to a new vertex z, which is linked to no $M(v)$ except $M(v_0)$ (thus the interval of each v_i intersecting $M(v_i)$, for $i \neq 0$, won't intersect z). We add three $K_{5,3}$, H_1, H_2 and H_3 : two vertices of the S_5 of H_1 are linked to z, a third one is linked to one vertex of the S_5 of H_2, one vertex of the S_5 of H_3 is linked to z, and all vertices of H_3 to v_0.

To explain this construction in detail, we study the realization of G', if we suppose it is a (balanced) 2-interval graph, and we prove that it leads us to find a Hamiltonian cycle in G.

As the realization of H_1 and H_2 are two contiguous blocks of intervals then one of their extremities must intersect. As z is linked to two disjoint vertices of H_1, both intervals of z are used to realize those intersections. But one interval of z that we call z_r, also has to intersect one vertex of H_3 which is not linked to H_1, so z_r intersects the second extremity of the block H_1 (the first extremity being occupied by the extremity of H_2). And as z_r intersects only one interval of H_3, it must be the extremity of H_3. The other interval of z is contained in the block H_1, thus can't intersect $M(v_0)$ neither all the vertices v_i, so all those 2-intervals intersect z_r. And as none of them intersect H_3 except v_0, all of them except v_0 have an interval contained in z_r, that we call $v_{i,g}$. The other interval of each v_i is linked to a $K_{5,3}$ so it has one extremity occupied by $K_{5,3}$, and the other one is free.

Conversely, if G has a Hamiltonian cycle, then it is possible to find a H-representation, such that all the constraints induced by the edges of G' are respected, as illustrated with the realization in Figure 6. We have already proved that this realization can be balanced. □

Proof of Property 4. In the following, as we only considering the interval of v_l^i or v_r^i located at one extremity of the block X_i, and not the one inside, we will use v_l^i and v_r^i to denote those extremity intervals. For each vertex v_i, we call $v_{i,l}$ its left interval and $v_{i,r}$ its right interval. We do the same for v'_i, and call $l(I)$ the left extremity of any interval I.

We prove by induction that the graph K'_x is $(x+1, x+1)$-interval but not (x, x)-interval, and that for any unit 2-interval realization, there exists an order $\sigma \in \mathcal{S}_x$ such that :

- either $l(v_{\sigma(x),l}) < \ldots < l(v_{\sigma(1),l}) < l(v'_{\sigma(x),l}) < \ldots < l(v'_{\sigma(1),l})$ and $l(v'_{\sigma(x),r}) < \ldots < l(v'_{\sigma(1),r}) < l(v_{\sigma(x),r}) < \ldots < l(v_{\sigma(1),r})$,
- or the symmetric case: $l(v_{\sigma(1),l}) < \ldots < l(v_{\sigma(x),l}) < l(v'_{\sigma(1),l}) < \ldots < l(v'_{\sigma(x),l})$ and $l(v'_{\sigma(1),r}) < \ldots < l(v'_{\sigma(x),r}) < l(v_{\sigma(1),r}) < \ldots < l(v_{\sigma(x),r})$.

Those two equalities correspond in fact to the "two stairways structure" which appears in Figure 7.

Base case : we study all possible unit 2-interval realizations of K'_2 to prove that one of the expected inequalities is always true. We also prove that K'_2 has no (2,2)-interval realization.

First recall that realizations of X_i subgraphs can only be blocks of contiguous intervals. The edge between v_r^2 and v_l^3 forces the two blocks of X_2 and X_3 to be contiguous, with intervals v_l^2 and v_r^3 at their extremities. Each 2-interval v'_i must intersect both v_l^2 and v_r^3, so one of its intervals intersects v_l^2 and the other intersects v_r^3. Thus, one same interval of v'_i can not intersect both a and b which are disjoint, so a intersects one interval of v'_i (say the one intersecting v_l^2, the other case being treated symmetrically) and b intersects the other one (so, the one intersecting v_r^3). Each v_i has to intersect both a and b, so it has to intersect a with its first interval and b with the second. But 2-interval v_i must also intersect v_r^1 and v_l^4 which are both disjoint and disjoint to a and b. So one interval of each v_i must intersect v_r^1 and the other one must intersect v_l^4.

So we have shown that any unit 2-interval realization of K'_2 has the following aspect (or the symmetric) : the extremity of the block X_1 intersecting all v_i which intersect a (or b) which intersects all v'_i, which intersect the extremity X_2 (or X_3) which intersects the extremity of X_3 (or X_2), which intersects all v'_i, which intersect b (or a), which intersects all v_i, which intersect the extremity of X_4.

Now we suppose, by contradiction, that there exists a (2,2)-interval realization of K'_2. v_r^1 is an interval of length 2, but one of its two parts of length one has to intersect an element of X_1. The other has to intersect both v_1 and v_2. As neither v_1 nor v_2 can intersect other intervals of X_1, then the first interval of v_1 and v_2 is the same interval. By proceeding the same way on X_4 and v_l^4, we obtain that the second interval of v_1 and v_2 is the same interval, so v_1 and v_2 should correspond to the same 2-interval: it contradicts with the fact that vertices v_1 and v_2 have a different neighborhood. So K'_2 has no (2,2)-interval realization.

