



**HAL**  
open science

## Dual billiards, Fagnano orbits and regular polygons

Serge Troubetzkoy

► **To cite this version:**

| Serge Troubetzkoy. Dual billiards, Fagnano orbits and regular polygons. 2007. hal-00139790v1

**HAL Id: hal-00139790**

**<https://hal.science/hal-00139790v1>**

Preprint submitted on 3 Apr 2007 (v1), last revised 28 Mar 2008 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# DUAL BILLIARDS, FAGNANO ORBITS AND REGULAR POLYGONS.

SERGE TROUBETZKOY

ABSTRACT. We study the notion of Fagnano orbits for dual polygonal billiards. We used them to characterize regular polygons and we study the iteration of the developing map.

In this article we consider the dual notion of two results on polygonal billiards. We begin by describing these original results. DeTemple and Robertson [2] have shown that a closed convex polygon  $P$  is regular if and only if  $P$  contains a periodic billiard path  $Q$  similar to  $P$ . The second result is about the dynamics of the so called pedal map related to billiards in a triangle  $P$ . The three altitudes of  $P$  intersect the opposite sides (or their extensions) in three points called the feet. These three points form the vertices of a new triangle  $Q$  called the pedal triangle  $P^{(1)}$  of the triangle  $P^{(0)}$  (Figure 1). It is well known that for acute triangles the pedal triangle form a period three billiard orbit often referred to as the Fagnano orbit, i.e., the polygon  $Q$  is inscribed in  $P$  and satisfies the usual law of geometric optics: “the angle of incidence equals the angle of reflection” or equivalently (this is a theorem) that the pedal triangle has least perimeter among all inscribed triangles. The name Fagnano is used since in 1775 J.F.F. Fagnano gave the first proof the variational characterization. In a sequence of elegant and entertaining articles, J. Kingston and J. Synge [4], P. Lax [5], P. Ungar [9] and J. Alexander [1] studied the dynamics of the pedal map given by iterating this process.

There is a dual notion to billiards, called dual or outer billiards. The game of dual billiards is played outside the billiard table. Suppose the table is a polygon  $P$  and that  $x$  is a points outside  $P$  and not on the

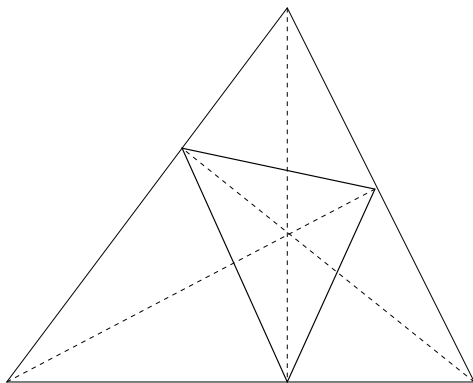


FIGURE 1. A pedal triangle.

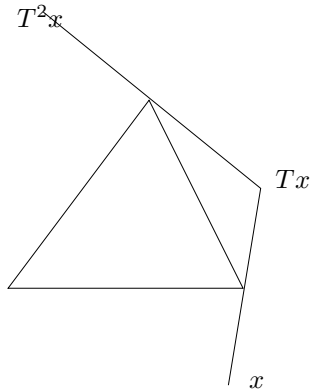


FIGURE 2. The polygonal dual billiard map.

continuation of any of  $P$ 's sides. There are two support lines to  $P$  through  $x$ , choose the right one as viewed from  $x$ , and reflect  $x$  in the support vertex of  $P$  to obtain  $x$ 's image under the dual billiard map (see Figure 2). The point  $x$  has an infinite orbit if none of its images belongs to the continuation of a side of the polygon. Here we allow self intersecting polygons which we will refer to as star shaped polygons.

Dual billiards have been extensively studied by S. Tabachnikov (see [6],[7],[8] and the reference therein). Corresponding to the fact that the Fagnano orbit hits each side of the triangle call a periodic dual billiard orbit of period  $n$  for an  $n$ -gon Fagnano if the orbit reflects from the vertices in a cyclic manner. Fagnano orbit for dual billiards were introduced by Tabachnikov in [7]. In this article he studied a certain variational property analogous to one in the pedal case studied by Gutkin [3].

Motivated by the notion of pedal triangles we introduce here dedal  $n$ -gons. An  $n$ -gon  $Q$  is called a dedal  $n$ -gon of the  $n$ -gon  $P$  if reflecting the vertex  $w_i$  of  $Q$  in the vertex  $z_i$  of  $P$  yields the vertex  $w_{i+1}$ . There is no requirement that the sides of  $Q$  touch  $P$  only at a vertex. From the definition it is clear that if the Fagnano orbit of an  $n$ -gon  $P$  exists then it is automatically a dedal  $n$ -gon.

