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► **To cite this version:**

Vincent Cossart, Olivier Piltant. RESOLUTION OF SINGULARITIES OF THREEFOLDS IN POSITIVE CHARACTERISTIC II. *Journal of Algebra*, 2009, 321 (7), pp.1836-1976. 10.1016/j.jalgebra.2008.11.030 . hal-00139445v2

**HAL Id: hal-00139445**

**<https://hal.science/hal-00139445v2>**

Submitted on 17 Nov 2008

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# RESOLUTION OF SINGULARITIES OF THREEFOLDS IN POSITIVE CHARACTERISTIC II.

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*Dédié à Baptiste le petit ange.*

## Brief survey of resolution of singularities in positive characteristic.

Resolution of singularities for algebraic varieties defined over a field of positive characteristic has been a long standing open problem. Several programs for proving Resolution in all dimensions have appeared in the last few years. Although none of them has been completed to this date, encouraging partial results have been obtained.

Building on classical material mostly due to Zariski and Abhyankar, as well as on subsequent works due to Hironaka, Giraud, and on Cossart's thesis, we present a complete proof of Resolution of singularities for algebraic varieties of dimension three defined over any ground field  $k$  which is differentially finite over a perfect field  $k_0$  of characteristic  $p > 0$  (this restriction means that  $\Omega_{k/k_0}^1$  has finite dimension).

Three basic approaches to the Resolution of singularities of curves appeared in the nineteenth Century. A possibly singular germ of irreducible curve can be viewed alternately as: a covering of a regular germ (Riemann), an integral domain  $D$  of dimension one, essentially of finite type over the ground field (Dedekind), or a geometric object  $C$  defined by variables and equations vanishing at a certain order at the singular point (M. Noether). Corresponding approaches to the study of the singularity respectively consist in: studying the local fundamental group of the pointed line, the normalization of  $R$ , or the effect of a quadratic transform on the order of the equations. While the last two approaches give a proof of Resolution which is characteristic free, the first one does not, due to the failure of the Puiseux theorem in positive characteristic.

For surfaces (in positive characteristic), the question was raised, but not solved by Zariski. Zariski systematically introduced valuation theory in the study of singularities. In particular, he introduced the Local Uniformization problem, a weaker form of Resolution of singularities at the center of a given valuation. He proved that Local Uniformization implies Resolution for surfaces of arbitrary characteristic.

The first proof was given by Abhyankar [A1] in 1956. Namely, all characteristic zero proofs at that time used either the structure of the local fundamental group of the complement of a normal crossings curve in a regular surface (Jung [J], Walker [W]), or the Tschirnhausen transform killing the degree  $m - 1$  term in a polynomial of degree  $m$  by a linear change of variable. The

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(\*) Partially supported by program Ayudas a la Movilidad de Profesores of the University of Valladolid

corresponding first approach fails in characteristic  $p > 0$  for similar reasons as in dimension one, and the second fails if  $m$  is divisible by  $p$ . Abhyankar's idea was to use ramification theory of valuations to reduce the Local Uniformization of valuations to that of Artin-Schreier coverings of a regular germ of surface. Then he could prove directly Local Uniformization in this case by controlling the behaviour under quadratic transform of a certain monomial in the Newton polygon of the singularity. All of these ideas have exerted a strong influence on our proof [CP1].

Refinement and extensions of Abhyankar's theorem include the proof of Embedded Resolution (Abhyankar [A2], Hironaka [CGO, Appendix]) for surfaces, and the extension of Resolution of singularities to any excellent surface (Lipman [L]).

In dimension three, the only result available up to date was also due to Abhyankar [A2] in 1966. However, this result was restricted to algebraically closed ground fields and excluded the small characteristics  $p = 2, 3, 5$ . The reason of this last restriction comes from an argument of Albanese used by Abhyankar: one first constructs a birationally equivalent model to the singularity which has multiplicity at most 6. When  $p > 6$ , Tschirnhausen transforms can be applied successfully to the local equations of this model, and one essentially reduces to Embedded Resolution of surfaces, previously proved by Abhyankar.

In his characteristic zero proof, Hironaka [H] introduced the very important notion of directrix. This is the minimal space of linear forms necessary to write down the initial forms of the equation of a singularity. Its importance is due to the following: when Hironaka's main invariant (Hilbert-Samuel function  $HS$ ) does not strictly drop along his blowing up procedure, the dimension  $\tau$  of this space does not decrease. Even more, the directrix appears as the space of initial forms of the equations of a *regular* space  $W$  which has maximal contact with the singularity; there is a new singularity defined on  $W$  whose resolution will make  $(HS, -\tau)$  strictly decrease for the lexicographic ordering. The latter point is crucial in Hironaka's proof.

This is no more true in characteristic  $p > 0$ . Giraud [Gi1] showed that the appropriate space to be considered in positive characteristic is the Ridge. The Ridge is the minimal space of *additive* forms (linear combinations of  $p^\alpha$ -powers of variables with scalar coefficients) necessary to write down the initial forms of the equations. He proves in [Gi2] the existence of a certain space  $W$  with the above properties except one:  $W$  is no more regular, and Hironaka's proof does not extend.

Another important tool introduced by Hironaka [H1] is the characteristic polyhedron. This generalizes Newton polygons of plane curves to all dimensions and to arbitrary subschemes. Given a germ of subscheme, Hironaka's construction consists in choosing a concrete projection, transverse to the directrix, which minimizes the induced projection of the Newton polyhedron associated with the subscheme. The image of the Newton polyhedron by the resulting projection is called the characteristic polyhedron. Hironaka used this tool to build his proof of Embedded Resolution for surfaces. Cossart [5], Moh [Mo] used it to prove Resolution of singularities for threefolds with local equation  $y^p + f(u_1, u_2, u_3) = 0$ ,  $f \in k[[u_1, u_2, u_3]]$ .

A new approach to the Resolution problem was initiated by de Jong [dJ]. He proved that any algebraic variety has a covering which admits Resolution. The composed map is called an alteration and induces a finite extension of the function field. One can furthermore get this extension to be separable if the ground field is perfect. This method leads to a new proof of (birational) Resolution of singularities in characteristic zero (Bogomolov-Pantev [BP] and Abramovich-de Jong [AdJ]), which unfortunately does not extend to positive characteristic.

The Local Uniformization problem has been attacked by Spivakovsky, Teissier [T] and F.V. Kuhlmann [K], [KK]. Spivakovsky began a systematic study of the graded algebra associated with a valuation  $\nu$  centered in the local ring  $R$  of a singularity. This graded ring  $\text{gr}_\nu R$  is not Noetherian

in general, but has a simple algebraic structure. When  $k$  is algebraically closed and equal to the residue field of the valuation,  $\text{gr}_\nu R$  can be defined by countably many generators and countably many binomial relations.

Teissier considers a specialization of  $\text{Spec} R$  to  $\text{gr}_\nu R$ , analogous to the specialization to the normal cone. Then resolving the special fiber by a toric morphism should extend to a resolution of the general fiber, which is the singularity itself.

Using his structure theorems about valued fields, Kuhlmann succeeded in giving a valuation theoretic proof of the valuative version of de Jong's theorem. He also pointed out the essential difficulty caused by the defect in finite extensions of valued fields. This defect is nontrivial when the fundamental inequality of ramification theory  $n \geq \sum e_i f_i$  is not an equality.

In the last few years, Hironaka [H4], Kawanoue [Ka] and Villamayor [V] initiated programs to resolve singularities in positive characteristic. One common tool in these approaches is that of differential operators of higher order (these already appear in Hironaka and Giraud). These operators act on regular functions of the ambient space of a given subscheme. One can define an associated graded algebra  $\mathfrak{P}(E)$ , where  $E = (I, b)$  is an idealistic exponent,  $I$  the ideal of the subscheme,  $b \geq 0$  a weight. Then the approaches differ. Hironaka proves the finiteness of  $\mathfrak{P}(E)$ . Kawanoue saturates in addition  $\mathfrak{P}(E)$  by taking roots and obtains a different finiteness theorem for the resulting algebra. Bravo and Villamayor (in preparation) construct by projection from  $\mathfrak{P}(E)$  an "elimination algebra" on a *regular* space of smaller dimension than that of the ambient space of the singularity.

## INTRODUCTION

The purpose of this article is to prove the main theorem below. This will prove conjecture **3.1** of [CP1]. Let us recall a definition.

**Definition.** *Let  $k$  be a field characteristic  $p > 0$  and  $S$  be a regular local ring of dimension three, essentially of finite type over  $k$  and such that  $K := QF(S)$  has transcendence degree 3 over  $k$ . An Artin-Schreier (resp. purely inseparable) singularity of dimension three over  $S$  is the spectrum of a local ring  $\bar{R}$  of the form*

$$\bar{R} := (S[X]/(h))_{(X, u_1, u_2, u_3)}, \quad h := X^p - g^{p-1}X + f, \quad (1)$$

where  $m_S := (u_1, u_2, u_3)$  is the maximal ideal of  $S$ ,  $h$  is irreducible over the quotient field of  $S$  and  $f, g \in m_S$ ,  $g \neq 0$  (resp.  $f \in m_S$ ,  $g = 0$ ). We denote by  $\eta$  the morphism

$$\eta : X_0 := \text{Spec}(\bar{R}) \longrightarrow \text{Spec}(S).$$

**Main theorem.** *Let  $k$  be a field of positive characteristic which is differentially finite over a perfect field  $k_0$ , i.e.  $\Omega_{k/k_0}^1$  has finite dimension.*

*Let  $S$  be a regular local ring of dimension three, essentially of finite type over  $k$  and such that  $K := QF(S)$  has transcendence degree 3 over  $k$ . Let  $\bar{R}$  be an Artin-Schreier or purely inseparable singularity of dimension three over  $S$ .*

*Let  $K := QF(S)$  and  $L := QF(\bar{R})$  (in particular  $L/K$  is a finite field extension).*

Then, each  $k$ -valuation  $\mu$  of  $L$  dominating  $\bar{R}$  and satisfying properties (i) and (ii) below has a local uniformization:

- (i)  $\mu$  has rank one and  $\kappa(\mu)/\kappa(S)$  is algebraic;
- (ii)  $\mu$  is the unique extension of its restriction to  $K$ .

In [CP1], it is proved that this theorem gives an affirmative answer to conjecture [CP1, **3.1**] that we recall below.

**Theorem.** *Let  $k$  be a field of positive characteristic which is differentially finite over a perfect field  $k_0$  and  $Z/k$  be a reduced quasiprojective scheme of dimension three with singular locus  $\Sigma$ . There exists a projective morphism  $\pi : \tilde{Z} \rightarrow Z$ , such that*

- (i)  $\tilde{Z}$  is regular.
- (ii)  $\pi$  induces an isomorphism  $\tilde{Z} \setminus \pi^{-1}(\Sigma) \simeq Z \setminus \Sigma$ .
- (iii)  $\pi^{-1}(\Sigma) \subset \tilde{Z}$  is a divisor with strict normal crossings.

We let  $R := S[X]_{(X, m_S)}$ ,  $X_0 = \text{Spec}(R/(h))$ ,  $x_0$  his closed point,  $\mathfrak{M} = (X, m_S)$ , and  $k(x_0) = R/\mathfrak{M}$  is a finitely generated field extension of  $k$ . We denote by  $(u_1, u_2, u_3)$  a regular system of parameters (r.s.p. for short) of  $S$ , so  $\mathfrak{M} = (X, u_1, u_2, u_3)$ .

We denote by  $\Sigma(X_0)$  (resp.  $\Sigma_p(X_0)$ ) the singular locus (resp. the locus of multiplicity  $p$ ) of  $X_0$ , omitting reference to  $X_0$  when the context is clear.

Let  $R_\mu$  be the valuation ring of  $\mu$  and  $\mathfrak{M}_\mu$  its maximal ideal. By assumption (i), the group of values of  $\mu$  is *Archimedean*, and the center  $y$  of  $\mu$  in any model  $Y/k$  of  $L$  is always a closed point. Note the following consequence of assumption (ii):  $(X_0, x_0)$  is analytically irreducible. Namely,  $(X_0, x_0)$  is analytically reduced because  $S$  is an excellent ring ([Ma] section 32); any two distinct irreducible factors of  $h$  in  $\hat{R}$  induce distinct extensions of  $R_\mu \cap S$  to  $R/(h) = \bar{R}$ .

From Hironaka's theory of maximal contact [Gi2], and from resolution of singularities in dimension two, it is enough to build some local hypersurface model  $(X', x')$  of  $K(X_0)$  such that  $\mu$  is centered in  $(X', x')$  and  $x' \notin \Sigma_p(X')$ . When such a model has been constructed, we say that "the local uniformization problem is solved for  $\mu$ ". This model will be constructed by a sequence of birational transformations which are either blowing ups of  $X_0$  along regular centers (chapter 1, **II.4.6** and **II.5.1**) or blowing ups along regular cylinders over the base  $\text{Spec}S$  (chapter 1, **III**).

A complete proof of the existence of such  $(X', x')$  is given in the following four chapters.

**Along all this article, we assume that**

$$\text{ord}_{x_0}(h) = p.$$

## GENERAL OVERVIEW OF THE PROOF

In chapter 1, we introduce our main invariant  $\iota(x) := (\text{ord}_x h, \Omega(x))$  at any point  $x \in X_0$ . To begin with, it can be assumed that  $X$  is nonsingular away from  $\eta^{-1}(E)$ ,  $E$  a divisor with normal crossings. So  $\text{Spec}(S)$  is endowed with a logarithm structure  $\Omega_{S|k_0}^1(\log E)$ . Then  $\Omega(x) = (\omega(x), \omega'(x)) \in \mathbf{N} \times \{1, 2, 3\}$  is built up from certain Jacobian ideals  $J(f, E)$  [5] when the Hironaka characteristic polyhedron  $\Delta(h; u_1, u_2, u_3; X) \subseteq \mathbb{R}_{\geq 0}^3$  associate with (1) is minimal [H1].

The case  $\omega(x) = 0$  is easily dealt with in **II.4.6** by a simple combinatorial algorithm (Hironaka's game). When  $\omega(x) > 0$ , the refinement  $\omega'(x) \in \{1, 2, 3\}$  essentially stores in the information that  $\omega(x)$  is computed from  $g$  ( $\omega'(x) = 1$ ), from  $f$  alone ( $\omega'(x) = 2$ ) or from both  $f$  and  $g$  ( $\omega'(x) = 3$ )

in equation (1). The main point is our definition of *permissible* blowing ups for  $\Omega(x)$  and which is non-increasing by such blowing ups (section **II.5**). Our notion of permissible blowing ups is more restrictive than that of Hironaka (i.e. regular and equimultiple centers for the hypersurface  $X_0$ ). The main difficulty which is overcome here is to get some hold on the transformation laws for Hironaka's characteristic polyhedra under blowing ups which are permissible in our sense. The cases  $\omega'(x) = 3$  (theorem **II.5.6**) and  $\omega'(x) = 1$  (theorems **I.1** and **I.2.7** of chapter 2) are easily dispatched once this behavior has been understood. In these cases, there exists some form of maximal contact for  $\iota(x)$ .

In chapter 2, we begin the definition of a secondary invariant  $\kappa(x) \in \{0, 1, 2, 3, 4, 5, 6\}$  which is a *multivalued* function. The case  $\kappa(x) \in \{0, 1\}$  (**I.2.3** and **I.2.5**) corresponds to Abhyankar's good points: some reasonable algorithm makes  $\Omega(x)$  drop.

In the remaining case  $\omega'(x) = 2$ , we cannot produce any notion of maximal contact for  $\iota(x)$  in general. The proof is then casuistic: we define  $\kappa(x) \in \{2, 3, 4, 5, 6\}$  in terms of the equation (1) and the strategy is to drop the smallest value of the multi-valued function  $(\iota(x), \kappa(x))_{\text{lex}}$ . The proof goes from chapter 2 section **II** to the end of this article.

Section **II** of chapter 2 recollects several cases where  $\kappa(x) \in \{0, 1\}$ .

In chapter 3 (resp. chapter 4), we define and resolve  $\kappa(x) \in \{2, 3, 4\}$  (resp.  $\kappa(x) \in \{5, 6\}$ ). Roughly speaking, the cases  $\kappa(x) \in \{2, 3, 4\}$  (resp.  $\kappa(x) \in \{5, 6\}$ ) mean that our Jacobian ideal  $J(f, E)$  is transverse (resp. tangent) to  $E$  in a certain sense.

It is worth noting that in the case  $\kappa(x) = 3$ , we are lead to use *nonpermissible* blowing ups to prepare the base  $\text{Spec}S$ .

We now include a detailed summary of the proof.

**CHAPTER 1: Invariants and blowing ups.**

In this chapter, we define our main invariant  $\iota(x_0) := (\text{ord}_{x_0}(h), \Omega(x_0))$  at the center  $x_0 \in X_0 := \text{Spec}(\bar{R})$  of  $\mu$ . We have  $\Omega(x_0) := (\omega(x_0), \omega'(x_0)) \in \mathbb{N} \times \{1, 2, 3\}$  (definition **II.4**).

The case  $\omega(x_0) = 0$  is resolved by a simple combinatorial algorithm. This means that **we can make  $\text{ord}_{x_0}(h)$  strictly decrease at the center of  $\mu$  (II.4.6).**

When  $\omega(x_0) \geq 1$ , we define a notion of permissible center and prove that  $\iota(x_0)$  does not increase by permissible blowing-ups (**II.5**).

Finally, it is proved that  $\iota(x_0)$  can be strictly decreased when  $\omega'(x_0) = 3$  (**II.5.6**).

**I.** It can be assumed that  $X_0$  is regular away from  $\eta^{-1}(E)$ ,  $E$  normal crossings divisor on  $\text{Spec}(S)$  and  $\text{g.c.d.}_E(f, g^p) \neq 1$ .

**II.1 to II.1.5.** Introduction of Hironaka's characteristic polyhedron  $\Delta(h; u_1, u_2, u_3; X) \subset \mathbb{R}_{\geq 0}^3$  at any point  $x \in X_0$  not necessarily closed. Associated invariants:  $\delta(x) \in \frac{1}{p}\mathbb{N}$ ,  $H(x) := \text{g.c.d.}_E(f, g^p)$  and the directrix.

**II.2 to II.2.3.** Hironaka's characteristic polyhedron  $\Delta(h; u_1, u_2, u_3; X) \subset \mathbb{R}_{\geq 0}^3$  requires coordinates in  $\widehat{R}$  the formal completion of  $R$ . However, its initial face and our invariant  $\delta(x)$  can be computed with coordinates in  $R$ , whenever  $\eta(Y)$  ( $Y = \{\bar{x}\}$ ) has normal crossings with  $E$  (proposition **II.2.2**).

**II.3 to II.4.3.** We define  $J(f, E)$ , the ideal generated by the coefficients of  $H(x)^{-1}df \in \Omega_{S/k_0}^1(\log E)$ . The main invariant is:

$$\omega(x) := \min\{\text{ord}_x(J(f, E), H(x)^{-1}g^p)\},$$

where the minimum is taken over all possible  $(X, u_1, u_2, u_3)$  computing Hironaka's characteristic polyhedron and  $E \subset \text{div}(u_1 u_2 u_3)$  (**II.4**). Another invariant is

$$\epsilon(x) := \text{ord}_x(H(x)^{-1}f, H(x)^{-1}g^p),$$

and we always have  $\epsilon(x) - 1 \leq \omega(x) \leq \epsilon(x)$  (**II.3.3**).

The invariant  $\text{ord}_x(H(x)^{-1}f, H(x)^{-1}g^p)$  does not depend on the choice of  $(X, u_1, u_2, u_3)$  as above, but  $\text{ord}_x(J(f, E), H(x)^{-1}g^p)$  may depend on it (**II.3.3.1** and **II.3.3.2**). This phenomenon is encoded in  $\omega'(x) \in \{1, 2, 3\}$ : when  $\text{ord}_x(J(f, E), H(x)^{-1}g^p)$  depends on choices of coordinates, we let  $\omega'(x) = 3$ . Otherwise, we let  $\omega'(x) = 1$  if  $\omega(x) = \text{ord}_x(H(x)^{-1}g^p)$  and  $\omega'(x) = 2$  if  $\omega(x) < \text{ord}_x(H(x)^{-1}g^p)$  (**II.4**). We denote

$$\Omega(x) = (\omega(x), \omega'(x)).$$

**II.4.4 to II.4.7.** The case  $\omega(x_0) = 0$  is solved. The set  $W := \{x \in \Sigma_p \mid \omega(x) \geq 1\}$  is Zariski closed of dimension at most 1 (**II.4.7**).

**We assume that  $\omega(x_0) \geq 1$  from this point on.**

**II.5 to II.5.2.** We define permissible centers for our invariant  $\Omega$ . A center  $Y \subset X_0$  with generic point  $y$  is permissible if it is permissible in Hironaka's sense ( $Y$  regular and  $\text{ord}_{x_0}(h) = \text{ord}_Y(h) = p$ ),  $\eta(y)$  has normal crossings with  $E$  and satisfies one of the following conditions:

- (i)  $\epsilon(x_0) = \epsilon(y)$  (first kind)
- (ii)  $\omega(x_0) = \epsilon(x_0) - 1 = \epsilon(y)$  plus some extra transversality condition (second kind).

This is defined in **II.5.1**. The point  $x_0$  is always permissible of the first kind.

**II.5.3** and **II.5.3.1**. The condition  $H(x_0) \neq 1$  of **I** has an essential consequence: If  $(X, u_1, u_2, u_3)$  computes Hironaka's characteristic polyhedron and  $E \subset \text{div}(u_1 u_2 u_3)$ ,  $e : X_1 \rightarrow X_0$  is the blowing up along a permissible (in Hironaka's sense) center  $Y$ , then  $\text{ord}_{x'}(h') < p$ , whenever  $x'$  is not on the strict transform of  $\text{div}(X)$  where  $h'$  is the strict transform of  $h$  (**II.5.3**(i)).

The condition  $H(x_0) \neq 1$  of **I** is not stable under permissible blowing up, but we can solve directly the local uniformization problem for  $\mu$  whenever we lose this condition after performing a permissible blowing up (**II.5.3**(ii)).

**We assume that  $H(x_0) \neq 1$  from this point on.**

**Important remark.** In spite of this statement, we emphasize that there exists in general no choice of  $X$  in such way that  $\text{div}(X)$  has maximal contact with respect to the invariant  $\iota$ . In particular, one needs to minimize polyhedra after each performed blowing up.

**II.5.3.2.** This theorem is fundamental and repeatedly used along this article; resolution in dimension 2 heavily relies on this result. Whenever one performs a permissible blowing up  $e : X_1 \rightarrow X_0$  and pick  $x' \in X_1$  on the strict transform of  $\text{div}(u_1)$ , we can apply this result to get an estimate on  $\omega(x')$ .

