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► **To cite this version:**

Hyam Abboud, Toni Sayah. Upwind discretisation of a time-dependent two dimensional grade-two fluid model. 2007. hal-00141016

HAL Id: hal-00141016

<https://hal.science/hal-00141016>

Preprint submitted on 11 Apr 2007

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Upwind discretisation of a time-dependent two dimensional grade-two fluid model

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9th April 2007

Abstract

In this paper, we propose a finite-element scheme for solving numerically the equations of a transient two dimensional grade-two fluid non-Newtonian Rivlin-Ericksen fluid model. This system of equations is considered an appropriate model for the motion of a water solution of polymers. As expected, the difficulties of this problem arise from the transport equation. As one of our aims is to derive unconditional a priori estimates from the discrete analogue of the transport equation, we stabilize our scheme by adding a consistent stabilizing term. We use the $\mathbb{P}_2 - \mathbb{P}_1$ Taylor-Hood finite-element scheme for the velocity \mathbf{v} and the pressure p , and the discontinuous \mathbb{P}_1 finite element for an auxiliary variable \mathbf{z} . The error is of the order of $h^{3/2} + k$, considering that the discretization of the transport equation loses inevitably a factor $h^{1/2}$.

Keywords Grade-two fluid, non-linear problem, incompressible flow, time and space discretizations.

1 Introduction

This article is devoted to the numerical solution of the equation of a grade two fluid non-Newtonian Rivlin-Ericksen fluid ([16]) :

$$\frac{\partial}{\partial t}(\mathbf{u} - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + \mathbf{curl}(\mathbf{u} - \alpha \Delta \mathbf{u}) \times \mathbf{u} + \nabla p = \mathbf{f} \text{ in }]0, T[\times \Omega, \quad (1)$$

with the incompressibility condition :

$$\operatorname{div} \mathbf{u} = 0 \text{ in }]0, T[\times \Omega, \quad (2)$$

where the velocity $\mathbf{u} = (u_1, u_2, 0)$,

$$\operatorname{div} \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \operatorname{curl} \mathbf{u} = (0, 0, \operatorname{curl} \mathbf{u}), \quad \operatorname{curl} \mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},$$

here \mathbf{f} denotes an external force, ν the viscosity and α is a constant normal stress modulus.

This model is considered an appropriate one for the motion of water solutions of polymers ([7]). The case $\alpha = 0$ represents the transient Navier-Stokes problem. Here, p is not the pressure, but the formula which gives the pressure from \mathbf{u} and p is complex. To simplify, we refer to p as the “pressure” in the sequel. According to Dunn and Fosdick’s work [8], in order to be consistent with thermodynamics, a grade-two fluid must satisfy $\alpha \geq 0$ and $\nu \geq 0$. The reader can refer to [7] for a discussion on the sign of α .

The equations of a grade two fluid model have been studied by many authors (Videmann gives in [17] a very extensive list of references), but the best construction of solutions for the problem, with homogeneous Dirichlet boundary conditions and mildly smooth data, is given by Ouazar [15] and by Cioranescu and Ouazar [3], [4]. They prove existence of solutions, with H^3 regularity in space, by looking for a velocity \mathbf{u} such that

$$\mathbf{z} = \operatorname{curl}(\mathbf{u} - \alpha \Delta \mathbf{u})$$

has L^2 regularity in space, introducing \mathbf{z} as an auxiliary variable and discretizing the equations of motion by Galerkin’s method in the basis of the eigenfunctions of the operator $\operatorname{curl} \operatorname{curl}(\mathbf{u} - \alpha \Delta \mathbf{u})$. This choice allows one to recover estimates from the transport equation in two dimensions

$$\alpha \frac{\partial \mathbf{z}}{\partial t} + \nu \mathbf{z} + \alpha \mathbf{u} \cdot \nabla \mathbf{z} = \nu \operatorname{curl} \mathbf{u} + \alpha \operatorname{curl} \mathbf{f}, \quad (3)$$

whenever $\operatorname{curl} \mathbf{f}$ belongs to $L^2(\Omega)^3$. In this case, $\mathbf{z} = (0, 0, z)$ with $z = \operatorname{curl}(\mathbf{u} - \alpha \Delta \mathbf{u})$. Hence, \mathbf{z} is necessarily orthogonal to \mathbf{u} .

In this article, we propose finite-element schemes for solving numerically the equation of a two dimensional grade-two fluid model. Defining \mathbf{z} as above, the equations of motion becomes :

$$\frac{\partial}{\partial t}(\mathbf{u} - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + z \times \mathbf{u} + \nabla p = \mathbf{f}, \quad (4)$$

and

$$\alpha \frac{\partial z}{\partial t} + \nu z + \alpha \mathbf{u} \cdot \nabla z = \nu \operatorname{curl} \mathbf{u} + \alpha \operatorname{curl} \mathbf{f}, \quad (5)$$

the Dirichlet boundary condition :

$$\mathbf{u} = \mathbf{0} \quad \text{on }]0, T[\times \partial \Omega, \quad (6)$$

and the initial conditions :

$$\mathbf{u}(x, t) = \mathbf{0}, \quad \text{and} \quad z(x, t) = 0. \quad (7)$$

This problem is analyzed by Girault and Saadouni [10]. If we want to derive an unconditional a

priori estimate for the discrete analogue of (3), we add to the left-hand side of this last equation a stabilizing, consistent term, so it becomes

$$\alpha \frac{\partial z}{\partial t} + \nu z + \alpha \mathbf{u} \cdot \nabla z + \frac{\alpha}{2} (\operatorname{div} \mathbf{u}) z = \nu \operatorname{curl} \mathbf{u} + \alpha \operatorname{curl} \mathbf{f}. \quad (8)$$

In this work, we propose to discretize this last equation, as Girault and Scott did in [11], by an upwind scheme based on the discontinuous Galerkin method of degree one introduced by Lesaint and Raviart in [12]. Let X_h , M_h and Z_h be the discrete spaces for the velocity and the pressure. We approximate the velocity and the pressure by the standard $\mathbb{P}_2 - \mathbb{P}_1$ Taylor-Hood scheme, where \mathbb{P}_k denotes the space of polynomials of degree k in two variables. Also, in each element of the triangulation, z_h^n is a polynomial of degree one, without continuity requirement on interelement boundaries. Our discrete system corresponding to (4) and (8) is :

Find $\mathbf{u}_h^{n+1} \in X_h, p_h^{n+1} \in M_h$ and $z_h^{n+1} \in Z_h$ such that

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \frac{1}{k} (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}_h) + \frac{\alpha}{k} (\nabla (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n), \nabla \mathbf{v}_h) + \nu (\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) \\ + (z_h^n \times \mathbf{u}_h^{n+1}, \mathbf{v}_h) - (p_h^{n+1}, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \end{aligned} \quad (9)$$

$$\begin{aligned} \forall \theta_h \in Z_h, \frac{\alpha}{k} (z_h^{n+1} - z_h^n, \theta_h) + \nu (z_h^{n+1}, \theta_h) + c(\mathbf{u}_h^{n+1}; z_h^{n+1}, \theta_h) \\ = \nu (\operatorname{curl} \mathbf{u}_h^{n+1}, \theta_h) + \alpha (\operatorname{curl} \mathbf{f}^{n+1}, \theta_h), \end{aligned} \quad (10)$$

where $c(\mathbf{u}_h^{n+1}; z_h^{n+1}, \theta_h)$ is the discrete non-linear part of the transport equation and the functions of X_h vanish on $\partial\Omega$. This system is linearized in the sense that in (9), knowing z_h^n , we calculate \mathbf{u}_h^{n+1} and p_h^{n+1} with a linear equation. Then, we calculate z_h^{n+1} with the second linear equation (10). For both the velocity and pressure discretizations, the error is of order $h^{3/2}$ and k . This is the best that can be achieved, considering that the discretization of the transport equation loses inevitably a factor $h^{1/2}$. Other finite elements can be used, cf. Crouzeix and Raviart [6], Brezzi and Fortin [2] and Girault and Raviart [9].

Now, we recall some notation and basic functional results. As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a, b[$ with values in a functional space, say X (cf. Lions and Magenes [13]). More precisely, let $\| \cdot \|_X$ denote the norm of X ; then for any r , $1 \leq r \leq \infty$, we define

$$L^r(a, b; X) = \{f \text{ measurable in }]a, b[; \int_a^b \| f(t) \|_X^r dt < \infty\}$$

equipped with the norm

$$\| f \|_{L^r(a, b; X)} = \left(\int_a^b \| f(t) \|_X^r dt \right)^{1/r},$$

with the usual modifications if $r = \infty$. It is a Banach space if X is a Banach space.

Let (k_1, k_2) denote a pair of non-negative integers, set $|k| = k_1 + k_2$ and define the partial derivative ∂^k by

$$\partial^k v = \frac{\partial^{|k|} v}{\partial \mathbf{x}_1^{k_1} \partial \mathbf{x}_2^{k_2}}.$$

We denote by :

$$W^{m,r}(\Omega) = \{v \in L^r(\Omega); \partial^k v \in L^r(\Omega) \forall |k| \leq m\},$$

This space is equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \left[\sum_{|k|=m} \int_{\Omega} |\partial^k v|^r d\mathbf{x} \right]^{1/r},$$

and is a Banach space for the norm

$$\|v\|_{W^{m,r}(\Omega)} = \left[\sum_{0 \leq |k| \leq m} |v|_{W^{k,r}(\Omega)}^r d\mathbf{x} \right]^{1/r}.$$

When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. In particular, the scalar product of $L^2(\Omega)$ is denoted by (\cdot, \cdot) .

Similarly, $L^2(a, b; H^m(\Omega))$ is a Hilbert space and in particular $L^2(a, b; L^2(\Omega))$ coincides with $L^2(\Omega \times]a, b[)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\mathbf{u} = (u_1, u_2)$; then we set

$$\|\mathbf{u}\|_{L^r(\Omega)} = \left[\int_{\Omega} \|\mathbf{u}(\mathbf{x})\|^r d\mathbf{x} \right]^{1/r},$$

where $\|\cdot\|$ denotes the Euclidean vector norm.

For functions that vanish on the boundary, we define for any $r \geq 1$

$$W_0^{1,r}(\Omega) = \{v \in W^{1,r}(\Omega); v|_{\partial\Omega} = 0\}$$

and recall Sobolev's imbeddings in two dimensions: for each $r \in [2, \infty[$, there exists a constant S_r such that

$$\forall \mathbf{v} \in H_0^1(\Omega), \quad \|\mathbf{v}\|_{L^r(\Omega)} \leq S_r |\mathbf{v}|_{H^1(\Omega)}, \quad (11)$$

where

$$|\mathbf{v}|_{H^1(\Omega)} = \|\nabla \mathbf{v}\|_{L^2(\Omega)}. \quad (12)$$

When $r = 2$, (11) reduces to Poincaré's inequality and S_2 is Poincaré's constant.

The case $r = \infty$ is excluded and is replaced by: for any $r > 2$, there exists a constant M_r such that

$$\forall v \in W_0^{1,r}(\Omega), \quad \|v\|_{L^\infty(\Omega)} \leq M_r |v|_{W^{1,r}(\Omega)}. \quad (13)$$

We have also in dimension 2,

$$\|g\|_{L^4(\Omega)} \leq 2^{1/4} \|g\|_{L^2(\Omega)}^{1/2} \|\nabla g\|_{L^2(\Omega)}^{1/2}. \quad (14)$$

Owing to Poincaré's inequality, the seminorm $|\cdot|_{H^1(\Omega)}$ is a norm on $H_0^1(\Omega)$ and we use it to define the dual norm:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v \rangle}{|v|_{H^1(\Omega)}},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Also, we introduce the space:

$$L_0^2(\Omega) = \{q \in L^2(\Omega); \int_{\Omega} q d\mathbf{x} = 0\},$$

2 The exact problem

Let Ω be a bounded polygon in two dimensions with boundary $\partial\Omega$ and let $]0, T[$ be a given time-interval. We want to find a vector velocity \mathbf{u} , a scalar pressure p and an auxiliary scalar function z solution of

$$\frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + z \times \mathbf{u} + \nabla p = \mathbf{f} \text{ in }]0, T[\times\Omega, \quad (15)$$

$$\alpha\frac{\partial z}{\partial t} + \nu z + \alpha\mathbf{u} \cdot \nabla z = \nu \operatorname{curl}\mathbf{u} + \alpha \operatorname{curl}\mathbf{f} \text{ in }]0, T[\times\Omega, \quad (16)$$

$$\mathbf{u} = \mathbf{0} \text{ on }]0, T[\times\partial\Omega, \quad (17)$$

$$\mathbf{u}(x, t) = \mathbf{0} \text{ and } z(x, t) = 0, \quad (18)$$

where $\mathbf{z} \times \mathbf{u} = (-zu_2, zu_1)$. Here $\nu > 0$ and $\alpha > 0$ are given constants.

