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To cite this version:

Serge Troubetzkoy. Dual billiards, Fagnano orbits and regular polygons. The American Mathematical Monthly, 2009, 116, pp.251-260. hal-00139790v3

HAL Id: hal-00139790

https://hal.science/hal-00139790v3

Submitted on 28 Mar 2008

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hal-00139790, version 3 - 28 Mar 2008

Dual Billiards, Fagnano Orbits, and Regular Polygons.

Serge Troubetzkoy

In this article we consider the dual version of two results on polygonal billiards. We begin by describing these original results. The first result is about the dynamics of the so-called pedal map related to billiards in a triangle P. The three altitudes of P intersect the opposite sides (or their extensions) in three points called the feet. These three points form the vertices of a new triangle Q called the pedal triangle of the triangle P (Figure 1). It is well known that for acute triangles the pedal triangle forms a period-three billiard orbit often referred to as the Fagnano orbit, i.e., the polygon Q is inscribed in P and satisfies the usual law of geometric optics (the angle of incidence equals the angle of reflection) or equivalently (this is a theorem) the pedal triangle has least perimeter among all inscribed triangles. The name Fagnano is used since in 1775 J. F. F. Fagnano gave the first proof of the variational characterization. In a sequence of elegant and entertaining articles, J. Kingston and J. Synge [5], P. Lax [6], P. Ungar [11], and J. Alexander [1] studied the dynamics of the pedal map given by iterating this process. The second result, due to DeTemple and Robertson [3], is that a closed convex polygon P is regular if and only if P contains a periodic billiard path Q similar to P.

There is a dual notion to billiards, called dual or outer billiards. The game of dual billiards is played outside the billiard table. Suppose the table is a polygon P and that z is a point outside P and not on the continuation of any of P's

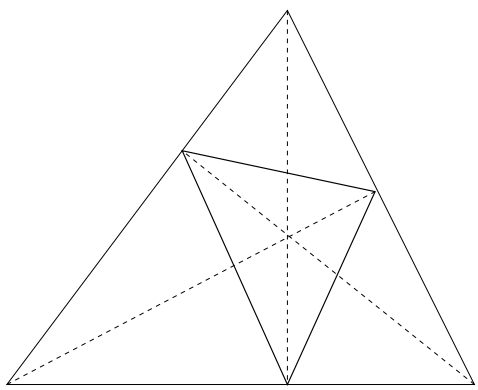

Figure 1: A pedal triangle.

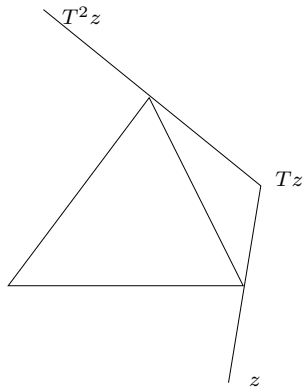

Figure 2: The polygonal dual billiard map.

sides. A line L is a *support line* of P if it intersects the boundary ∂P of P and P lies entirely in one of the two regions into which the line L divides the plane. There are two support lines to P through z; choose the right one as viewed from z. If z is not on the continuation of a side of the convex hull of P then this support line intersects P at a single point which we call the *support vertex* of z. Reflect z in its support vertex to obtain z's image under the dual billiard map denoted by T (see Figure 2). The map T^n is defined for all n at the point z if none of its images belongs to the continuation of a side of the convex hull of the polygon. Dual billiards have been extensively studied by S. Tabachnikov (see [7]–[10] and the reference therein).

Throughout the article we will identify the polygon having vertices $z_i \in \mathbb{C}$, $i = 1, \ldots, n$, ordered cyclically with the point $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. In particular a polygon for us is an oriented object. The notion of polygon includes self-intersecting polygons (which we call star shaped polygons) and geometrically degenerate n-gons: those with a side of length 0, or an angle of π or 2π at a vertex. Corresponding to the fact that the Fagnano billiard orbit hits each side of the triangle, we call an n-periodic dual billiard orbit consisting of n points consecutively reflected in the vertices $z_1, z_2, \ldots, z_n$ a *Fagnano* dual billiard orbit. Fagnano orbits for dual billiards were introduce by Tabachnikov in [8], where he studied a certain variational property analogous to one in the pedal case studied by Gutkin [4].