**II.5.4** to **II.5.5**. Proof that  $\iota(x') := (\text{ord}_{x'}(h'), \Omega(x')) \leq \iota(x_0)$  if  $e : X_1 \rightarrow X_0$  is a permissible blowing up. Any point  $x' \in X_1$  with  $e(x') = x_0$  and  $\iota(x') = \iota(x_0)$  is said to be very near  $x_0$ .

**II.5.6** to **II.5.7**. We prove that  $\iota(x_0)$  can be strictly decreased after a finite number of closed point blowing ups when  $\omega'(x_0) = 3$ .

**II.6** to **II.6.2**. If  $H(x_0) \neq 1$ , the function  $\omega$  is uppersemicontinuous on  $\Sigma_p(X_0)$ .

**We assume that  $\omega'(x_0) \leq 2$  from this point on.**

**III** In those situations analysed in chapter 3 **II** below, we use nonpermissible blowing ups of a special type described in this section.

**Important remark.** In all that follows, it is always assumed that  $\Delta(h; u_1, u_2, u_3; X)$  is minimal. Chapter 3 **II** and our definitions of  $\kappa(x) = 2, 3, 4, 5, 6$  (chapters 3 and 4) rely on the shape of the expansion of  $f$  under this assumption.

## CHAPTER 2: A few easy cases.

*Note that  $X$  now denotes some iterated blowing up of  $X_0$  and  $x \in X$  the center of  $\mu$  in  $X$ .*

*We prove that  $\iota(x_0)$  can be strictly decreased by a finite number of permissible blowing ups when  $\omega'(x_0) = 1$  (theorem **I** and **I.2.7**).*

*We introduce a new invariant  $\kappa$  with  $\kappa(x) \in \{0, 1, \dots, 6\}$ . The case  $\kappa(x) \leq 1$  corresponds to Abhyankar's good points and we give a few examples with  $\kappa(x) \leq 1$  (**I.2.3** and **I.2.5**) and section **II**.*

**I.1** to **I.2.7**. To solve the case  $\omega'(x) = 1$ , we argue on the number of irreducible components of the normal crossing divisor  $H(x)^{-1}g^p$ . This is easy (theorem **I.1**) unless this number is 1, say  $\text{div}(H(x)^{-1}g^p)_{\text{red}} = \text{div}(u_1)$ . In this remaining case, we project Hironaka's characteristic polyhedron  $\Delta(h; u_1, u_2, u_3; X) \subset \mathbb{R}_{\geq 0}^3$  on the  $(u_2, u_3)$  plane and use Hironaka's invariants for dimension 2 resolution of singularities which we control under permissible blowing ups (**I.2** to **I.2.2**). This works straightforwardly because there is maximal contact with  $\text{div}(u_1)$  for the invariant  $\iota$ .

We introduce the multivalued function  $\kappa$ . We have  $\kappa(x) = 0$  (resp.  $\kappa(x) = 1$ ) if  $\iota(x)$  strictly drops after performing a finite number of closed point blowing ups (resp. of permissible blowing ups) (definitions **I.2.3** and **I.2.5**).

We prove that  $\omega'(x) = 1$  implies  $\kappa(x) \leq 1$  using the above invariants (**I.2.4** to **I.2.7**).

**We assume that  $\omega'(x) = 2$  from this point on.**

**II.1.1 to II.1.4.** We define a notion of directrix adapted to our invariant  $\iota$  and permissible blowing ups: if  $e : X_1 \rightarrow X$  is a permissible blowing up of the first kind, any  $x' \in X_1$  with  $\iota(x') = \iota(x)$  ( $x'$  very near  $x$ ) maps to the projective space  $\text{PDir}(x)$  associated to the directrix  $\text{VDir}(x)$ . We denote by  $\tau'(x)$  the codimension of  $\text{VDir}(x)$ .

**II.1, II.3 and II.4.** These examples of  $\kappa(x) \leq 1$  are fundamental preliminaries of chapter 3: they will be used to prove further on that the case  $\kappa(x) \leq 2$  is stable by blowing up closed points.

**II.5.** This lemma classifies some cases when  $\tau'(x) = 3$ , hence  $\kappa(x) = 0$  by **II.1.4**.

### CHAPTER 3: Resolution when there is transverseness.

*This chapter is devoted to some cases where  $\text{VDir}(x)$  is not contained in  $\langle \{U_i \mid \text{div}(u_i) \subset E\} \rangle$ , which we refer to as the “transverse case”. Unfortunately, transverseness is not even stable by blowing up closed points.*

We introduce subcases of transverseness called  $\kappa(x) = 2$  (**I**, definition **I.1**),  $\kappa(x) = 3$  (**II**, definition **II.1.1**),  $\kappa(x) = 4$  (**III**, definition **III.2**).

We have that  $(\iota(x), \kappa(x))$  can be strictly decreased in each of these cases: theorem **I.8** for  $\kappa(x) = 2$ ; definition **II.1.3** and **II.7** for  $\kappa(x) = 3$ ; definitions **III.3** and **III.4**, propositions **III.5** and **III.6** for  $\kappa(x) = 4$ .

#### I Resolution of the case $\kappa(x) = 2$ .

**I.1.** Definition of  $\kappa(x) = 2$ . We have  $\kappa(x) = 2$  if:

(i)  $\epsilon(x) = \omega(x)$  and  $\text{VDir}(x) \not\subset \langle \{U_i \mid \text{div}(u_i) \subset E\} \rangle$ .

Unfortunately, this is not stable by blowing up closed points, so we have to include as well the following case:

(ii)  $\epsilon(x) = 1 + \omega(x)$  plus some cross derivative condition.

**I.1.1 to I.4.** Using those “easy examples” of  $\kappa(x) \leq 1$  in chapter 2, one reduces  $\kappa(x) = 2$  to three subcases (\*1),(\*2) and (\*3) of definition **I.1.2** (corollary **I.3**).

**We assume that this condition (\*) holds up to the end of section I** (remark **I.4**).

**I.5 to I.6.3.** By (\*), we have  $E \subseteq \text{div}(u_1 u_2)$ . We project Hironaka’s characteristic polyhedron  $\Delta(h; u_1, u_2, u_3; X) \subset \mathbb{R}_{\geq 0}^3$  on the  $(u_1, u_2)$  plane. This projection can be minimized over all choices of the  $X$  and  $u_3$  coordinates, and we call this minimizing “well preparedness of variables” (definitions **I.5.1.1** and **I.5.1.2**).

Then our invariants are essentially Hironaka’s invariants for dimension two resolution of singularities computed for the projection of  $\Delta(h; u_1, u_2, u_3; X)$  (section **I.5**). The main invariant is denoted  $\gamma(x) \in \mathbb{N}$  (definition **I.5.2.3**).

Existence of well prepared variables is proved in proposition **I.6**.

**I.7.** Well prepared coordinates may be chosen in such a way that every very near point  $x' \in X_1$  in the blowing up  $X_1 \rightarrow X$  of  $x \in X$  lies on the strict transform of  $\text{div}(u_3)$  or  $\kappa(x) \leq 1$ .

**Important remark.** In spite of this statement, we emphasize that there exists in general no choice of  $(X, u_3)$  in such way that  $\text{div}(u_3)$  has maximal contact with respect to our invariants. In particular, one needs to well prepare again coordinates after each performed blowing up.

**I.8.** Statement of the main theorem:  $\iota(x)$  can be strictly decreased whenever  $\kappa(x) = 2$ .

**I.8.1.** Proof of **I.8** when  $\gamma(x) = 0$ .

**We assume that  $\gamma(x) \geq 1$  up to the end of section I.**

**Important remark.** The following is proved below. Let  $e : X_1 \longrightarrow X$  be the blowing up of  $x$ . Then, either  $\kappa(x) \leq 1$  or

$$\kappa(x') = \kappa(x) = 2, \text{ (*) holds at } x' \text{ and } \gamma(x') \leq \gamma(x). \quad (\Gamma)$$

Unfortunately, none of cases (\*1),(\*2) or (\*3) is stable by blowing up closed points, nor eventually gets stable by blowing up enough; this produces extra technical difficulties.

**I.8.2 to I.8.3.** Control of  $\gamma(x)$  under blowing up the closed point  $x$  in case (\*1) or (\*2) and when  $x'$  belongs to the first chart of the blowing up  $X_1$  (i.e. away from the strict transform of  $\text{div}(u_1)$ ).

( $\Gamma$ ) is proved in **I.8.3(a)(b)(c)(d)**. All other statements in **I.8.3** give sharper estimates of the invariants which appear along the proof. These estimates will be used to study the cases where  $\gamma(x') = \gamma(x)$ .

Some very special case (Dis) appears when  $\omega(x) = p = 2$  which requires further work (theorem **I.10** below).

**I.8.4 to I.8.6.** In some (\*3) cases, blowing up a permissible curve of the second kind gives  $\kappa(x) \leq 1$ . This result is included there to simplify **I.8.7** and **I.8.9**.

**I.8.6.1 to I.8.7.** Control of  $\gamma(x)$  under blowing up the closed point  $x$  in case (\*3) and when  $x'$  belongs to the first chart of the blowing up  $X_1$ .

( $\Gamma$ ) is proved in **I.8.7(a)(b)(d)**. All other statements in **I.8.7** give sharper estimates of the invariants which appear along the proof. These estimates will be used to study the cases where  $\gamma(x') = \gamma(x)$ .

**I.8.8.** Control of  $\gamma(x)$  under blowing up the closed point  $x$  in case (\*1) or (\*2) and when  $x'$  is at infinity in the blowing up  $X_1$ : proof of ( $\Gamma$ ) in this case.

**I.8.9.** Control of  $\gamma(x)$  under blowing up the closed point  $x$  in case (\*3) and when  $x'$  is at infinity in the blowing up  $X_1$ : proof of ( $\Gamma$ ) in this case.

**I.9.** Proof of **I.8** when  $\gamma(x) = 1$  plus some extra condition in case (\*3).

**I.10.** Proof of **I.8** when  $\gamma(x) = 2$ ,  $x$  is in case (\*1) plus some extra condition. This theorem deals with those special cases that appeared in **I.8.3(e)** and **I.8.7(b)** (in particular the case (Dis)).

**Important remark.** Synthesis of **I.8.2 to I.10**.

The invariant  $\gamma(x_i)$  takes a constant value  $\gamma(\mu)$  for  $i \gg 0$  where

$$X_o \longleftarrow X_1 \longleftarrow \dots \longleftarrow X_i \longleftarrow \dots$$

is the quadratic sequence along  $\mu$  and  $x_i \in X_i$  its center, or  $\kappa(x) \leq 1$ .

**I.11 to I.11.4.** If  $\gamma(\mu) = 1$  or if  $\gamma(\mu) \geq 3$ , the estimates on  $\gamma(x)$  proved above imply the existence of a formal curve  $\mathcal{C} \subset X$ ,  $\mathcal{C} \not\subset E$  whose strict transform contains all the  $x_i$ 's,  $i \geq 0$ . This is a contradiction since  $\mathcal{C}$  is not contained in the singular locus  $\Sigma$  of  $X$  (remember that  $\Sigma \subseteq \eta^{-1}(E)$ , chapter 1 **I**).

If  $\gamma(\mu) = 2$ , we reduce successively in **I.11.4** to lemmas **I.11.1**, **I.11.2** and **I.11.3**. We eventually get the existence of a formal curve  $\mathcal{C} \subset X$  as above - a contradiction - unless  $\kappa(x) \leq 1$ .

## II Resolution of the case $\kappa(x) = 3$ .

**II.1 and II.1.1.** Definition of  $\kappa(x) = 3$ . This other important case of transverseness which is not contained in  $\kappa(x) = 2$  goes as follows:

$$E \subseteq \text{div}(u_1 u_2) \text{ and } (H(x))^{-1} \frac{\partial f}{\partial u_3} \equiv (u_3^{\omega(x)}) \pmod{(u_1, u_2)}.$$

We may have  $\epsilon(x) = \omega(x)$  or  $\epsilon(x) = 1 + \omega(x)$ . Necessarily  $1 + \omega(x) \neq 0 \pmod{p}$ .

We project Hironaka's characteristic polyhedron  $\Delta(h; u_1, u_2, u_3; X) \subset \mathbb{R}_{\geq 0}^3$  on the  $(u_1, u_2)$  plane. This projection can be minimized over all choices of the  $X$  and  $u_3$  coordinates and we call this minimizing "well preparedness of variables".

Then our invariants are essentially Hironaka's invariants for dimension two resolution of singularities computed for the projection of  $\Delta(h; u_1, u_2, u_3; X)$ . The main invariant is denoted  $\gamma(x) \in \mathbb{N} - \{0\}$  (**II.1**).

Existence of well prepared variables is proved in **II.1**.

**Important remarks.**

Unfortunately,  $\kappa(x) = 3$  is not stable by blowing up a closed point. We will use certain non permissible blowing ups in this section which preserve the Artin-Schreier or purely inseparable structure of  $h$ , the assumption  $\eta(\Sigma) \subseteq E$ ,  $H(x) \neq 1$  and do not increase  $(\iota(x), \kappa(x))_{\text{lex}}$ .

The strategy is as follows: list some cases where  $\kappa(x) = 3$  can be reduced to  $\kappa(x) \leq 2$  by permissible blowing ups (**II.1.2** to **II.2.2**). Then, in the general case of  $\kappa(x) = 3$ , perform a sequence of nonpermissible blowing ups to reduce to one of these previous cases (**II.3** to **II.7**).

**II.1.2** to **II.2.2**. Well prepared coordinates may be chosen in such a way that every very near point  $x' \in X_1$  in the blowing up  $X_1 \rightarrow X$  of  $x \in X$  lies on the strict transform of  $\text{div}(u_3)$  (**II.1.2**).

We have  $\kappa(x) \leq 1$  in some cases where the projection of Hironaka's characteristic polyhedron has only one vertex (**II.2**).

**II.3**. If  $\kappa(x) = 3$  and  $e : X_1 \rightarrow X$  is the blowing up along  $\eta^{-1}(V(u_1, u_2))$  and  $x \in e^{-1}(x)$  then,  $\iota(x) \leq \iota(x')$ ,  $\kappa(x') \leq \kappa(x) = 3$  and  $\gamma(x') \leq \gamma(x)$ . Statements (ii) to (vii) study the behaviour of auxiliary invariants.

**II.4** to **II.5**. Proof that  $\kappa(x) \leq 1$  in some cases where  $\gamma(x) = 1$ .

**II.6**. Sharpening of **II.2**: we have  $\kappa(x) \leq 1$  whenever the projection of Hironaka's characteristic polyhedron has only one vertex.

**II.7**. Those nonpermissible blowing ups considered in **II.3** allow to drop the invariant  $\gamma(x)$  to a minimal value  $\gamma(\mu)$ . The case  $\gamma(\mu) \geq 2$  is dispatched on **II.7.1**, **II.7.2**. The case  $\gamma(\mu) = 1$  is reduced to three different cases **II.7.4**, **II.7.5** or **II.7.6**.

**III End of transverseness.**

**III.1** to **III.2**. The most general case of transverseness is defined in **III.2**. Once  $\kappa(x) = 2$  and  $\kappa(x) = 3$  have been solved, the only remaining case of transverseness is:

$$E = \text{div}(u_1) \text{ and } \text{ord}_{(u_2, u_3)}(H(x)^{-1} \frac{\partial f}{\partial u_3} \text{ mod } (u_1)) = \omega(x).$$

This remaining case is reduced to lemma **III.1**. Note that the assumptions of **III.1** do not imply  $\kappa(x) \leq 4$  rather there is an algorithm to reduce this case to  $\kappa(x) \leq 3$ .

**III.3** to the end. Assume  $\kappa(x) = 4$ ,  $x$  is good if a finite sequence of permissible blowing ups makes  $(\iota(x), \kappa(x))$  strictly drop (**III.3**).

Assume that  $\kappa(x) = 4$ . We let  $\bar{\tau}(x) := \tau(\text{cl}_{\omega(x)} J(f, E) + (\{U_i \mid \text{div}(u_i) \subseteq E\})) \geq 2$  (**III.4**).

Assume that  $\kappa(x) = 4$ . There are two different cases:  $\bar{\tau}(x) = 2$  (**III.5**) or  $\bar{\tau}(x) = 3$  (**III.6**). In both cases,  $x$  is good.

**Important remark.**  $\kappa(x) = 4$  is not stable under permissible blowing ups; in a sequence of permissible blowing ups  $X = X_0 \leftarrow X_1 \leftarrow \dots \leftarrow X_i \leftarrow \dots \leftarrow X_n$ , it may happen that for  $0 < i < n$ ,  $\kappa(x) > 4$ , but then,  $x_i$  will verify the assumptions of **III.1**.

**CHAPTER 4: Resolution when there is tangency**

Tangency is the complement case of transverseness, i.e. all remaining cases are called tangent. In **III**, tangency is reduced to the case  $\kappa(x) = 5$ . When  $\kappa(x) = 5$ , we prove that  $(\iota(x), \kappa(x))_{\text{lex}}$  can be strictly decreased by permissible blowing ups (theorem **II.1** and definition **I.1**).

**I.1** Definition of  $\kappa(x) = 5$ . We have  $\kappa(x) = 5$  if  $\text{div}(u_1) \subseteq E$  and one of the following conditions holds:

$$\epsilon(x) = \omega(x), (H(x)^{-1}f) \equiv (u_1^{\omega(x)}) \pmod{(u_2, u_3)},$$

$$\epsilon(x) = 1 + \omega(x), E \subseteq \text{div}(u_1 u_2) \text{ and } (H(x)^{-1} \frac{\partial f}{\partial u_3}) \equiv (u_1^{\omega(x)}) \pmod{(u_2, u_3)}.$$

$x$  is said to be good if a finite sequence of permissible blowing ups makes  $(\iota(x), \kappa(x))_{\text{lex}}$  strictly drop.

**I.2** to **I.2.4**. We project Hironaka's characteristic polyhedron  $\Delta(h; u_1, u_2, u_3; X)$  on the  $(u_2, u_3)$  plane.

Then our invariants are essentially Hironaka's invariants for dimension two resolution of singularities computed for the projection of  $\Delta(h; u_1, u_2, u_3; X)$  (section **I.5**). The main invariant is denoted  $\gamma(x) \in \mathbb{N}$  (definition **I.2.4**).

**I.3** to **I.3.3**. If  $e : X_1 \rightarrow X$  is the blowing up of a closed point and  $x' \in e^{-1}(x)$ , then  $(\iota(x'), \kappa(x')) \leq_{\text{lex}} (\iota(x), \kappa(x))$  and, if equality holds,  $x'$  lies on the strict transform of  $\text{div}(u_1)$  (maximal contact) (**I.3**).

Same result as in **I.3** for the blowing up of a permissible curve (with some extra assumption when  $\epsilon(x) = 1 + \omega(x)$ ) (**I.3.3**).

**I.4** to **I.4.1**. If  $e : X_1 \rightarrow X$  is the blowing up of a closed point with  $\omega(x) = \epsilon(x)$  and  $x' \in e^{-1}(x)$  in the first chart with  $(\iota(x'), \kappa(x')) = (\iota(x), \kappa(x))$ , then  $\gamma(x') \leq \gamma(x)$  and control of auxiliary invariants.

**I.4.2**. When  $\omega(x) = \epsilon(x)$  and the projection of  $\Delta(h; u_1, u_2, u_3; X)$  has only one vertex plus some extra conditions,  $x$  is good.

**I.5**. If  $e : X_1 \rightarrow X$  is the blowing up of the closed point with  $1 + \omega(x) = \epsilon(x)$  and  $x' \in e^{-1}(x)$  in the first chart with  $(\iota(x'), \kappa(x')) = (\iota(x), \kappa(x))$ , then  $\gamma(x') \leq \gamma(x)$  and control of auxiliary invariants.

**I.5.1**. Some cases where  $x$  is good and  $\gamma(x) \leq 1$ ,  $1 + \omega(x) = \epsilon(x)$ .

**I.5.2** and **I.5.3**. Blowing up of the closed point  $x$ , case where  $x'$  is at infinity:  $(\iota(x'), \kappa(x'), \gamma(x')) \leq (\iota(x), \kappa(x), \gamma(x))$  and control of auxiliary invariants ( $\omega(x) = \epsilon(x)$  in **I.5.2**,  $1 + \omega(x) = \epsilon(x)$  in **I.5.3**).

**I.6**. Some cases where  $x$  is good and  $\gamma(x) \leq 1$ .

**I.7**. Sharpening of **I.4.2**: when  $\omega(x) = \epsilon(x)$  and the projection of  $\Delta(h; u_1, u_2, u_3; X)$  has only one vertex,  $x$  is good.

## II End of the case $\kappa(x) = 5$

**II.1**. Statement of the theorem. Let  $x \in \Sigma_p$  with  $\kappa(x) = 5$ , then  $x$  is good.

We perform the quadratic sequences along  $\mu$ . We suppose that  $(\iota(x), \kappa(x))_{\text{lex}}$  does not strictly decrease. As usual, there is a minimal value  $\gamma(\mu)$  of  $\gamma(x_i)$  for  $i \gg 0$ .

**II.2**. Proof of **II.1** when  $m(x_i) = 2$  for  $i \gg 0$ .

**II.3**. Proof of **II.1** when  $m(x_i) = 3$  for  $i \gg 0$ .

**Till the end of II, we assume that neither the assumption of II.2, nor II.3 is satisfied.**

**II.4 to II.4.2.** For  $i \gg 0$ ,  $x_{i+1}$  is rational over  $x_i$  and  $\gamma(\mu) \leq 2$ . Furthermore  $\gamma(\mu) \leq 1$  implies  $x$  is good.

**II.4.3.** Proof of **II.1** when  $\gamma(\mu) = 2$ .

**III End of the proof of the main theorem.**

**III.1.**  $\kappa(x) = 6$  means “no expansion of  $h$  gives  $\kappa(x) \in \{2, 3, 4, 5\}$ ”.

$x$  is said to be good if the quadratic sequence along  $\mu$  makes  $(\iota(x), \kappa(x))_{\text{lex}}$  strictly drop.

**III.2.** Main result:  $\kappa(x) = 6$  implies  $x$  is good. The end is devoted to its proof.

**III.3.**  $\kappa(x) = 6$  implies  $\tau'(x) = 2$ . furthermore, if  $e : X_1 \rightarrow X$  is the blowing up of the closed point  $x$ , and  $(\iota(x'), \kappa(x')) = (\iota(x), \kappa(x))$ , then  $x'$  is rational over  $x$ .

**III.4 to III.5.1.** If  $\kappa(x) = 6$  and (either  $E = \text{div}(u_1)$ , or  $\text{VDir}(x) \not\subseteq \langle \{U_i | \text{div}(u_i) \subseteq E \} \rangle$ ), then  $x$  is good.