Suppose  $Q(w_1, \dots, w_n)$  is the dedal polygon of  $P(z_1, \dots, z_n)$ . The developing map  $\mu : Q \rightarrow P$  is defined by the equation

$$(1) \quad z_i = (w_i + w_{i+1})/2$$

(see Figures 3, 5 and 6). Note that throughout this article the subscripts will be taken modulo  $n$  without explicit mention. The characteristic polynomial of  $\mu$  is

$$(1 - 2x)^n - (-1)^n.$$

Its eigenvalues are  $(1+q^i)/2$  for  $i = 0, 1, \dots, n-1$  where  $q := \exp(2\pi i/n)$ . All the eigenvalues differ from zero except for the  $n/2$ nd eigenvalue for  $n$  even. The  $i$ th eigenvector is  $X_i := (1, q^i, q^{2i}, \dots, q^{(n-1)i})$ . If  $i$  divides

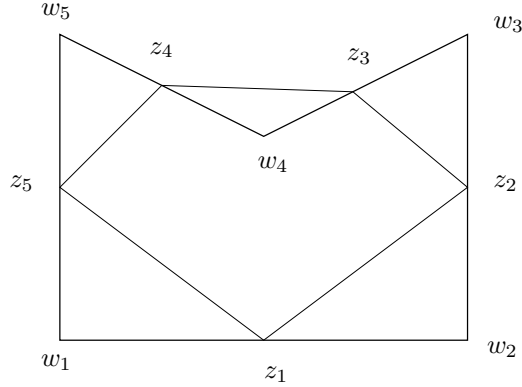


FIGURE 3. A pentagon without a Fagnano orbit, the unique dedal pentagon is pictured.

$n$  then  $X_i$  is a polygon with  $n/i$  sides. None the less for the sake of clarity (avoiding stating special cases) we will think of this as an  $n$ -gon which is traced  $i$  times. An exception to this rule is the case  $n$  even and  $i = n/2$ . In this case  $X_{n/2}$  is a segment which we do not consider as a polygon.

The following proposition clarifies the existence question of dedal polygons.

**Proposition 1.** *a) Suppose that  $n \geq 3$  is odd. Then for any  $n$ -gon  $P$  there is a unique dedal  $n$ -gon  $Q$ .*

*b) If  $n \geq 3$  is even dedal  $n$ -gons exist if and only if the vertices of  $P$  satisfy  $z_1 - z_2 + z_3 - \dots - z_n = 0$ . If this equation is satisfied then there is a unique dedal  $n$ -gon  $Q_0$  in the space  $X_{n/2}^\perp$ . The space  $D := \{Q_0 + sX_{n/2} : s \in \mathbb{C}\}$  consists of the dedal  $n$ -gons of  $P$ . In particular for each  $i \in \{1, \dots, n\}$  every point  $w \in \mathbb{C}$  is the  $i$ th vertex of a unique dedal  $n$ -gon  $Q_i(w)$ .*

**Proof of Proposition 1.** If  $n \geq 3$  is odd then the map  $\mu$  is invertible with

$$(2) \quad w_i = z_i - z_{i+1} + z_{i+2} - \dots + z_{i-1},$$

and thus the dedal polygon exists and is unique.

On the other hand if  $n \geq 3$  is even then the map  $\mu$  is not invertible. The eigenspace of the zero eigenvalue is one-complex dimensional and is generated by the vector  $X_{n/2} := (1, -1, 1, -1, \dots, 1, -1)$ . Thus dedal polygons exist if and only if the vector  $\vec{z} = (z_1, z_2, \dots, z_n)$  is orthogonal to the vector  $X_{n/2}$ , i.e., it satisfies  $X_{n/2} \cdot \vec{z} = 0$ , or equivalently

$$(3) \quad z_1 - z_2 + z_3 - \dots - z_n = 0.$$

The uniqueness of  $Q_0$  follows since the map  $\mu$  is invertible on the space  $X_{n/2}^\perp$ . The statement on the set  $D$  follows immediately since  $X_{n/2}$  is the kernel of  $\mu$ . Let  $Q_0 := (w_1^0, \dots, w_n^0)$ . For each  $i$  any point  $w \in \mathbb{C}$