Motivated by the notion of pedal triangles, we introduce here dedal n-gons. An n-gon Q is called a *dedal* n-gon of the n-gon P if reflecting the vertex w_i of Q in the vertex z_i of P yields the vertex w_{i+1}. (Throughout the article all subscripts will be taken modulo n without explicit mention.) There is no requirement that the sides of Q touch P only at a vertex. From the definition it is clear that if a Fagnano orbit of an n-gon P exists then it is a dedal n-gon. A nondegenerate polygon can have a degenerate dedal polygon and vice-versa. Examples are shown in Figures 3 and 4. The degeneracy not shown cannot occur: i.e., it is impossible for two consecutive vertices of P to coincide if Q is

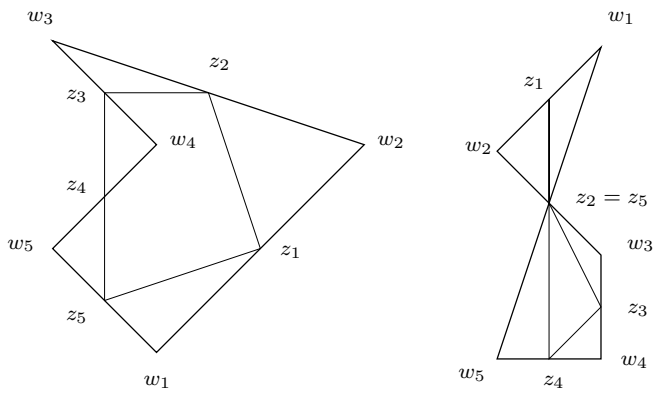

Figure 3: Nondegenerate dedal pentagons corresponding to degenerate pentagons, with angle π and 2π.

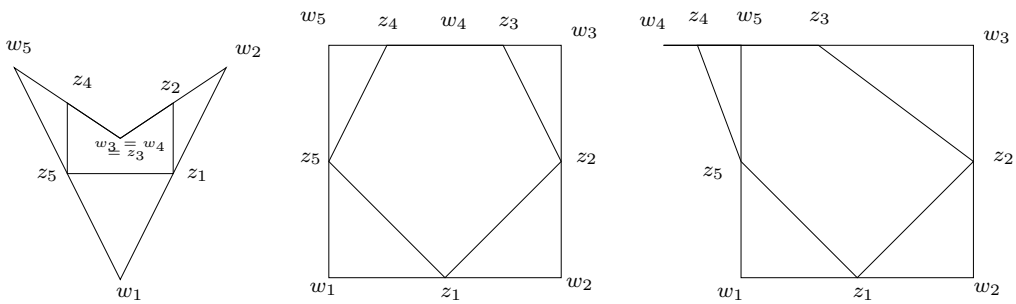

Figure 4: Degenerate dedal polygons: loss of a vertex, angle π, and angle 2π.

nondegenerate. We will not dwell on this aspect.

In this article we study the dedal n-gon(s) Q of an n-gon P. Our main results are the following. If n is odd then its dedal n-gon exists and is unique. For n even we give a necessary and sufficient condition for the existence of dedal n-gons and describe the space of all dedal n-gons of P. Then we go on to characterize regular and affinely regular n-gons by similarity to their dedal n-gons. Finally we give a complete description of the dynamics of the *developing map* $\mu(Q) := P$. The proofs of all our results boil down to some linear algebra of the dedal map.

After we wrote this article one of the anonymous referees pointed out that iteration of the developing map has already been studied by Berlekamp, Gilbert, and Sinden in 1965 [2]. They answer a question they attribute to G. R. MacLane, namely, they prove that for almost every polygon Q there exists an $M \geq 1$ such that $\mu^M(Q)$ is convex. Note that the image of a convex polygon is convex; thus this implies that for almost every Q there exists an M such that $\mu^m(Q)$ is convex for all $m \geq M$.