**III.5.2 and III.5.3.** We reduce to the case  $\text{div}(u_1 u_2) \subseteq E$ ,  $\text{VDir}(x) \subseteq \langle \{U_i | \text{div}(u_i) \subseteq E \} \rangle$  and  $\epsilon(x) = \omega(x)$ .

**III.6 to III.6.3.** Proof in this last case.

## CHAPTER 1: invariants and blowing ups.

In this chapter, we define our main invariant  $\Omega(x_0) \in \mathbb{N} \times \{1, 2, 3\}$  in **II.4**. When  $\Omega(x_0)$  is minimal, resolution is easily obtained by some combinatorial blowing up process in **II.4.6**. Otherwise, some notion of permissible blowing up w.r.t. this invariant is given in **II.5.1**.

The main results **II.5.4** and **II.6** respectively deal with the behavior of  $\Omega(x_0)$  by permissible blowing ups and upper-semicontinuity of  $\Omega(x_0)$  on  $\Sigma_p(X_0)$ . Before going that far, we introduce techniques of characteristic polyhedra due to Hironaka and differential invariants due to the first author and to his advisor Jean Giraud in section **II**, after performing some preparation of the singular locus  $\Sigma(X_0)$  in section **I**. Those nonpermissible blowing ups used in this article are described in **III**.

### I Preparation of the singular locus.

In the Artin-Schreier case  $g \neq 0$ , we can suppose that  $g$  is a monomial, i.e. that there exists a reduced normal crossings divisor  $E \subset \text{Spec}S$ , such that

$$g = \gamma \prod_{\text{div}(u_i) \subseteq E} u_i^{\beta_i}, \quad (1)$$

with  $\beta_i \geq 0$  and  $\gamma \in S$  invertible. Indeed, apply [CP] **8.1** with  $f_0 := g$ . The integers  $1 \leq e \leq j \leq 3$  are defined by:  $E = \text{div}(u_1 \cdots u_j)$ ,  $\beta_1 \cdots \beta_e \neq 0$  and  $\beta_{e+1} = \cdots = \beta_j = 0$ . We let  $E_0 := \text{div}(u_1 \cdots u_e)$ . Note that  $\eta: X_0 \rightarrow \text{Spec}S$  is étale away from  $\eta^{-1}E_0$ . In particular  $\Sigma \subseteq \eta^{-1}E_0$ .

In the inseparable case  $g = 0$ , let  $df$  be the image of  $f$  in  $\Omega_{S/k_0}^1$  (where  $k$  is differentially finite over the perfect field  $k_0$ ). We have  $df \neq 0$  since  $f \notin S^p$  (remember that  $h$  is irreducible). We pick  $\varphi \in S$ ,  $\varphi \neq 0$ , such that  $df$  does not vanish away from the set  $\{\varphi = 0\}$ . By [CP] **8.1**, it can be assumed that  $f_0 := \varphi$  is a monomial whose support is defined to be  $E = \text{div}(u_1 \cdots u_j)$ . We let  $\beta_i = \infty$  for  $1 \leq i \leq j$  and  $E_0 := E$  in this case. Like in the Artin-Schreier case, we have  $\Sigma \subseteq \eta^{-1}E_0$ . See **II.3.1** for an important consequence of the assumption.

Also note that, if  $g = 0$ , or if  $g \neq 0$  and  $s \in E_0$ , the fiber ring  $\text{Spec}(k(s) \otimes_S R/(h))$  is *local*, i.e.  $\eta^{-1}(s)$  is a single point.

From now on, we suppose that (1) holds if  $g \neq 0$ , or that  $E$  is defined as above if  $g = 0$  and that (unless stated otherwise) r.s.p.'s  $(u_1, u_2, u_3)$  of  $S$  are chosen according to the above convention on  $E_0$  and  $E$ . We then say that  $(u_1, u_2, u_3)$  is *adapted* to  $E$ . We let

$$f = \prod_{\text{div}(u_i) \subseteq E} u_i^{\alpha_i} f_0, \quad (2)$$

where  $f_0 \in S$  is not divisible by any  $u_i$  with  $\text{div}(u_i) \subseteq E$ . Finally, note that since  $(f, g) \subseteq m_S$ , it can also be assumed that  $(f, g) \subseteq (u_i)$  for some  $i$ ,  $1 \leq i \leq j$ .

### II The invariant $\omega$ .

As said before, we suppose that the center of  $\mu$  in  $X_0$  is a closed point, but, to prove some semi-continuity theorems, we have to define our invariants also at all points. In this section,  $x \in X_0$  is not necessarily the center  $x_0$  of  $\mu$ :  $x$  is a point such that  $x_0 \in \overline{\{x\}}$ . We always assume that  $x_0 \in \Sigma_p$  in this section, but do not necessarily assume that  $x \in \Sigma_p$ .

**II.1 Notations.** Let  $(u_1, \dots, u_n)$ ,  $n \leq 3$  be a r.s.p. of  $S_{\eta(x)}$  and  $E_x$  (resp.  $E_{0,x}$ ) be the stalk of  $E$  (resp.  $E_0$ ) at  $\eta(x)$ . Then  $(X, u_1, \dots, u_n)$  is a system of coordinates at  $x$ .

The associated polyhedron  $\Delta(h; u_1, \dots, u_n; X) \subseteq \mathbb{R}_{\geq 0}^n$  (projection on the  $(u_1, \dots, u_n)$ -space of  $1/p$  times the Newton polyhedron of  $h$  from the point  $(0, \dots, 0, 1)$ ) can be minimized by a “translation on  $X$ ”, i.e. by replacing  $X$  by  $Z := X + \psi$ ,  $\psi \in \widehat{S_{\eta(x)}}$  as in [H1]. This translation does not modify  $g$ , nor the vanishing locus of  $df \in \Omega_{S/k_0}^1$  if  $g = 0$ . Also note that  $\Delta(h; u_1, \dots, u_n; X) \neq \emptyset$  by definition if  $g \neq 0$ , and because  $S[X]/(h)$  is analytically reduced. For a given system of coordinates  $(X, u_1, \dots, u_n)$ , we denote

$$\delta(X, u_1, \dots, u_n) := \inf\{x_1 + \dots + x_n \mid (x_1, \dots, x_n) \in \Delta(h; u_1, \dots, u_n; X)\} < \infty.$$

We also maintain the writing  $f = \prod_{\operatorname{div}(u_i) \subseteq E_x} u_i^{\alpha_i} f_0$  of **I(2)**, allowing  $f_0 \in \widehat{S_{\eta(x)}}$  for arbitrary coordinates  $(X, u_1, \dots, u_n)$  on  $\widehat{S_{\eta(x)}}[X]$ .

**II.1.1 Definition of  $\delta$ .** *Assume that the polyhedron  $\Delta(h; u_1, \dots, u_n; X)$  is minimal. Then  $\delta(x) < \infty$  is defined by:*

$$\delta(x) := \inf\left\{\operatorname{ord}_{\eta(x)}(g), \frac{\operatorname{ord}_{\eta(x)}(f)}{p}\right\} \in \frac{1}{p}\mathbb{N}.$$

**II.1.2 Definition of  $d_i$ .** *Assume that the polyhedron  $\Delta(h; u_1, \dots, u_n; X)$  is minimal. Let  $d_i(x) := \inf\{\beta_i, \frac{\alpha_i}{p}\}$  for  $\operatorname{div}(u_i) \subseteq E_x$  and  $H(x) := \prod_{\operatorname{div}(u_i) \subseteq E_x} u_i^{p d_i(x)} \in S_{\eta(x)}$ .*

We extend the notation by setting  $d_i(x) = 0$  for  $\operatorname{div}(u_i) \not\subseteq E_x$ . Although all numerical invariants which have been associated with  $x$  depend on  $f$ , hence on the choice of minimal coordinates, we will see in **II.2** that they usually do not.

**II.1.3 Definition.** *Let  $k$  be a field,  $S_1$  a  $k$ -vector space of finite dimension and  $S = k[S_1]$  be the symmetric algebra. Let  $V := \operatorname{Spec} S$  and  $I$  be a homogeneous ideal of  $S$  which defines a cone  $C := \operatorname{Spec}(S/I)$ . Let  $\mathfrak{F}$  be the following subfunctor of the functor represented by  $V$ : for every  $k$ -scheme  $k'$ ,*

$$\mathfrak{F}(k') = \{v \in V(k') \mid L_v(C \times_k k') \subset C \times_k k'\},$$

where  $L_v$  is the translation defined by  $v$ , i.e.

$$L_v : V \times_k k' \longrightarrow V \times_k k', \quad L_v(v') = v + v'.$$

This functor is represented by a closed group subscheme  $F/k$  of  $V$  which is also a cone and called the ridge of  $V$ . The scheme  $F$  has for equations homogeneous additive polynomials with coefficients in  $k$ . By a theorem of Hironaka, the ridge is the larger group subscheme of  $V$  which leaves  $C$  stable by translations. See [Gil, prop 1.5.4] and [Gi2 1.5].

**II.1.4 Definition.** *With notations as above, the directrix  $\operatorname{VDir}(I)$  of  $C$  is the smallest  $k$ -vector subspace  $W$  of  $S_1$  such that  $I = (k[W] \cap I)S$ . We also denote  $\operatorname{IDir}(I) := \operatorname{VDir}(I)S$ ,  $\operatorname{PDir}(I) := \operatorname{Proj}(S/\operatorname{IDir}(I))$ .*

By Hironaka’s quoted theorem, the directrix is the smallest  $k$ -vector subspace  $\operatorname{Dir}(I)$  of  $S_1$  which generates an ideal containing the ideal of the ridge. If  $x \in X_0$ , we will denote by  $C_h(x) := \operatorname{Proj}(\operatorname{gr}_{m_x} S_x / \operatorname{in}_x h)$  the tangent cone of  $X_0$  at  $x$ .

**II.1.5 Notation.** *We denote by  $\tau(I)$  the codimension of  $\operatorname{VDir}(I)$  in  $S_1$ .*

**II.2 Proposition.** *The integer  $\delta(x)$  does not depend on any choice of coordinates  $(X, u_1, \dots, u_n)$  such that  $\Delta(h; u_1, \dots, u_n; X)$  is minimal.*

If  $\text{div}(u_1) \subseteq E_{0,x}$ ,  $d_1(x)$  does not depend on any choice of coordinates  $(X, u_1, \dots, u_n)$  such that  $E_{0,x} \subseteq \text{div}(u_1 \cdots u_n)$  and  $\Delta(h; u_1, \dots, u_n; X)$  is minimal. For any such choice of coordinates, the following holds: if  $V(h, u_1) = V(X, u_1)$ , we have  $d_1(x) = d_1(y) = \delta(y) > 0$ , where  $y$  is the generic point of  $V(X, u_1)$ ; we have  $d_1(x) = 0$  otherwise.

Proof. See [6] for a stronger form of this proposition that will not be needed here. Let us first prove that  $\delta(x)$  does not depend on the choice of coordinates minimizing the polyhedron of  $h$ . If there exists coordinates  $(X', u'_1, \dots, u'_n)$  such that  $\delta(X', u'_1, \dots, u'_n) > \delta(X, u_1, \dots, u_n)$ , then if we write  $h = X'^p - X'g^{p-1} + f'$ , we have  $f', g^p \in m_{S_{\eta(x)}}^{p\delta(X', u'_1, \dots, u'_n)}$ . So  $\delta(X', u_1, \dots, u_n) \geq \delta(X', u'_1, \dots, u'_n) > \delta(X, u_1, \dots, u_n)$ :  $\Delta(h; u_1, \dots, u_n; X)$  is not minimal, and this proves the first part of the proposition.

Note that the finite map  $V(h, u_1) \rightarrow \text{div}(u_1)$  is either an isomorphism or purely inseparable of degree  $p$  whenever  $\text{div}(u_1) \subseteq E_0$ .

If  $0 < d_1(x)$ , then  $(h, u_1)_{\text{red}} = (X, u_1)$ . By the lemma below, the value of  $d_1(x)$  computed in a given system of coordinates such that the polyhedron  $\Delta(h; u_1, \dots, u_n; X)$  is minimal satisfies  $d_1(x) = d_1(y) = \delta(y)$  where  $y$  is the generic point of  $V(X, u_1)$  and  $\Delta(h; u_1; X) = [\delta(y), +\infty[ \subset \mathbb{R}^+$  is minimal. But  $\delta(y)$  does not depend on the choice of minimal coordinates by the first part of the proposition and the conclusion follows.

If  $d_1(x) = 0$  and if  $V(h, u_1) \rightarrow \text{div}(u_1)$  is an isomorphism, then  $(h, u_1)_{\text{red}} = (X + \psi, u_1)$ ,  $\psi \in S$ . We replace  $X$  by  $X + \psi$ , and get that  $\Delta(h; u_1, \dots, u_n; X + \psi)$  does not contain  $\Delta(h; u_1, \dots, u_n; X)$ , which contradicts the minimality hypothesis. So  $d_1(x) = 0$  is equivalent to:  $V(h, u_1) \rightarrow \text{div}(u_1)$  is not an isomorphism (hence purely inseparable of degree  $p$ ).

**II.2.1 Lemma** (Semi-continuity of the characteristic polyhedron). *Assume that the polyhedron  $\Delta(h; u_1, \dots, u_n; X)$  is minimal. Let  $Y := V(X, u_1, \dots, u_i) \subset \text{Spec}R$  and  $y$  be the generic point of  $Y$  for some  $i$ ,  $1 \leq i \leq n$ . The polyhedron  $\Delta(h; u_1, \dots, u_i; X)$  is minimal, where  $h$  is viewed as an element of  $(\widehat{S_{\eta(x)}} \otimes_{S_{\eta(x)}} \widehat{S_{\eta(y)}})[X]$ . We have the equivalences:*

- (i)  $\inf\{x_1 + \dots + x_i \mid (x_1, x_2, \dots, x_n) \in \Delta(h; u_1, u_2, \dots, u_n; X)\} \geq 1 \Leftrightarrow Y \subseteq \Sigma_p$ ;
- (ii)  $\inf\{x_1 + \dots + x_i \mid (x_1, x_2, \dots, x_n) \in \Delta(h; u_1, u_2, \dots, u_n; X)\} > 1 \Leftrightarrow Y \subseteq \Sigma_p$  and  $\tau(I_Y S_{\eta(y)}) = 1$ .

Proof. In fact the polyhedron  $\Delta(h; u_1, \dots, u_i; X)$  is the image of  $\Delta(h; u_1, \dots, u_n; X)$  by the projection  $\pi$  on the  $(x_1, \dots, x_i)$ -space, so a vertex  $v = (x_1, \dots, x_i)$  of  $\Delta(h; u_1, \dots, u_i; X)$  is the projection  $\pi(w)$  of at least one vertex  $w = (x_1, \dots, x_n)$  of  $\Delta(h; u_1, \dots, u_n; X)$ .

Let us prove the first assertion. We write

$$h = X^p - X \sum_{x \in \mathcal{E}} \epsilon_x u_1^{(p-1)x_1} \cdots u_n^{(p-1)x_n} + \sum_{x \in \mathcal{E}} \phi_x u_1^{px_1} \cdots u_n^{px_n},$$

where

- (a)  $x = (x_1, \dots, x_n)$ ,  $\mathcal{E} \subset \Delta(h; u_1, \dots, u_n; X)$ ;
- (b)  $\mathcal{E}$  is finite and contains all vertices of  $\Delta(h; u_1, \dots, u_n; X)$ ;
- (c) each of  $\epsilon_x$  and  $\phi_x$  is either invertible or identically zero in  $\widehat{S_{\eta(x)}}$ .

Let  $v = (x_1, \dots, x_i)$  be a vertex of  $\Delta(h; u_1, \dots, u_i; X)$  and let us look at

$$X^p - X \sum_{x, \pi(x)=v} \epsilon_x u_1^{(p-1)x_1} \cdots u_n^{(p-1)x_n} + \sum_{x, \pi(x)=v} \phi_x u_1^{px_1} \cdots u_n^{px_n}.$$

If  $\sum_{x, \pi(x)=v} \epsilon_x u_1^{(p-1)x_1} \cdots u_n^{(p-1)x_n} \neq 0$ , then  $v$  is not solvable in  $\Delta(h; u_1, \dots, u_i; X)$ . We claim that if one vertex  $w = (x_1, \dots, x_i, w_{i+1}, \dots, w_n)$  with  $\pi(w) = v$  is not solvable over  $\widehat{S_{\eta(x)}}$ ,

then  $v$  is not solvable in  $\Delta(h; u_1, \dots, u_i; X)$ . Namely, otherwise we have  $x_1, \dots, x_i \in \mathbb{N}$  and we can find

$$\psi \in T := (\widehat{S_{\eta(x)}} / (u_1, \dots, u_i) \widehat{S_{\eta(x)}}) \otimes_{(S_{\eta(x)} / (u_1, \dots, u_i) S_{\eta(x)})} k(y)$$

such that

$$\psi^p \equiv \sum_{x, \pi(x)=v} \phi_x u_{i+1}^{px_{i+1}} \dots u_n^{px_n} \bmod (u_1, \dots, u_i) \widehat{S_{\eta(x)}} \otimes_{S_{\eta(x)}} \widehat{S_{\eta(y)}}.$$

Since  $S$  is regular and excellent, the fiber ring  $T$  is an integral domain ([Ma] section 32). Therefore there exists  $a \in \widehat{S_{\eta(x)}}$ ,  $b \in S_{\eta(x)}$  with  $ab \notin (u_1, \dots, u_i) \widehat{S_{\eta(x)}}$  such that

$$b^p \sum_{x, \pi(x)=v} \phi_x u_{i+1}^{px_{i+1}} \dots u_n^{px_n} \equiv a^p \bmod (u_1, \dots, u_i) \widehat{S_{\eta(x)}}.$$

Since  $w$  is a vertex of  $\Delta(h; u_1, \dots, u_n; X)$ , we must have

$$w_{i+1}, \dots, w_n \in \mathbb{N}, \text{ and } \phi_w \bmod m_{\widehat{S_{\eta(x)}}} \in k(x)^p,$$

so  $w$  is solvable: a contradiction. The last two equivalences are now straightforward.

At this point, we remark that  $d_i(x) = 0$  whenever  $\text{div}(u_i) \not\subseteq E_{0,x}$ . Otherwise,  $d_i(x)$  does not depend on the choice of  $x$  such that  $\text{div}(u_i) \subseteq E_{0,x}$  and is determined by **II.2**. From now on, we thus relax the notation by writing  $d_i$  instead of  $d_i(x)$ .

**II.2.2 Proposition.** *Let  $N \in \mathbb{N}$ ,  $\delta(x_0) \leq N < +\infty$  (resp.  $N = +\infty$ ). There exists  $X \in R$  (resp.  $X \in \widehat{R}$ ) such that  $E \subseteq \text{div}(u_1 u_2 u_3)$  and*

- (i)  $d_i = \inf\{x_i | (x_1, x_2, x_3) \in \Delta(h; u_1, u_2, u_3; X)\}$  whenever  $\text{div}(u_i) \subseteq E$ ;
- (ii) no vertex  $v = (x_1, x_2, x_3)$  of  $\Delta(h; u_1, u_2, u_3; X)$  with  $x_1 + x_2 + x_3 < N$  is solvable. In particular,  $\delta(x_0) = \inf\{x_1 + x_2 + x_3 | (x_1, x_2, x_3) \in \Delta(h; u_1, u_2, u_3; X)\}$ .

Furthermore, if for some r.s.p.  $(X', u_1, u_2, u_3)$  of  $R$ ,  $Y := V(X', u_1, u_2) \subseteq \eta^{-1}E$  is such that  $\eta(Y) = V(u_1, u_2)$  has normal crossings with  $E$ , we can choose  $X \in R$  (resp.  $X \in \widehat{R}$ ) such that moreover  $\Delta(h; u_1, u_2; X)$  is minimal and  $Y = V(X, u_1, u_2)$ .

Proof. We first point out that the formal version ( $X \in \widehat{R}$ ) is a consequence of **II.2.1** except when  $Y$  has been specified.

We start with some r.s.p.  $(X', u_1, u_2, u_3)$  of  $R$  such that  $E \subseteq \text{div}(u_1 u_2 u_3)$  and a writing  $h = X'^p - X'g^{p-1} + f \in S[X]$ ,  $f = \prod_{\text{div}(u_i) \subseteq E} u_i^{a_i} f_0 \in S$ , with  $a_i$  maximal for each  $i$ .

**II.2.2.1** Suppose that  $d_1 \neq \inf\{x_1 | (x_1, x_2, x_3) \in \Delta(h; u_1, u_2, u_3; X')\}$ . Then  $d_1 > 0$ , so that  $\text{div}(u_1) \subseteq E_0$  and  $V(h, u_1)$  is regular by **II.2**. If  $a_1 = 0$ , we can choose  $\psi \in S$  such that  $(h, u_1)_{\text{red}} = (X' + \psi, u_1)$  and get  $a_1 > 0$  after changing  $X'$  to  $X' + \psi$ . Note that  $u_2$  divides  $\psi$  if  $\text{div}(u_2) \subseteq E_0$  and  $a_2 > 0$ . Also note that if  $h \in (X', u_1, u_2)$  then  $\psi \in (X', u_1, u_2) \cap S = (u_1, u_2)$ . Finally, if  $h \in (X', u_2, u_3)$  then we can choose  $\psi \in (u_2, u_3)S$  satisfying  $f \equiv \psi^p \bmod (u_1)$  and  $(h, u_1)_{\text{red}} = (X' + \psi, u_1)$ . Therefore  $V(X', u_2, u_3) = V(X' + \psi, u_2, u_3)$  in this case. In other terms, it can be assumed that  $a_i > 0$  whenever  $\text{div}(u_i) \subseteq E_0$  and  $d_i > 0$ ; moreover we still have  $Y = V(X', u_1, u_2)$  in the new variables if  $Y = V(X', u_1, u_2)$  has been specified.