A straightforward formulation of (15)–(18) is :

Find $(\mathbf{u}(t), p(t), z(t)) \in L^\infty(0, T; H_0^1(\Omega)^2) \times L^2(0, T; L_0^2(\Omega)) \times L^\infty(0, T; L^2(\Omega))$, $\mathbf{u}' \in L^2(0, T; H_0^1(\Omega)^2)$ such that

$$\forall \mathbf{v} \in H_0^1(\Omega), \quad (\mathbf{u}'(t), \mathbf{v}) + \alpha(\nabla\mathbf{u}'(t), \nabla\mathbf{v}) + \nu(\nabla\mathbf{u}(t), \nabla\mathbf{v}) \quad (19)$$

$$+(z(t) \times \mathbf{u}(t), \mathbf{v}) - (p(t), \operatorname{div} \mathbf{v}) = (\mathbf{f}(t), \mathbf{v}) \text{ in } \Omega \times]0, T[,$$

$$\forall q \in L_0^2(\Omega), \quad (q(t), \operatorname{div} \mathbf{u}(t)) = 0, \quad (20)$$

$$\alpha\frac{\partial z}{\partial t} + \nu z + \alpha\mathbf{u} \cdot \nabla z = \nu \operatorname{curl}\mathbf{u} + \alpha \operatorname{curl}\mathbf{f} \text{ in }]0, T[\times\Omega, \quad (21)$$

$$\mathbf{u}(0) = \mathbf{0} \quad \text{and } z(0) = 0 \text{ in } \Omega. \quad (22)$$

The following theorem is established in [10]:

Theorem 2.1. *Let Ω be a lipschitz polygon. For all $\nu > 0$ and $\mathbf{f} \in L^2(0, T; L^2(\Omega)^2)$ such that $\operatorname{curl}\mathbf{f} \in L^2(0, T; L^2(\Omega))$, (15)–(18) has at least one solution (\mathbf{u}, z, p) that satisfies the following estimates :*

$$\|z\|_{L^\infty(0, T; L^2(\Omega))} \leq \sqrt{2}\frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(0, T; L^2(\Omega)^2)} + \frac{|\alpha|}{\nu}\|\operatorname{curl}\mathbf{f}\|_{L^2(0, T; L^2(\Omega))},$$

$$\|\mathbf{u}\|_{L^\infty(0, T; H^1(\Omega)^2)} \leq \frac{S_2}{\nu}\|\mathbf{f}\|_{L^2(0, T; L^2(\Omega)^2)},$$

$$\|p\|_{L^2(0, T; L^2(\Omega))} \leq \frac{1}{\beta}(S_2\|\mathbf{f}\|_{L^2(0, T; L^2(\Omega)^2)} + S_4^2\|\mathbf{u}\|_{L^\infty(0, T; H^1(\Omega)^2)}\|z\|_{L^\infty(0, T; L^2(\Omega))}).$$

3 A discontinuous upwind scheme

Let $h > 0$ be a discretization parameter and let \mathcal{T}_h be a regular family of triangulation of $\bar{\Omega}$, consisting of triangles κ with maximum mesh size h : There exists a constant σ_0 , independent

of h , such that $\forall \kappa \in \mathcal{T}_h$, $\frac{h_\kappa}{\rho_\kappa} \leq \sigma_0$, where h_κ is the diameter and ρ_κ is the diameter of the ball inscribed in κ . We introduce $\rho_{min} = \min_{\kappa} \rho_\kappa$. As usual, the triangulation is such that any two triangles are either disjoint or share a vertex or a complete side.

We first recall how upwinding can be achieved by the discontinuous Galerkin approximation introduced in [12]. Let Z_h be the discontinuous finite-element space :

$$Z_h = \{\theta_h \in L^2(\Omega); \forall \kappa \in \mathcal{T}_h, \theta_h|_\kappa \in \mathbb{P}_1\}.$$

There exists an approximation operator, [5], $R_h \in \mathcal{L}(W^{1,p}(\Omega); Z_h \cap C^0(\overline{\Omega}))$ such that for any $p \geq 1$, for $m = 0, 1$ and $0 \leq l \leq 1$

$$\forall z \in W^{l+1,p}(\Omega), \quad |R_h(z) - z|_{W^{m,p}(\Omega)} \leq Ch^{l+1-m} |z|_{W^{l+1,p}(\Omega)}.$$

Let \mathbf{u}_h be a discrete velocity in $H_0^1(\Omega)^2$, and for each triangle κ , let

$$\partial\kappa_- = \{x \in \partial\kappa; \alpha \mathbf{u}_h \cdot \mathbf{n} < 0\},$$

where \mathbf{n} denotes the unit exterior normal to $\partial\kappa$. Note that, for all triangles κ of \mathcal{T}_h , $\partial\kappa_-$ only involves interior segments of \mathcal{T}_h because $\mathbf{u}_h = 0$ on $\partial\Omega$. Then, the non-linear term $\alpha[(\mathbf{u} \cdot \nabla z, \theta) + \frac{1}{2}(\text{div } \mathbf{u} z, \theta)]$ is approximated by

$$\begin{aligned} c(\mathbf{u}_h^{n+1}; z_h^{n+1}, \theta_h^{n+1}) &= \frac{\alpha}{2} \int_{\Omega} \text{div } \mathbf{u}_h^{n+1} z_h^{n+1} \theta_h^{n+1} d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \left(\int_{\kappa} \alpha(\mathbf{u}_h^{n+1} \cdot \nabla z_h^{n+1}) \theta_h^{n+1} d\mathbf{x} \right. \\ &\quad \left. + \int_{\partial\kappa_-} |\alpha \mathbf{u}_h^{n+1} \cdot \mathbf{n}| (z_{h,\text{int}}^{n+1} - z_{h,\text{ext}}^{n+1}) \theta_{h,\text{int}}^{n+1} ds \right). \end{aligned}$$

The subscript int (resp. ext) refers to the trace on the segment $\partial\kappa$ of the function taken inside (resp. outside) κ . Note that in the above sum, the boundary integrations act in fact over complete interior segments.

On the other hand, let us recall the standard Taylor-Hood discretization of the velocity and pressure. The discrete space of the pressure is :

$$M_h = \{q_h \in H^1(\Omega) \cap L_0^2(\Omega); \forall \kappa \in \mathcal{T}_h, q_h \in \mathbb{P}_1\}.$$

There exists an operator $r_h \in \mathcal{L}(L_0^2(\Omega); M_h)$ such that for $0 \leq l \leq 2$,

$$\forall q \in H^l(\Omega) \cap L_0^2(\Omega), \quad \|r_h(q) - q\|_{L^2(\Omega)} \leq Ch^l \|q\|_{H^l(\Omega)}.$$

The discrete velocity space is :

$$X_h = \{\mathbf{v}_h \in C^0(\overline{\Omega}); \forall \kappa \in \mathcal{T}_h, \mathbf{v}_h|_\kappa \in \mathbb{P}_2, \mathbf{v}_h|_{\partial\Omega} = \mathbf{0}\},$$

and let

$$V_h = \{\mathbf{v}_h \in X_h; (q_h, \text{div } \mathbf{v}_h) = 0 \quad \forall q \in M_h\}.$$

There exists an operator $P_h \in \mathcal{L}(H_0^1(\Omega)^2; X_h)$, such that

$$\left\{ \begin{array}{l} \forall \mathbf{v} \in H_0^1(\Omega)^2, \quad \forall \kappa \in \mathcal{T}_h, \quad \forall q_h \in M_h, \quad \int_{\kappa} q_h \operatorname{div}(P_h(\mathbf{v}) - \mathbf{v}) d\mathbf{x} = 0, \\ \text{for all number } p \geq 2; \quad \forall \mathbf{v} \in H_0^1(\Omega)^2, \quad \|P_h(\mathbf{v}) - \mathbf{v}\|_{L^p(\Omega)} \leq Ch^{2/p} |\mathbf{v}|_{H^1(\Omega)}, \\ \text{for all number } p \geq 2, 1 \leq s \leq 3, m = 0 \text{ or } 1, \quad \forall \mathbf{v} \in [W^{s,p}(\Omega) \cap H_0^1(\Omega)]^2, \\ \|P_h(\mathbf{v}) - \mathbf{v}\|_{W^{m,p}(\Omega)} \leq Ch^{s-m} |\mathbf{v}|_{W^{s,p}(\Omega)}. \end{array} \right. \quad (23)$$

We take $\mathbf{f}^{n+1}(\mathbf{x}) = \frac{1}{k} \int_{t^n}^{t^{n+1}} \mathbf{f}(t, \mathbf{x}) dt$. Then the discrete system corresponding to the formulation (19)–(22) is :

Given $(\mathbf{u}_h^0, z_h^0) = (\mathbf{0}, 0)$ and $z_h^n \in Z_h$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in X_h \times M_h$ such that :

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \quad \frac{1}{k}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}_h) + \frac{\alpha}{k}(\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n), \nabla \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) \\ + (\mathbf{z}_h^n \times \mathbf{u}_h^{n+1}, \mathbf{v}_h) - (p_h^{n+1}, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \end{aligned} \quad (24)$$

$$\forall q_h \in M_h, \quad (q_h, \operatorname{div} \mathbf{u}_h^{n+1}) = 0. \quad (25)$$

Once we have \mathbf{u}_h^{n+1} , we compute z_h^{n+1} by solving the system :

$$\begin{aligned} \forall \theta_h \in Z_h, \quad \frac{1}{k}(z_h^{n+1} - z_h^n, \theta_h) + \nu(z_h^{n+1}, \theta_h) + c(\mathbf{u}_h^{n+1}; z_h^{n+1}, \theta_h) \\ = \nu(\operatorname{curl} \mathbf{u}_h^{n+1}, \theta_h) + \alpha(\operatorname{curl} \mathbf{f}^{n+1}, \theta_h). \end{aligned} \quad (26)$$

In order to prove the existence of solutions of (24)–(26), let us recall the following identity established by Lesaint and Raviart [12] :

Lemma 3.1. *For all $\mathbf{v}_h^n \in X_h$, z_h^n and θ_h^n in Z_h , we have*

$$\begin{aligned} c(\mathbf{v}_h^n; z_h^n, \theta_h^n) &= \sum_{\kappa \in \mathcal{T}_h} \left(- \int_{\kappa} \alpha(\mathbf{v}_h^n \cdot \nabla \theta_h^n) z_h^n d\mathbf{x} + \int_{\partial \kappa_-} \alpha |\mathbf{v}_h^n \cdot \mathbf{n}| z_{h,ext}^n (\theta_{h,ext}^n - \theta_{h,int}^n) ds \right) \\ &\quad - \frac{\alpha}{2} \int_{\Omega} (\operatorname{div} \mathbf{v}_h^n) \theta_h^n z_h^n d\mathbf{x}. \end{aligned}$$

For $\theta_h^n \in H^1(\Omega)$, we have

$$c(\mathbf{v}_h^n; z_h^n, \theta_h^n) = - \int_{\Omega} \alpha(\mathbf{v}_h^n \cdot \nabla \theta_h^n) z_h^n d\mathbf{x} - \frac{\alpha}{2} \int_{\Omega} (\operatorname{div} \mathbf{v}_h^n) \theta_h^n z_h^n d\mathbf{x}.$$