Suppose $Q(w_1, \dots, w_n)$ is a dedal polygon of $P(z_1, \dots, z_n)$. The definition implies that

$$z_i = (w_i + w_{i+1})/2$$

(see Figures 3, 4, and 5). The linear transformation $\mu : \mathbb{C}^n \to \mathbb{C}^n$ given by $\mu(w_1, \dots, w_n) = (z_1, \dots, z_n)$ is called the *developing map.*

The characteristic polynomial of μ is

$$(1 - 2x)^n - (-1)^n.$$

Its eigenvalues are $(1 + q^i)/2$ for $i = 0, 1, \dots, n-1$, where $q := \exp(2\pi i/n)$. All the eigenvalues differ from zero except for the $(n/2)$th eigenvalue when n is even. The ith eigenvector is $X_i := (1, q^i, q^{2i}, \dots, q^{(n-1)i})$. The vectors X_i form a basis of our space of polygons. If i divides n then X_i is a polygon with n/i sides. Nonetheless for the sake of clarity (avoiding stating special cases) we will think of this as an n-gon which is traced i times; for example if $n = 6$ then $X_2 = (1, q^2, q^4, q^6, q^8, q^{10}) = (1, q^2, q^4, 1, q^2, q^4)$ traces the triangle $(1, q^2, q^4)$ twice. An exception to this rule is the case n even and $i = n/2$. In this case $X_{n/2}$ is a segment which we do not consider as a polygon.

The following proposition clarifies the existence of dedal polygons.

Proposition 1 *a) Suppose that $n \geq 3$ is odd. Then for any n-gon P there is a unique dedal n-gon Q.*
b) If $n \geq 3$ is even then dedal n-gons exist if and only if the vertices of P satisfy $z_1 - z_2 + z_3 - \cdots - z_n = 0$. If this equation is satisfied then there is a unique dedal n-gon Q_0 in the space $X_{n/2}^{\perp} := \{z \in \mathbb{C}^n : \ z \cdot X_{n/2} = 0\}$. The set $D := \{Q_0 + sX_{n/2} : \ s \in \mathbb{C}\}$ consists of the dedal n-gons of P. In particular for each $i \in \{1, \dots, n\}$ every point $w \in \mathbb{C}$ is the ith vertex of a unique dedal n-gon $Q_i(w)$.

We remark that the condition $z_1 - z_2 + z_3 - \cdots - z_n = 0$ means that the center of mass of the even vertices coincides with the center of mass of the odd vertices.
Proof of Proposition 1. If $n \geq 3$ is odd then the map μ is invertible with

$$w_i = z_i - z_{i+1} + z_{i+2} - \cdots + z_{i-1},$$

and thus the dedal polygon exists and is unique.

On the other hand, if $n \geq 3$ is even then the map μ is not invertible. The kernel is one (complex) dimensional and is generated by the vector $X_{n/2} := (1, -1, 1, -1, \dots, 1, -1)$. Dedal polygons exist if and only if $z = (z_1, z_2, \dots, z_n)$ is in the range of μ, i.e., the space spanned by the X_i for $i \neq n/2$. The range and kernel of μ are orthogonal since if $i \neq n/2$ then

$$X_i \cdot X_{n/2} = (1 + q^{2i} + q^{4i} + \cdots + q^{(n-2)i}) - (q^i + q^{3i} + \cdots + q^{(n-1)i}) = 0 - 0 = 0.$$

Thus z is in the range of μ if and only if it satisfies $z \cdot X_{n/2} = 0$, or equivalently

$$z_1 - z_2 + z_3 - \cdots - z_n = 0. \tag{1}$$

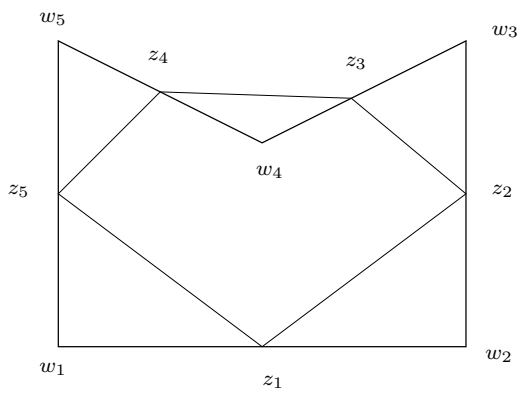

Figure 5: A pentagon without a Fagnano orbit. The unique dedal pentagon is pictured.