The polyhedron  $\Delta(h; u_1; X') = [a_1/p, +\infty[ \subset \mathbb{R}^+$  is not minimal, so  $\delta(X', u_1) = a_1/p \in \mathbb{N}$ ; the initial form of  $h$  with respect to the unique vertex  $\delta(X', u_1)$  is  $X'^p + U_1^{p\delta(X', u_1)} \Psi^p$  with  $\Psi \in QF(S/(u_1))$ . Since  $f \in S$ , we actually have  $\Psi \in S/(u_1)$ . Since  $\prod_{\text{div}(u_i) \subseteq E_0} u_i^{a_i}$  divides  $f$ , we can lift  $\Psi$  to  $\psi \in S$  in such a way that  $\prod_{\text{div}(u_i) \subseteq E_0} u_i^{a_i}$ ,  $i \neq 1$  divides  $\psi^p$ . Replacing  $X'$  with  $X'' := X' + u_1^{a_1/p} \psi$ , we get  $\Delta(h; u_1; X'') \subset \Delta(h; u_1; X')$  and  $\Delta(h; u_i; X'') \subseteq \Delta(h; u_i; X')$  whenever  $\text{div}(u_i) \subseteq$

$E_0$ ,  $i \neq 1$ . Moreover we can choose  $\psi \in S$  such that  $Y = V(X'', u_1, u_2)$  if  $Y = V(X', u_1, u_2) \subset \eta^{-1}E$  has been specified, as in the previous paragraph. By induction on  $\sum a_i$ , we achieve an expression  $h = X'^p - X'g^{p-1} + f \in S[X]$  with  $d_i = \inf\{x_i | (x_1, x_2, x_3) \in \Delta(h; u_1, u_2, u_3; X')\}$  whenever  $\text{div}(u_i) \subseteq E$ , that is,  $\Delta(h; u_i; X')$  is minimal whenever  $\text{div}(u_i) \subseteq E$  so that (i) holds.

**II.2.2.2** Now consider the case where  $Y := V(X', u_1, u_2) \subseteq \eta^{-1}E$  has been specified. The polyhedron  $\Delta(h; u_1, u_2; X')$  is the image of  $\Delta(h; u_1, u_2, u_3; X')$  by the projection  $\pi$  on the space of the  $(x_1, x_2)$ . Suppose that a vertex  $v = (x_1, x_2)$  of  $\Delta(h; u_1, u_2; X')$  is solvable, i.e. does not belong to the minimal polyhedron. This means that there exist  $x_1, x_2 \in \mathbb{N}$  such that the Newton polyhedron of  $f - (\mu u_1^{x_1} u_2^{x_2})^p$  is strictly contained in that of  $f$ , where  $\mu \in S$  is invertible in  $S_{(u_1, u_2)}$ . By the same argument as above, we can ensure  $\mu^p \in u_3^{pd_3} S$  in case  $\text{div}(u_3) \subseteq E$ . After changing  $X'$  with  $X' + \mu u_1^{x_1} u_2^{x_2}$ , we dissolve  $v$  without losing (i).

We claim that this algorithm is finite: if not, it would mean that we dissolve an infinite number of vertices with  $x_1$  or  $x_2$  minimal, say  $x_1$ . Since the polyhedron  $\Delta(h; u_i; X')$  is minimal whenever  $\text{div}(u_i) \subseteq E$ , we get that  $\text{div}(u_1) \not\subseteq E$ , and therefore  $\text{div}(u_2) \subseteq E$ . First suppose that  $x_1 = 0$ . Necessarily, we have  $g = 0$ , and there exists  $\psi \in S$  such that  $f \equiv \psi^p \pmod{u_1 S}$ . We can assume as above that  $\psi^p \in u_i^{pd_i} S$  whenever  $\text{div}(u_i) \subseteq E$ . Also we have  $\psi \in (u_1, u_2)S$ , since  $f \in (u_1, u_2)S$ . We then replace  $X'$  with  $X' + \psi$  to get  $x_1 > 0$  without losing (i). Now, assuming  $x_1 > 0$ , we have  $(f, g^p) \subseteq (u_1^p)$ , so that  $V(X', u_1) \subseteq \Sigma$ . Therefore  $\text{div}(u_1) \subseteq \eta(\Sigma) \subseteq E$  by definition of  $E$  in section **I**: a contradiction.

**II.2.2.3** We suppose that  $d_i = \inf\{x_i | (x_1, x_2, x_3) \in \Delta(h; u_1, u_2, u_3; X')\}$  whenever  $\text{div}(u_i) \subseteq E$  and that  $\Delta(h; u_1, u_2; X')$  is minimal if  $V(X', u_1, u_2) \subseteq \eta^{-1}E$  has been specified. Then, we start Hironaka's algorithm of vertex dissolution: if there is a vertex  $v = (x_1, x_2, x_3)$  of  $\Delta(h; u_1, u_2, u_3; X')$  which is solvable, there exists  $\lambda \in S$  invertible giving a translation on  $X'$ , say  $X = X' + \lambda u_1^{x_1} u_2^{x_2} u_3^{x_3}$ , with  $\Delta(h; u_1, u_2, u_3; X) \subset \Delta(h; u_1, u_2, u_3; X')$  and  $v$  is not a vertex of  $\Delta(h; u_1, u_2, u_3; X)$ . Note that by projection  $\Delta(h; u_i; X) \subseteq \Delta(h; u_i; X')$  whenever  $\text{div}(u_i) \subseteq E$ , and  $\Delta(h; u_1, u_2; X) \subseteq \Delta(h; u_1, u_2; X')$  if  $Y = V(X', u_1, u_2)$  is specified. Consequently, these projections are still minimal. Furthermore, if  $h \in (X', u_1, u_2)$ , we have  $V(X, u_1, u_2) = V(X', u_1, u_2)$ . We will get the algebraic version ( $X \in R$ ) of (ii) after a finite number of steps. In order to get  $\Delta(h; u_1, u_2, u_3; X)$  minimal, we may need infinitely many steps. Then we construct a series  $X = X' + \sum_x \lambda_x u_1^{x_1} u_2^{x_2} u_3^{x_3} \in \widehat{R}$  and this proves the formal version of the proposition.

**II.2.3 Definition.** We call *initial face* or *first face* of  $\Delta(h; u_1, u_2, u_3; X)$  (not necessarily minimal) the face of equation  $x_1 + x_2 + x_3 = \delta(X, u_1, u_2, u_3)$ .

**II.3 Adapted Jacobian ideals.** We remind that  $k_0$  is a perfect subfield of  $k$  and that  $\Omega_{k/k_0}^1$  has finite dimension. Then, for any r.s.p.  $(u_1, u_2, u_3)$  of  $S$ , a basis  $(d\lambda_1, \dots, d\lambda_s)$  of  $\Omega_{S/k_0}^1$  may be chosen so that

$$u_1 = \lambda_1, u_2 = \lambda_2, u_3 = \lambda_3. \quad (1)$$

The derivations  $\frac{\partial}{\partial \lambda_1}, \dots, \frac{\partial}{\partial \lambda_s}$  are defined by  $\frac{\partial \lambda_j}{\partial \lambda_i} = \delta_{i,j}$ ,  $1 \leq i, j \leq s$ . If (1) is true for  $d\lambda_1, \dots, d\lambda_s$ , we can take

$$\lambda_i \in S, \quad 4 \leq i \leq s, \quad (2)$$

such that  $S$  is unramified over  $k_0(\lambda_4, \dots, \lambda_s)[u_1, u_2, u_3]$ . In this case, the derivations  $\frac{\partial}{\partial \lambda_i}$ ,  $4 \leq i \leq s$  are so-called "derivations relative to constants".

For  $x \in \text{Spec} S$ , we extend the definition of  $H$  in **II.1.2** by writing  $H(x)$  for  $H(\tilde{x})$ , where  $\tilde{x}$  is any point of  $X_0$  such that  $\eta(\tilde{x}) = x$ . If  $x \in E_0$ ,  $\tilde{x}$  is uniquely determined as pointed out in **I**. On the other hand,  $H(x) = 1$  if  $x \notin E_0$ .

Let  $Y \subseteq E_0$  be a regular closed subset of  $\text{Spec}S$ , having normal crossings with  $E$ . We denote  $I(Y) = (\{u_i\}_{i \in I})$ ,  $I \subseteq \{1, 2, 3\}$ , where  $(u_1, u_2, u_3)$  is adapted to  $E$ . We let  $I_E := I \cap \{1, \dots, j\}$  (see **I** for the definition of  $j$ ) and  $I^c := \{1, 2, 3\} \setminus I$ .

Let us denote  $\mathcal{D} := \{D \in \text{Der}_{k_0}(S) \mid D(I(E)_{\text{red}}) \subseteq I(E)_{\text{red}}\}$ , where  $\text{Der}_{k_0}(S) = (\Omega_{S/k_0}^1)^\vee$  denotes the  $S$ -module of  $k_0$ -derivations of  $S$  into itself. We let  $\mathcal{D}(Y) := \{D \in \mathcal{D} \mid D(I(Y)) \subseteq I(Y)\}$ . More generally, for every point (not necessarily closed)  $x \in \text{Spec}S$ , we let  $\mathcal{D}(x) := \{D \in \mathcal{D} \mid D(I(x)) \subseteq I(x)\}$ . If  $\tilde{x} \in X_0$ , we also write  $\mathcal{D}(\tilde{x})$  for  $\mathcal{D}(x)$ , where  $\eta(\tilde{x}) = x$ .

It is easy to see that if (1) is true for  $d\lambda_1, \dots, d\lambda_s$ , then

$$\mathcal{D} = \sum_{1 \leq i_1 \leq j} S u_{i_1} \frac{\partial}{\partial u_{i_1}} + \sum_{j+1 \leq i_2 \leq 3} S \frac{\partial}{\partial u_{i_2}} + \sum_{4 \leq i_3 \leq s} S \frac{\partial}{\partial \lambda_{i_3}}, \quad (3)$$

$$\mathcal{D}(Y) = \sum_{1 \leq i_1 \leq j} S u_{i_1} \frac{\partial}{\partial u_{i_1}} + \sum_{i_2 \in I \setminus I_E} I(Y) \frac{\partial}{\partial u_{i_2}} + \sum_{i_3 \in I^c} S \frac{\partial}{\partial u_{i_3}} + \sum_{4 \leq i_4 \leq s} S \frac{\partial}{\partial \lambda_{i_4}}. \quad (4)$$

In particular, if  $Y = \{\eta(x_0)\}$ ,

$$\mathcal{D}(x_0) = \sum_{1 \leq i_1 \leq j} S u_{i_1} \frac{\partial}{\partial u_{i_1}} + \sum_{j+1 \leq i_2 \leq 3} (u_1, u_2, u_3) \frac{\partial}{\partial u_{i_2}} + \sum_{4 \leq i_3 \leq s} S \frac{\partial}{\partial \lambda_{i_3}}. \quad (5)$$

At every point (not necessarily closed)  $z \in \text{Spec}S$ , we let  $\mathcal{J}(f, E)_z := (\mathcal{D}f)S_z$ ,  $\mathcal{J}(f, E, Y)_z := (\mathcal{D}(Y)f)S_z$  and  $\mathcal{J}(f, E, x)_z := (\mathcal{D}(x)f)S_z$ . These definitions make  $\mathcal{J}(f, E)$ ,  $\mathcal{J}(f, E, Y)$  and  $\mathcal{J}(f, E, x)$  into sheaves of ideals on  $\text{Spec}S$  and we will usually omit the subscript  $z$  when the context is clear. Clearly,  $\mathcal{J}(f, E, Y)$  and  $\mathcal{J}(f, E, x)$  are subsheaves of  $\mathcal{J}(f, E)$ ,  $\mathcal{J}(f, E)_z$  and  $\mathcal{J}(f, E, Y)_z$  (resp.  $\mathcal{J}(f, E, x)_z$ ) coincide with  $S_z$  for  $z \notin E$  (resp.  $z \notin E \cup \{x\}$ ). If  $f = \prod_{\text{div}(u_i) \subseteq E} u_i^{\alpha_i} f_0$ , formula (3) shows that the monomial  $\prod_{\text{div}(u_i) \subseteq E} u_i^{\alpha_i}$  divides  $\mathcal{J}(f, E)$ ,  $\mathcal{J}(f, E, x)$  and  $\mathcal{J}(f, E, Y)$ . We let

$$J(f, E) := H(x)^{-1} \mathcal{J}(f, E), \quad J(f, E, x) := H(x)^{-1} \mathcal{J}(f, E, x), \quad J(f, E, Y) := H(x)^{-1} \mathcal{J}(f, E, Y).$$

The above definitions of  $\mathcal{J}(f, E)_z$ ,  $\mathcal{J}(f, E, Y)_z$  and  $\mathcal{J}(f, E, x)_z$  also make sense for  $f \in \widehat{S}_z$ . In the special case when the polyhedron  $\Delta(h; u_1, \dots, u_n; X)$  is minimal, we will use the further notation

$$\nu(x) := \text{ord}_x J(f, E), \quad \alpha(x) := \text{ord}_x J(f, E, x), \quad \epsilon(x) := \inf\{\text{ord}_x J(f, E, x), \text{ord}_x(H(x)^{-1} g^p)\}.$$

We also write  $\mathcal{J}(f, E, \tilde{x})$ ,  $J(f, E, \tilde{x})$ ,  $\nu(\tilde{x})$ ,  $\alpha(\tilde{x})$ ,  $\epsilon(\tilde{x})$  to mean  $\mathcal{J}(f, E, x)$ ,  $J(f, E, x)$ ,  $\nu(x)$ ,  $\alpha(x)$ ,  $\epsilon(x)$  respectively, where  $\eta(\tilde{x}) = x$ .

Note that all of these ideals depend on a choice of the variable  $X$  in  $S[X]$ , since  $f$  does. If we make a translation on  $X$ , i.e. if we change  $X$  to  $Z = X - \theta$ ,  $\theta \in S$ ,  $f$  is changed into  $f_Z = f + \theta^p - \theta g^{p-1}$ . If  $g = 0$ , we have  $J(f, E) = J(f_Z, E)$ ,  $J(f, E, x) = J(f_Z, E, x)$  and  $J(f, E, Y) = J(f_Z, E, Y)$ . In this purely inseparable case, we have:

**II.3.1 Lemma.** *Assume that  $g = 0$ . Then  $J(f, E)$ ,  $J(f, E, x)$  and  $J(f, E, Y)$  do not depend on any choice of coordinates  $(X, u_1, u_2, u_3)$  on  $R$ , even if  $\Delta(h; u_1, u_2, u_3)$  is not minimal.*

*Moreover, there exists a closed subset  $W$  (resp.  $W_x, W_Y$ ) of  $E$  of dimension at most one such that  $J(f, E)_z = S_z$  (resp.  $J(f, E, x)_z = S_z$ ,  $J(f, E, Y)_z = S_z$ ) whenever  $z \notin W$  (resp.  $z \in E \setminus W_x$ ,  $z \notin W_Y$ ).*

*Proof.* The first part of the lemma has already been pointed out above. From our conventions in **I**,  $\mathcal{J}(f, E)_z$  and  $\mathcal{J}(f, E, Y)_z$  coincide with  $S_z$  for  $z \notin E$ . In case  $Y$  (resp.  $\overline{\{x\}}$ ) is an irreducible component of  $E$ , we have  $J(f, E, Y) = J(f, E)$  (resp.  $J(f, E, x) = J(f, E)$ ).

Therefore to prove the lemma, since  $J(f, E)$  and  $J(f, E, x)$  (resp. and  $J(f, E, Y)$ ) coincide outside  $\overline{\{x\}}$  (resp.  $Y$ ), it is enough to reach  $\dim V(J(f, E)) \leq 1$ .

If  $\operatorname{div}(u_1) \subseteq E$ , it can be assumed by **II.2.2** that  $f = u_1^{pd_1} f_0$ , where  $u_1$  does not divide  $f_0$ , and either (i)  $d_1 \notin \mathbb{N}$ , or (ii)  $f_0 \bmod(u_1) \notin QF(S/(u_1))^p$ . In case (i),  $\operatorname{ord}_{u_1}(u_1 \frac{\partial f}{\partial u_1}) = pd_1$ , and in case (ii),  $\operatorname{ord}_{u_1}(\frac{\partial f}{\partial \lambda_i}) = pd_1$  for some  $i \neq 1$ . Therefore  $u_1$  does not divide  $J(f, E)$ . This proves that  $\dim V(J(f, E)) \leq 1$  and the proof is complete.

We now turn to the Artin-Schreier version of **II.3.1**. The important point now is that  $J(f, E)$ ,  $J(f, E, x)$  and  $J(f, E, Y)$  do depend on the choice of coordinates  $(X, u_1, u_2, u_3)$  on  $R$ , even with  $\Delta(h; u_1, u_2, u_3)$  minimal. However, the order of these ideals is better behaved. This is made precise in **II.3.3** below and leads to the definition of our main invariant in **II.4** below. The Artin-Schreier version of the set  $W$  in **II.3.1** is defined in **II.4.7**.

**II.3.2 Notations.** Let  $x \in \Sigma$ . Then  $\delta(x) > 0$  and  $R_x := \mathcal{O}_{X,x} = (S_{\eta(x)}[X])_{(X, u_1, \dots, u_n)}$ . With notations and conventions as in **II.1**, we define the monomial valuation  $v_\delta$  on  $R_x$  by:  $v_\delta(X) = 1$ ,  $v_\delta(u_i) = \frac{1}{\delta(x)}$ ,  $1 \leq i \leq n$ . In particular,  $v_\delta(g) = (\sum_{\operatorname{div}(u_i) \subseteq E_{0,x}} \beta_i) / \delta(x)$ . When  $\Delta(h; u_1, \dots, u_n; X)$  is minimal, we define  $\operatorname{in}_\delta h \in \operatorname{gr}_{v_\delta} R_x = k(x)[X, U_1, \dots, U_n]$  as follows:

$$\operatorname{in}_\delta h = X^p - X\bar{\gamma}^{p-1} \prod_{\operatorname{div}(u_i) \not\subseteq E_x} \bar{u}_i^{(p-1)\beta_i} \prod_{\operatorname{div}(u_i) \subseteq E_x} U_i^{(p-1)\beta_i} + \prod_{\operatorname{div}(u_i) \not\subseteq E_x} \bar{u}_i^{pd_i} \prod_{\operatorname{div}(u_i) \subseteq E_x} U_i^{pd_i} \Psi$$

the initial form of  $h$  for the valuation  $v_\delta$ . Here,  $\bar{\gamma} \in k(x)$ ,  $\bar{\gamma} = 0$  if  $\operatorname{ord}_{\eta(x)} g > \delta(x)$ , and  $\Psi \in k(x)[X, U_1, \dots, U_n]_{p\delta(x) - \operatorname{ord}_{\eta(x)} H(x)}$ , degrees counted w.r.t.  $v_\delta$ .

Given  $\theta \in S_{\eta(x)}$  and  $d = \operatorname{ord}_{\eta(x)} \theta$  (resp.  $d < \operatorname{ord}_{\eta(x)} \theta$ ), we denote by  $\operatorname{cl}_d \theta$  the initial form of  $\theta$  (resp. zero) in  $\operatorname{gr}_{(u_1, \dots, u_n)} S_{\eta(x)} = k(x)[U_1, \dots, U_n]$ . Similarly, if  $I \subset S_{\eta(x)}$  and  $d \leq \operatorname{ord}_{\eta(x)} I$ , we denote

$$\operatorname{cl}_d I := \operatorname{Vect}(\{\operatorname{cl}_d \theta\}_{\theta \in I}).$$

Consistently with the previous paragraph, when  $\Delta(h; u_1, \dots, u_n; X)$  is minimal, we usually denote

$$\Phi := \operatorname{cl}_{p\delta(x)} f, \quad \Psi := \operatorname{cl}_{\epsilon(x)} (H(x)^{-1} f).$$

**II.3.3 Proposition.** *With hypotheses and notations of II.1,  $\epsilon(x)$  does not depend on choices of coordinates  $(X, u_1, \dots, u_n)$  such that  $\Delta(h; u_1, \dots, u_n; X)$  is minimal, and we have*

$$\epsilon(x) = p(\delta(x) - \sum_{\operatorname{div}(u_i) \subseteq E_x} d_i).$$

For  $x \in \Sigma$ , exactly one of the following three properties holds for all possible r.s.p.'s  $(X, u_1, \dots, u_n)$  of  $\widehat{R}_x$  such that  $E_x \subseteq \operatorname{div}(u_1 \cdots u_n)$  and  $\Delta(h; u_1, \dots, u_n; X)$  minimal:

- (i) we have  $v_\delta(g) = 1$ , and either  $v_\delta(f) > p$  or  $(v_\delta(f) = p$  and  $\Phi \in k(x)[\{U_i \mid \operatorname{div}(u_i) \subseteq E\}])$ . In this case,  $\epsilon(x) = \operatorname{ord}_{\eta(x)}(H(x)^{-1} g^p)$ ,  $\nu(x) \geq \epsilon(x)$  and  $\operatorname{cl}_{p\delta(x)} f \in k(x)[\{U_i \mid \operatorname{div}(u_i) \subseteq E\}]$ ;
- (ii) we have  $v_\delta(g) > 1$ . In this case,  $\nu(x)$  is independent of  $(X, u_1, \dots, u_n)$  and  $\nu(x) \leq \epsilon(x)$ . Moreover,  $\operatorname{cl}_{\nu(x)} J(f, E)$  (resp.  $\operatorname{cl}_{\epsilon(x)} J(f, E, x)$ ) is independent of  $(X, u_1, \dots, u_n)$  if  $\nu(x) < \epsilon(x)$  (resp.  $\nu(x) = \epsilon(x)$ );
- (iii) we have  $v_\delta(g) = 1$ ,  $v_\delta(f) = p$  and  $\Phi \notin k(x)[\{U_i \mid \operatorname{div}(u_i) \subseteq E\}]$ .

Proof. The equality  $\operatorname{ord}_{\eta(x)} H(x) + \epsilon(x) = p\delta(x)$  is clear from the definitions, so  $\epsilon(x)$  is an invariant by **II.2**.

Assume that we are in case (i) for some r.s.p.  $(X, u_1, \dots, u_n)$ . We emphasize that  $v_\delta(f)$  and  $\text{cl}_{p\delta(x)}f$  are not at all stable under change of  $X$  such that  $\Delta(h; u_1, \dots, u_n; X)$  is minimal. In fact, we can replace  $X$  by any  $Z := X - \theta$  such that the Newton polyhedron of  $\theta$  is a subset of  $\Delta(h; u_1, \dots, u_n; X)$  and get  $\Delta(h; u_1, \dots, u_n; Z) = \Delta(h; u_1, \dots, u_n; X)$ ,  $f$  being changed into  $f_Z := f + \theta^p - \theta g^{p-1}$ . Let

$$\Theta := \text{cl}_{\delta(x)}\theta \in k(x)[U_1, \dots, U_j],$$

where  $E_x = \text{div}(u_1 \cdots u_j)$  by assumption. Then

$$\Phi_Z := \text{cl}_{p\delta(x)}f_Z = \Phi + \Theta^p - \Theta G^{p-1},$$

where  $G := \text{cl}_{\delta(x)}g \in k(x)[U_1, \dots, U_j]$ . Note that if  $\nu_\delta(\theta) > \delta(x)$ , then  $\Phi_Z = \Phi$  but in general  $\Phi_Z \neq \Phi$ .