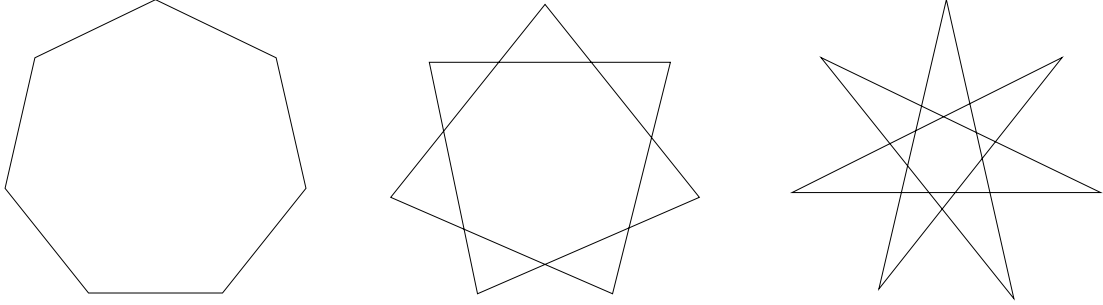


FIGURE 4. Up to orientation there are three regular 7-gons.

can be uniquely expressed as  $w_i^0 + s$  for some  $s \in \mathbb{C}$ .  $\square$

Note that the dedal polygon  $Q$  of a convex polygon  $P$  is a Fagnano orbit if and only if  $Q$  is convex. In particular Fagnano orbits always exist for triangles, but not for polygons with more sides. An example of a pentagon without a Fagnano orbit is given in Figure 3.

$X_1$  and  $X_{n-1}$  are the usual regular  $n$ -gon in counter clockwise and clockwise orientation (Figure 4a). If  $i$  and  $n$  are relatively prime and  $i \notin \{1, n-1\}$  then  $X_i$  is star shaped and we also call it regular (Figures 4b and c). Finally if  $i$  divides  $n$  and  $i \neq n/2$  then  $X_i$  is naturally a regular  $n/i$ -gon which we will think of as (a multiple cover of) a regular  $n$ -gon. A simple calculation (see [7]) shows that the vectors  $X_i$  form a basis of our space of polygons. In analog to DeTemple and Robertson's result we have

**Theorem 2.** *Fix  $n \geq 3$ . An  $n$ -gon  $P$  is regular if and only if it has a dedal polygon  $Q$  which is similar to  $P$ .*

We remark that if  $n$  is odd then  $Q$  is the unique dedal polygon of  $P$  while if  $n$  is even then  $Q$  is the unique dedal polygon  $Q_0 \in X_{n/2}^\perp$ .

**Proof.** Suppose  $P$  is regular, i.e.,  $P = \text{const } X_j$  where  $j \neq n/2$  if  $n$  is even. If  $n$  is odd let  $Q$  be the unique dedal polygon of  $P$ . If  $n$  is even then since  $P$  is regular it satisfies (3) and thus we can choose  $Q = Q_0$ . In both cases let  $Q = \sum a_i X_i$ . Since  $P = \mu(Q) = \sum \frac{1+q^i}{2} a_i X_i$  we have  $a_i = 0$  for  $i \neq j$  and thus  $Q = a_j X_j$  is similar to  $P$ .

Conversely suppose that  $Q = \sum a_i X_i$  is a dedal polygon of  $P = \sum b_i X_i$  and that  $P$  and  $Q$  are similar, i.e.,  $a_i = \text{const } b_i$ . Note that side  $Q$  exists either  $n$  is odd or that  $n$  is even and  $P$  satisfies (3).

Since  $P = \mu(Q)$  we have  $\frac{1+q^i}{2} a_i = b_i$  for  $i = 1, \dots, n$ . Thus for each  $i$  such that  $a_i \neq 0$  we have  $\frac{1+q^i}{2} = \text{const}$  and it follows  $q^i = q^j$  if  $a_i$  and  $a_j \neq 0$ . Since  $q^i \neq q^j$  if  $i \neq j$  it follows that only a single  $a_j$  is non-null and thus  $P = b_j X_j$  and  $Q = a_j X_j$  are regular. Note that if  $n$  is even then  $i \neq n/2$  since in this case  $(1+q^i)/2 = 0$  and thus  $b_i = a_i = 0$ , i.e.,