Alternatively, to see this note that

$$\begin{aligned} z_1 + z_3 + \cdots + z_{n-1} &= \tfrac{1}{2}\left((w_1 + w_2) + (w_3 + w_4) + \cdots + (w_{n-1} + w_n)\right) \\ &= \tfrac{1}{2}\left((w_2 + w_3) + (w_4 + w_5) + \cdots + (w_n + w_1)\right) = z_2 + z_4 + \cdots + z_n. \end{aligned}$$

The uniqueness of Q_0 follows since the map μ is invertible on the space $X_{n/2}^{\perp}$. The statement about the set D follows immediately since $X_{n/2}$ is the kernel of μ. Let $Q_0 := (w_1^0, \ldots, w_n^0)$. For each i, any point $w \in \mathbb{C}$ can be uniquely expressed as $w_i^0 + s$ for some $s \in \mathbb{C}$. $\square$

Suppose that a polygon Q without self intersection is a dedal polygon of P. If Q is convex then it is clearly a Fagnano orbit of P since by convexity the polygon P must be contained in Q. On the other hand if Q is not convex then it cannot be a Fagnano orbit since P cannot be contained in Q. In particular Fagnano orbits always exist for triangles, but not for polygons with more sides. An example of a pentagon without a Fagnano orbit is given in Figure 5. Although not every polygon has a Fagnano orbit, it does have a periodic orbit that is nearly as simple. Namely, consider the second iteration T^2 of the dual billiard map. Connecting the consecutive points of a periodic orbit of T^2 yields a polygon. Cutler has shown that the map T^2 has a periodic orbit which lies outside any compact neighborhood of P and the polygon constructed from the orbit makes a single turn about P [9]. The existence of a periodic orbit for the usual billiard in an arbitrary polygon remains open.

The set of polygons with center of mass at the origin is

$$\mathsf{C} := \{(w_1, \ldots, w_n) \in \mathbb{C}^n : \ w_1 + \cdots + w_n = \ 0\}.$$

The eigenvector $X_0 = (1, 1, \ldots, 1)$ represents a polygon which degenerates to a point. The set C is the orthogonal complement of X_0, i.e., $\mathsf{C} = \{w = (w_1, \ldots, w_n) \in \mathbb{C}^n : \ w \cdot X_0 = 0\}$. We will express polygons in the eigenbasis, i.e., we write $P = \sum_{i=0}^{n-1} a_i X_i$; the coefficients a_i are complex numbers.

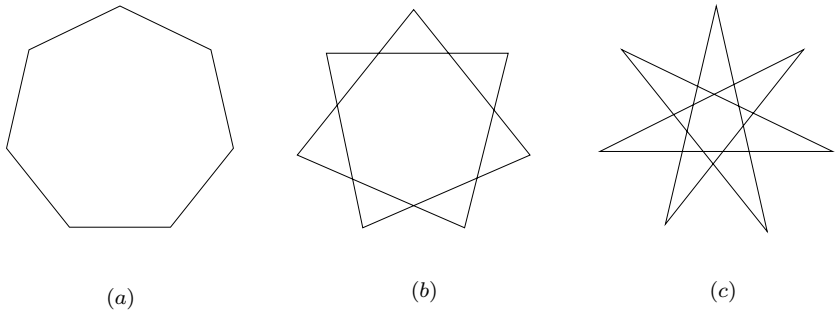

Figure 6: Up to orientation there are three regular 7-gons.

Since the map μ preserves the center of mass, throughout the rest of the article we assume that the center of mass is at the origin, i.e., $a_0 = 0$.

X_1 and X_{n-1} are the usual regular n-gon in counterclockwise and clockwise orientation (Figure 6a). If i and n are relatively prime and $i \notin \{1, n-1\}$ then X_i is star shaped and we also call it regular (Figures 6b and c). Finally if i divides n and $i \neq n/2$ then X_i is naturally a regular n/i-gon which, as mentioned above, we will think of as (a multiple cover of) a regular n-gon.

Two (unoriented) polygons are called *similar* if all corresponding angles are equal and all distances are increased (or decreased) in the same ratio. Since we study oriented ordered polygons we additionally want the marking of the vertices of P and Q to correspond, in which case we will call P and Q *$\star$-similar*. Thus two polygons $P = \sum_{i=1}^{n-1} b_i X_i$ and $Q = \sum_{i=1}^{n-1} a_i X_i$ are $\star$-similar if there exists a nonzero complex constant ℓ such that $b_i = \ell\, a_i$ for all i. We will also write this as $P = \ell\, Q$. Note that if P and Q are $\star$-similar then they are similar. On the other hand if P and Q are similar then P is $\star$-similar to a cyclic permutation $Q^{(k)} := (w_k, w_{k+1}, w_{k+2}, \ldots, w_{k-1})$ of Q or a cyclic permutation $\bar{Q}^{(k)} := (w_k, w_{k-1}, w_{k-2}, \ldots, w_{k+1})$ of Q with the opposite orientation.