For $\theta_h^n = z_h^n \in Z_h$ we have

$$c(\mathbf{v}_h^n; z_h^n, z_h^n) = \frac{1}{2} \sum_{\kappa \in \mathcal{T}_h} \int_{\partial \kappa_-} |\alpha \mathbf{v}_h^n \cdot \mathbf{n}| (z_{h,ext}^n - z_{h,int}^n)^2 ds.$$

Theorem 3.2. *Given $\mathbf{f}^{n+1} \in L^2(\Omega)^2$ with $\operatorname{curl} \mathbf{f}^{n+1} \in L^2(\Omega)$, for all $(\mathbf{u}_h^n, z_h^n) \in X_h \times Z_h$, there exists a unique solution $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \mathbf{z}_h^{n+1})$ of problem (24)–(26) that belongs to $X_h \times M_h \times Z_h$.*

Proof. On the one hand, for $\mathbf{z}_h^n \in Z_h$, it is clear that problem (24)–(25) has a unique solution $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ as a consequence of the coerciveness of the corresponding bilinear form on $X_h \times X_h$. On the other hand, the last lemma proves that the bilinear form corresponding to the equation (26) is also coercive on $Z_h \times Z_h$. Then (26) has a unique solution. \square

Theorem 3.3. *We assume that $\mathbf{f} \in L^2(0, T; L^2(\Omega)^2)$ with $\operatorname{curl} \mathbf{f} \in L^2(0, T; L^2(\Omega))$. The solution of the problem (24)–(26) satisfies :*

$$\begin{aligned} \|\mathbf{u}_h\|_{L^\infty(0, T; H^1(\Omega)^2)} &\leq C_1 \|\mathbf{f}\|_{L^2(0, T; L^2(\Omega)^2)}, \\ \|z_h\|_{L^\infty(0, T; L^2(\Omega))}^2 &\leq C_2 \|\mathbf{f}\|_{L^2(0, T; L^2(\Omega)^2)}^2 + C_3 \|\operatorname{curl} \mathbf{f}\|_{L^2(0, T; L^2(\Omega))}^2, \\ \|p_h\|_{L^2(0, T; L^2(\Omega))}^2 &\leq C_4 \|\mathbf{f}\|_{L^2(0, T; L^2(\Omega)^2)}^2 + C_5 \|\mathbf{u}_h\|_{L^\infty(0, T; H^1(\Omega)^2)}^2 + C_6 \|z_h\|_{L^\infty(0, T; L^2(\Omega))}^2 \|\mathbf{u}_h\|_{L^\infty(0, T; H^1(\Omega)^2)}^2, \end{aligned}$$

where $C_i, i = 1, \dots, 6$ are positive constants that depend on Ω and T .

Proof. On the one hand, we take $\mathbf{v}_h = \mathbf{u}_h^{n+1}$ in (24) and we obtain :

$$\begin{aligned} \frac{1}{2} \|\mathbf{u}_h^{n+1}\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\mathbf{u}_h^n\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} |\mathbf{u}_h^{n+1}|_{H^1(\Omega)}^2 - \frac{\alpha}{2} |\mathbf{u}_h^n|_{H^1(\Omega)}^2 + \nu k |\mathbf{u}_h^{n+1}|_{H^1(\Omega)}^2 \\ \leq \frac{k\varepsilon}{2} \|\mathbf{f}^{n+1}\|_{L^2(\Omega)}^2 + \frac{k\mathcal{S}_2^2}{2\varepsilon} |\mathbf{u}_h^{n+1}|_{H^1(\Omega)}^2. \end{aligned}$$

We choose $\varepsilon = \frac{\mathcal{S}_2^2}{2\nu}$ and sum over $n = 0, \dots, i$. We obtain :

$$\frac{1}{2} \|\mathbf{u}_h^i\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} |\mathbf{u}_h^i|_{H^1(\Omega)}^2 \leq \sum_{n=1}^i \frac{k\mathcal{S}_2^2}{4\nu} \|\mathbf{f}^n\|_{L^2(\Omega)}^2 \leq \sum_{n=1}^N \frac{k\mathcal{S}_2^2}{4\nu} \|\mathbf{f}^n\|_{L^2(\Omega)}^2.$$

This implies the first estimate :

$$\|\mathbf{u}_h\|_{L^\infty(0, T; H^1(\Omega)^2)}^2 = \sup_{0 \leq i \leq N} |\mathbf{u}_h^i|_{H^1(\Omega)}^2 \leq \frac{\mathcal{S}_2^2}{2\nu\alpha} \|\mathbf{f}\|_{L^2(0, T; L^2(\Omega)^2)}^2.$$

On the other hand, we choose $\theta_h = z_h^{n+1}$ in (26), use the third relation in Lemma 3.1 and we obtain :

$$\begin{aligned} \frac{\alpha}{2} \|z_h^{n+1}\|_{L^2(\Omega)}^2 - \frac{\alpha}{2} \|z_h^n\|_{L^2(\Omega)}^2 + \nu k \|z_h^{n+1}\|_{L^2(\Omega)}^2 &\leq \frac{k\nu}{2\varepsilon_1} \|\operatorname{curl} \mathbf{u}_h^{n+1}\|_{L^2(\Omega)}^2 \\ &+ \frac{k\nu\varepsilon_1}{2} \|z_h^{n+1}\|_{L^2(\Omega)}^2 + \frac{k\alpha}{2\varepsilon_2} \|\operatorname{curl} \mathbf{f}^{n+1}\|_{L^2(\Omega)}^2 + \frac{k\alpha\varepsilon_2}{2} \|z_h^{n+1}\|_{L^2(\Omega)}^2, \end{aligned}$$

taking $\varepsilon_1 = 1$, $\varepsilon_2 = \frac{\nu}{\alpha}$ and summing over $n = 0, \dots, i$, this becomes :

$$\alpha \|z_h^i\|_{L^2(\Omega)}^2 \leq \nu T \|\mathbf{u}_h\|_{L^\infty(0, T; H^1(\Omega)^2)}^2 + \frac{\alpha^2}{\nu} \|\operatorname{curl} \mathbf{f}\|_{L^2(0, T; L^2(\Omega))}^2.$$

Then we obtain the second estimate :

$$\|z_h\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq \frac{\nu T}{\alpha} \|\mathbf{u}_h\|_{L^\infty(0, T; H^1(\Omega)^2)}^2 + \frac{\alpha}{\nu} \|\operatorname{curl} \mathbf{f}\|_{L^2(0, T; L^2(\Omega))}^2.$$

The third estimate is obtained in two steps: First, we take the function test $\mathbf{v}_h^{n+1} = \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{k}$ in (24). We obtain :

$$\begin{aligned} & \frac{1}{k^2} \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_{L^2(\Omega)}^2 + \frac{\alpha}{k^2} \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_{H^1(\Omega)}^2 + \frac{\nu}{2} \|\mathbf{u}_h^{n+1}\|_{H^1(\Omega)}^2 + \frac{\nu}{2k^2} \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_{H^1(\Omega)}^2 \\ & \leq \frac{1}{2\varepsilon_1} \|\mathbf{f}^{n+1}\|_{L^2(\Omega)}^2 + \frac{\varepsilon_1}{2k^2} \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon_2} \|z_h^n\|_{L^2(\Omega)}^2 \|\mathbf{u}_h^{n+1}\|_{L^4(\Omega)}^2 + \frac{S_4^2 \varepsilon_2}{2k^2} \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_{H^1(\Omega)}^2. \end{aligned}$$

Then by choosing $\varepsilon_1 = 2$ and $\varepsilon_2 = \frac{\alpha}{2S_4^2}$ we obtain :

$$\left\| \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{k} \right\|_{H^1(\Omega)}^2 \leq \frac{1}{2\alpha} \|\mathbf{f}^{n+1}\|_{L^2(\Omega)}^2 + \frac{2S_4^2}{\alpha^2} \|z_h^n\|_{L^2(\Omega)}^2 \|\mathbf{u}_h^{n+1}\|_{L^4(\Omega)}^2 + \frac{2\nu^2}{\alpha^2} \|\mathbf{u}_h^{n+1}\|_{H^1(\Omega)}^2.$$

Next, owing that the pair (X_h, M_h) satisfies a uniform discrete inf-sup condition, we associate with $p_h^{n+1} \in M_h$ the function $\mathbf{v}_h \in X_h$ defined by

$$\begin{cases} \forall \mathbf{w}_h \in V_h, & (\nabla \mathbf{v}_h, \nabla \mathbf{w}_h) = 0, \\ \forall q_h \in M_h, & (\operatorname{div} \mathbf{v}_h, q_h) = (p_h^{n+1}, q_h), \\ \|\mathbf{v}_h\|_{H^1(\Omega)} \leq \frac{1}{\beta} \|p_h^{n+1}\|_{L^2(\Omega)}, \end{cases} \quad (27)$$

we substitute this \mathbf{v}_h into (24) and we obtain :

$$\begin{aligned} \|p_h^{n+1}\|_{L^2(\Omega)}^2 & \leq \frac{\mathcal{P} + \alpha}{2\varepsilon_1} \left\| \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{k} \right\|_{H^1(\Omega)}^2 + \frac{\mathcal{P} + \alpha}{2\beta^2} \varepsilon_1 \|p_h^{n+1}\|_{L^2(\Omega)}^2 + \frac{\nu}{2\varepsilon_2} \|\mathbf{u}_h^{n+1}\|_{H^1(\Omega)}^2 \\ & \quad + \frac{\nu\varepsilon_2}{2\beta^2} \|p_h^{n+1}\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon_3} \|z_h^n\|_{L^2(\Omega)}^2 \|\mathbf{u}_h^{n+1}\|_{L^4(\Omega)}^2 \\ & \quad + \frac{S_4^2 \varepsilon_3}{2\beta^2} \|p_h^{n+1}\|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon_4} \|\mathbf{f}^{n+1}\|_{L^2(\Omega)}^2 + \frac{\mathcal{P}^2 \varepsilon_4}{2\beta^2} \|p_h^{n+1}\|_{L^2(\Omega)}^2. \end{aligned}$$

By choosing $\varepsilon_1 = \frac{\beta^2}{4(\mathcal{P} + \alpha)}$, $\varepsilon_2 = \frac{\beta^2}{4\nu}$, $\varepsilon_3 = \frac{\beta^2}{4S_4^2}$ and $\varepsilon_4 = \frac{\beta^2}{4\mathcal{P}^2}$ and summing over n from 0 to $N - 1$, we obtain the third estimate. \square

4 Error estimates

Theorem 4.1. *Under the assumptions $\mathbf{u} \in L^\infty(0, T; W^{1,4}(\Omega)^2) \cap L^2(0, T; H^3(\Omega)^2)$, $\mathbf{u}' \in L^2(0, T; H^3(\Omega)^2)$, $p \in L^2(0, T; H^2(\Omega))$, $z \in L^\infty(0, T; L^2(\Omega))$ and $z' \in L^2(0, T; L^2(\Omega))$, there exist positive constants C and C' that depend on \mathbf{u}, z, Ω and T such that :*

$$\begin{aligned} & \frac{1}{2} \|\mathbf{u}_h^N - \mathbf{u}(t^N)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\nabla \mathbf{u}_h^N - \nabla \mathbf{u}(t^N)\|_{L^2(\Omega)}^2 + \frac{\nu}{2} \sum_{n=0}^{N-1} k \|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})\|_{H^1(\Omega)}^2 \\ & \leq C(h^4 + k^2) + C' \sum_{n=0}^{N-1} k \|z_h^{n+1} - z(t^{n+1})\|_{L^2(\Omega)}^2. \end{aligned} \quad (28)$$

Proof. We consider (19), choose the function test $\mathbf{v}_h^{n+1} = \mathbf{u}_h^{n+1} - P_h \mathbf{u}(t^{n+1})$, integrate from t^n to t^{n+1} and take the difference between this and (24) multiplied by k . We obtain :

$$\begin{aligned} & \left((\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - \mathbf{u}(t^n)), \mathbf{v}_h^{n+1} \right) + \alpha \left(\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - \nabla(\mathbf{u}_h^n - \mathbf{u}(t^n)), \nabla \mathbf{v}_h^{n+1} \right) \\ & + \nu \left(k \nabla \mathbf{u}_h^{n+1} - \int_{t^n}^{t^{n+1}} \nabla \mathbf{u}(t) dt, \nabla \mathbf{v}_h^{n+1} \right) - \left(k p_h^{n+1} - \int_{t^n}^{t^{n+1}} p(t) dt, \operatorname{div} \mathbf{v}_h^{n+1} \right) \\ & + \left(k z_h^n \times \mathbf{u}_h^{n+1} - \int_{t^n}^{t^{n+1}} z(t) \times \mathbf{u}(t) dt, \mathbf{v}_h^{n+1} \right) = 0. \end{aligned}$$

Let us treat the terms in the left-hand side of this equation that we denote $(a_i), i = 1, \dots, 5$.