We now prove (i) and begin with some r.s.p.  $(X, u_1, \dots, u_n)$  at  $x$  satisfying (i). We perform the following coordinate changes:

- (a)  $(u_1, \dots, u_n) \mapsto (v_1, \dots, v_n)$ , with  $E_x = \text{div}(u_1 \cdots u_j) = \text{div}(v_1 \cdots v_j)$ ;
- (b)  $X \mapsto Z := X - \theta$  such that  $\Delta(h; v_1, \dots, v_n; Z)$  is minimal.

After the change (a), the expansion of  $h$  remaining  $h = X^p - Xg^{p-1} + f$  (with  $\delta(x)$  unchanged by **II.2**), we minimize  $\Delta(h; v_1, \dots, v_n; X)$  in  $X$  by successive translations on  $X$  of the form  $X_{\mathbf{a}} := X - \lambda_{\mathbf{a}}v_1^{a_1} \cdots v_n^{a_n}$ , where  $\mathbf{a} := (a_1, \dots, a_n)$  is a solvable vertex of  $\Delta(h; v_1, \dots, v_n; X)$ . In particular,  $(\lambda_{\mathbf{a}}v_1^{a_1} \cdots v_n^{a_n})^p$  is a monomial in the expansion of  $f$ , so  $a_{j+1} = \cdots = a_n = 0$  whenever  $a_1 + a_2 + \cdots + a_n = \delta(x)$  by assumption (i). In this case, let  $\bar{\lambda}_{\mathbf{a}} \in k(x)$  be the residue of  $\lambda_{\mathbf{a}}$  and let  $\bar{\lambda}_{\mathbf{a}} := 0$  otherwise. The translation on  $X$  changes  $\Phi$  into

$$\Phi_{\mathbf{a}} = \Phi + (\bar{\lambda}_{\mathbf{a}}V_1^{a_1} \cdots V_j^{a_j})^p - \bar{\lambda}_{\mathbf{a}}V_1^{a_1} \cdots V_j^{a_j}G^{p-1},$$

where  $G \in k(x)[V_1, \dots, V_j]$  is the initial form of  $g$ . Therefore  $\Phi_{\mathbf{a}} \in k(x)[V_1, \dots, V_j]$  and  $x$  is still in case (i) w.r.t. the r.s.p.  $(X_{\mathbf{a}}, v_1, \dots, v_n)$ . Let  $Z := X - \sum_{\mathbf{a}} \widehat{\lambda_{\mathbf{a}}} v_1^{a_1} \cdots v_n^{a_n}$  be obtained by this minimizing process. Now, for any  $X' \in \widehat{S_{\eta(x)}}[X]$  such that  $(X', v_1, \dots, v_n)$  is a r.s.p. of  $\widehat{R}_x$  and the polyhedron  $\Delta(h; v_1, \dots, v_n; X') = \Delta(h; v_1, \dots, v_n; Z)$  is minimal, we have  $X' = \gamma'Z - \theta$ ,  $\gamma'$  invertible, the Newton polyhedron of  $\theta$  is a subset of  $\Delta(h; v_1, \dots, v_n; Z)$ , so  $\text{cl}_{\delta(x)}\theta \in k(x)[V_1, \dots, V_j]$  and after this new change of variable,  $x$  is still in case (i).

Finally by **II.3**(1), if  $\text{cl}_{p\delta(x)}f \in k(x)[V_1, \dots, V_j]$ , then

$$\nu(x) = \text{ord}_{\eta(x)}J(f, E) \geq p\delta(x) - \text{ord}_{\eta(x)}H(x) = \text{ord}_{\eta(x)}(H(x)^{-1}g^p) = \epsilon(x).$$

In case (ii), we have  $\text{in}_\delta(h) = X^p + \Phi$  with  $\Phi \notin (k(x)[U_1, \dots, U_n])^p$ , since  $\Delta(h; u_1, \dots, u_n; X)$  is minimal. If we change parameters  $(u_1, \dots, u_n) \mapsto (v_1, \dots, v_n)$  as in (a) above, then  $X \mapsto Z$  as in (b) above to get  $\Delta(h; v_1, \dots, v_n; Z)$  minimal, the above computations show that the translations on  $X$  change  $\Phi$  into

$$\Phi_Z = \text{cl}_{p\delta(x)}f_Z = \Phi + \left(\sum \bar{\lambda}_{\mathbf{a}}V_1^{a_1} \cdots V_n^{a_n}\right)^p$$

for certain values of  $\bar{\lambda}_{\mathbf{a}} \in k(x)$ . For any  $X' \in \widehat{S_{\eta(x)}}[X]$  such that  $(X', v_1, \dots, v_n)$  is a r.s.p. of  $\widehat{R}_x$  and the polyhedron  $\Delta(h; v_1, \dots, v_n; X') = \Delta(h; v_1, \dots, v_n; Z)$  is minimal,  $\Phi_Z$  gets changed into  $\bar{\gamma}'^{-p}(\Phi_Z + \Theta^p)$ , where  $0 \neq \bar{\gamma}' \in k(x)$ .

Since  $\Phi \notin (k(x)[U_1, \dots, U_n])^p$ ,  $\nu(x) = \text{ord}_{\eta(x)}J(f, E)$  is unaffected by all above coordinate changes and is equal to  $\epsilon(x) - 1$  or  $\epsilon(x)$ . In the former (resp. latter) case,  $\text{cl}_{\nu(x)}J(f, E)$  (resp.  $\text{cl}_{\epsilon(x)}J(f, E, x)$ ) has been multiplied by  $\bar{\gamma}'^{-p}$  along the above process and this proves (ii).

Since cases (i), (ii) and (iii) are mutually exclusive for any fixed r.s.p.  $(X, u_1, \dots, u_n)$  of  $\widehat{R}_x$ , which must belong to one of them, the independence on  $(X, u_1, \dots, u_n)$  of (iii) is also proved. We produce two examples showing the possible ambiguity on  $\nu(x)$  in case (iii).

**II.3.3.1 First example** of ambiguous case. We take  $E = \text{div}(u_1 u_2)$  and

$$h = X^p - X(u_1 u_2)^{2(p-1)} + \sum_{i \neq 0 \pmod{p}} u_1^{3p-i} u_2^i u_3^p + \sum_{j \neq 0 \pmod{p}} u_1^{4p-j} u_2^j.$$

An easy computation shows that  $\Delta(h; u_1, u_2, u_3; X)$  is minimal and that  $\delta(x_0) = 4$ ,  $g = (u_1 u_2)^2$ ,  $H(x_0) = (u_1 u_2)$ ,  $\nu(x_0) = 4p - 2 = \epsilon(x_0)$ .

Now we change  $X$  into  $Z := X - u_1^2 u_2 u_3$ . Then

$$h = Z^p - Z(u_1 u_2)^{2(p-1)} + \sum_{i \neq 0 \pmod{p}} u_1^{3p-i} u_2^i u_3^p + \sum_{j \neq 0 \pmod{p}} u_1^{4p-j} u_2^j + u_1^{2p} u_2^p u_3^p - u_1^2 u_2 (u_1 u_2)^{2(p-1)} u_3.$$

An easy computation shows that the exponents  $(2p, p, p)$  and  $(2p, 2p - 1, 1)$  appearing in  $u_1^{2p} u_2^p u_3^p - u_1^2 u_2 (u_1 u_2)^{2(p-1)} u_3$  lie in the interior of the convex hull of the exponents of

$$\sum_{i \neq 0 \pmod{p}} u_1^{3p-i} u_2^i u_3^p + \sum_{j \neq 0 \pmod{p}} u_1^{4p-j} u_2^j.$$

Therefore  $\Delta(h; u_1, u_2, u_3; Z)$  is still minimal. On the other hand, we now have

$$\text{ord}_{x_0} \left( \frac{\partial}{\partial u_3} (u_1^2 u_2 (u_1 u_2)^{2(p-1)} u_3) \right) = 4p - 1,$$

so  $\nu(x_0) = 4p - 3 = \epsilon(x_0) - 1$  w.r.t. the r.s.p.  $(Z, u_1, u_2, u_3)$ .

**II.3.3.2 Second example** of ambiguous case. We take  $\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2)$ ,  $a(1) + \epsilon(x) \equiv 0 \pmod{p}$ ,  $a(2) \equiv 0 \pmod{p}$ . If  $E = \text{div}(u_1)$ , we take  $a(2) = 0$ . Let

$$h = X^p - X \gamma^{p-1} u_1^{(p-1) \frac{a(1)+\epsilon(x_0)}{p}} u_2^{(p-1) \frac{a(2)}{p}} + u_1^{a(1)} u_2^{a(2)} \left( \sum_{0 \leq j \leq \frac{\epsilon(x_0)}{p}} \lambda_j u_1^{\epsilon(x_0) - jp} u_3^{jp} + \psi \right),$$

where  $\gamma \in S$  is invertible,  $\lambda_j \in k(x_0)$  and  $\text{ord}_{\eta(x_0)} \psi > \epsilon(x_0)$ . Let  $j_0 := \sup\{j | \lambda_j \neq 0\}$ . We assume that  $\lambda_{j_0} \notin k(x_0)^p$  and  $j_0 > 0$ . In particular,  $k(x_0)$  is not a perfect field and  $\epsilon(x_0) \geq p$ .

An easy computation shows that  $\Delta(h; u_1, u_2, u_3; X)$  is minimal, that its initial face is the segment with ends

$$a = \left( \frac{a(1) + \epsilon(x_0)}{p}, \frac{a(2)}{p}, 0 \right), \quad b = \left( \frac{a(1) + \epsilon(x_0) - j_0}{p}, \frac{a(2)}{p}, \frac{j_0}{p} \right), \quad (1)$$

and that  $H(x_0) = u_1^{a(1)} u_2^{a(2)}$ ,  $(H(x_0)^{-1} g^p) = (u_1^{\epsilon(x_0)})$  and  $\nu(x_0) = \epsilon(x_0)$ .

Now we change  $X$  into  $Z = X - u_1^{\frac{a(1)+\epsilon(x_0)}{p} - (j_0 - i)} u_2^{\frac{a(2)}{p}} u_3^{j_0 - i}$ ,  $i = 0$  or  $1$  defined by  $j_0 - i \not\equiv 0 \pmod{p}$ . Then

$$h = Z^p - \gamma^{p-1} Z u_1^{(p-1) \frac{a(1)+\epsilon(x_0)}{p}} u_2^{(p-1) \frac{a(2)}{p}} + f_Z,$$

where

$$f_Z = u_1^{a(1)} u_2^{a(2)} \left( \sum_{0 \leq j \leq \frac{\epsilon(x_0)}{p}} \lambda_j u_1^{\epsilon(x_0) - jp} u_3^{jp} + u_1^{\epsilon(x_0) - p(j_0 - i)} u_3^{p(j_0 - i)} - \gamma^{p-1} u_1^{\epsilon(x_0) - (j_0 - i)} u_3^{j_0 - i} + \psi \right).$$

An easy computation shows that  $\Delta(h; u_1, u_2, u_3; Z) = \Delta(h; u_1, u_2, u_3; X)$  is minimal and that  $\nu(x_0) = \epsilon(x_0) - 1$  w.r.t. the r.s.p.  $(Z, u_1, u_2, u_3)$ .

**II.4 Definition** (invariants  $\Omega$  and  $\tau$ ). *With hypotheses and notations of II.1, for any (not necessary closed)  $x \in \Sigma_p$ , let  $\Omega(x) = (\omega(x), \omega'(x)) \in \mathbb{N} \times \{1, 2, 3\}$  be defined as follows:*

$$\omega(x) := \inf\{\text{ord}_{\eta(x)}(J(f_X, E), H(x)^{-1}g^p)\},$$

the infimum being taken over all possible  $f_X$ 's corresponding to all possible r.s.p.'s  $(X, u_1, \dots, u_n)$  of  $\widehat{R}_x$  such that  $E_x \subseteq \text{div}(u_1 \cdots u_n)$  and  $\Delta(h; u_1, \dots, u_n; X)$  is minimal. The refinement  $\omega'(x)$  is given by

$$\omega'(x) = 1 \text{ if } \omega(x) = \text{ord}_{\eta(x)}(H(x)^{-1}g^p),$$

$$\omega'(x) = 2 \text{ if } \omega(x) < \text{ord}_{\eta(x)}(H(x)^{-1}g^p) \text{ and for every such r.s.p., } \text{ord}_{\eta(x)}J(f_X, E) = \omega(x).$$

$$\omega'(x) = 3 \text{ if } \omega(x) < \text{ord}_x(H(x)^{-1}g^p) \text{ and for some such r.s.p., } \text{ord}_{\eta(x)}J(f_X, E) = 1 + \omega(x).$$

We denote by  $\tau(x)$  the natural number:

$$\tau(x) = \max\{\tau(\text{cl}_{\omega(x)}(J(f_X, E, x), H(x)^{-1}g^p))\} \text{ if } \omega(x) = \epsilon(x),$$

$$\tau(x) = \max\{\tau(\text{cl}_{\omega(x)}J(f_X, E))\} \text{ if } \omega(x) = \epsilon(x) - 1,$$

where in both cases, the maximum is also taken over all possible  $f_X$ 's corresponding to all possible r.s.p.'s computing  $\omega(x)$ .

With the help and notation of II.3.3, we can now explicit several different cases:

(i)  $v_\delta(g) = 1$ , and either  $v_\delta(f) > p$  or  $(v_\delta(f) = p$  and  $\Phi \in k(x)[\{U_i \mid \text{div}(u_i) \subseteq E\}])$ : we have  $\Omega(x) = (\epsilon(x), 1)$ ;

(ii)  $v_\delta(g) > 1$ : we have  $\Omega(x) = (\nu(x), 2)$ ;

(iii)  $v_\delta(g) = 1$ ,  $v_\delta(f) = p$ ,  $\Phi \notin k(x)[\{U_i \mid \text{div}(u_i) \subseteq E\}]$  and  $\text{ord}_{\eta(x)}J(f, E)$  depends on the r.s.p. computing  $\omega(x)$ : we have  $\Omega(x) = (\epsilon(x) - 1, 3)$ ;

(iv)  $v_\delta(g) = 1$ ,  $v_\delta(f) = p$ ,  $\Phi \notin k(x)[\{U_i \mid \text{div}(u_i) \subseteq E\}]$  and for every r.s.p. computing  $\omega(x)$ ,  $\text{ord}_{\eta(x)}J(f, E) = \epsilon(x)$ : we have  $\Omega(x) = (\epsilon(x), 1)$ . In this case,

$$\Phi \in k(x)[\{U_i, U_j^p \mid \text{div}(u_i) \subseteq E, \text{div}(u_j) \not\subseteq E\}];$$

(v)  $v_\delta(g) = 1$ ,  $v_\delta(f) = p$ ,  $\Phi \notin k(x)[\{U_i \mid \text{div}(u_i) \subseteq E\}]$  and for every r.s.p. computing  $\omega(x)$ ,  $\text{ord}_{\eta(x)}J(f, E) = \epsilon(x) - 1$ : we have  $\Omega(x) = (\epsilon(x) - 1, 2)$ . In this case,

$$\Phi \notin k(x)[\{U_i, U_j^p \mid \text{div}(u_i) \subseteq E, \text{div}(u_j) \not\subseteq E\}].$$

So in the ambiguous case (iii), we give to  $\omega(x)$  the least possible value w.r.t. choices of r.s.p.'s, but keep this ambiguity in mind by setting  $\omega'(x) = 3$  maximal. The philosophy is that the property  $v_\delta(g) = 1$  is helpful for the local uniformization process only in cases (i) and (iv), i.e.  $\omega'(x_0) = 1$ .

By II.3.3(i), case (i) above is independent of the r.s.p. such that  $E_x \subseteq \text{div}(u_1 \cdots u_n)$  and  $\Delta(h; u_1, \dots, u_n; X)$  is minimal.

By II.3.3(ii), the definition of  $\tau(x)$  in case (ii) above is independent of the r.s.p. such that  $E_x \subseteq \text{div}(u_1 \cdots u_n)$  and  $\Delta(h; u_1, \dots, u_n; X)$  is minimal. On the other hand, in cases (iii)-(v),  $\tau(x)$

may depend on the r.s.p. used to compute  $\omega(x)$ . We will make use of the invariant  $\tau(x)$  only in cases (iv) and (v).

**II.4.1 Proposition.** *If  $x \in \Sigma_p$  and  $\overline{\{\eta(x)\}}$  is the intersection of components of  $E$ , then  $\omega(x) = \epsilon(x)$ .*

Proof. Since  $x$  is the intersection of components of  $E$ , we have  $J(f, E) = J(f, E, x)$ . Therefore if  $v_\delta(g) = 1$ ,  $x$  belongs to case **II.4(i)**, so  $\omega(x) = \epsilon(x)$ . If  $v_\delta(g) > 1$ , then  $\omega(x) = \nu(x) = \text{ord}_{\eta(x)}(H(x)^{-1}f) = \epsilon(x)$ .

**II.4.2 Proposition.** *For any  $x \in \Sigma_p$ , we have  $\omega(x) \leq \nu(x) \leq \alpha(x) \leq 1 + \nu(x)$  and  $\omega(x) \leq \epsilon(x)$ .*

Proof. This follows from the inclusions  $(m_{\eta(x)} \cap S)J(f, E) \subseteq J(f, E, x) \subseteq J(f, E)$  and the definitions, whether  $\alpha(x)$  and  $\nu(x)$  do or do not depend on the r.s.p. such that  $E_x \subseteq \text{div}(u_1 \cdots u_n)$  and  $\Delta(h; u_1, \dots, u_n; X)$  is minimal.

**II.4.3 Remark.** *In cases **II.4(iii)**, **(iv)**, we have  $\omega(x) \geq p$ .*

Proof. In these cases,  $\Phi \notin k(x)[\{U_i \mid \text{div}(u_i) \subseteq E\}]$ . Moreover, there exists a choice of  $(X, u_1, \dots, u_n)$  with  $E_x \subseteq \text{div}(u_1 \cdots u_n)$  and  $\Delta(h; u_1, \dots, u_n; X)$  minimal such that for each  $i$ , with  $\text{div}(u_i) \not\subseteq E$ , we have  $\deg \frac{\partial \Phi}{\partial U_i} \geq \deg \Phi$ . Therefore  $\frac{\partial \Phi}{\partial U_i} = 0$ ; these  $U_i$ 's appear in the expansion of  $\Phi$  with exponents divisible by  $p$ , and at least one of them effectively appears by definition of cases **II.4(iii)**, **(iv)**.

**II.4.4. Proposition and Definition.** *If  $x_0 \in \Sigma_p$  and  $\omega(x_0) = 0$ , then  $(X_0, x_0)$  is said to be a quasi-ordinary singularity. In this case, exactly one of the following properties holds for all possible r.s.p.'s  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  such that  $E \subseteq \text{div}(u_1 u_2 u_3)$  and  $\Delta(h; u_1, u_2, u_3; X)$  is minimal:*

- (i) *we have  $(H(x_0)) = (g^p)$ . In this case,  $\delta(x_0) = \text{ord}_{\eta(x_0)} g$  and  $\Delta(h; u_1, u_2, u_3; X)$  has only one vertex  $(d_1, d_2, d_3)$ ;*
- (ii) *we have  $(g^p) \subset \mathcal{J}(f, E) = (H(x_0)) = (f)$ . In this case,  $\delta(x_0) = (\text{ord}_{\eta(x_0)} f)/p$  and the polyhedron  $\Delta(h; u_1, u_2, u_3; X)$  has only one vertex  $(d_1, d_2, d_3)$ ;*
- (iii) *we have  $(f, g^p) \subset \mathcal{J}(f, E) = (H(x_0))$ . In this case,  $\delta(x_0) = (1 + \text{ord}_{\eta(x_0)} f)/p$  and  $v := H(x_0)^{-1}f$  is a regular parameter of  $\widehat{S}$ , transverse to  $E$ . If we choose indices  $i = 1, 2$  such that  $E \subseteq \text{div}(u_1 u_2)$  and  $(u_1, u_2, v)$  is a r.s.p. of  $\widehat{S}$ , then  $\Delta(h; u_1, u_2, v; X)$  is minimal. If  $g \neq 0$  (resp.  $g = 0$ ),  $\Delta(h; u_1, u_2, v; X)$  has two vertices (resp. one vertex)  $(d_1, d_2, \frac{1}{p})$  and  $(\beta_1, \beta_2, 0)$  (resp.  $(d_1, d_2, \frac{1}{p})$ ).*

Proof. The condition  $\omega(x_0) = 0$  is equivalent to  $(\mathcal{J}(f, E), g^p) = (H(x_0))$  for all (cf. **II.4.3**) r.s.p.'s  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  such that  $E \subseteq \text{div}(u_1 u_2 u_3)$  and  $\Delta(h; u_1, u_2, u_3; X)$  is minimal.

If  $(g^p) = (H(x_0))$ , we are in case (i) and all assertions in (i) are clear.

If  $(g^p) \subset (H(x_0))$ , then  $\mathcal{J}(f, E) = (H(x_0))$ . If  $(f) = (H(x_0))$ , we are in case (ii) and all subsequent assertions are clear. Otherwise,  $H(x_0)$  divides strictly  $f$  and we are in case (iii). Since there is a derivation  $D \in \mathcal{D}$  such that  $(Df) = (H(x_0))$ , we can choose a parameter  $v$ ,  $\text{div}(v) \not\subseteq E$ , transverse to  $E$ , such that  $D = \frac{\partial}{\partial v}$ ,  $E \subseteq \text{div}(u_1 u_2)$ ,  $(X, u_1, u_2, v)$  is a r.s.p. of  $\widehat{R}$  and  $f = vH(x_0)$ . Then  $f$  is a monomial and defines the vertex  $(d_1, d_2, \frac{1}{p})$  which is not solvable because the third coordinate is not an integer. If  $g = 0$ , this is the only vertex. If  $g \neq 0$ ,  $g$  defines another vertex  $(\beta_1, \beta_2, 0)$  which is not solvable by definition.

**II.4.5 Proposition.** *Let  $x \in \Sigma_p$ . Assume that there exists a r.s.p.  $(X, u_1, \dots, u_n)$  of  $R_x$ , with  $E_x \subseteq \text{div}(u_1 \cdots u_n)$ , such that  $h = X^p - Xg^{p-1} + f \in S_{\eta(x)}[X]$  satisfies  $(\mathcal{J}(f, E), g^p) = (H(x))$  and  $H(x)$  divides  $f$ . Then  $\omega(x) = 0$ .*

Proof. Note that we do not have any minimality assumption on  $\Delta(h; u_1, \dots, u_n; X)$ .

If  $(H(x)) = (g^p)$ , then  $\Delta(h; u_1, \dots, u_n; X)$  has only one vertex  $(d_1, d_2, d_3)$  which is not solvable by definition, hence minimal. So  $\omega(x) = 0$  in this case.