In analogy to DeTemple and Robertson's result we have:

Theorem 2 *Fix $n \geq 3$. An n-gon P is regular if and only if it has a dedal polygon Q which is $\star$-similar to P.*

Note that if n is odd then Q is the unique dedal polygon of P, while if n is even then Q is the unique dedal polygon $Q_0 \in X_{n/2}^{\perp}$.

Proof. Suppose P is regular, i.e., there is a nonzero complex constant ℓ such that $P = \ell\, X_j$, where $j \neq n/2$ if n is even. If n is odd then let Q be the unique dedal polygon of P. If n is even then since P is regular it satisfies (1). Thus P has dedal polygons and we choose $Q = Q_0 \in X_{n/2}^{\perp}$. In both cases let $Q = \sum a_i X_i$. Since $P = \mu(Q) = \sum \frac{1+q^i}{2} a_i X_i$, we have $a_i = 0$ for $i \neq j$, i.e., $Q = a_j X_j$. Thus Q is $\star$-similar to $P = \ell\, X_j$.

Conversely suppose that $Q = \sum a_i X_i$ is a dedal polygon of $P = \sum b_i X_i$ and that P and Q are $\star$-similar, i.e., there is a nonzero complex constant ℓ such that $a_i = \ell\, b_i$. Since $P = \mu(Q)$ we have $\frac{1+q^i}{2} a_i = b_i$ for $i = 1, \ldots, n$. Combining this

with the $\star$-similarity yields $\frac{1+q^i}{2} = 1/\ell$ (a constant) for each i such that $a_i \neq 0$. It follows $q^i = q^j$ if $a_i \neq 0$ and $a_j \neq 0$. Since $q^i \neq q^j$ for $i \neq j$ it follows that only a single a_j is nonnull. Therefore $Q = a_j X_j$ and thus $P = \ell^{-1} a_j X_j$, and both are regular. Note that if n is even then $i \neq n/2$, since if $i = n/2$ then $(1+q^i)/2 = 0$ and thus $b_i = a_i = 0$. □

Of course we would like to have the analog of Theorem 2 with the usual notion of similarity. We call an n-gon *affinely regular* if there exists a j ($j \neq n/2$ when n is even) such that P is in the subspace A_j generated by X_j and X_{n-j}. This is equivalent to P being the image of a regular n-gon by an affine transformation, which explains the name.

Theorem 3 *a) Suppose $n \geq 3$ is odd. An n-gon P is affinely regular if and only if it has a dedal polygon Q which is similar to P.*
b) If $n \geq 4$ is even then an n-gon P appears in the following list of affinely regular polygons if and only if it has a dedal polygon Q which is similar to P.

i) Regular n-gons

ii) Any $P \in A_j$ such that there exists $k \in \{1, , \ldots, n\} \setminus \{n/2\}$ such that n divides $j(2k-1)$

iii) Any $P = b_j X_j + b_{n-j} X_{n-j} \in A_j$ such that there exists $k \in \{1, \ldots, n\}$ with $b_j/b_{n-j} = \pm q^{j(k+3/2)}$

All triangles are affinely regular. Berlekamp et al. noticed that every triangle is similar to its dedal triangle [2]. One would like to know if there are other special properties of the polygons in the list.
Proof. Suppose $Q^{(1)} = Q = \sum_{i=1}^{n-1} a_i X_i$. Then it is a simple exercise in linear algebra to verify that $Q^{(k)} = \sum a_i q^{i(k-1)} X_i$ and that $\bar{Q}^{(k)} = \sum a_{n-i} q^{i(k+1)} X_i$. Thus the similarity of P to Q is equivalent to the existence of a $k \in \{1, \ldots, n\}$ and a nonzero complex constant ℓ such that $b_i = \ell\, a_i q^{i(k-1)}$ for $i = 1, \ldots, n-1$ or $b_i = \ell\, a_{n-i} q^{i(k+1)}$ for $i = 1, \ldots, n-1$.

The structure of the proof is as follows. We group the even and odd cases together and start by proving that the various classes of affinely regular polygons are similar to their dedal polygons. Then we turn to the converse.