For the first term, we insert $P_h \mathbf{u}(t^{n+1})$ and $P_h \mathbf{u}(t^n)$ and we split (a_1) into two terms that we treat separately. The first part is as follows :

$$(a_{1,1}) = \frac{1}{2} \|\mathbf{v}_h^{n+1}\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\mathbf{v}_h^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{L^2(\Omega)}^2,$$

and the second part is as follows :

$$\begin{aligned} |(a_{1,2})| &= \left| \left(\int_{t^n}^{t^{n+1}} (P_h \mathbf{u}'(\tau) - \mathbf{u}'(\tau)) d\tau, \mathbf{v}_h^{n+1} \right) \right| \\ &\leq \frac{1}{2\varepsilon_1} C h^4 \|\mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^2(\Omega)^2)}^2 + \frac{S_2 \varepsilon_1}{2} k \|\mathbf{v}_h^{n+1}\|_{H^1(\Omega)}^2. \end{aligned}$$

We treat the second term (a_2) as the first one and we obtain :

$$(a_{2,1}) = \frac{\alpha}{2} \|\mathbf{v}_h^{n+1}\|_{H^1(\Omega)}^2 - \frac{\alpha}{2} \|\mathbf{v}_h^n\|_{H^1(\Omega)}^2 + \frac{\alpha}{2} \|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{H^1(\Omega)}^2,$$

and

$$|(a_{2,2})| \leq \frac{C\alpha}{2\varepsilon_2} h^4 \|\mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^3(\Omega)^2)}^2 + \frac{\varepsilon_2 \alpha}{2} k \|\mathbf{v}_h^{n+1}\|_{H^1(\Omega)}^2.$$

For the third term (a_3) , we insert $\nabla P_h \mathbf{u}(t^{n+1})$ and $\nabla P_h \mathbf{u}(t)$ and we split it into three parts that are treated successively as follows :

$$(a_{3,1}) = \nu k \|\mathbf{u}_h^{n+1} - P_h \mathbf{u}(t^{n+1})\|_{H^1(\Omega)}^2,$$

$$\begin{aligned} |(a_{3,2})| &= \nu \left| \left(\int_{t^n}^{t^{n+1}} \nabla P_h(\mathbf{u}(t^{n+1}) - \mathbf{u}(t)) dt, \nabla \mathbf{v}_h^{n+1} \right) \right| = \nu \left| \left(\int_{t^n}^{t^{n+1}} \nabla P_h \mathbf{u}'(\tau) (\tau - t^n) d\tau, \nabla \mathbf{v}_h^{n+1} \right) \right| \\ &\leq \frac{\nu \varepsilon_3}{2\sqrt{3}} k \|\mathbf{v}_h^{n+1}\|_{H^1(\Omega)}^2 + \frac{\nu C^2 k^2}{2\varepsilon_3 \sqrt{3}} \|\mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^2(\Omega)^2)}^2, \end{aligned}$$

and

$$\begin{aligned} |(a_{3,3})| &= \nu \left| \left(\nabla \int_{t^n}^{t^{n+1}} (P_h \mathbf{u}(t) - \mathbf{u}(t)) dt, \nabla \mathbf{v}_h^{n+1} \right) \right| \\ &\leq \frac{\nu C_2}{2\varepsilon_4} h^4 \|\mathbf{u}\|_{L^2(t^n, t^{n+1}; H^3(\Omega)^2)}^2 + \frac{\nu \varepsilon_4}{2} k \|\mathbf{v}_h^{n+1}\|_{H^1(\Omega)}^2. \end{aligned}$$

To study the fourth term, we use the fact that $\int_{\Omega} p_h^{n+1} \operatorname{div}(P_h \mathbf{u}(t^{n+1}) - \mathbf{u}(t^{n+1})) = 0$,

$\operatorname{div} \mathbf{u}(t^{n+1}) = 0$ and $\int_{\Omega} p_h^{n+1} \operatorname{div} \mathbf{u}_h^{n+1} = 0$ and we obtain :

$$\begin{aligned} |(a_4)| &= \left| \left(\int_{t^n}^{t^{n+1}} (r_h p(t) - p(t)) dt, \operatorname{div} \mathbf{v}_h^{n+1} \right) \right| \\ &\leq \frac{C_1}{2\varepsilon_5} h^4 \|p\|_{L^2(t^n, t^{n+1}; H^2(\Omega))}^2 + \frac{\varepsilon_5}{2} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2. \end{aligned}$$

Finally, for the last term (a_5) , we have $(z_h^n \times \mathbf{u}_h^{n+1}, \mathbf{v}_h^{n+1}) = (z_h^n \times P_h \mathbf{u}(t^{n+1}), \mathbf{v}_h^{n+1})$, because $(a \times b, b) = 0$. But

$$\begin{aligned} z_h^n \times P_h \mathbf{u}(t^{n+1}) - z(t) \times \mathbf{u}(t) &= (z_h^n - z(t^n)) \times P_h \mathbf{u}(t^{n+1}) + z(t^n) \times P_h(\mathbf{u}(t^{n+1}) - \mathbf{u}(t)) \\ &\quad + z(t^n) \times (P_h \mathbf{u}(t) - \mathbf{u}(t)) + (z(t^n) - z(t)) \times (\mathbf{u}(t) - \mathbf{u}(t^n)) + (z(t^n) - z(t)) \times \mathbf{u}(t^n), \end{aligned}$$

than (a_5) is split into five parts that we treat successively.

The first part is as follows :

$$\begin{aligned} |(a_{5,1})| &= \left| \int_{t^n}^{t^{n+1}} ((z_h^n - z(t^n)) \times P_h \mathbf{u}(t^{n+1}), \mathbf{v}_h^{n+1}) dt \right| \\ &\leq \frac{S_4^2 \varepsilon_6}{2} \|P_h \mathbf{u}\|_{L^\infty(0, T; H^1(\Omega)^2)}^2 k \|z_h^n - z(t^n)\|_{L^2}^2 + \frac{S_4^2}{2\varepsilon_6} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 \\ &\leq \frac{S_4^2 \varepsilon_6}{2} \|\mathbf{u}\|_{L^\infty(0, T; W^{1,4}(\Omega)^2)}^2 k \|z_h^n - z(t^n)\|_{L^2}^2 + \frac{S_4^2}{2\varepsilon_6} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2. \end{aligned}$$

The second part is as follows :

$$\begin{aligned} |(a_{5,2})| &= \left| (z(t^n) \times \int_{t^n}^{t^{n+1}} (P_h \mathbf{u}(t^{n+1}) - P_h \mathbf{u}(t)) dt, \mathbf{v}_h^{n+1}) \right| \\ &= \left| (z(t^n) \times \int_{t^n}^{t^{n+1}} P_h \mathbf{u}'(\tau) (\tau - t^n) d\tau, \mathbf{v}_h^{n+1}) \right| \\ &\leq \frac{S_4^2}{2\sqrt{3}} \left(\frac{C'''}{\varepsilon_7} \|z\|_{L^\infty(0, T; L^2(\Omega))}^2 \|\mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^2(\Omega)^2)}^2 k^2 + k \varepsilon_7 |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 \right). \end{aligned}$$

For the third part, we have

$$\begin{aligned} |(a_{5,3})| &= \left| \int_{t^n}^{t^{n+1}} (z(t^n) \times (P_h \mathbf{u}(t) - \mathbf{u}(t)), \mathbf{v}_h^{n+1}) dt \right| \\ &\leq S_4^2 \|z(t^n)\|_{L^2(\Omega)} |\mathbf{v}_h^{n+1}|_{H^1(\Omega)} \int_{t^n}^{t^{n+1}} |P_h \mathbf{u}(t) - \mathbf{u}(t)|_{H^1(\Omega)} dt \\ &\leq \frac{C_2 S_4^2}{2\varepsilon_8} \|z\|_{L^\infty(0, T; L^2(\Omega))}^2 \|\mathbf{u}\|_{L^2(t^n, t^{n+1}; H^3(\Omega)^2)}^2 h^4 + \frac{C_2 S_4^2 \varepsilon_8}{2} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2. \end{aligned}$$

The fourth part is treated as follows :

$$\begin{aligned}
|(a_{5,4})| &= \left| \left(\int_{t^n}^{t^{n+1}} (z(t^n) - z(t)) \times (\mathbf{u}(t) - \mathbf{u}(t^n)), \mathbf{v}_h^{n+1} \right) dt \right| \\
&= \left| \int_{t^n}^{t^{n+1}} \left(\left(\int_{t^n}^t z'(\tau) d\tau \right) \times \left(\int_{t^n}^t \mathbf{u}'(\tau) d\tau \right), \mathbf{v}_h^{n+1} \right) dt \right| \\
&\leq \frac{S_4^2 \varepsilon_9}{2\sqrt{2}} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 + \frac{S_4^2 k^3}{2\sqrt{2}\varepsilon_9} \|\mathbf{u}'\|_{L^2(0,T;H^1(\Omega)^2)}^2 \|z'\|_{L^2(0,T;L^2(\Omega))}^2.
\end{aligned}$$

Finally, for the last part, we have

$$\begin{aligned}
|(a_{5,5})| &= \left| \left(\int_{t^n}^{t^{n+1}} (z(t^n) - z(t)) \times \mathbf{u}(t^n), \mathbf{v}_h^{n+1} \right) dt \right| \\
&= \left| \left(\left(\int_{t^n}^{t^{n+1}} z'(t)(t - t^{n+1}) dt \right) \times \mathbf{u}(t^n), \mathbf{v}_h^{n+1} \right) \right| \\
&\leq \frac{S_4^2 \varepsilon_{10}}{2\sqrt{3}} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 + \frac{S_4^2}{2\sqrt{3}\varepsilon_{10}} k^2 \|z'\|_{L^2(t^n, t^{n+1}; L^2(\Omega))}^2 \|\mathbf{u}\|_{L^\infty(0,T;H^1(\Omega)^2)}^2
\end{aligned}$$

At the end, (28) follows easily after the decomposition

$$(a_{1,1}) + (a_{2,1}) + (a_{3,1}) \leq |(a_{1,2})| + |(a_{2,2})| + |(a_{3,2})| + |(a_{3,3})| + |(a_4)| + |(a_5)|,$$

the sum over $n = 1, \dots, N-1$, a suitable choice of ε_i , $i = 1, \dots, 10$ and by using the properties of P_h in :

$$|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|_{H^1(\Omega)} \leq |\mathbf{u}_h^{n+1} - P_h \mathbf{u}(t^{n+1})|_{H^1(\Omega)} + |P_h \mathbf{u}(t^{n+1}) - \mathbf{u}(t^{n+1})|_{H^1(\Omega)}.$$

□

We define ρ_h as the L^2 projection of z onto \mathbb{P}_1 in each triangle κ : for $z \in L^2(\Omega)$,

$$\forall q \in \mathbb{P}_1, \int_{\kappa} (\rho_h(z) - z) q d\mathbf{x} = 0.$$

This operator has locally the same accuracy as R_h .