Otherwise,  $(g^p) \subset (H(x)) = \mathcal{J}(f, E)$ . Then we may have to do a translation  $Z := X - \theta$  to make  $\Delta(h; u_1, \dots, u_n; Z)$  minimal. The Newton polyhedron of  $\theta$  is a subset of  $\Delta(h; u_1, \dots, u_n; X)$ . Since  $H(x)$  divides  $f$ ,  $H(x)$  divides  $\theta^p$ . But  $H(x)$  strictly divides  $g^p$ , so that there exists  $u_i$ ,  $\text{div}(u_i) \subseteq E_x$ , such that  $u_i H(x)$  divides  $\theta g^{p-1}$ . Now after changing  $X$  to  $Z$ ,  $f$  being changed into  $f_Z = f + \theta^p - \theta g^{p-1}$ , we have  $\mathcal{J}(f_Z, E) \equiv \mathcal{J}(f, E) \pmod{(u_i H(x))}$  and we get  $\mathcal{J}(f_Z, E) = (H(x))$  as required.

**II.4.6 Theorem.** *If  $x_0 \in \Sigma_p$  and  $\omega(x_0) = 0$ , the local uniformization problem is solved for  $(X_0, x_0)$ .*

*Proof.* We first pick a r.s.p.  $(X, u_1, u_2, u_3)$  of  $R$  satisfying **II.2.2** with  $N > \delta(x_0) + 1$ . Then  $h = X^p - Xg^{p-1} + f$  and  $H(x_0)$  divides  $f$ .

If  $(H(x_0)) = (g^p)$ , we have  $(\mathcal{J}(f, E), g^p) = (H(x_0))$ . Otherwise, we have  $(g^p) \subset (H(x_0))$ . By **II.4.4(ii)** and (iii), there exists  $Z = X - \theta$ ,  $\theta \in \widehat{R}$  such that  $\Delta(h; u_1, u_2, u_3; Z)$  is minimal,  $h = Z^p - Zg^{p-1} + f_Z$  with  $(f_Z) = (H(x_0))$  or  $(f_Z = v_Z H(x_0))$ ,  $\text{div}(v_Z)$  regular and transverse to  $E$ ). In the latter case, the vertex  $(d_1, d_2, \frac{1}{p})$  is a vertex of  $\Delta(h; u_1, u_2, u_3; X)$  by **II.2.2**.

We thus have either  $(f) = (f_Z) = (H(x_0))$ , in which case  $\Delta(h; u_1, u_2, u_3; X)$  is already minimal, or  $(f_Z = v_Z H(x_0))$  and  $f = v H(x_0)$  for some  $v \in m_S$ ,  $\text{div}(v)$  regular and transverse to  $E$ . In particular, we always have  $(\mathcal{J}(f, E), g^p) = (H(x_0))$ , although  $\Delta(h; u_1, u_2, u_3; X)$  is not necessarily minimal.

Assuming  $x_0 \in \Sigma_p$  and  $\omega(x_0) = 0$ , we write  $H(x_0) = \prod_{i=1}^3 u_i^{a(i)}$ ,  $a(i) = 0$  if  $i \geq e + 1$ , and apply the following (globally defined) algorithm:

(i) if  $\text{ord}_{\eta(x_0)} H(x_0) \geq p$ , there exists a nonempty subset  $I \subseteq \{1, \dots, e\}$  such that

$$Y := V(X, \{u_i\}_{i \in I}) \subseteq \Sigma_p,$$

i.e.  $\sum_{i \in I} a(i) \geq p$ . We let  $\pi : X_1 \rightarrow X_0$  be the blowing up of  $Y$ , with  $(e - |I|, \sum_{i \in I} a(i))$  maximal for the lexicographical ordering;

(ii) if  $\text{ord}_{\eta(x_0)} H(x_0) < p$ , we have  $f = v H(x_0)$  and  $\delta(x_0) = 1$ . We let  $X_1 \rightarrow X_0$  be the blowing up of

$$Y := V(X, \{u_i \mid a(i) > 0\}, v).$$

In both cases, we take  $E_1$  to be the reduced inverse image of  $E$  in  $X_1$ . If the center  $x'$  of  $\mu$  in  $X_1$  verifies  $x' \in \Sigma_p(X_1)$ , we define  $H(x')$  and  $\omega(x')$  w.r.t.  $E_1$ . We claim that such  $x' \in \Sigma_p(X_1)$  lies on the strict transform of  $X = 0$ , has  $\omega(x') = 0$  and  $\text{ord}_{\eta(x')} H(x') < \text{ord}_{\eta(x_0)} H(x_0)$ , where  $\eta' : (X_1, x') \rightarrow \text{Spec} S'$  is induced by  $\pi$ . The theorem will follow by descending induction on  $\text{ord}_{\eta(x_0)} H(x_0) = a(1) + a(2) + a(3)$ .

*Case 1:*  $(H(x_0)) = (g^p)$ . Then  $V(X, \{u_i \mid a(i) > 0\}) \subseteq \Sigma_p$ , so we are in case (i) above. If  $\text{ord}_{\eta(x_0)} g = 1$ , say  $g = \gamma u_1$ ,  $X_1$  is the blowing up along  $V(X, u_1)$  and is regular. Otherwise,  $X^p$  is in the ideal of the ridge of  $h$ , so  $x' \in \Sigma_p(X_1)$  lies on the strict transform of  $X = 0$ . We change indices so that  $Y = V(X, u_1, \dots, u_k)$ , where  $k \leq e$ . By symmetry, we need only look at the chart of origin the point with r.s.p.  $(X' = \frac{X}{u_1}, u'_1 = u_1, \dots, u'_k = \frac{u_k}{u_1}, \{u'_i = u_i\}_{k+1 \leq i \leq 3})$ . Let  $h' = u_1^{-p} h = X'^p - X' g'^{p-1} + f'$  be the strict transform of  $h$ . Then  $g'^p = u_1^{-p} g^p = \gamma^p u_1^{a(1)+\dots+a(k)-p} u_2^{a(2)} u_3^{a(3)}$  and, since  $g^p$  divides  $f$ ,  $g'^p$  divides  $f'$ . So  $\omega(x') = 0$  by **II.4.5**. Since  $|I|$  is minimal,  $a(2) + \dots + a(k) < p$ , so

$$\text{ord}_{\eta'(x')} H(x') \leq a(1) + \dots + a(k) - p + a(2) + a(3) < a(1) + a(2) + a(3) = \text{ord}_{\eta(x_0)} H(x_0)$$

as required.

*Case 2:*  $(g^p) \subset (H(x_0)) = \mathcal{J}(f, E)$  and (i) above holds. We change indices so that  $Y = V(X, u_1, \dots, u_k)$  and once more distinguish two cases.

First assume that  $\text{ord}_{\eta(x_0)} f = p$ . We have  $f = \gamma_1 H(x_0)$ ,  $\gamma_1$  invertible, and  $Y = V(X, \{u_i \mid a(i) > 0\})$ . After rearranging indices, it can be assumed that  $Y = V(X, u_1, \dots, u_k)$  as above. By symmetry, we need only look at the chart of origin the point with r.s.p.  $(X' = \frac{X}{u_1}, u'_1 = u_1, \dots, u'_k = \frac{u_k}{u_1}, \{u'_i = u_i\}_{k+1 \leq i \leq 3})$  (the remaining point at infinity is not on the strict transform of  $X_0$ ). Let  $h' = u_1^{-p} h = X'^p - X' g'^{p-1} + f'$  be the strict transform of  $h$ , where  $u'_1$  divides  $g'^{p-1}$ . If  $u'_i$  is not invertible at  $\eta'(x')$  for some  $i$ ,  $2 \leq i \leq k$ , we have  $0 < \text{ord}_{\eta'(x')} f' < p$ ; otherwise  $u'_2 \cdots u'_k$  is invertible at  $\eta'(x')$ . In this case, we have  $\frac{\partial f'}{\partial u'_2}$  invertible if  $k \geq 2$ ; if  $k = 1$ , then  $\bar{\gamma}_1 \notin k(x)^p$  because the vertex  $(1, 0, 0)$  of  $\Delta(h; u_1, u_2, u_3; X)$  is not solvable and  $x' = (X' + \bar{\gamma}_1, u'_1, u'_2, u'_3)$  is a regular point of  $X_1$ . Therefore  $x' \notin \Sigma_p(X_1)$  in all cases.

Assume now that  $\text{ord}_{\eta(x_0)} f > p$ . Then every  $x' \in \Sigma_p(X_1)$  mapping to  $x_0$  lies on the strict transform of  $X = 0$ . By symmetry, we need only look at the chart of origin the point with r.s.p.  $(X' = \frac{X}{u_1}, u'_1 = u_1, \dots, u'_k = \frac{u_k}{u_1}, \{u'_i = u_i\}_{k+1 \leq i \leq 3})$ . Let  $h' = u_1^{-p} h = X'^p - X' g'^{p-1} + f'$  be the strict transform of  $h$ . Note that

$$u_1^{-p} H(x_0) = u_1^{a(1)+\dots+a(k)-p} \prod_{2 \leq i \leq e} u_i^{a(i)}$$

divides  $f'$  and strictly divides  $g'^p$ . We claim that

$$H(x') = u_1^{-p} H(x_0). \quad (1)$$

If (1) is true, we get  $\omega(x') = 0$  by **II.4.5**, then  $\text{ord}_{\eta'(x')} H(x') < \text{ord}_{\eta(x_0)} H(x_0)$  as in the previous case when  $(H(x_0)) = (g^p)$ . Let us prove (1). Indeed, for every  $D \in \mathcal{D}_1 := \mathcal{D}(E_1)$ ,  $Df' =: \lambda_D u_1^{a(1)+\dots+a(k)-p} u_2^{a(2)} u_3^{a(3)}$ , with at least one  $\lambda_D$  invertible. Furthermore,  $u_1^{-p} H(x_0)$  strictly divides  $g'^p$ , so when performing a translation  $Z' := X' - \theta$  on  $X'$  in order to get  $\Delta(h'; u'_1, u'_2, u'_3; Z')$  minimal (i.e.  $f'$  is replaced by  $f'_{Z'} = f' + \theta^p - \theta g'^{p-1}$  where the Newton polyhedron of  $\theta$  is contained in  $\Delta(h'; u'_1, u'_2, u'_3; X')$ ),  $u_1^{-p} H(x_0)$  strictly divides  $\theta g'^{p-1}$ . Therefore we have  $Df'_{Z'} = \lambda_{D, Z'} u_1^{a(1)+\dots+a(k)-p} u_2^{a(2)} u_3^{a(3)}$ , with at least one  $\lambda_{D, Z'}$  invertible, and this proves (1).

*Case 3:*  $(g^p) \subset (H(x_0)) = \mathcal{J}(f, E)$  and (ii) above holds. Then  $f = vH(x_0)$  has order  $p$ , we blow up  $Y = V(X, \{u_i \mid a(i) > 0\}, v)$  and  $H(x_0)$  strictly divides  $g^p$ . If  $\text{ord}_{\eta(x_0)} g = 1$ , say  $g = \gamma u_1$ , we must have  $H(x_0) = u_1^{p-1}$  and  $x' \notin \Sigma_p(X_1)$ . Otherwise  $X^p$  is in the ideal of the ridge of  $h$ , so every point  $x' \in \Sigma_p(X_1)$  mapping to  $x_0$  lies on the strict transform of  $X = 0$ . We relabel indices so that  $Y = V(X, u_1, \dots, u_k)$  (one of the  $u_i$ 's being  $v$ ) and by symmetry, we need only consider the chart of origin the point with r.s.p.  $(X' = \frac{X}{u_1}, u'_1 = u_1, \dots, u'_k = \frac{u_k}{u_1}, \{u'_i = u_i\}_{k+1 \leq i \leq 3})$ . Let  $h' = u_1^{-p} h = X'^p - X' g'^{p-1} + f'$  be the strict transform of  $h$ . Then  $f' = \gamma_1 u_2^{a(2)} u_3^{a(3)}$ , with  $\gamma_1$  invertible. By definition of  $Y$ , we had  $a(1) > 0$ , so  $\text{ord}_{\eta'(x')} (h') \leq a(2) + a(3) = p - a(1) < p$  and this proves that  $x' \notin \Sigma_p(X_1)$ .

**II.4.7 Theorem.** *The set  $W := \{x \in \Sigma_p \mid \omega(x) \geq 1\}$  is Zariski closed and of dimension at most one.*

*Proof.* By **II.2.2**, there exists a r.s.p.  $(X, u_1, u_2, u_3)$  of  $R$  such that  $h = X^p - Xg^{p-1} + f$ ,  $H(x_0)$  divides  $f$  and  $\delta(x_0) = \inf\{\text{ord}_{x_0} g, (\text{ord}_{x_0} f)/p\}$ .

First assume that  $\omega(x_0) = 0$ . The r.s.p.  $(X, u_1, u_2, u_3)$  then satisfies the assumptions in **II.4.5** (see the beginning of the proof of theorem **II.4.6**), and these assumptions are stable by localizing at any  $x \in \Sigma_p$ . Therefore  $W = \emptyset$  in this case.

Assume now that  $\omega(x_0) \geq 1$ . It is sufficient to prove that there exists a Zariski closed subset  $Z \subseteq \Sigma_p$  of dimension at most one such that  $W \subseteq Z$ . By **II.2.2**, and by definition of  $H(x_0)$  and  $\mathcal{D}$ , the ideals  $(f, g^p)$  and  $(\mathcal{J}(f, E), g^p)$  coincide, and are equal to  $H(x_0)$ , when localized at the generic point of any component  $\text{div}(u_i) \subseteq E_0$ . Therefore there exists  $Z \subseteq E_0$  of dimension at most one such that

$$(\mathcal{J}(f, E), g^p)_x = (f, g^p)_x = H(x_0)_x$$

whenever  $x \in E_0 \setminus Z$ . By **II.4.5**, we have  $W \subseteq Z$  as required.

**II.5 Permissible blowing ups.** Theorem **II.4.6** settles the easy case of quasi-ordinary singularities, which are dealt with by combinatorial blowing ups in the same way as in characteristic zero. To reduce to quasi-ordinary singularities, we need some notion of permissible blowing up which is well behaved w.r.t. our main invariant  $\Omega(x_0)$ .

**II.5.1 Definition.** Assume that  $x_0 \in \Sigma_p$  and that  $\omega(x_0) \geq 1$ . A Zariski closed subset  $Y \subseteq \Sigma_p$ , with generic point  $y$ , is a permissible blowing up center if  $Y$  is regular at  $x_0$ ,  $\eta(Y)$  has normal crossing with  $E$ , and if one of the following conditions holds:

- (i)  $\epsilon(x_0) = \epsilon(y)$ , (first kind of permissible blowing up), or
- (ii)  $\nu(x_0) = \epsilon(x_0) - 1 = \text{ord}_{\eta(x_0)}(J(f, E, Y), H(x_0)^{-1}g^p) = \text{ord}_{\eta(y)}(J(f, E, Y), H(x_0)^{-1}g^p) = \epsilon(y)$ , where  $f$  is given by some choice of the r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  such that  $E \subseteq \text{div}(u_1 u_2 u_3)$  and  $\Delta(h; u_1, u_2, u_3; X)$  is minimal (second kind of permissible blowing up).

The first (resp. second) type of permissible blowing up is studied in **II.5.2**, **II.5.4.2** and **II.5.4.3** (resp. **II.5.4.4**) below.

**II.5.1.1 Proposition.** Assume that  $x_0 \in \Sigma_p$  and that  $\omega(x_0) \geq 1$ . Any permissible center has dimension at most one and is contained in  $E_0$ . Moreover,  $\{x_0\}$  is a permissible center (of the first kind).

Proof. We have  $\epsilon(y) = 0$  if  $y$  is the generic point of a component of  $E_0$ . If  $\overline{\{y\}}$  were permissible, definition **II.5.1** would imply  $\omega(x_0) = 0$ . All other statements are obvious.

**II.5.2 Theorem.** For each permissible center  $Y$  of the first kind, there exists a r.s.p.  $(u_1, u_2, u_3)$  of  $S$  and  $X \in \widehat{R}$  such that  $E \subseteq \text{div}(u_1 u_2 u_3)$ ,  $\Delta(h; u_1, u_2, u_3; X)$  is minimal,  $Y = V(X, \{u_i\}_{i \in I})$  for some  $I \subseteq \{1, 2, 3\}$ , and whose associated  $f$  satisfies one of the following conditions:

- (i)  $\omega(x_0) = \epsilon(x_0) = \text{ord}_{\eta(x_0)}(J(f, E, Y), H(x_0)^{-1}g^p) = \text{ord}_{\eta(y)}(J(f, E, Y), H(x_0)^{-1}g^p) = \epsilon(y)$ ,
- (ii)  $1 + \omega(x_0) = \epsilon(x_0) = \text{ord}_{\eta(x_0)}(J(f, E, Y), H(x_0)^{-1}g^p) = \text{ord}_{\eta(y)}(J(f, E, Y), H(x_0)^{-1}g^p) = \epsilon(y)$ .

Proof. The statement is trivial if  $Y = \{x_0\}$ . Otherwise,  $Y$  is a curve by **II.5.1.1**. Since  $Y \subseteq \Sigma_p$  and  $\eta(Y)$  has normal crossing with  $E$ , it is of the form  $Y = V(Z, u_1, u_2)$  for some r.s.p.  $(Z, u_1, u_2, u_3)$  of  $R$  such that  $E \subseteq \text{div}(u_1 u_2 u_3)$ .

The formal version of **II.2.2** implies that there exists a r.s.p.  $X \in \widehat{R}$  such that  $Y = V(X, u_1, u_2)$  and both of  $\Delta(h; u_1, u_2, u_3; X)$ ,  $\Delta(h; u_1, u_2; X)$  minimal. The associated  $f$  thus computes  $\epsilon(x_0)$  and  $\epsilon(y)$  at the same time. The statement now follows easily from formulæ **II.3**(4) and (5) and the definitions.

**II.5.2.1 Proposition** (H. Hironaka). Assume that  $x_0 \in \Sigma_p$  and that  $\omega(x_0) \geq 1$ . Let  $e : X_1 \rightarrow X_0$  be a permissible blowing up and  $x' \in e^{-1}(x_0)$  be the center of  $\mu$  in  $X_1$ . Then  $(X_1, x')$  has multiplicity at most  $p$ .

Proof. Since  $Y \subseteq \Sigma_p$  and  $Y$  is regular at  $x_0$ , it is a permissible center in Hironaka's sense, and the assertion is classical.

**II.5.3 Proposition.** *Assume that  $x_0 \in \Sigma_p$  and that  $\omega(x_0) \geq 1$ . Let  $e : X_1 \rightarrow X_0$  be a permissible blowing up with center  $Y$  and  $x' \in e^{-1}(x_0)$  be the center of  $\mu$  in  $X_1$ . The following holds:*

- (i) *if  $H(x_0) \neq 1$  and  $x' \in \Sigma_p(X_1)$ , then  $x'$  lies on the strict transform of  $\{X = 0\}$ , whenever  $Y$  is expressed as  $Y = V(X, \{u_i\}_{i \in I})$ , with  $E \subseteq \text{div}(u_1 u_2 u_3)$  and  $\Delta(h; u_1, u_2, u_3; X)$  minimal;*
- (ii) *if  $H(x_0) \neq 1$  and  $H(x') = 1$ , then the local uniformization problem is solved for  $(X_0, x_0)$ .*

Proof. Assertion (i) is an easy consequence of [H2, thm. 3, p. 331]: with notations as in *loc.cit.*, if  $x' \in \Sigma_p(X_1)$ , then  $x'$  lies on the projective space  $\mathcal{B}_{\mathbf{P}, x'} \subseteq e^{-1}(x_0)$  which verifies

$$\mathcal{B}_{\mathbf{P}, x'} \subseteq \text{PDir}(\text{in}_{x_0} h) \quad (1)$$

except if  $p = 2$  and there exists a r.s.p.  $(Z, v_1, v_2, v_3)$  of  $R$  such that

$$\text{in}_{x_0} h = Z^2 + uV_1^2 + vV_2^2 + uvV_3^2, \quad (2)$$

where  $u, v \in k(x_0)$  and  $[k(x_0)^2(u, v) : k(x_0)^2] = 4$ . Since  $H(x_0) \neq 1$  by assumption, we have

$$I := \text{in}_{x_0} h = X^2 + U_1(\bar{\gamma}X + \lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 U_3),$$

for some  $\lambda_1, \lambda_2, \lambda_3 \in k(x_0)$  and  $\text{div}(u_1) \subseteq E_0$ . Since  $\Delta(h; u_1, u_2, u_3; X)$  is minimal, we have  $\text{in}_{x_0} h = X^2$  if  $\delta(x_0) > 1$ ; if  $\delta(x_0) = 1$ , either  $\text{in}_{x_0} h \in k(x_0)[X, U_1]$  or  $\frac{\partial I}{\partial U_1} \neq 0$ , so (2) does not hold and this proves (i).

We now prove (ii). Since  $Y \subseteq \Sigma_p$ , we have  $\delta(y) \geq 1$ . Let  $t = 0$  be a local equation of  $E' := e^{-1}(x_0)$  at  $x'$  and  $h' := t^{-p}h = X'^p - g'^{p-1}X' + f'$  be the local equation of  $X_1$  at  $x'$ , where  $X' := X/t$  by (i).

If  $\delta(y) > 1$ , we have  $(f', g') \subseteq (t)$  so  $H(x') \neq 1$ . If  $\delta(y) = \delta(x_0) = 1$ , it can be assumed that  $(f, g) \subseteq (u_1)$  since  $H(x_0) \neq 1$ , where  $\text{div}(u_1) \subseteq E_0$ . Then  $\text{PDir}(\text{in}_{x_0} h) \subseteq V(X, U_1)$  in (1), so  $u_1/t$  is a regular parameter at  $x'$  and divides  $H(x')$ , hence  $H(x') \neq 1$ .

Assume now that  $\delta(x_0) > \delta(y) = 1$ . In particular,  $Y$  is a permissible curve. After renumbering variables, it can be assumed that  $Y = V(X, u_1, u_2)$ ,  $\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2 u_3)$ . If  $H(x_0) \subseteq (u_3)$  we have  $H(x') \subseteq (u_3)$  as well, so from now on, after possibly renumbering again variables, it can be assumed that  $H(x_0) = u_1^{a(1)} u_2^{a(2)}$  with  $a(1) > 0$ . By **II.2** and **II.3.3**, this implies

$$\epsilon(x_0) - \epsilon(y) = p(\delta(x_0) - \delta(y)) > 0,$$

hence  $Y$  is permissible of the second kind. In particular  $E \subseteq \text{div}(u_1 u_3)$  by **II.5.1(ii)** and we have  $a(2) = 0$ . We get  $\epsilon(x_0) = p + 1 - a(1)$  by **II.3.3**. Finally by permissibility of the second kind, we actually have  $E = \text{div}(u_1)$  and there is an expression

$$\Phi := \text{in}_{x_0} f = U_1^{a(1)} (U_3 \Phi_3(U_1, U_2) + \Phi_0(U_1, U_2)),$$

with  $0 \neq \Phi_3 \in k(x_0)[U_1, U_2]_{p-a(1)}$  and  $\Phi_0 \in k(x_0)[U_1, U_2]_{p+1-a(1)}$ .