Suppose that n is odd and $P = b_j X_j + b_{n-j} X_{n-j}$ is affinely regular. Let $Q = Q^{(1)} = \sum a_i X_i$ be the unique dedal polygon of P. Since $P = \mu(Q) = \sum \frac{1+q^i}{2} a_i X_i$, we have $a_i = 0$ for $i \notin \{j, n-j\}$. We claim that there is a nonzero complex constant ℓ such that $P = \ell\, Q^{((n+3)/2)}$. To see this note that

$$1 + q^j = q^j(1+q^{-j}) = q^{j(n+1)}(1+q^{-j}) = q^{j(n+1)/2} q^{j(n+1)/2}(1+q^{-j})$$

or

$$\frac{1+q^j}{2q^{j(n+1)/2}} = \frac{1+q^{-j}}{2q^{-j(n+1)/2}}.$$

Thus $P = \ell\, Q^{((n+3)/2)}$ for $\ell := (1+q^j)/(2q^{j(n+1)/2})$.

The case n is even and P of class (ii) is similar. Consider the dedal polygon $Q = Q_0 \in X_{n/2}^{\perp}$. Again $P = \mu(Q)$ implies that $a_i = 0$ if $i \notin \{j, n-j\}$. We claim that there is a nonzero complex constant ℓ such that $P = \ell\, Q^{(k+1)}$. Since n divides $j(2k-1)$ we have

$$1 + q^j = q^j(1 + q^{-j}) = q^{j+j(2k-1)}(1 + q^{-j}) = q^{kj}q^{kj}(1 + q^{-j})$$

or

$$\frac{1 + q^j}{2q^{kj}} = \frac{1 + q^{-j}}{2q^{-kj}}.$$

Thus $P = \ell\, Q^{(k+1)}$ for $\ell := (1 + q^j)/(2q^{kj})$.

The case n even and P regular has already been treated in Theorem 2; therefore it remains to treat case (iii) of n even. We suppose P is of this form and want to show that P is similar to $\bar{Q}^{(k)}$, i.e., that there is a nonzero complex constant ℓ such that $b_i = \ell\, a_{n-i} q^{i(k+1)}$ for all i. As before $P = \mu(Q)$ implies that $a_i = 0$ if $i \notin \{j, n-j\}$. Combining the two relations, $b_i = \frac{1+q^i}{2} a_i$ and $b_j/b_{n-j} = \pm q^{j(k+3/2)}$ yields

$$b_j = \pm q^{j(k+3/2)} b_{n-j} = \pm q^{j(k+3/2)} \frac{1 + q^{-j}}{2} a_{n-j} = \pm q^{j(k+1)} \frac{q^{j/2} + q^{-j/2}}{2} a_{n-j}$$

and

$$b_{n-j} = \pm q^{-j(k+3/2)} b_j = \pm q^{-j(k+3/2)} \frac{1 + q^j}{2} a_j = \pm q^{-j(k+1)} \frac{q^{j/2} + q^{-j/2}}{2} a_j.$$

Choosing $\ell := \pm(q^{j/2} + q^{-j/2})/2 \in \mathbb{R}$ yields $P = \ell\, \bar{Q}^{(k)}$.

We turn now to the converse. We will first prove that for n even or odd if $Q = \sum a_i X_i$ is similar to $P = \sum b_i X_i$ then P is affinely regular.

We first treat that case when $P = \ell\, Q^{(k+1)}$ for some k and $\ell \in \mathbb{C} \setminus \{0\}$, i.e., $b_i = \ell\, a_i q^{ik}$. Since $P = \mu(Q)$ we have $\frac{1+q^i}{2} a_i = b_i$ for $i = 1, \ldots, n-1$. Note that if n is even then $(1 + q^{n/2})/2 = 0$, and thus $b_{n/2} = \frac{1+q^{n/2}}{2} a_{n/2} = 0$ and $a_{n/2} = \ell^{-1} q^{-nk/2} b_{n/2}$ is zero as well. For each i such that $a_i \neq 0$, combining the two relations between a_i and b_i yields $q^{-ik} \frac{1+q^i}{2} = \ell$. If only a single a_j is nonnull, then Q is regular, and since P is similar to Q it is regular as well. Now suppose that a_i and a_j are nonnull. Then the previous equation implies that

$$(1 + q^i)/(1 + q^j) = q^{k(i-j)}. \tag{2}$$

Taking absolute values yields $|(1 + q^i)| = |(1 + q^j)|$, which implies $i = \pm j$. Thus P is affinely regular.