Theorem 4.2. *We suppose that there exists a constant $\gamma > 0$ such that $k \leq \gamma h$. In addition to the assumptions of Theorem 4.1, we assume that $\mathbf{u} \in L^\infty(0, T; W^{1,\infty}(\Omega)^2)$, $z \in L^\infty(0, T; W^{1,4}(\Omega))$ and $z' \in L^\infty(0, T; L^4(\Omega)) \cap L^2(0, T; W^{1,4}(\Omega))$. Denoting $\theta_h^{n+1} = z_h^{n+1} - \rho_h z(t^{n+1})$ we have :*

$$\begin{aligned}
\sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} c(\mathbf{u}_h^{n+1}; \rho_h z(t^{n+1}) - z(t), \theta_h^{n+1}) dt &\leq L_1(h^3 + k^2) + L_3 \sum_{n=0}^{N-1} k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2 \\
+ L_2 \sum_{n=0}^{N-1} k |\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|_{H^1(\Omega)}^2 &+ \frac{\alpha}{2} \sum_{\kappa \in \mathcal{T}_h} \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| (\theta_{h,ext}^{n+1} - \theta_{h,int}^{n+1})^2 ds dt
\end{aligned} \tag{29}$$

where L_i are constants that only depend on \mathbf{u}, z, Ω, T and arbitrary coefficients $\varepsilon_i (i = 1, \dots, 8)$

Proof. Owing to Lemma 3.1 and denoting $\xi_h = \rho_h(z(t^{n+1}))$, we have :

$$\begin{aligned} \int_{t^n}^{t^{n+1}} c(\mathbf{u}_h^{n+1}; \rho_h z(t^{n+1}) - z(t), \theta_h^{n+1}) dt &= \alpha \sum_{\kappa \in \mathcal{T}_h} \left[- \int_{t^n}^{t^{n+1}} \int_{\kappa} (\mathbf{u}_h^{n+1} \cdot \nabla \theta_h^{n+1})(\xi_h - z(t)) dx dt \right. \\ &+ \left. \int_{t^n}^{t^{n+1}} \int_{\partial \kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| (\xi_h - z(t))^{\text{ext}} (\theta_{h,\text{ext}}^{n+1} - \theta_{h,\text{int}}^{n+1}) ds dt \right] \\ &- \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\Omega} \text{div}(\mathbf{u}_h^{n+1} - \mathbf{u}(t)) (\xi_h - z(t)) \theta_h^{n+1} dx dt \end{aligned}$$

In the left-hand side, we denote the terms by (d_i) , $i = 1 \dots, 3$. as $\nabla(z_h - \rho_h(z))$ is a constant vector, the first term (d_1) , for any constant vector \mathbf{c} , can be treated as :

$$\begin{aligned} \left| \int_{t^n}^{t^{n+1}} \int_{\kappa} (\mathbf{u}_h^{n+1} \cdot \nabla \theta_h^{n+1})(\xi_h - z(t)) dx dt \right| &\leq \alpha \left| \int_{t^n}^{t^{n+1}} \int_{\kappa} (\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) \nabla \theta_h^{n+1} (\xi_h - z(t)) dx dt \right| \\ &+ \alpha \left| \int_{t^n}^{t^{n+1}} \int_{\kappa} (\mathbf{u}(t^{n+1}) - \mathbf{c}) \nabla \theta_h^{n+1} (\xi_h - z(t^{n+1})) dx dt \right. \\ &\quad \left. + \int_{t^n}^{t^{n+1}} \int_{\kappa} \mathbf{u}(t^{n+1}) \nabla \theta_h^{n+1} (z(t^{n+1}) - z(t)) dx dt \right|. \end{aligned}$$

With

$$\begin{aligned} &\alpha \left| \int_{t^n}^{t^{n+1}} \int_{\kappa} (\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) \nabla \theta_h^{n+1} (\xi_h - z(t)) dx dt \right| \\ &\leq \frac{\alpha S_4}{\rho_k} \|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})\|_{H^1(\kappa)} \|\theta_h^{n+1}\|_{L^2(\kappa)} \left\{ c_1 k \|\xi_h - z(t^{n+1})\|_{L^4(\kappa)} + \frac{c_2 k^2}{\sqrt{2}} \|z'\|_{L^\infty(0,T;L^4(\kappa))} \right\} \\ &\leq \alpha c_3 S_4 (\sigma_0 \|z\|_{L^\infty(0,T;W^{1,4}(\kappa))} + \gamma \|z'\|_{L^\infty(0,T;L^4(\kappa))}) \left(\frac{k}{2\varepsilon_1} \|\theta_h^{n+1}\|_{L^2(\kappa)}^2 + \frac{\varepsilon_1 k}{2} \|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})\|_{H^1(\kappa)}^2 \right), \end{aligned}$$

and

$$\begin{aligned} &\alpha \left| \int_{t^n}^{t^{n+1}} \int_{\kappa} (\mathbf{u}(t^{n+1}) - \mathbf{c}) \nabla \theta_h^{n+1} (\xi_h - z(t^{n+1})) dx dt + \int_{t^n}^{t^{n+1}} \int_{\kappa} \mathbf{u}(t^{n+1}) \nabla \theta_h^{n+1} (z(t^{n+1}) - z(t)) dx dt \right| \\ &= \alpha \left| \int_{t^n}^{t^{n+1}} \int_{\kappa} (\mathbf{u}(t^{n+1}) - \mathbf{c}) \nabla \theta_h^{n+1} (\xi_h - z(t^{n+1})) dx dt - \int_{t^n}^{t^{n+1}} \int_{\kappa} \theta_h^{n+1} \mathbf{u}(t^{n+1}) \nabla (z(t^{n+1}) - z(t)) dx dt \right| \\ &\leq \|\theta_h^{n+1}\|_{L^2(\kappa)} \left\{ \frac{\alpha c_3}{\rho_k} k h^{3/2} \|z(t^{n+1})\|_{W^{1,4}(\kappa)} \|\mathbf{u}(t^{n+1}) - \mathbf{c}\|_{L^\infty(\kappa)} + \|\mathbf{u}\|_{L^\infty(0,T;\kappa)} \frac{\alpha k^{3/2}}{\sqrt{3}} \|z'\|_{L^2(t^n, t^{n+1}; H^1(\kappa))} \right\} \\ &\leq \alpha c_4 \left(\sigma_0 k h^{3/2} \|\theta_h^{n+1}\|_{L^2(\kappa)} \|\mathbf{u}(t^{n+1})\|_{W^{1,\infty}(\kappa)} \|z\|_{L^\infty(0,T;W^{1,4}(\kappa))} \right. \\ &\quad \left. + \frac{k^{3/2}}{\sqrt{3}} \|\theta_h^{n+1}\|_{L^2(\kappa)} \|z'\|_{L^2(t^n, t^{n+1}; H^1(\kappa))} \|\mathbf{u}\|_{L^\infty(0,T;\kappa)} \right) \\ &\leq \alpha c_5 \|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\kappa)^2)} \left(\frac{k}{2\varepsilon_2} \|\theta_h^{n+1}\|_{L^2(\kappa)}^2 + \frac{\varepsilon_2}{2} (k^2 \|z'\|_{L^2(t^n, t^{n+1}; H^1(\kappa))}^2 + k h^3 \|z\|_{L^\infty(t^n, t^{n+1}; W^{1,4}(\kappa))}^2) \right). \end{aligned}$$

For the second part (d_2), we write :

$$\begin{aligned} & \alpha \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| (\xi_h - z(t))^{\text{ext}} (\theta_{h,\text{ext}}^{n+1} - \theta_{h,\text{int}}^{n+1}) ds dt \\ & \leq \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| (\theta_{h,\text{ext}}^{n+1} - \theta_{h,\text{int}}^{n+1})^2 ds dt + \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \left((\xi_h - z(t))^{\text{ext}} \right)^2 ds dt \end{aligned}$$

We keep the first term in the right-hand side of this inequality. The second term can be written as follows :

$$\begin{aligned} & \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \left((\xi_h - z(t))^{\text{ext}} \right)^2 ds dt \\ & \leq \alpha \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \left((\xi_h - z(t^{n+1}))^{\text{ext}} \right)^2 ds dt + \alpha \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \left((z(t^{n+1}) - z(t))^{\text{ext}} \right)^2 ds dt \end{aligned}$$

with

$$\begin{aligned} & \left| \alpha \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \left((\xi_h - z(t^{n+1}))^{\text{ext}} \right)^2 ds dt \right| \\ & \leq \alpha c_6 k \left(\|\xi_h - z(t^{n+1})\|_{L^4(\partial\kappa_-)}^2 \|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})\|_{L^2(\partial\kappa_-)} + \|\xi_h - z(t^{n+1})\|_{L^2(\partial\kappa_-)}^2 \|\mathbf{u}(t^{n+1})\|_{L^\infty(\omega_k)} \right) \\ & \leq \alpha c_7 k \left(h^{3/2} \|z(t^{n+1})\|_{W^{1,4}(\omega_k)}^2 \|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})\|_{H^1(\omega_k)} + h^3 \|z(t^{n+1})\|_{W^{1,4}(\omega_k)}^2 \|\mathbf{u}(t^{n+1})\|_{L^\infty(\omega_k)} \right) \\ & \leq \alpha \tilde{c}_7 h^3 k \left(\|z\|_{L^\infty(0,T;W^{1,4}(\omega_k))}^2 \|\mathbf{u}\|_{L^\infty(0,T;\omega_k)} + \frac{1}{2\varepsilon_3} \|z\|_{L^\infty(0,T;W^{1,4}(\omega_k))}^2 \right) + \frac{\varepsilon_3}{2} k \|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})\|_{H^1(\omega_k)}^2, \end{aligned}$$

where ω_k denotes the union of triangles adjacent to κ and

$$\begin{aligned} & \left| \alpha \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \left((z(t^{n+1}) - z(t))^{\text{ext}} \right)^2 ds dt \right| \\ & \leq \alpha \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \int_{t^n}^{t^{n+1}} \left[(t^{n+1} - t) \left(\int_t^{t^{n+1}} |z'(\tau)|^2 d\tau \right)^{\text{ext}} \right] dt ds \\ & \leq \alpha \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| \left(\int_{t^n}^{t^{n+1}} |z'(\tau)|^2 d\tau \right)^{\text{ext}} \int_{t^n}^{t^{n+1}} (t^{n+1} - t) dt ds \\ & \leq \frac{\alpha k^2}{2} \left\{ \int_{\partial\kappa_-} |(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) \cdot \mathbf{n}| \left(\int_{t^n}^{t^{n+1}} |z'(\tau)|^2 d\tau \right)^{\text{ext}} ds \right. \\ & \quad \left. + \int_{\partial\kappa_-} |\mathbf{u}(t^{n+1}) \cdot \mathbf{n}| \left(\int_{t^n}^{t^{n+1}} |z'(\tau)|^2 d\tau \right)^{\text{ext}} ds \right\} \\ & \leq \alpha c_8 \left(\frac{k}{2\varepsilon_4} \|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})\|_{H^1(\omega_k)}^2 + \frac{\varepsilon_4 k^3}{2} \|z'\|_{L^2(t^n, t^{n+1}; W^{1,4}(\omega_k))}^2 \right. \\ & \quad \left. + k^2 \|\mathbf{u}\|_{L^\infty(0,T;H^1(\omega_k)^2)} \|z'\|_{L^2(t^n, t^{n+1}; W^{1,4}(\omega_k))}^2 \right) \end{aligned}$$