In computing  $h'$ , it can be assumed that  $t = u_1$  is a local equation of  $E' = \text{div}(u'_1)$  at  $x'$  if  $H(x') \neq 1$ . Then  $x'$  belongs to the chart with origin  $(X' := X/u_1, u'_1 = u_1, u'_2 := u_2/u_1, u_3)$ . Then

$$u_1^{-p} f = u_3 \Phi_3(1, u'_2) + u'_1 (\Phi_0(1, u'_2) + \varphi') \quad (3)$$

for some  $\varphi' \in (u'_1, u'_2)$ . Since  $\Phi_3 \neq 0$ , we have  $x' \notin \Sigma_p(X_1)$  unless  $a(1) = 1$  and  $\text{ord}_{\eta'(x')} \Phi_3(1, u'_2) = p - 1$ . Therefore  $x'$  is rational over  $x_0$  and after possibly replacing  $(X, u_1, u_2, u_3)$  with a r.s.p.

$(Z, u_1, v_2, u_3)$  such that  $\Delta(h; u_1, v_2, u_3; Z)$  is minimal, it can be assumed in (3) that  $\Phi_3 = \lambda U_2^{p-1}$  for some  $\lambda \neq 0$ . Summing up, we get

$$h' = X'^p - \gamma^{p-1} u_1'^{b'(p-1)} X' + \lambda u_3 u_2'^{p-1} + u_1' \phi', \quad (4)$$

with  $E' = \text{div}(u_1')$ ,  $b' \geq 1$  and  $\text{ord}_{\eta'(x')} \phi' \geq p-1$ , where  $x' = (X', u_1', u_2', u_3)$  if  $x' \in \Sigma_p(X_1)$ . Let  $e' : X_2 \rightarrow X_1$  be the blowing up of  $X_1$  along  $x'$  and  $x'' \in e'^{-1}(x')$  be the center of  $\mu$  in  $X_2$ . We consider two cases:

*Case 1:*  $b = 1$  or  $\text{ord}_{\eta'(x')} \phi' = p-1$ . Then  $\text{VDir}(\text{in}_{x_0} h) = \langle X', U_1', U_2', U_3 \rangle$ , so  $\text{PDir}(\text{in}_{x_0} h) = \emptyset$ . Since (2) does not hold,  $x'' \notin \Sigma_p(X_2)$  by (1).

*Case 2:*  $b > 1$  and  $\text{ord}_{\eta'(x')} \phi' > p-1$ . Then  $\text{VDir}(\text{in}_{x_0} h) = \langle X', U_2', U_3 \rangle$ , so again by (1),  $x'' \in \Sigma_p(X_2)$  only if  $x'' = (X'' = X'/u_1', u_1'' = u_1', u_2'' = u_2'/u_1', u_3'' = u_3/u_1')$ . Hence  $h'' := u_1''^{-p} h'$  satisfies (4) w.r.t. the r.s.p.  $(X'', u_1'', u_2'', u_3'')$ .

We iterate the above argument. Let

$$X_1 \leftarrow X_2 \leftarrow \cdots X_{n-1} \leftarrow X_n \leftarrow \cdots$$

be the quadratic sequence along  $\mu$ , i.e.  $X_i$  is the blowing up along the center  $x_{i-1}$  of  $\mu$  in  $X_{i-1}$  for  $i \geq 2$ , where  $x_1 := x'$ . Assume that  $x_i \in \Sigma_p(X_i)$  for all  $i \geq 1$ . Then  $x_{i-1}$  is in the above case 2, hence lies on the strict transform  $Y_i$  in  $X_i$  of the curve  $Y' := V(X', u_2', u_3)$ . By standard arguments, we have  $Y_i \subseteq \Sigma_p(X_i)$  for  $i \gg 0$ . But this implies that  $\eta'(Y') \subseteq E_1$ , in contradiction with our conventions in **I**. Therefore  $x_i \notin \Sigma_p(X_i)$  for some  $i \geq 1$  and (ii) is proved.

**II.5.3.1 Remark.** Theorem **II.5.3** plays an essential role in our approach. Namely, we may assume that  $H(x_0) \neq 1$  (last line of section **I**). *This additional assumption will be maintained up to the end of this article.*

Let  $e : X' \rightarrow X_0$  be a *permissible* blowing up. By **II.5.3**(i) and (ii), in order to reduce the multiplicity of the strict transform  $h' = t^{-p} h$  of  $h = X^p - X g^{p-1} + f$  at the center  $x'$  of  $\mu$  in  $X'$  ( $t$  being a local equation of the exceptional divisor at  $x'$ ), we may assume that  $h' = X'^p - X' g'^{p-1} + f'$  and that  $H(x') \neq 1$  where  $X' = X/t$ ,  $(f', g'^p) = t^{-p}(f, g^p)$ .

However, since our main invariant  $\Omega(x_0)$  can be read off  $(f, g^p)$  only when  $\Delta(h; u_1, u_2, u_3; X)$  is minimal, we need to relate some minimal  $\Delta(h'; v_1', v_2', v_3'; Z')$  at  $x'$  to a given minimal polygon  $\Delta(h; u_1, u_2, u_3; X)$  at  $x_0$ . The following elementary result is essential for this purpose.

**II.5.3.2 Theorem.** *Let  $F(U_2, U_3) \in k(x_0)[U_2, U_3]$  be a homogeneous polynomial of degree  $i \geq 0$ , and  $a, b \in \mathbb{N}$  be such that  $U_2^a U_3^b F(U_2, U_3) \notin (k(x_0)[U_2, U_3])^p$ .*

*Let  $x' \in \text{Speck}(x_0)[\frac{U_3}{U_2}]$  be a closed point with ideal  $(v := P(1, \frac{U_3}{U_2}))$ ,  $P \in k(x_0)[U_2, U_3]$  a nonzero homogeneous irreducible polynomial of degree  $d := [k(x') : k(x_0)]$ , unitary in  $U_3$ .*

*Let  $A \in T' := k(x_0)[U_2, \frac{U_3}{U_2}]_{(U_2, v)}$  be such that  $U_2^{a+b+i}$  (resp.  $U_2^{a+b+i} v^b$ ) divides  $A^p$  in  $T'$  if  $P \neq U_3$  (resp.  $P = U_3$ ). There exists an integer  $e \geq 0$  such that*

$$U_2^{a+b+i} \left(\frac{U_3}{U_2}\right)^b F\left(1, \frac{U_3}{U_2}\right) + A^p \equiv U_2^{a+b+i} \left(\frac{U_3}{U_2}\right)^b \gamma v^e \pmod{(U_2^{a+b+i+1} T')}, \quad (1)$$

*with  $\gamma$  invertible in  $T'$ . We have the following estimates for  $e$ :*

- (i) *if  $P \neq U_3$  (resp.  $P = U_3$ ), we have  $e \leq \frac{i}{d} + 1$  (resp.  $e \leq i$ );*
- (ii) *if  $P \neq U_3$ , then  $e < p(1 + \lfloor \frac{i}{pd} \rfloor)$  (equivalently: for every  $N \in \mathbb{N}$  such that  $\frac{i}{pd} < N$ , we have  $e < Np$ );*

(iii) if  $i \geq 1$  and  $b = 0$ , then  $e \leq i$ ;

(iv) if  $i \geq 2$  and  $b = 0$ , there exists at most one  $x'$  as above with  $e = i$ . If such an  $x'$  exists, we have  $P(U_2, U_3) = \lambda U_2 + U_3$  for some  $\lambda \in k(x_0)$ . In this case,  $U_2^{-(a-1)} \frac{\partial U_2^a F}{\partial U_2}$  and all  $\frac{\partial F}{\partial \lambda_i}$ 's,  $4 \leq i \leq s$ , are multiples of  $(\lambda U_2 + U_3)^i$ ,  $\frac{\partial F}{\partial U_3}$  is a multiple of  $(\lambda U_2 + U_3)^{i-1}$ . In particular,  $x'$  is rational over  $x_0$ .

Proof. The existence of some integer  $e \geq 0$  satisfying (1) is clear from the assumptions. We prove the estimates in (i), (ii), (iii), then prove (iv).

Since  $U_2^a U_3^b F(U_2, U_3) \notin (k(x_0)[U_2, U_3])^p$ , there exists a derivation  $D \in \text{Der}_{k_0}(k(x_0)[U_2, U_3])$ ,  $D$  preserving degrees of homogeneous polynomials, such that

$$D(U_2^a U_3^b F(U_2, U_3)) = U_2^a U_3^b F_D(U_2, U_3),$$

with  $F_D \neq 0$  a homogeneous polynomial of degree  $i$ . We pick  $D$  in such a way that  $\text{ord}_v F_D(1, \frac{U_3}{U_2})$  is minimal. With conventions as in **II.3** on  $\text{Der}_{k_0}(k)$ , we can take

$$D \in \{U_2 \frac{\partial}{\partial U_2}, U_3 \frac{\partial}{\partial U_3}, \{\frac{\partial}{\partial \lambda_i}\}_{4 \leq i \leq s}\}. \quad (2)$$

Since  $D((U_2, U_3)k(x_0)[U_2, U_3]) \subseteq (U_2, U_3)k(x_0)[U_2, U_3]$ , we have  $D(U_2 T') \subseteq U_2 T'$ . So in  $T'$  there exists a derivation  $D' \in \{U_2 \frac{\partial}{\partial U_2}, \frac{\partial}{\partial v}, \{\frac{\partial}{\partial \lambda_i}\}_{4 \leq i \leq s}\}$ , where  $(dU_2, dv, \{d\lambda_i\}_{4 \leq i \leq s})$  is a basis of  $\Omega_{T'/k_0}^1$ , such that

$$D' \left( U_2^{a+b+i} \left( \frac{U_3}{U_2} \right)^b F \left( 1, \frac{U_3}{U_2} \right) + A^p \right) = u U_2^{a+b+i} \left( \frac{U_3}{U_2} \right)^b F_D \left( 1, \frac{U_3}{U_2} \right), \quad (3)$$

with  $u \in T'$  invertible. We consider two cases:

*Case 1:*  $D' \in \{U_2 \frac{\partial}{\partial U_2}, \{\frac{\partial}{\partial \lambda_i}\}_{4 \leq i \leq s}\}$ . Then by (3) the integer  $e$  in (1) satisfies

$$e \leq \text{ord}_{x'} F_D \left( 1, \frac{U_3}{U_2} \right) \leq \frac{\deg F_D}{d} = \frac{i}{d}.$$

This proves (i), (ii) and (iii) in this first case.

*Case 2:* (3) is satisfied only by  $D' = \frac{\partial}{\partial v}$ . Then by (3), the integer  $e$  in (1) satisfies

$$e \leq 1 + \text{ord}_v F_D \left( 1, \frac{U_3}{U_2} \right) \leq 1 + \frac{\deg F_D}{d} = 1 + \frac{i}{d}, \quad (4)$$

and this proves (i) if  $P \neq U_3$ . If some inequality is strict, we also get (i) if  $P = U_3$  as well as (iii) for every  $P$  (note the trivial fact that  $1 + \frac{i}{d} \leq i$  whenever  $(i, d \geq 2)$ ).

If  $P \neq U_3$ , we have  $e \not\equiv 0 \pmod{p}$ , since

$$D' \left( \left( \frac{U_3}{U_2} \right)^b F \left( 1, \frac{U_3}{U_2} \right) + \frac{A^p}{U_2^{a+b+i}} \right) \equiv \frac{\partial \gamma' v^e}{\partial v} \pmod{U_2 T'},$$

where  $\gamma'$  is a unit. Then (ii) follows from (4) and the trivial inequalities

$$\frac{i}{d} < 1 + \left\lfloor \frac{i}{d} \right\rfloor \leq p \left( 1 + \left\lfloor \frac{i}{pd} \right\rfloor \right)$$

for  $i \geq 0$  and  $d \geq 1$ .

We now assume that  $e = 1 + \frac{i}{d}$ . In particular  $i \neq 2$  if  $d \geq 2$ . If  $d \geq 2$  and  $i \geq 2$ , we have  $1 + \frac{i}{d} \leq i$ , which proves (iii) when  $d \geq 2$ .

Suppose now that  $d = 1$ . Then  $v = \frac{U_3}{U_2} + \lambda$ ,  $\lambda \in k(x_0)$ ,  $D' = \frac{\partial}{\partial v} = \frac{\partial}{\partial U_3}$ . The only possibility for the derivation  $D$  in (2) is  $D = U_3 \frac{\partial}{\partial U_3}$ ; hence  $\lambda \neq 0$  and this proves (i) and (iii) when  $P = U_3$ .

There remains to prove (iii) when  $d = 1$  and  $P \neq U_3$ . Then

$$D(U_2^a F(U_2, U_3)) = U_2^a F_D(U_2, U_3) = U_2^a U_3 G(U_2, U_3),$$

with  $G \neq 0$  a homogeneous polynomial of degree  $i - 1$ . In (3), we now have  $F_D(1, \frac{U_3}{U_2}) = (v - \lambda)G(1, \frac{U_3}{U_2})$ , and therefore get the sharper estimate

$$e \leq 1 + \text{ord}_v G(1, \frac{U_3}{U_2}) \leq 1 + \deg G = i,$$

thus contradicting the assumption  $e = 1 + i$ .

Let us finally prove (iv), so  $i \geq 2$ . If  $d \geq 2$ , we have the inequality  $e \leq 1 + \frac{i}{d} < i$  except possibly if  $d = i = 2$ . In this last case,  $(U_3 \frac{\partial F}{\partial U_3})(1, \frac{U_3}{U_2})$  is a unit in  $T'$ , so that we get  $e \leq 1$  in this case. We can therefore assume that  $x'$  is rational over  $x_0$ . Then  $v = \lambda + \frac{U_3}{U_2}$ ,  $\lambda \in k(x_0)$ . Note that  $(dU_2, dv, \{d\lambda_i\}_{4 \leq i \leq s})$  is then a basis of  $\Omega_{T'/k_0}^1$ . We denote by  $u_2$  the image of  $U_2$  in  $T'$  to avoid confusion in what follows

Since

$$dv = d\left(\frac{U_3}{U_2} + \lambda\right) = d\frac{U_3}{U_2} + \sum_{4 \leq i \leq s} a_i d\lambda_i, \quad dU_3 = u_2 d\frac{U_3}{U_2} + \frac{U_3}{U_2} du_2,$$

with  $a_i \in k(x_0)$ ,  $4 \leq i \leq s$ , we get the formulæ

$$\begin{aligned} u_2 \frac{\partial \circ f}{\partial u_2} &= u_2 \frac{\partial}{\partial U_2} + U_3 \frac{\partial}{\partial U_3}, & \frac{\partial \circ f}{\partial v} &= u_2 \frac{\partial}{\partial U_3}, \\ \frac{\partial \circ f}{\partial \lambda_i} &= \frac{\partial}{\partial \lambda_i} - a_i u_2 \frac{\partial}{\partial U_3} \end{aligned}$$

in  $\text{Der}_{k_0}(k(x_0)[U_2, U_3])$ , where  $f : k(x_0)[U_2, U_3] \rightarrow T'$  is the natural map.

Then, if  $\lambda \neq 0$ , the (weak) transform of

$$I := U_2^{-a} \left( U_2 \frac{\partial U_2^a F(U_2, U_3)}{\partial U_2}, U_3 \frac{\partial U_2^a F(U_2, U_3)}{\partial U_3}, \left\{ \frac{\partial U_2^a F(U_2, U_3)}{\partial \lambda_i} \right\}_{4 \leq i \leq s} \right)$$

is

$$I' := u_2^{-a-i} \left( u_2 \frac{\partial u_2^{a+i} F(1, \frac{U_3}{U_2})}{\partial u_2}, \frac{\partial u_2^{a+i} F(1, \frac{U_3}{U_2})}{\partial v}, \left\{ \frac{\partial u_2^{a+i} F(1, \frac{U_3}{U_2})}{\partial \lambda_i} \right\}_{4 \leq i \leq s} \right).$$

Since we have  $e = i$ , all of these derivatives in  $I'$  are multiples of  $v^i$ , except  $u_2^{-a-i} \frac{\partial u_2^{a+i} F(1, \frac{U_3}{U_2})}{\partial v}$  which is a multiple of  $v^{i-1}$ . So in  $I$ , the corresponding derivatives are multiples of  $(\lambda U_2 + U_3)^i$ , except  $U_2^{-a} U_3 \frac{\partial U_2^a F(U_2, U_3)}{\partial U_3}$  which is a multiple of  $U_3 (\lambda U_2 + U_3)^{i-1}$ .

If  $\lambda = 0$ , we have  $a_i = 0$ ,  $4 \leq i \leq s$ , in the previous formulæ, and  $I'$  becomes

$$I' := u_2^{-a-i} \left( u_2 \frac{\partial u_2^{a+i} F(1, \frac{U_3}{U_2})}{\partial u_2}, U_3 \frac{\partial u_2^{a+i} F(1, \frac{U_3}{U_2})}{\partial \frac{U_3}{U_2}}, \left\{ \frac{\partial u_2^{a+i} F(1, \frac{U_3}{U_2})}{\partial \lambda_i} \right\}_{4 \leq i \leq s} \right).$$

Since we have  $e = i$ , all of these derivatives in  $I'$  are multiples of  $(\frac{U_3}{U_2})^i$ . So in  $I$ , all derivatives are multiples of  $U_3^i$ .

So if we have equality in (iii), either all the derivatives in  $I$  are multiples of  $U_3^i$  and the only possible  $x'$  has parameter  $v = \frac{U_3}{U_2}$ , or they are all multiples of  $(\lambda U_2 + U_3)^i$  (except  $U_2^{-a} U_3 \frac{\partial U_2^a F(U_2, U_3)}{\partial U_3}$  which is a multiple of  $U_3(\lambda U_2 + U_3)^{i-1}$ ), for some  $\lambda \in k(x_0) - \{0\}$ . Since  $i \geq 2$  and  $U_2^a F(U_2, U_3)$  is not a  $p^{\text{th}}$ -power,  $\lambda$  is uniquely determined and gives as only possible  $x'$  the point with parameter  $v = \lambda + \frac{U_3}{U_2}$ . We get (iv).

We now come to the main result of this chapter: our main invariant  $(\text{ord}_{x_0} h, \Omega(x_0))$  does not increase above  $x_0$  when performing a permissible blowing up.

**II.5.4 Theorem.** *Assume that  $x_0 \in \Sigma_p$ ,  $\omega(x_0) \geq 1$  and  $H(x_0) \neq 1$ . Let  $e : X_1 \rightarrow X_0$  be the blowing up along a permissible center  $Y$  and  $x' \in e^{-1}(x_0)$  be the center of  $\mu$  in  $X_1$ . If  $x' \in \Sigma_p(X_1)$ , then  $\Omega(x') \leq \Omega(x_0)$ , where  $\Omega(x')$  is computed w.r.t.  $E' := e^{-1}(E)_{\text{red}}$ .*

Proof. The proof is long and needs to study all different cases, depending on the kind of permissible blowing up and on the different values of  $\text{ord}_{\eta(y)} g^p$ .

By **II.5.2**, there exists a r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  such that  $Y = V(X, \{u_i\}_{i \in I})$ ,  $E \subseteq \text{div}(u_1 u_2 u_3)$ , and both of  $\Delta(h; u_1, u_2, u_3; X)$  and  $\Delta(h; \{u_i\}_{i \in I}; X)$  are minimal. If the blowing up is of the second kind, any r.s.p. satisfying the condition in definition **II.5.1** (ii) automatically has these properties. We have  $|I| = 2$  or  $3$  by **II.5.1.1**. Also,  $\delta(y) \geq 1$  by **II.2.1**. We keep conventions on indices as in **II.3** as well as the writings  $f = \prod_{\text{div}(u_i) \subseteq E} u_i^{\alpha_i} f_0$  of **II.1** and  $\Psi = \text{cl}_{\epsilon(x_0)}(H(x_0)^{-1} f)$  of **II.3**.

By **II.5.3**(i),  $e^{-1}(Y) = \text{div}(t)$ , where  $t = u_i$  for some  $i \in I$ . Let  $h' := t^{-p} h$  be the strict transform of  $h$  at  $x'$ :

$$h' = X'^p - X' t^{(p-1)((\sum_{i \in I_E} \beta_i) - 1)} \tilde{g}^{p-1} + t^{(\sum_{i \in I} \alpha_i) + \alpha(y) - p} \tilde{f} =: X'^p - X' g'^{(p-1)} + f', \quad (1)$$

where  $\tilde{f}$  (resp.  $\tilde{g}$ ) is the strict transform of  $f$  (resp.  $g$ ),  $y$  is the generic point of  $Y$  and  $X' := X/t$ .

Let  $u'_j := u_j/t$ ,  $j \in I$  ( $u'_i = 1$  for  $j = i$ ) and  $I_i := I \setminus \{i\}$ . Then  $(X', t, \{u'_i\}_{i \in I_i}, \{u_r\}_{r \in I^c})$  is a system of coordinates at  $x'$ . Let  $\eta' : (X_1, x') \rightarrow S[\{u'_i\}_{i \in I_i}]$  be the induced map. We denote  $S' := S[\{u'_i\}_{i \in I_i}]_{\eta'(x')}$  and  $R' := S'[X']_{x'}$ . If  $x' \in \Sigma_p(X_1)$ ,  $\Omega(x')$  is thus defined. Note that  $u'_j$  may be a unit for some  $j \in I_i$ . In any case: *the polyhedron  $\Delta(h'; v_1, v_2, v_3; X')$  where  $(X', v_1, v_2, v_3)$  is a r.s.p. of  $R'$  adapted to  $E' = e^{-1}(E)_{\text{red}}$  is not in general minimal.*

Let us recall the transformation laws given in [5, I.E.1]:

$$\mathcal{J}(f, E')_{\eta'(x')} = \mathcal{J}(f, E, Y) S',$$

where  $Y$  is the center of the blowing up, the only hypothesis in this formula being that  $Y$  is regular, and that  $\eta(Y)$  has normal crossings with  $E$ . Since we factor out the  $p^{\text{th}}$ -power  $t^p$  in  $h$ , we get

$$\mathcal{J}(f', E')_{\eta'(x')} = t^{-p} \mathcal{J}(f, E, Y) S'. \quad (2)$$

If the blowing up is of the first kind, then  $\Psi \in k(x_0)[\{U_i\}_{i \in I}]$  and

$$\tilde{f} \equiv M \Psi(\{u'_i\}_{i \in I}) \text{ mod } M(t, u_r). \quad (3)$$

where  $M := (\prod_{i \in I_E} u_i^{p d_i}) u_r^{p d_r}$  ( $d_i$  is defined in **II.1.2**) and  $\{r\} = I^c$ .