If n is even we need to conclude more. Note that $(1 + q^{-j})/(1 + q^j) = q^{-j}$. Thus taking $i = -j$ in (2) implies $1 = q^{j(1-2k)}$. Thus $j(2k-1)$ is a multiple of n, i.e., we are in case (ii) of the list.

Finally suppose that $P = \mu(Q)$ and Q are similar but have the opposite orientation, i.e., $P = \ell\, \bar{Q}^{(k)}$ for some $\ell \in \mathbb{C} \setminus \{0\}$, or equivalently $b_i = \ell\, a_{n-i} q^{i(k+1)}$.

Combing this with $\frac{1+q^i}{2}a_i = b_i$ yields $\frac{1+q^i}{2}a_i = \ell\, a_{n-i}q^{i(k+1)}$. If n is even and $i = n/2$ then this equation implies that $a_{n/2} = 0$. For all other cases it implies that a_i and a_{n-i} are simultaneous zero or nonzero. For any i such that they are nonzero there are two such equations, and they imply

$$\frac{a_i}{a_{n-i}} = \frac{2\,\ell\, q^{i(k+1)}}{1+q^i} = \frac{1+q^{-i}}{2\,\ell\, q^{-i(k+1)}}. \tag{3}$$

This in turn yields

$$4\,\ell^2 = (1+q^i)\cdot(1+q^{n-i}) = 2 + q^i + q^{n-i} := f(i). \tag{4}$$

It is easy to see that $f(i) = f(j)$ if and only if $j = i$ or $j = n-i$. Since the left-hand side of Equation (4) does not depend on i, there is exactly one pair (a_j, a_{n-j}) of nonzero coefficients, i.e., Q is affinely regular. Since P is similar to Q it is also affinely regular.

Finally suppose that n is even. Equations (3) and (4) imply that for each j there exists $k \in \{1, 2\ldots, n\}$ such that

$$\frac{b_j}{b_{n-j}} = \pm\frac{\sqrt{f(j)}q^{j(k+1)}}{1+q^{-j}} = \pm\sqrt{\frac{1+q^j}{1+q^{-j}}}q^{j(k+1)} = \pm q^{j(k+3/2)},$$

i.e., we are in case (iii) of the list. □

Lemma 4 *If Q is affinely regular then $\mu^n(Q)$ is $\star$-similar to Q.*

Proof. Suppose $Q = a_jX_j + a_{n-j}X_{n-j}$. Then

$$\mu^n(Q) = \Big(\frac{1+q^j}{2}\Big)^n a_jX_j + \Big(\frac{1+q^{n-j}}{2}\Big)^n a_{n-j}X_{n-j}.$$

But

$$\begin{aligned}(1+q^{n-j})^n &= \sum_{k=0}^{n}\binom{n}{k}q^{(n-j)k} = \sum_{k=0}^{n}\binom{n}{n-k}q^{-jk} \\ &= \sum_{i=0}^{n}\binom{n}{i}q^{-j(n-i)} = \sum_{i=0}^{n}\binom{n}{i}q^{ji} = (1+q^j)^n.\end{aligned} \tag{5}$$

Thus $\mu^n(Q) = \ell\, Q$ with $\ell = (\frac{1+q^j}{2})^n$. □

In analogy to the billiard results stated in the introduction we now study the dynamics of μ. On the space C the dynamics are not very interesting: $\mu^m(Q) \to (0,\ldots,0)$ as $m \to \infty$ for all $Q \in \mathsf{C}$. To get a somewhat more interesting behavior notice that $\star$-similarity is an equivalence relation. Let $[Q] := \{\ell Q :\ \ell \in \mathbb{C}\setminus\{0\}\}$ denote the equivalence class of Q. By identifying $\star$-similar polygons we obtain

the quotient space $\hat{\mathsf{C}} := \{[Q] : \ Q \in \mathsf{C}\}$. If Q_1 and Q_2 are $\star$-similar then so are $\mu(Q_1)$ and $\mu(Q_2)$, and thus the map μ defines a map $\hat{\mu}$ of $\hat{\mathsf{C}}$ to itself.