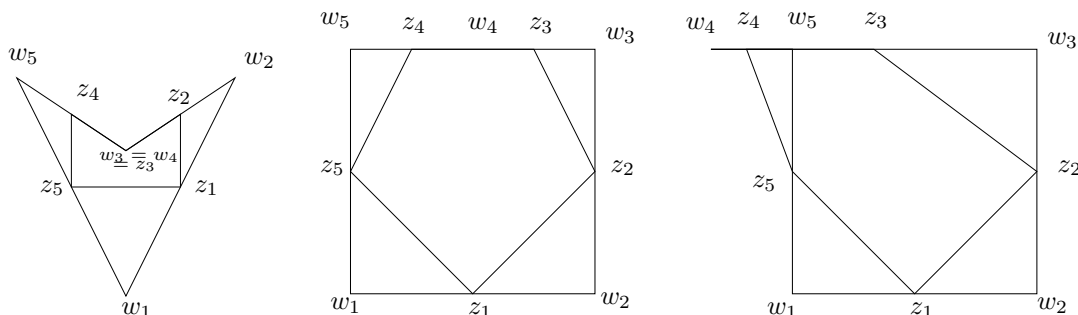


FIGURE 5. Degenerate dedal polygons: loss of a vertex, angle  $\pi$ , angle  $2\pi$ .

$P$  and  $Q$  are not 2-gons. □

The map  $\mu$  is well defined for any point  $(w_1, \dots, w_n) \in \mathbb{C}^n$ . Since the dual billiard is affine invariant we can assume that the center of mass of the polygons is at the origin: i.e., that  $w_1 + w_2 + \dots + w_n = 0$ . This reduces the complex dimension by 1. The map  $\mu$  preserves the center of mass. The eigenvector  $X_0 = (1, 1, \dots, 1)$  corresponds to polygons which degenerate to a point. The reduced space is orthogonal to this eigenvector since we have fixed the center of mass to be the origin. We have already mentioned that the polygons  $X_j$  are multiple covers when  $j$  divides  $n$ .  $n$ -gons can degenerate in other ways. If  $w_i = w_{i+1}$  for some  $i$  then  $Q$  has  $(n - 1)$  sides, however for the sake of consistency we will think of it as an  $n$ -gon with the  $i$ th side of length 0. The other degeneracies are polygons with adjacent edges with angle  $\pi$  or  $2\pi$ . We will call all of these degenerate objects  $n$ -gons, the notion of dual billiard and dedal  $n$ -gon make sense in this setting. These degeneracies hold for a complex codimension 1 subspace.

The dedal polygon of a non-degenerate  $n$ -gon can be degenerate in all the ways described above. A dedal polygon will have a side of length 0 if  $w_i = z_i$  for some  $i$  and thus  $w_i = w_{i+1}$ . An example of a pentagon whose dedal polygon degenerates to a quadrilateral in this way is shown in Figure 5a. Another way to degenerate is that two adjacent edges have angle  $\pi$  or  $2\pi$ . This happens if a vertex of the dedal polygon lies on the continuation of a side of the polygon. Example are shown in Figures 5b and c.

The polygon  $\mu(Q)$  can also degenerate for a non-degenerate dedal  $n$ -gon  $Q$ . This happens if two mid-points of three consecutive sides of  $Q$  are colinear with angle  $\pi$  (Figure 6a) or  $2\pi$  (Figure 6b). It is impossible for two consecutive vertices of  $\mu(Q)$  to coincide if  $Q$  is non-degenerate. The space of  $n$ -gons  $Q$  with degenerate  $\mu(Q)$  is a complex co-dimension 1 subspace.

Consider the space  $\Delta$  of  $n$ -gons up to similarity.

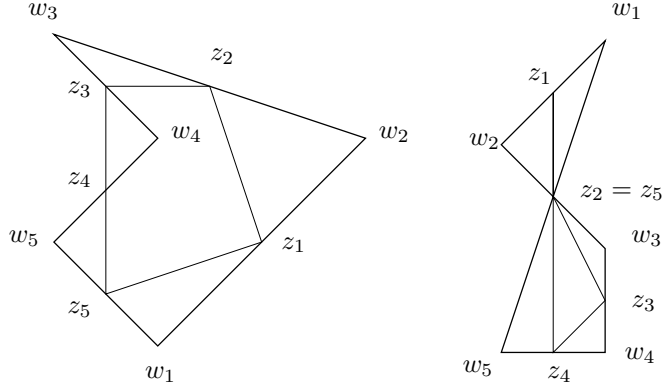


FIGURE 6. Non-degenerate dedal pentagon corresponding to a degenerate pentagons, a) angle  $\pi$  and b)  $2\pi$ .

As notice by Tabachnikov, [7], since  $|\lambda_1| = |\lambda_{n-1}| > |\lambda_i|$  for  $i = 2, \dots, n-2$  the subspace generated by the eigenvectors  $X_1$  and  $X_{n-1}$ , i.e., the affinely regular  $n$ -gons, is an attractor for the developing map. Its basin of attraction is all polygons except for those which lie in the subspace spanned by  $X_i$  with  $i = 2, \dots, n-2$ .