To obtain the expected inequalities, we have to analyze the possible positions of all v_i and v'_i. We only treat the first two inequalities as the second case is symmetric.

Suppose that $l(v_{2,l}) < l(v_{1,l})$. As v_1 and v'_1 are non adjacent, then interval $v_{1,l}$ is strictly on the left of $v'_{1,l}$, so $v_{2,l}$ is strictly on the left of $v'_{1,l}$. Thus those two intervals do not intersect. But v_2 and v'_1 are adjacent, so v_2 and v'_1 must have intersecting right intervals. But then we have $l(v'_{2,r}) < l(v'_{1,r}) < l(v_{2,r}) < l(v_{1,r})$, and the right intervals of v'_2 and v_1 can not intersect. We deduce their left intervals intersect, so $l(v_{2,l}) < l(v_{1,l}) < l(v'_{2,l}) < l(v'_{1,l})$.

If we suppose that $l(v_{1,l}) < l(v_{2,l})$, we get as well that $l(v'_{1,r}) < l(v'_{2,r}) < l(v_{1,r}) < l(v_{2,r})$ and $l(v_{1,l}) < l(v_{2,l}) < l(v'_{1,l}) < l(v'_{2,l})$. So for any unit 2-interval realization of K'_2 there exists an order $\sigma = 12$ or $\sigma = 21$ such that:

- either $l(v_{\sigma(2),l}) < l(v_{\sigma(1),l}) < l(v'_{\sigma(2),l}) < l(v'_{\sigma(1),l})$ and $l(v'_{\sigma(2),r}) < l(v'_{\sigma(1),r}) < l(v_{\sigma(2),r}) < l(v_{\sigma(1),r})$,
- or the symmetric inequalities.

Recursion: suppose that for some x, K'_{x-1} is not $(x-1,x-1)$-interval but is (x,x)-interval, and that any (x,x)-interval realization verifies one of the expected inequalities.

Graph K'_{x-1} is an induce subgraph of $K'_x = (V,E)$: $K'_{x-1} = K'_x[V \setminus \{v_x, v'_x\}]$. So by the induction hypothesis, there exists an order $\sigma \in \mathcal{S}_{x-1}$ such that for any unit 2-interval realization of K'_x :

- either $l(v_{\sigma(x-1),l}) < \ldots < l(v_{\sigma(1),l}) < l(v'_{\sigma(x-1),l}) < \ldots < l(v'_{\sigma(1),l})$ and $l(v'_{\sigma(x-1),r}) < \ldots < l(v'_{\sigma(1),r}) < l(v_{\sigma(x-1),r}) < \ldots < l(v_{\sigma(1),r})$,

- or the symmetric case: $l(v_{\sigma(1),l}) < \ldots < l(v_{\sigma(x-1),l}) < l(v'_{\sigma(1),l}) < \ldots < l(v'_{\sigma(x-1),l})$ and $l(v'_{\sigma(1),r}) < \ldots < l(v'_{\sigma(x-1),r}) < l(v_{\sigma(1),r}) < \ldots < l(v_{\sigma(x-1),r})$.

The position of v_x and v'_x remains to be determined. We treat only the case where the first two inequalities are true, as the second case is symmetric.

As v_x and v^1_r are adjacent, and $v'_{\sigma(x-1)}$ and v^1_r are not, then $l(v^1_r) < l(v_{x,l}) < l(v'_{\sigma(x-1),l})$. So we define j the following way: $v_{\sigma(j),l}$ is the leftmost interval such that $l(v_{x,l}) \leq l(v_{\sigma(j),l})$. if there is none, we say $j = 0$. Then we call $\sigma' \in \mathcal{S}_x$ the permutation defined by:

$$\begin{cases} \sigma'(i) = \sigma(i) \text{ if } i < j, \\ \sigma'(j+1) = x, \\ \sigma'(i) = \sigma(i-1) \text{ if } i > j. \end{cases}$$

Then we directly get inequalities:

- $l(v^1_r) < l(v_{\sigma'(x),l}) < \ldots < l(v_{\sigma'(j+1),l}) \leq l(v_{x,l}) < l(v_{\sigma'(j-1),l}) < \ldots < l(v_{\sigma'(1),l}) < l(v'_{\sigma'(x),l}) < \ldots < l(v'_{\sigma'(j+1),l}) < l(v'_{\sigma'(j-1),l}) < \ldots < l(v'_{\sigma'(1),l})$
- $l(v'_{\sigma'(x),r}) < \ldots < l(v'_{\sigma'(j+1),r}) < l(v'_{\sigma'(j-1),r}) < \ldots < l(v'_{\sigma'(1),r}) < l(v_{\sigma'(x),r}) < \ldots < l(v_{\sigma'(j+1),r}) < l(v_{\sigma'(j-1),r}) < \ldots < l(v_{\sigma'(1),r})$

We obtain the expected inequalities by reasoning the same way as in the end of the base case.

So in particular we have $l(v_{\sigma(x),l}) < \ldots < l(v_{\sigma(1),l})$ and v^1_r must intersect all those v_i for $i \in [\![1, x]\!]$, but also an interval of X_1 which intersects none of the v_i. So it must have length $x + 1$, thus K'_x is not a (x, x)-interval graph

Conclusion: As the base case and the recursion has been proved, expected properties of the graph K'_x are true for any $x \geq 2$. □