We remark that the equivalence class $[Q]$ of a polygon $Q = \sum a_i X_i$ can be represented by the vector $(a_1, \dots, a_{n-1}) \in \mathbb{C}^{n-1} \setminus \{(0, \dots, 0)\}$ with the identification $(a_1, \dots, a_{n-1}) \equiv (\ell a_1, \dots, \ell a_{n-1})$ with $\ell \in \mathbb{C} \setminus \{0\}$. Thus the quotient space $\hat{\mathsf{C}}$ is naturally identified with the complex projective space $\mathbb{CP}^{n-2}$.

Let S be the unit sphere in C, i.e., the set of all $Q = \sum a_i X_i \in \mathsf{C}$ such that $\sum |a_i|^2 = 1$. Each equivalence class $[Q]$ intersects the unit sphere S in a circle, i.e., $[Q] \cap \mathsf{S} = \{\ell Q : |\ell| = 1/\sum |a_i|^2\}$. We define $\mathrm{dist}([P], [Q]) = \inf\{d(P_0, Q_0) : P_0 \in [P] \cap \mathsf{S}, Q_0 \in [Q] \cap \mathsf{S}\}$, where $d((a_1, \dots, a_{n-1}), (b_1, \dots, b_{n-1})) = (\sum |a_i - b_i|^2)^{1/2}$ is the Euclidean distance on S. It is a simple exercise to verify that this defines a metric on $\hat{\mathsf{C}}$.

A $\hat{\mu}$-invariant set $A \subset \hat{\mathsf{C}}$ is an *exponential attractor* with basin B for $\hat{\mu}$ if there exists $c, \gamma > 0$ such that $\mathrm{dist}(\hat{\mu}^{\mathrm{m}}([\mathrm{Q}]), \mathrm{A}) \leq \mathrm{c} \exp(-\gamma \mathrm{m})$ for all $m \geq 0$ and all $[Q] \in B$. If $B = \hat{\mathsf{C}}$ then we say that A is a *global* exponential attractor.

For each $j \in \{1, \dots, \lceil n/2 \rceil - 1\}$ let $B_j \subset \mathsf{C}$ be the subspace generated by $\{X_j, X_{j+1}, \dots, X_{n-j}\}$. Let $B_{\lceil n/2 \rceil} := \emptyset$, and $\mathrm{Aff} := \cup_{j=1}^{\lceil n/2 \rceil - 1} A_j$. For any subset $D \subset \mathsf{C}$ let $\hat{D} := \{[P] : P \in D\}$. We will only use this notation for sets D which are maximal in the sense that if $\hat{D} = \hat{E}$ then $E \subset D$.

Theorem 5 *For each $j \in \{1, \dots, \lceil n/2 \rceil - 1\}$ the set $\hat{A}_j$ is an exponential attractor with basin $\hat{B}_j \setminus \hat{B_{j+1}}$ and thus $\hat{\mathrm{Aff}}$ is a global exponential attractor for $\hat{\mu}$. The map $\hat{\mu}$ is n-periodic (i.e., $\hat{\mu}^n = Id$) on each $\hat{A}_j$, and thus on $\hat{\mathrm{Aff}}$.*

Proof. Suppose $Q \in B_j \setminus B_{j+1}$. Equation (5) implies that the polygon $\mu^m(Q) = \sum_{i=1}^{n-1} (\frac{1+q^i}{2})^m a_i X_i = \sum_{i=j}^{n-j} (\frac{1+q^i}{2})^m a_i X_i$ is $\star$-similar to

$$\left(\frac{2}{1+q^j}\right)^m \mu^m(Q) = a_j X_j + \sum_{i=j+1}^{n-j-1} \left(\frac{1+q^i}{1+q^j}\right)^m a_i X_i + \left(\frac{1+q^{n-j}}{1+q^j}\right)^m a_{n-j} X_{n-j}.$$

Since $|(1+q^i)/(1+q^j)| < 1$ the terms in the middle sum are exponentially small. Thus $\hat{\mu}^m([Q])$ is exponentially close to the equivalence class $[P_m] \in \hat{A}_j$ of the polygon $P_m := a_j X_j + \left[(1+q^{n-j})/(1+q^j)\right]^m a_{n-j} X_{n-j}$. The fact that $\hat{\mathrm{Aff}}$ is a global attractor follows from this since $\hat{\mathsf{C}}$ is the union of the $\hat{B}_j$. Lemma 4 immediately implies that $\hat{\mu}$ is periodic on each $\hat{A}_j$. $\square$

Acknowledgments. Many thanks to Sergei Tabachnikov and the two anonymous referees for helpful remarks.

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