For the third part (d_3), we insert respectively in the divergence term and the term in z , $\pm \mathbf{u}(t^{n+1})$ and $\pm z(t^{n+1})$. We get four parts that we treat as follow :

$$\begin{aligned} |(d_{3,1})| &= \left| \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\Omega} \operatorname{div}(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1}))(\xi_h - z(t^{n+1}))\theta_h^{n+1} dxdt \right| \\ &\leq \frac{\alpha}{2} \sigma_0 C_9 \|z\|_{L^\infty(0,T;W^{1,4}(\Omega))} \left(\frac{k}{2\varepsilon_5} |\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|_{H^1(\Omega)}^2 + \frac{\varepsilon_5}{2} k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

$$\begin{aligned} |(d_{3,2})| &= \left| \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\Omega} \operatorname{div}(\mathbf{u}(t^{n+1}) - \mathbf{u}(t))(\xi_h - z(t^{n+1}))\theta_h^{n+1} dxdt \right| \\ &\leq \frac{\alpha}{2} C_{10} \sigma_0 k^{3/2} \|z\|_{L^\infty(0,T;W^{1,4}(\Omega))} \|\theta_h^{n+1}\|_{L^2(\Omega)} \|\mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^1(\Omega)^2)} \\ &\leq \frac{\alpha}{2} C_{10} \sigma_0 \|z\|_{L^\infty(0,T;W^{1,4}(\Omega))} \left(\frac{k^2}{2\varepsilon_6} \|\mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^1(\Omega)^2)}^2 + \frac{\varepsilon_6}{2} k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

$$\begin{aligned} |(d_{3,3})| &= \left| \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\Omega} \operatorname{div}(\mathbf{u}(t^{n+1}) - \mathbf{u}(t))(z(t^{n+1}) - z(t))\theta_h^{n+1} dxdt \right| \\ &\leq \frac{\alpha}{2} C_{11} \frac{k^2}{\rho_{\min}} \|z'\|_{L^2(t^n, t^{n+1}; L^4(\Omega))} \|\theta_h^{n+1}\|_{L^2(\Omega)} \|\mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^1(\Omega)^2)} \\ &\leq \frac{\alpha}{2} C_{11} \sigma_0 \gamma \|\mathbf{u}'\|_{L^\infty(0,T; H^1(\Omega)^2)} \left(\frac{k^2}{2\varepsilon_7} \|z'\|_{L^2(t^n, t^{n+1}; L^4(\Omega))}^2 + \frac{\varepsilon_7}{2} k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

and

$$\begin{aligned} |(d_{3,4})| &= \left| \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\Omega} \operatorname{div}(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1}))(z(t^{n+1}) - z(t))\theta_h^{n+1} dxdt \right| \\ &\leq \frac{\alpha}{2} C_{13} \frac{k^{3/2}}{\rho_{\min}} \|\theta_h^{n+1}\|_{L^2(\Omega)} |\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|_{H^1(\Omega)} \|z'\|_{L^2(t^n, t^{n+1}; L^4(\Omega))} \\ &\leq \frac{\alpha}{2} C_{13} \gamma \|z'\|_{L^\infty(0,T; L^4(\Omega))} \left(\frac{k}{2\varepsilon_8} \|\theta_h^{n+1}\|_{L^2(\Omega)}^2 + \frac{\varepsilon_8}{2} k |\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|_{H^1(\Omega)}^2 \right). \end{aligned}$$

We deduce the result after summing over $n = 0, \dots, N-1$

□

Theorem 4.3. *With the same assumptions of Theorem 4.2, we have :*

$$\sum_{n=1}^{N-1} k \|z_h^{n+1} - z(t^{n+1})\|_{L^2(\Omega)}^2 \leq F_1(h^3 + k^2) + F_2 \sum_{n=0}^{N-1} k |\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|_{H^1(\Omega)}^2, \quad (30)$$

where F_i are constants that only depend on \mathbf{u}, z, Ω and T .

Proof. We consider (21), take the test function $\theta_h = \theta_h^{n+1} = z_h^{n+1} - \rho_h z(t^{n+1})$, integrate from t^n

to t^{n+1} and subtract (26) multiplied by k . We obtain :

$$\begin{aligned}
& \alpha \left((z_h^{n+1} - z(t^{n+1})) - (z_h^n - z(t^n)), \theta_h^{n+1} \right) + \nu \left(\int_{t^n}^{t^{n+1}} (z_h^{n+1} - z(t)) dt, \theta_h^{n+1} \right) \\
& + \left\{ k c(\mathbf{u}_h^{n+1}; z_h^{n+1}, \theta_h^{n+1}) - \alpha \int_{t^n}^{t^{n+1}} (\mathbf{u}(t) \nabla z(t) + \frac{1}{2} \operatorname{div} \mathbf{u}(t) z(t), \theta_h^{n+1}) dt \right\} \\
& = \nu \left(\int_{t^n}^{t^{n+1}} (\operatorname{curl} \mathbf{u}_h^{n+1} - \operatorname{curl} \mathbf{u}(t)) dt, \theta_h^{n+1} \right)
\end{aligned} \tag{31}$$

Let us treat each term of this equation that we denote by $(b_i), i = 1, \dots, 4$.

For the first term, we follow the same steps as for the term (a_1) in the Theorem 4.1. We obtain :

$$(b_{1,1}) = \frac{\alpha}{2} \|\theta_h^{n+1}\|_{L^2(\Omega)}^2 - \frac{\alpha}{2} \|\theta_h^n\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\theta_h^{n+1} - \theta_h^n\|_{L^2(\Omega)}^2,$$

and

$$\begin{aligned}
|(b_{1,2})| &= \alpha \left| \left(\int_{t^n}^{t^{n+1}} (\rho_h z'(\tau) - z'(\tau)) d\tau, \theta_h^{n+1} \right) \right| \\
&\leq \frac{C\alpha h^3}{2\varepsilon_9} \|z'\|_{L^2(t^n, t^{n+1}; W^{1,4}(\Omega))}^2 + \frac{\alpha\varepsilon_9}{2} k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

For the second term (b_2) , we write :

$$z_h^{n+1} - z(t) = z_h^{n+1} - \rho_h z(t^{n+1}) + \rho_h z(t^{n+1}) - \rho_h z(t) + \rho_h z(t) - z(t),$$

and we obtain three parts that we treat successively.

The first one gives : $(b_{2,1}) = \nu k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2$.

The second part is bounded as follows :

$$\begin{aligned}
|(b_{2,2})| &= \nu \left| \left(\int_{t^n}^{t^{n+1}} (\rho_h z(t^{n+1}) - \rho_h z(t)) dt, \theta_h \right) \right| = \nu \left| \left(\int_{t^n}^{t^{n+1}} \rho_h z'(t) (t - t^n) dt, \theta_h^{n+1} \right) \right| \\
&\leq \frac{\nu k^2}{2\sqrt{3}\varepsilon_{10}} \|z'\|_{L^2(t^n, t^{n+1}; H^1(\Omega))}^2 + \frac{\varepsilon_{10}}{2\sqrt{3}} k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2,
\end{aligned}$$

and the last part is bounded as follows :

$$\begin{aligned}
|(b_{2,3})| &= \nu \left| \left(\int_{t^n}^{t^{n+1}} (\rho_h z(t) - z(t)) dt, \theta_h^{n+1} \right) \right| \\
&\leq \frac{\nu h^3}{2\varepsilon_{11}} \|z\|_{L^2(t^n, t^{n+1}; W^{1,4}(\Omega))}^2 + \frac{\nu k \varepsilon_{11}}{2} \|\theta_h^{n+1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

The third term can be written as follows :

$$\begin{aligned}
(b_3) &= \int_{t^n}^{t^{n+1}} c(\mathbf{u}_h^{n+1}; z_h^{n+1}, \theta_h^{n+1}) dt - \alpha \int_{t^n}^{t^{n+1}} \left(\mathbf{u}(t) \nabla z(t) + \frac{1}{2} \operatorname{div} \mathbf{u}(t) z(t), \theta_h^{n+1} \right) dt \\
&= \int_{t^n}^{t^{n+1}} c(\mathbf{u}_h^{n+1}; \theta_h^{n+1}, \theta_h^{n+1}) dt + \int_{t^n}^{t^{n+1}} c(\mathbf{u}_h^{n+1}; \rho_h z(t^{n+1}) - z(t), \theta_h^{n+1}) dt \\
&+ \int_{t^n}^{t^{n+1}} c(\mathbf{u}_h^{n+1}; z(t), \theta_h^{n+1}) dt - \alpha \int_{t^n}^{t^{n+1}} \left(\mathbf{u}(t) \nabla z(t) + \frac{1}{2} \operatorname{div} \mathbf{u}(t) z(t), \theta_h^{n+1} \right) dt.
\end{aligned}$$

Owing to Lemma 3.1 and denoting $\xi_h = \rho_h(z(t^{n+1}))$, (b_3) becomes :

$$(b_3) = \frac{\alpha}{2} \sum_{\kappa \in \mathcal{T}_h} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa_-} |\mathbf{u}_h^{n+1} \cdot \mathbf{n}| (\theta_{h,\text{ext}}^{n+1} - \theta_{h,\text{int}}^{n+1})^2 ds dt + \int_{t^n}^{t^{n+1}} c(\mathbf{u}_h^{n+1}; \rho_h z(t^{n+1}) - z(t), \theta_h^{n+1}) dt \\ + \alpha \int_{t^n}^{t^{n+1}} \int_{\Omega} (\mathbf{u}_h^{n+1} - \mathbf{u}(t)) \nabla z(t) \theta_h^{n+1} dx dt + \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\Omega} \operatorname{div}(\mathbf{u}_h^{n+1} - \mathbf{u}(t)) z(t) \theta_h^{n+1} dx dt.$$

We divide (b_3) into four terms $(b_{3,i}), i = 1, \dots, 4$. We keep the term $(b_{3,1})$ in the left-hand side of (31). The second term $(b_{3,2})$ is bounded as in the previous theorem.

For the third part $(b_{3,3})$, we have :

$$|(b_{3,3})| = \left| \alpha \int_{t^n}^{t^{n+1}} \int_{\Omega} (\mathbf{u}_h^{n+1} - \mathbf{u}(t)) \nabla z(t) \theta_h^{n+1} dx dt \right| \\ \leq \alpha C_{14} \|z\|_{L^\infty(0,T;W^{1,4}(\Omega))} \left(\frac{k}{2\varepsilon_{12}} \|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})\|_{H^1(\Omega)}^2 + \frac{\varepsilon_{12}}{2} k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2 \right) \\ + \alpha C_{15} \|z\|_{L^\infty(0,T;W^{1,4}(\Omega))} \left(\frac{k^2}{2\varepsilon_{13}} \|\mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^1(\Omega)^2)}^2 + \frac{\varepsilon_{13}}{2} k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2 \right),$$

and the last part of the third term (b_3) is :

$$|(b_{3,4})| = \left| \frac{\alpha}{2} \int_{t^n}^{t^{n+1}} \int_{\Omega} \operatorname{div}(\mathbf{u}_h^{n+1} - \mathbf{u}(t)) z(t) \theta_h^{n+1} dx dt \right| \\ \leq \frac{\alpha}{2} C_{16} \|z\|_{L^\infty(0,T;L^\infty(\Omega))} \left(\frac{k}{2\varepsilon_{14}} \|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})\|_{H^1(\Omega)}^2 + \frac{\varepsilon_{14}}{2} k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2 \right) \\ + \frac{\alpha}{2} C_{17} \|z\|_{L^\infty(0,T;L^\infty(\Omega))} \left(\frac{k^2}{2\varepsilon_{15}} \|\mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^1(\Omega)^2)}^2 + \frac{\varepsilon_{15}}{2} k \|\theta_h^{n+1}\|_{L^2(\Omega)}^2 \right).$$

For the last term (b_4) , we split it into two parts, as follows : Using $\|\operatorname{curl} \mathbf{u}_h\|_{L^2(\Omega)}^2 \leq 2\|\mathbf{u}_h\|_{H^1(\Omega)}^2$, we have

$$|(b_{4,1})| = \left| \nu \left(\int_{t^n}^{t^{n+1}} (\operatorname{curl} \mathbf{u}_h^{n+1} - \operatorname{curl} \mathbf{u}(t^{n+1})) dt, \theta_h^{n+1} \right) \right| \\ \leq \frac{\nu k}{\varepsilon_{16}} \|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})\|_{H^1(\Omega)}^2 + \frac{\nu \varepsilon_{16} k}{2} \|z_h^{n+1} - \rho_h z(t^{n+1})\|_{L^2(\Omega)}^2,$$

and

$$|(b_{4,2})| = \left| \nu \left(\int_{t^n}^{t^{n+1}} (\operatorname{curl} \mathbf{u}(t^{n+1}) - \operatorname{curl} \mathbf{u}(t)) dt, \theta_h^{n+1} \right) \right| \\ \leq \frac{\nu k^2}{\sqrt{3}\varepsilon_{17}} \|\mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^1(\Omega)^2)}^2 + \frac{\nu \varepsilon_{17} k}{\sqrt{3}} \|z_h^{n+1} - \rho_h z(t^{n+1})\|_{L^2(\Omega)}^2.$$

Collecting all these results, we obtain :

$$(b_{1,1}) + (b_{2,1}) + (b_{3,1}) \leq |(b_{1,2})| + |(b_{2,2})| + |(b_{2,3})| + \left| \sum_{i=2}^4 (b_{3,i}) \right| + |(b_4)|.$$