**Theorem 3.** *For any affinely regular  $n$ -gon, the developing map is  $2n$  periodic.*

**Proof.** Suppose that  $Q$  is affinely regular, i.e.,  $Q = aX_1 + bX_2$ . Then  $\mu^{2n}(Q) = \left(\frac{1+q}{2}\right)^{2n} aX_1 + \left(\frac{1+q^{n-1}}{2}\right)^{2n} bX_2$ . Now  $(1+q)^n = \sum_{k=0}^n \binom{n}{k} q^k$ . If  $n$  is odd then  $(1+q)^n = \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} (q^k + q^{-k})$  while if  $n$  is even then  $(1+q)^n = \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} (q^k + q^{-k}) + \binom{n}{\frac{n}{2}} q^{\frac{n}{2}}$ .

In both cases the sums of the roots of unity in the parentheses are real and the first half are positive the second half are negative. Since the coefficients are strictly increasing the sum is a negative real number and thus  $\left(\frac{1+q}{|1+q|}\right)^n = -1$  and  $\left(\frac{1+q}{|1+q|}\right)^{2n} = 1$ . The same holds for  $q^{n-1}$ . Thus  $\mu^{2n}(Q)$  and  $Q$  are similar.  $\square$

If  $n \geq 3$  is odd we call the inverse map  $f := \mu^{-1}$  the dedal map. Since  $|\lambda_{\frac{n-1}{2}}| = |\lambda_{\frac{n+1}{2}}| < |\lambda_i|$  for the other  $i$  the subspace generated by the eigenvectors  $X_{\frac{n-1}{2}}$  and  $X_{\frac{n+1}{2}}$ , i.e., the *affinely regular  $n$ -stars*, is an attractor for the dedal map. Its basin of attraction is all polygons except for those which lie in the subspace spanned by  $X_i$  with  $i \neq \frac{n-1}{2}, \frac{n+1}{2}$ .

**Theorem 4.** *If  $n$  is odd then the dedal map is  $2n$  periodic on the space of affinely regular  $n$ -stars.*

The proof of this theorem is essentially the same of that of Theorem 3 with  $q$  replaced by  $q^{\frac{n-1}{2}}$ .

We remark that the  $n$ -gons not in the basin of attraction (for both the developing map and the dedal map) have themselves an 2 dimensional attractor, the set of  $n$ -gons with the next largest modulus of the eigenvalue, on which again the motion is  $2n$  periodic. This set has a co-dimension 4 basin of attraction, and this decomposition can be iterated to yield co-dimension 6, 8, ... basins of attraction. This process stops at the repeller of the one map which is the attractor of the other.

**Acknowledgments.** Many thanks to Sergei Tabachnikov for helpful remarks.

#### REFERENCES

- [1] J.C. Alexander, *The symbolic dynamics of the sequence of pedal triangles*, Math. Mag. 66 (1993) 147–158.
- [2] D. DeTemple and J. Robertson, *A billiard path characterisation of regular polygons*, Math. Mag. 54 (1981) 73–75.
- [3] E. Gutkin, *Two applications of calculus to triangular billiards*, Amer. Math. Monthly 104 (1997) 618–622.
- [4] J. Kingston and J. Synge, *The sequence of pedal triangles*, Amer. Math. Monthly 95 (1988) 609–620.
- [5] P. Lax, *The ergodic character of sequences of pedal triangles*, Amer. Math. Monthly 97 (1990) 377–381.
- [6] S. Tabachnikov, *Billiards*, “Panoramas et Syntheses”, Soc. Math. France (1995).
- [7] S. Tabachnikov, *Fagnano orbits of polygonal dual billiards*, Geom. Dedicata 77 (1999) 279–286.
- [8] S. Tabachnikov and F. Dogru, *Dual billiards*, Math. Intelligencer 27 (2005) 18–25.
- [9] P. Ungar, *Mixing property of the pedal mapping*, Amer. Math. Monthly 97 (1990) 898–900.

CENTRE DE PHYSIQUE THORIQUE, FEDERATION DE RECHERCHES DES UNITES DE MATHÉMATIQUE DE MARSEILLE, INSTITUT DE MATHÉMATIQUES DE LUMINY AND, UNIVERSIT DE LA MDITERRANE, LUMINY, CASE 907, F-13288 MARSEILLE CEDEX 9, FRANCE

*E-mail address:* `troubetz@iml.univ-mrs.fr`

*URL:* `http://iml.univ-mrs.fr/~troubetz/`