Then (30) follows easily after the sum over $n = 1, \dots, N-1$, a suitable choice of $\varepsilon_i, i = 1, \dots, 17$ and by applying a triangular inequality to $\|z_h^{n+1} - z(t^{n+1})\|_{L^2(\Omega)}$:

$$\|z_h^{n+1} - z(t^{n+1})\|_{L^2(\Omega)} \leq \|z_h^{n+1} - \rho_h z(t^{n+1})\|_{L^2(\Omega)} + \|\rho_h z(t^{n+1}) - z_h^{n+1}\|_{L^2(\Omega)}$$

and properties of P_h . \square

Corollary 4.4. *Under the assumptions of Theorem 4.1 and Theorem 4.3, and for k sufficiently small, there exist constants C_1, C_2 and C_3 independent of h and k such that :*

$$\sum_{n=0}^{N-1} k |\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|_{H^1(\Omega)}^2 \leq C_1 (h^3 + k^2), \quad (32)$$

$$\sum_{n=0}^{N-1} k \|z_h^{n+1} - z(t^{n+1})\|_{L^2(\Omega)}^2 \leq C_2 (h^3 + k^2), \quad (33)$$

and

$$\sup_n |\mathbf{u}_h^n - \mathbf{u}(t^n)|_{H^1(\Omega)}^2 \leq C_3 (h^3 + k^2). \quad (34)$$

Proof. On one hand, we consider (30). On the other hand, the only difference between this proof and that of Theorem 4.1 is the upper bound of the term $(a_{5,1})$. Here, using the inequality

$$\|\mathbf{u}\|_{L^4(\Omega)}^2 \leq C \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{u}\|_{L^2(\Omega)},$$

we have :

$$\begin{aligned} |(a_{5,1})| &\leq \frac{S_4 \varepsilon_6}{2} \|P_h \mathbf{u}\|_{L^\infty(0,T;H^1(\Omega)^2)}^2 k \|z_h^n - z(t^n)\|_{L^2(\Omega)}^2 + \frac{S_4}{4\tilde{\varepsilon}_6} k \|\mathbf{u}_h^{n+1} - P_h \mathbf{u}(t^{n+1})\|_{L^2(\Omega)}^2 \\ &+ \frac{S_4}{4\tilde{\varepsilon}_6} k |\mathbf{u}_h^{n+1} - P_h \mathbf{u}(t^{n+1})|_{H^1(\Omega)}^2. \end{aligned}$$

Then, using this result with (30) and after a suitable choice of $\varepsilon_i, i = 1, \dots, 10$ and $\tilde{\varepsilon}_6$, we obtain :

$$\begin{aligned} &\|\mathbf{u}_h^N - P_h \mathbf{u}(t^N)\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \|(\mathbf{u}_h^{n+1} - P_h(\mathbf{u}(t^{n+1}))) - (\mathbf{u}_h^n - P_h(\mathbf{u}(t^n)))\|_{L^2(\Omega)}^2 \\ &+ \alpha \|\mathbf{u}_h^N - P_h \mathbf{u}(t^N)\|_{H^1(\Omega)}^2 + \alpha \sum_{n=0}^{N-1} \|(\mathbf{u}_h^{n+1} - P_h \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - P_h \mathbf{u}(t^n))\|_{H^1(\Omega)}^2 \\ &+ \nu \sum_{n=0}^{N-1} k |\mathbf{u}_h^{n+1} - P_h \mathbf{u}(t^{n+1})|_{H^1(\Omega)}^2 \leq C(h^3 + k^2) + C' \sum_{n=0}^{N-1} k \|\mathbf{u}_h^{n+1} - P_h \mathbf{u}(t^{n+1})\|_{L^2(\Omega)}^2. \end{aligned}$$

Then by applying the discrete Gronwall lemma, we obtain, for k sufficiently small :

$$\|\mathbf{u}_h^N - \mathbf{u}(t^N)\|_{L^2(\Omega)}^2 + \nu \sum_{n=0}^{N-1} k |\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|_{H^1(\Omega)}^2 \leq C e^{C' k N} (h^3 + k^2),$$

and the results follow easily. \square

Remark 4.5. *If we have, for example,*

$$\| \mathbf{v}_h^m \|_{L^2(\Omega)}^2 + \sum_{n=0}^{m-1} \| \mathbf{v}_h^{n+1} - \mathbf{v}_h^n \|_{L^2(\Omega)}^2 + \nu \sum_{n=0}^{m-1} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 \leq C_1 + C_2 \sum_{n=0}^{m-1} k \| \mathbf{v}_h^{n+1} \|_{L^2(\Omega)}^2,$$

by writing,

$$\| \mathbf{v}_h^m \|_{L^2(\Omega)} \leq \| \mathbf{v}_h^m - \mathbf{v}_h^{m-1} \|_{L^2(\Omega)} + \| \mathbf{v}_h^{m-1} \|_{L^2(\Omega)},$$

we obtain

$$C_2 k \| \mathbf{v}_h^m \|_{L^2(\Omega)}^2 \leq 2C_2 k \| \mathbf{v}_h^m - \mathbf{v}_h^{m-1} \|_{L^2(\Omega)}^2 + 2C_2 k \| \mathbf{v}_h^{m-1} \|_{L^2(\Omega)}^2.$$

By assuming k sufficiently small such that $2C_2 k \leq 1$, we obtain :

$$\| \mathbf{v}_h^m \|_{L^2(\Omega)}^2 + \sum_{n=0}^{m-2} \| \mathbf{v}_h^{n+1} - \mathbf{v}_h^n \|_{L^2(\Omega)}^2 + \nu \sum_{n=0}^{m-1} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 \leq C_1 + 3C_2 \sum_{n=1}^{m-1} k \| \mathbf{v}_h^n \|_{L^2(\Omega)}^2,$$

then we can apply the discrete classic Gronwall lemma.

Theorem 4.6. *Under the assumptions of Corollary 4.4, we suppose that $p' \in L^2(0, T; L^2(\Omega))$. Then the pressure satisfies the following estimate :*

$$\begin{aligned} \sum_{n=0}^{N-1} k \| p_h^{n+1} - p(t^{n+1}) \|_{L^2(\Omega)}^2 &\leq \frac{1}{\beta^*} \left\{ C(h^3 + k^2) \right. \\ &\quad \left. + (\alpha + S_2^2) \sum_{n=0}^{N-1} k \left| \frac{(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - \mathbf{u}(t^n))}{k} \right|_{H^1(\Omega)}^2 \right\}, \end{aligned} \quad (35)$$

where the coefficients C and S_2 are respectively the inf-sup constant and Poincaré's constant and are independent of h and k .

Proof. We consider again the first equation of the proof of Theorem 4.1, insert $\pm kr_h p(t^{n+1})$ in the terms of the pression and we get :

$$\begin{aligned} &\int_{t^n}^{t^{n+1}} \left(p_h^{n+1} - r_h p(t^{n+1}), \operatorname{div} \mathbf{v}_h^{n+1} \right) dt = \left((\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - \mathbf{u}(t^n)), \mathbf{v}_h^{n+1} \right) \\ &+ \alpha \left(\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - \nabla(\mathbf{u}_h^n - \mathbf{u}(t^n)), \nabla \mathbf{v}_h^{n+1} \right) + \nu \left(\int_{t^n}^{t^{n+1}} \nabla(\mathbf{u}_h^{n+1} - \mathbf{u}(t)) dt, \nabla \mathbf{v}_h^{n+1} \right) \\ &+ \int_{t^n}^{t^{n+1}} \left((z_h^n \wedge \mathbf{u}_h^{n+1} - z(t) \wedge \mathbf{u}(t)) dt, \mathbf{v}_h^{n+1} \right) - \int_{t^n}^{t^{n+1}} \left(r_h p(t^{n+1}) - p(t), \operatorname{div} \mathbf{v}_h^{n+1} \right) dt. \end{aligned}$$

Owing to the inf-sup condition, $\forall q_h \in M_h$,

$$\exists \mathbf{w}_h \in V_h^\perp; (\operatorname{div} \mathbf{w}_h, q_h) = \|q_h\|_{L^2(\Omega)}^2 \text{ and } \|\nabla \mathbf{w}_h\|_{L^2(\Omega)} \leq \|q_h\|_{L^2(\Omega)},$$

and summing over $n = 0, \dots, N-1$, the left-hand side of this equation becomes

$$\sum_{n=0}^{N-1} k \| p_h^{n+1} - r_h p(t^{n+1}) \|_{L^2(\Omega)}^2. \text{ Let us treat the terms in the right-hand side of the equation.}$$

For the first term, we have

$$\begin{aligned}
& \left| \sum_{n=0}^{N-1} \left((\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - \mathbf{u}(t^n)), \mathbf{v}_h^{n+1} \right) \right| \\
& \leq S_2 \left(\sum_{n=0}^{N-1} k \left\| \frac{(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) - (\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n))}{k} \right\|_{L^2(\Omega)}^2 \right)^{1/2} \left(\sum_{n=0}^{N-1} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2} \\
& \leq S_2^2 \left(\sum_{n=0}^{N-1} k \left| \frac{(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) - (\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n))}{k} \right|_{H^1(\Omega)}^2 \right)^{1/2} \left(\sum_{n=0}^{N-1} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2},
\end{aligned}$$

and for the second,

$$\begin{aligned}
& \left| \alpha \sum_{n=0}^{N-1} \left(\nabla((\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - \mathbf{u}(t^n))), \nabla \mathbf{v}_h^{n+1} \right) \right| \\
& \leq \alpha \left(\sum_{n=0}^{N-1} k \left| \frac{(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) - (\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n))}{k} \right|_{H^1(\Omega)}^2 \right)^{1/2} \left(\sum_{n=0}^{N-1} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2}.
\end{aligned}$$

For the third term, we have :

$$\begin{aligned}
& \nu \left| \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}(t)), \nabla \mathbf{v}_h^{n+1}) dt \right| \\
& \leq \nu \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \left| (\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})), \nabla \mathbf{v}_h^{n+1}) \right| dt + \nu \sum_{n=0}^{N-1} \left| \int_{t^n}^{t^{n+1}} \left(\int_{t^n}^t \nabla \mathbf{u}'(\tau) d\tau, \nabla \mathbf{v}_h^{n+1} \right) dt \right| \\
& \leq \nu \left(C_1(h^3 + k^2)^{1/2} + C_2 k \|\mathbf{u}'\|_{L^2(0,T;H^1(\Omega)^2)} \right) \left(\sum_{n=0}^{N-1} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2}.
\end{aligned}$$

The fourth term is treated as the fifth term in the proof of Theorem 4.1 and by using the result of Theorem 4.3, the result is the following :

$$\left| \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \left((z_h^n \wedge \mathbf{u}_h^{n+1} - z(t) \wedge \mathbf{u}(t)), \nabla \mathbf{v}_h^{n+1} \right) dt \right| \leq C(h^3 + k^2)^{1/2} \left(\sum_{n=0}^{N-1} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2}.$$

Finally, the last term is treated as follows :

$$\begin{aligned}
& \int_{t^n}^{t^{n+1}} \left(r_h p(t^{n+1}) - p(t), \operatorname{div} \mathbf{v}_h^{n+1} \right) dt \\
& \leq \left| \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \left((r_h p(t^{n+1}) - r_h p(t)), \operatorname{div} \mathbf{v}_h^{n+1} \right) dt \right| + \left| \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \left((r_h p(t) - p(t)), \operatorname{div} \mathbf{v}_h^{n+1} \right) dt \right| \\
& \leq C \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (\tau - t^n) \|p'(\tau)\|_{L^2(\Omega)} |\mathbf{v}_h^{n+1}|_{H^1(\Omega)} d\tau + C_2 \sum_{n=0}^{N-1} h^2 k^{1/2} \|p\|_{L^2(t^n, t^{n+1}; H^2(\Omega))} |\mathbf{v}_h^{n+1}|_{H^1(\Omega)} \\
& \leq \left(\frac{C_1 k}{\sqrt{3}} \|p'\|_{L^2(0,T;L^2(\Omega))} + C_2 h^2 \|p\|_{L^2(0,T;H^2(\Omega))} \right) \left(\sum_{n=0}^{N-1} k |\mathbf{v}_h^{n+1}|_{H^1(\Omega)}^2 \right)^{1/2}.
\end{aligned}$$

Then (35) follows easily from these inequalities. \square

We still have to estimate $\left(\sum_{n=0}^{N-1} k \left| \frac{(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - \mathbf{u}(t^n))}{k} \right|_{H^1(\Omega)}^2\right)^{1/2}$.

We introduce a variant of Stokes projection as follows : $\forall(\mathbf{u}, p) \in V \times L_0^2(\Omega)$, $S_h \mathbf{u} \in V_h$ is defined by

$$\forall \mathbf{v}_h \in V_h, \quad \nu(\nabla(S_h \mathbf{u} - \mathbf{u}), \nabla \mathbf{v}_h) = -(p, \operatorname{div} \mathbf{v}_h), \quad (36)$$

Lemma 4.7. *Let $(\mathbf{u}, p) \in V \times L_0^2(\Omega)$. Then $S_h \mathbf{u}$ defined by (36) satisfies :*

$$|S_h \mathbf{u} - \mathbf{u}|_{H^1(\Omega)} \leq 2|P_h \mathbf{u} - \mathbf{u}|_{H^1(\Omega)} + \frac{1}{\nu} \|r_h p - p\|_{L^2(\Omega)}. \quad (37)$$

If, in addition, Ω is convex, there exists a constant C independent of h such that:

$$\|S_h \mathbf{u} - \mathbf{u}\|_{L^2(\Omega)} \leq Ch(|S_h \mathbf{u} - \mathbf{u}|_{H^1(\Omega)} + \|r_h p - p\|_{L^2(\Omega)}). \quad (38)$$

Theorem 4.8. *Under the assumptions of Theorem 4.6 and assuming $p' \in L^2(0, T; H^2(\Omega))$, we have :*

$$\sum_{n=0}^{N-1} k \left| \frac{(\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - \mathbf{u}(t^n))}{k} \right|_{H^1(\Omega)}^2 \leq C(h^3 + k^2). \quad (39)$$

Proof. We consider, once more, the first equation in the proof of Theorem 4.6, choose $\mathbf{v}_h^{n+1} \in V_h$, insert $r_h p(s)$ and $S_h \mathbf{u}' = (S_h \mathbf{u})'$ and we set $\mathbf{e}_h^n = \mathbf{u}_h^n - S_h \mathbf{u}(t^n)$. We obtain :

$$\begin{aligned} & \left((\mathbf{u}_h^{n+1} - S_h \mathbf{u}(t^{n+1})) - (\mathbf{u}_h^n - S_h \mathbf{u}(t^n)), \mathbf{v}_h^{n+1} \right) - \int_{t^n}^{t^{n+1}} \left(\mathbf{u}'(s) - S_h \mathbf{u}'(s), \mathbf{v}_h^{n+1} \right) ds \\ & + \alpha \left(\nabla(\mathbf{u}_h^{n+1} - S_h \mathbf{u}(t^{n+1})) - \nabla(\mathbf{u}_h^n - S_h \mathbf{u}(t^n)), \nabla \mathbf{v}_h^{n+1} \right) - \alpha \int_{t^n}^{t^{n+1}} \left(\nabla(\mathbf{u}'(s) - S_h \mathbf{u}'(s)), \nabla \mathbf{v}_h^{n+1} \right) ds \\ & + \nu \left(\int_{t^n}^{t^{n+1}} \nabla(\mathbf{u}_h^{n+1} - S_h \mathbf{u}(s)) ds, \nabla \mathbf{v}_h^{n+1} \right) + \nu \left(\int_{t^n}^{t^{n+1}} \nabla(S_h \mathbf{u}(s) - \mathbf{u}(s)) ds, \nabla \mathbf{v}_h^{n+1} \right) \\ & + \int_{t^n}^{t^{n+1}} \left((z_h^n \wedge \mathbf{u}_h^{n+1} - z(s) \wedge \mathbf{u}(s)) ds, \mathbf{v}_h^{n+1} \right) + \int_{t^n}^{t^{n+1}} \left(p(s), \operatorname{div} \mathbf{v}_h^{n+1} \right) ds = 0. \end{aligned}$$

We sum this above equation over $n = 0, \dots, N-1$ and we treat the terms denoted again $(a_i), i = 1, \dots, 6$ in the left-hand side. We take $\mathbf{v}_h^{n+1} = \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k}$.

The first term is composed of two parts as follows :

$$(a_1) = \sum_{n=0}^{N-1} (\mathbf{e}_h^{n+1} - \mathbf{e}_h^n, \mathbf{v}_h^{n+1}) - \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \left(\mathbf{u}'(s) - S_h \mathbf{u}'(s), \mathbf{v}_h^{n+1} \right) ds = (a_{1,1}) + (a_{1,2}),$$

where

$$\begin{aligned}
(a_{1,1}) &= \sum_{n=0}^{N-1} k \left\| \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k} \right\|_{L^2(\Omega)}^2, \\
|(a_{1,2})| &\leq \sum_{n=0}^{N-1} \|\mathbf{u}' - S_h \mathbf{u}'\|_{L^2(t^n, t^{n+1}; L^2(\Omega)^2)} \left(k \left\| \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k} \right\|_{L^2(\Omega)}^2 \right)^{1/2} \\
&\leq \frac{1}{2\varepsilon_1} \|\mathbf{u}' - S_h \mathbf{u}'\|_{L^2(0, T; L^2(\Omega)^2)}^2 + \frac{\varepsilon_1}{2} \sum_{n=0}^{N-1} k \left\| \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k} \right\|_{L^2(\Omega)}^2 \\
&\leq \frac{C}{2\varepsilon_1} h^4 (\|\mathbf{u}'\|_{L^2(0, T; H^2(\Omega)^2)}^2 + \|p'\|_{L^2(0, T; H^1(\Omega))}^2) + \frac{\varepsilon_1}{2} \sum_{n=0}^{N-1} k \left\| \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k} \right\|_{L^2(\Omega)}^2.
\end{aligned}$$

The second term (a_2) is treated as the first one. We have :

$$(a_2) = \alpha \sum_{n=0}^{N-1} (\nabla \mathbf{e}_h^{n+1} - \nabla \mathbf{e}_h^n, \nabla \mathbf{v}_h^{n+1}) - \alpha \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (\nabla \mathbf{u}'(s) - \nabla S_h \mathbf{u}'(s), \nabla \mathbf{v}_h^{n+1}) ds = (a_{2,1}) + (a_{2,2}),$$

with

$$\begin{aligned}
(a_{2,1}) &= \alpha \sum_{n=0}^{N-1} k \left| \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k} \right|_{H^1(\Omega)}^2, \\
|(a_{2,2})| &\leq \frac{C\alpha}{2\varepsilon_2} h^4 (\|\mathbf{u}'\|_{L^2(0, T; H^3(\Omega)^2)}^2 + \|p'\|_{L^2(0, T; H^2(\Omega))}^2) + \frac{\alpha\varepsilon_2}{2} \sum_{n=0}^{N-1} k \left\| \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k} \right\|_{H^1(\Omega)}^2.
\end{aligned}$$

The third term is treated as :

$$\begin{aligned}
|(a_3)| &= \nu \left| \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (\nabla(\mathbf{u}_h^{n+1} - S_h \mathbf{u}(s)), \nabla \mathbf{v}_h^{n+1}) ds \right| \\
&= \nu \left| \sum_{n=0}^{N-1} (\nabla \mathbf{e}_h^{n+1}, \nabla \mathbf{v}_h^{n+1}) + \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (\nabla S_h(\mathbf{u}(t^{n+1}) - \mathbf{u}(s)), \nabla \mathbf{v}_h^{n+1}) ds \right| \\
&= |(a_{3,1}) + (a_{3,2})|.
\end{aligned}$$

Using the relation $(a^{n+1} - a^n, a^{n+1}) = \frac{1}{2} \|a^{n+1}\|_{L^2(\Omega)}^2 - \frac{1}{2} \|a^n\|_{L^2(\Omega)}^2 + \frac{1}{2} \|a^{n+1} - a^n\|_{L^2(\Omega)}^2$ and by defining $S_h \mathbf{u}^0$ with $p = 0$, we obtain :

$$\begin{aligned}
|(a_{3,1})| &= \frac{\nu}{2} \sum_{n=0}^{N-1} k \left| \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k} \right|_{H^1(\Omega)}^2 + \frac{\nu}{2} \sum_{n=0}^{N-1} k \left(\left| \frac{\mathbf{e}_h^{n+1}}{k} \right|_{H^1(\Omega)}^2 - \left| \frac{\mathbf{e}_h^n}{k} \right|_{H^1(\Omega)}^2 \right) \\
&= \frac{\nu}{2} \sum_{n=0}^{N-1} k \left| \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k} \right|_{H^1(\Omega)}^2 + \frac{\nu}{2} k \left| \frac{\mathbf{e}_h^N}{k} \right|_{H^1(\Omega)}^2.
\end{aligned}$$

and

$$\begin{aligned}
|(a_{3,2})| &= \nu \left| \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (\nabla S_h(\mathbf{u}(t^{n+1}) - \mathbf{u}(s)), \nabla \mathbf{v}_h^{n+1}) ds \right| \\
&= \nu \left| \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} (\nabla S_h \mathbf{u}'(t)(t - t^n), \nabla \mathbf{v}_h^{n+1}) dt \right| \\
&\leq \nu \sum_{n=0}^{N-1} |\mathbf{v}_h^{n+1}|_{H^1(\Omega)} \int_{t^n}^{t^{n+1}} (t - t^n) |S_h \mathbf{u}'(t)|_{H^1(\Omega)} dt \\
&\leq \nu \sum_{n=0}^{N-1} k^{3/2} |\mathbf{v}_h^{n+1}|_{H^1(\Omega)} \|S_h \mathbf{u}'\|_{L^2(t^n, t^{n+1}; H^1(\Omega)^2)} \\
&\leq \frac{\nu}{2\varepsilon_3} \sum_{n=0}^{N-1} k \left| \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k} \right|_{H^1(\Omega)}^2 + \frac{\nu\varepsilon_3}{2} k^2 \|S_h \mathbf{u}'\|_{L^2(0, T; H^1(\Omega)^2)}^2.
\end{aligned}$$

Using the definition of S_h , we have

$$(a_4) + (a_6) = \nu \left(\int_{t^n}^{t^{n+1}} \nabla (S_h \mathbf{u}(s) - \mathbf{u}(s)) ds, \nabla \mathbf{v}_h^{n+1} \right) + \int_{t^n}^{t^{n+1}} (p(s), \operatorname{div} \mathbf{v}_h^{n+1}) ds = 0$$

Finally, the last term (a_5) is bounded as previously in Theorem 4.6 :

$$|(a_5)| \leq \frac{C\varepsilon_4}{2} (h^3 + k^2) + \frac{C}{2\varepsilon_4} \sum_{n=0}^{N-1} k \left| \frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{k} \right|_{H^1(\Omega)}^2.$$

Collecting these results, writing

$$(a_{1,1}) + (a_{2,1}) + (a_{3,1}) \leq |(a_{1,2})| + |(a_{2,2})| + |(a_{3,2})| + |(a_5)|,$$

choosing suitably $\varepsilon_i, i = 1, \dots, 4$ and by applying the following triangular inequality

$$|\mathbf{u}_h^{n+1} - \mathbf{u}(t^{n+1})|_{H^1(\Omega)} \leq |\mathbf{u}_h^{n+1} - S_h \mathbf{u}(t^{n+1})|_{H^1(\Omega)} + |S_h \mathbf{u}(t^{n+1}) - \mathbf{u}(t^{n+1})|_{H^1(\Omega)},$$

(39) follows easily. □

Theorem 4.9. *Under the assumptions of Theorem 4.6, there exists a constant C that does not depend on h and k such that*

$$\sum_{n=0}^{N-1} k \|p_h^{n+1} - p(t^{n+1})\|_{L^2(\Omega)}^2 \leq C(h^3 + k^2). \quad (40)$$

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