If the blowing up is of the second kind, the definition implies that  $Y$  is a curve and  $\Psi = \Psi_0(\{U_i\}_{i \in I}) + U_r \Psi_r(\{U_i\}_{i \in I})$ , where  $\{r\} = I^c$  is such that  $\text{div}(u_r) \not\subseteq E$ ,  $\Psi_r$  is (nonzero) homogeneous of degree  $\epsilon(x_0) - 1$  and  $\Psi_0 \in k(x_0)[\{U_i\}_{i \in I}]_{\epsilon(x_0)}$ . We have

$$\tilde{f} \equiv M u_r \Psi_r(\{u'_i\}_{i \in I}) \pmod{M(t, u_r^2)}. \quad (3')$$

In this case, it is easily seen that the function  $f \in S$  attached to any r.s.p.  $(X, u_1, u_2, u_3)$  of  $R$  satisfying the requirements stated in the beginning of the proof of this theorem produces a  $\Psi$  of the above form.

If  $\delta(y) > 1$ , we consider the valuation  $w$  of  $S'[X']$  centered at  $V(X', t)$  given by:  $w(X') = 1$ ,  $w(t) = \frac{1}{\delta(y)-1}$ . As

$$p\delta(y) = \inf\left\{p \sum_{i \in I_E} \beta_i, \alpha(y) + \sum_{i \in I} pd_i\right\},$$

we get  $w(h') = w(X'^p) = p$ , so vertices of the polyhedron  $\Delta(h'; v_1, v_2, v_3; X')$  where  $(X', v_1, v_2, v_3)$  is a r.s.p. of  $R'$  adapted to  $E'$  correspond to monomials with  $w$ -value at least  $p$ . Thus there is some  $Z' \in R'$  such that  $\Delta(h'; v_1, v_2, v_3; Z')$  is minimal, where  $Z' = X' - \theta'$ ,  $\theta' \in \widehat{S}'$ ,  $w(\theta') \geq 1$ . After changing  $X'$  to  $Z'$ , the degree zero term in  $Z'$  of  $h'$  is:

$$\phi := t^{(\sum_{i \in I} pd_i) + \alpha(y) - p} \tilde{f} + \theta'^p - \theta' t^{(p-1)((\sum_{i \in I_E} \beta_i) - 1)} \tilde{g}^{p-1}. \quad (4)$$

Formula (4) is also valid when  $\delta(y) = 1$ , where  $\theta' \in \widehat{S}'$  has no weight estimate.

**II.5.4.1 Lemma.** *We have  $(H(x')) = (t^{p(\delta(y)-1)} (\prod_{i \in I_E} u_i'^{pd_i}) u_r^{pd_r})$ .*

Proof. Indeed, as the exponents  $d_i$  are defined by the generic point of the corresponding component of the exceptional divisor (cf. **II.2**), the only question is to compute the exponent of  $t$  in  $g^p$  and  $\phi$ , the minimum will be the exponent of  $t$  in  $H(x')$ .

If  $\text{ord}_{\eta(y)} g^p \leq \text{ord}_{\eta(y)} f = \alpha(y)$ , then  $\epsilon(y) = \text{ord}_{\eta(y)} g^p$ . Formulæ(1) and (4) and **II.3.3** imply that

$$\text{ord}_t g^p = \sum_{i \in I_E} pd_i + \epsilon(y) - p = p(\delta(y) - 1),$$

$$\text{ord}_t \phi \geq \sum_{i \in I_E} pd_i + \epsilon(y) - p = p(\delta(y) - 1).$$

If  $\text{ord}_{\eta(y)} g^p > \text{ord}_{\eta(y)} f = \alpha(y)$ , then  $\epsilon(y) = \alpha(y)$ . Formula (1) implies that  $t^{(\sum_{i \in I_E} pd_i) + \epsilon(y) - p}$  divides  $\theta'^p$  in (4) and strictly divides  $g^p$ , so strictly divides  $\theta' g^{p-1} = \theta' t^{(p-1)((\sum_{i \in I_E} \beta_i) - 1)} \tilde{g}^{p-1}$ . Therefore

$$\text{ord}_t \phi = \sum_{i \in I_E} pd_i + \epsilon(y) - p = p(\delta(y) - 1)$$

and this completes the proof.

The theorem is a consequence of the following three lemmas which also classify the equality cases  $\Omega(x') = \Omega(x_0)$ . In **II.5.4.2** and **II.5.4.3**,  $(X, u_1, u_2, u_3)$  is any r.s.p. of  $\widehat{R}$  such that  $Y = V(X, \{u_i\}_{i \in I})$ ,  $E \subseteq \text{div}(u_1 u_2 u_3)$  and  $\Delta(h; u_1, u_2, u_3; X)$  is minimal (see comments in the beginning of the proof of this theorem).

**II.5.4.2 Lemma.** *If  $v_\delta(g) = 1$ ,  $\epsilon(x_0) = \epsilon(y)$  (first kind of permissible blowing-up) and  $x' \in \Sigma_p(X_1)$ , then  $\Omega(x') \leq \Omega(x_0)$ . Moreover, the following holds:*

- (i) the strict transform of  $\operatorname{div}(H(x_0)^{-1}g^p)$  is  $\operatorname{div}(H(x')^{-1}g'^p)$ . We have  $\Omega(x') < \Omega(x_0)$  if  $x'$  does not map to the strict transform of each component of  $\operatorname{div}(H(x_0)^{-1}g^p)$ ;
- (ii) if  $Y$  is a curve, there exists at most one  $x'$  such that  $\Omega(x') = \Omega(x_0)$ . Such  $x'$  satisfies

$$\eta'(x') \in \operatorname{PDir}(\operatorname{cl}_{\epsilon(x_0)}J(f, E, x_0)) \text{ (resp. } \eta'(x') \in \operatorname{PDir}(\operatorname{cl}_{\omega(x_0)}J(f, E))$$

- if  $\omega(x_0) = \epsilon(x_0)$  (resp.  $1 + \omega(x_0) = \epsilon(x_0)$ ), and has  $\omega'(x') = \omega'(x_0) \leq 2$ ;
- (iii) if  $Y = \{x_0\}$  and  $\operatorname{div}(H(x_0)^{-1}g^p)$  has two components, there exists at most one  $x'$  such that  $\Omega(x') = \Omega(x_0)$ . Such  $x'$  is the intersection of the exceptional divisor and of the strict transform of  $\operatorname{div}(H(x_0)^{-1}g^p)$ , and has either  $\omega'(x') = \omega'(x_0) = 2$ , or  $(\omega'(x_0) = 1$  and  $x_0$  belongs to case **II.4(i)**);
- (iv) if  $Y = \{x_0\}$  and  $\operatorname{div}(H(x_0)^{-1}g^p)$  has one component, then: if  $\omega'(x_0) = 2$  and  $\Omega(x') = \Omega(x_0)$ ,  $\eta'(x') \in \operatorname{PDir}(\operatorname{cl}_{\omega(x_0)}J(f, E))$ ; if  $\omega'(x_0) = 3$ , there exists at most one  $x'$  with  $\Omega(x') = \Omega(x_0)$ , and such  $x'$  is rational over  $x_0$ .

Proof of (i). We note that  $v_\delta(g) = 1$  is equivalent to  $\operatorname{ord}_{\eta(x_0)}(H(x_0)^{-1}g^p) = \epsilon(x_0)$ . By **II.5.4.1**,  $\operatorname{div}(H(x')^{-1}g'^p)$  is the strict transform of  $\operatorname{div}(H(x)^{-1}g^p)$  since  $\epsilon(x_0) = \epsilon(y)$ . So for every point  $x'$  above  $x_0$  not on the strict transform of all components of  $H(x_0)^{-1}g^p$ , we have

$$\epsilon(x') \leq \operatorname{ord}_{\eta'(x')} (H(x')^{-1}g'^p) < \operatorname{ord}_{\eta(x_0)} (H(x_0)^{-1}g^p) = \epsilon(x_0),$$

thus  $\omega(x') \leq \epsilon(x') \leq \epsilon(x_0) - 1 \leq \omega(x_0)$ . If  $\omega(x') = \omega(x_0)$ , then  $\omega(x') = \operatorname{ord}_{\eta'(x')} (H(x')^{-1}g'^p)$ , that is  $\omega'(x') = 1$  by definition **II.4**, and  $\omega(x_0) = \epsilon(x_0) - 1$ , which implies  $\omega'(x_0) \geq 2$  by definition **II.4**. Hence  $\Omega(x') < \Omega(x_0)$  and this ends the proof of (i).

Proof of (ii). Since  $Y$  is a curve and  $\epsilon(y) = \epsilon(x_0) = \operatorname{ord}_{\eta(x_0)}(H(x_0)^{-1}g^p)$ ,  $\operatorname{div}(H(x_0)^{-1}g^p)$  has at most two components. We consider two cases:

*Case 1:*  $\operatorname{div}(H(x_0)^{-1}g^p)$  has two reduced components, say is equal to  $\operatorname{div}(u_1u_2)$ . Since  $\epsilon(x_0) = \epsilon(y)$ , we have  $\operatorname{ord}_{\eta(x_0)}(H(x_0)^{-1}g^p) = \epsilon(x_0) = \operatorname{ord}_{\eta(y)}(H(x_0)^{-1}g^p)$ , so that  $Y = V(X, u_1, u_2)$ . There is no point  $x'$  on the strict transform of  $\operatorname{div}(u_1u_2)$ , so by (i),  $\Omega(x') < \Omega(x_0)$ .

*Case 2:*  $\operatorname{div}(H(x_0)^{-1}g^p)$  has only one reduced component. After possibly changing indices, we have  $(H(x_0)^{-1}g^p) = (u_1^{\epsilon(x_0)})$  and  $Y = V(X, u_1, u_2)$ . By (i), the only point  $x'$  to be considered has r.s.p.  $(\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, u_3)$  in  $R'$ . The polyhedron  $\Delta(h'; \frac{u_1}{u_2}, u_2, u_3; \frac{X}{u_2})$  is still minimal: if  $v' = (x_1, x_2, x_3)$  is a vertex of  $\Delta(h'; \frac{u_1}{u_2}, u_2, u_3; \frac{X}{u_2})$ , then  $v = (x_1, x_2 - x_1 + 1, x_3)$  is a vertex of  $\Delta(h; u_1, u_2, u_3; X)$  and  $\operatorname{in}_{v'}(h') = U_2^{-p} \operatorname{in}_v(h) \in k(x')[\frac{X}{U_2}, \frac{U_1}{U_2}, U_2, U_3]$  is not solvable (note that  $k(x') = k(x_0)$ ). We have  $\Psi \in k(x_0)[U_1, U_2]$  in (3), so  $H(x')^{-1}f' \equiv \Psi(\frac{u_1}{u_2}, 1) \pmod{(u_2, u_3)}$ .

If  $\Psi \in k(x_0)[U_1]$ , we have  $\Omega(x_0) = (\epsilon(x_0), 1)$  by **II.4**. On the other hand,

$$\omega(x') \leq \operatorname{ord}_{\eta'(x')} (H(x')^{-1}g'^p) = \epsilon(x_0) = \omega(x_0).$$

Therefore  $\Omega(x') < \Omega(x_0)$  if inequality is strict. If equality holds, we have by **II.4**  $\Omega(x') = (\omega(x'), 1) = \Omega(x_0)$ . Note that  $\operatorname{cl}_{\epsilon(x_0)}J(f, E, x_0) = k(x_0) \cdot U_1^{\epsilon(x_0)}$ , so  $\operatorname{IDir}(\operatorname{cl}_{\epsilon(x_0)}J(f, E, x_0)) = (U_1)$  as required.

If  $\Psi \notin k(x_0)[U_1]$ , we have

$$\omega(x') \leq \operatorname{ord}_{\eta'(x')} \Psi\left(\frac{u_1}{u_2}, 1\right) \leq \epsilon(x_0) - 1 < \epsilon(x_0) = \operatorname{ord}_{\eta'(x')} (H(x')^{-1}g'^p).$$

We have  $\Omega(x') < \Omega(x_0)$  unless  $\omega(x') = \omega(x_0) = \epsilon(x_0) - 1$ . In the latter case, we get  $\omega(x') = \epsilon(x')$ , so  $\Omega(x') = (\omega(x'), 2)$  by **II.4**. Since  $\omega(x_0) = \epsilon(x_0) - 1$ , we have  $\omega'(x_0) \geq 2$  and this proves that

$\Omega(x') \leq \Omega(x_0)$ . Equality holds only if  $\Psi = aU_1^{\omega(x_0)}U_2 + bU_1^{1+\omega(x_0)}$  for some  $0 \neq a \in k(x_0)$ ,  $b \in k(x_0)$ . Then  $\text{cl}_{\omega(x_0)}J(f, E) = k(x_0) \cdot U_1^{\omega(x_0)}$  and  $\text{IDir}(\text{cl}_{\epsilon(x_0)}J(f, E, x_0)) = (U_1)$  as required. This ends the proof of (ii).

Proof of (iii). After possibly changing indices, we have  $(H(x_0)^{-1}g^p) = (u_1^a u_2^b)$ ,  $a + b = \epsilon(x_0)$ ,  $a, b > 0$ . Then, by (i), the only point  $x'$  we have to consider has coordinates  $(\frac{X}{u_3}, \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3)$ . The polyhedron  $\Delta(h'; \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3; \frac{X}{u_3})$  is still minimal: if  $v' = (x_1, x_2, x_3)$  is a vertex of  $\Delta(h'; \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3; \frac{X}{u_3})$  then  $v = (x_1, x_2, x_3 - x_1 - x_2 + 1)$  is a vertex of  $\Delta(h; u_1, u_2, u_3; X)$  and  $\text{in}_{v'}(h') = U_3^{-p} \text{in}_v(h) \in k(x_0)[\frac{X}{U_3}, \frac{U_1}{U_3}, \frac{U_2}{U_3}, U_3]$  is not solvable (note that  $k(x') = k(x_0)$ ). We have  $\Psi \in k(x_0)[U_1, U_2, U_3]$  in (3) and  $H(x')^{-1}f' \equiv \Psi(\frac{u_1}{u_3}, \frac{u_2}{u_3}, 1) \pmod{u_3}$ .

If  $\Psi \in k(x_0)[U_1, U_2]$ , we have  $\Omega(x_0) = (\epsilon(x_0), 1)$  and  $x_0$  belongs to case **II.4(i)**. Then  $\omega(x') \leq \text{ord}_{\eta'(x')}(H(x')^{-1}g^p) = \epsilon(x_0)$  and we conclude that  $\Omega(x') \leq (\epsilon(x_0), 1)$  as in (ii).

If  $\Psi \notin k(x_0)[U_1, U_2]$ , then

$$\omega(x') \leq \text{ord}_{\eta'(x')}(\Psi(\frac{u_1}{u_3}, \frac{u_2}{u_3}, 1)) \leq \epsilon(x_0) - 1 < \epsilon(x_0) = \text{ord}_{\eta'(x')}(H(x')^{-1}g^p),$$

and we get  $\Omega(x') \leq \Omega(x_0)$  with equality only if  $\omega'(x_0) = 2$  as in the proof of (ii). This concludes the proof of (iii).

Proof of (iv). After possibly changing indices, we have  $(H(x_0)^{-1}g^p) = (u_1^{\epsilon(x_0)})$ . By (i), we only have to look at points  $x'$  on the strict transform of  $\text{div}(u_1)$ . At such a point,  $(H(x')^{-1}g^p) = ((\frac{u_1}{t})^{\epsilon(x_0)})$ , where  $t = 0$  is an equation of the exceptional divisor. So  $\omega(x') \leq \text{ord}_{\eta'(x')}(H(x')^{-1}g^p) = \epsilon(x_0)$ .

If  $\omega(x_0) = \epsilon(x_0)$ , we get  $\omega(x') \leq \omega(x_0)$ . Equality holds if and only if we have  $\omega(x') = \text{ord}_{\eta'(x')}(H(x')^{-1}g^p)$ , i.e.  $\omega'(x') = 1$ . This ends the proof of (iv) in the case  $\omega(x_0) = \epsilon(x_0)$ .

Let us now prove (iv) when  $\epsilon(x_0) = 1 + \omega(x_0)$ . In this case, we have  $\omega'(x_0) \geq 2$ ,  $\text{ord}_{\eta(x_0)}g^p = \text{ord}_{\eta(x_0)}f$ , and  $\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2)$ . First we note that, since  $(g^p) = (u_1^{a(1)} u_2^{a(2)} u_1^{\epsilon(x_0)})$ , where  $H(x_0) = u_1^{a(1)} u_2^{a(2)}$  ( $a(2) = 0$  if  $E = \text{div}(u_1)$ ), we have

$$a(1) + \epsilon(x_0) \equiv 0 \pmod{p}, \quad a(2) \equiv 0 \pmod{p}. \quad (5)$$

In particular,  $\delta(x_0) = \frac{a(1)+a(2)+\epsilon(x_0)}{p} \in \mathbb{N}$ .

We have  $\Psi \in k(x_0)[U_1, U_2, U_3]$ ,  $\Psi \notin k(x_0)[U_1, \dots, U_j]$  in (3) ( $j$  is the number of components of  $E$ ). We may suppose that  $U_3$  effectively appears in the expansion of  $\Psi$  and expand

$$\Psi = \sum_{0 \leq i \leq i_0} U_1^{\epsilon(x_0)-i} F_i(U_2, U_3), \quad (6)$$

where  $i_0 := \sup\{i | F_i \neq 0\}$ . We have  $i_0 > 0$  because  $\Psi \notin k(x_0)[U_1, \dots, U_j]$ .

Let us look at a point  $x'$  above  $x_0$  on the strict transform of  $\text{div}(u_1)$ , in the chart centered at the point of parameters  $\frac{X}{u_2} = X'$ ,  $\frac{u_1}{u_2} = u'_1$ ,  $u_2 = u'_2$ ,  $\frac{u_3}{u_2} = u'_3$ . Then we have

$$g'^{p-1} = \gamma^{p-1} u'_1^{\frac{p-1}{p}(a(1)+\epsilon(x_0))} u'_2^{(p-1)(\delta(x_0)-1)}$$

in (1) above, and

$$f' = u'_1^{a(1)} u'_2^{p(\delta(x_0)-1)} \left( \sum_{0 \leq i \leq i_0} u'_1^{\epsilon(x_0)-i} F_i(1, u'_3) + u'_2 \Sigma \right) \quad (7)$$

in (3) above, where  $\Sigma \in S'$ .

When  $u'_3$  is invertible at  $x'$ , we have to choose an irreducible homogeneous polynomial  $P \in k(x_0)[U_2, U_3]$ , unitary in  $U_3$  and such that, if we denote  $v' := P(1, u'_3)$ ,  $(X', u'_1, u'_2, v')$  is a r.s.p. of  $R'$ . Then, there is no reason for  $\Delta(h'; u'_1, u'_2, v'; X')$  to be minimal: we have to make a translation  $Z' := X' - \theta'$  to minimize this polyhedron.

To begin with, we compute the vertex  $w$  of  $\Delta(h'; u'_1, u'_2, v'; Z')$  with  $(x_1, x_2)$  minimal for the inverse lexicographical ordering.

As we change  $X'$  into  $Z' = X' - \theta'$ ,  $f'$  is changed into

$$\phi = f' + \theta'^p - \theta' g'^{p-1}.$$

Since the Newton polyhedron of  $\theta'$  is a subset of  $\Delta(h'; u'_1, u'_2, v'; X')$ , every  $(y_1, y_2, y_3)$  in this Newton polyhedron verifies (for the inverse lexicographical ordering)

$$(y_1, y_2) \geq \left( \frac{a(1) + \epsilon(x_0) - i_0}{p}, \delta(x_0) - 1 \right),$$

so every  $(z_1, z_2, z_3)$  in the Newton polyhedron of  $\theta' g'^{p-1}$  verifies

$$(z_1, z_2) \geq \left( \frac{a(1) + \epsilon(x_0) - i_0}{p} + \frac{p-1}{p}(a(1) + \epsilon(x_0)), p(\delta(x_0) - 1) \right).$$

In particular, all terms of order  $p(\delta(x_0) - 1)$  in  $u'_2$  in  $\theta' g'^{p-1}$  have order in  $u'_1$  strictly bigger than  $a(1) + \epsilon(x_0) - i_0$ , so the vertex  $w$  will be given by

$$u'_1{}^{a(1)} u'_2{}^{p(\delta(x_0)-1)} u'_1{}^{\epsilon(x_0)-i_0} F_{i_0}(1, u'_3) + \Theta'^p, \quad (8)$$

where  $\Theta' \in S'$  is zero or has order exactly  $(a(1) + \epsilon(x_0) - i_0)/p$  in  $u'_1$  and  $\delta(x_0) - 1$  in  $u'_2$ . We now consider two cases:

*Case 1:*  $i_0 \not\equiv 0 \pmod{p}$ . Then (5) implies that  $\Theta'$  is necessarily zero. So in this case

$$f' = u'_1{}^{a(1)+\epsilon(x_0)-i_0} u'_2{}^{p(\delta(x_0)-1)} (F_{i_0}(1, u'_3) + \Sigma'),$$

for some  $\Sigma' \in (u'_1, u'_2)$ . Case 1 splits into two subcases:

*Case 1a:*  $\text{ord}_{\eta'(x')} F_{i_0}(1, u'_3) < i_0$ . Then  $\epsilon(x') \leq \epsilon(x_0) - 1 = \omega(x_0)$  and we deduce that  $\Omega(x') \leq \Omega(x_0)$ . Equality holds only if  $\omega'(x_0) = 2$ ,  $\epsilon(x') = \omega(x') = \epsilon(x_0) - 1$ , and  $\text{ord}_{\eta'(x')} F_{i_0}(1, u'_3) = i_0 - 1$ . Then  $i_0 - 1 =: p\alpha$ ,  $\alpha \in \mathbb{N}$ , since  $\epsilon(x') = \omega(x')$  and we have  $F_{i_0}(U_2, U_3) = P(U_2, U_3)^{p\alpha} Q(U_2, U_3)$ , with  $\deg Q = 1$ . Now,

$$\epsilon(x') \leq \alpha + \epsilon(x_0) - i_0 = \epsilon(x_0) - 1 - (i_0 - 1)\left(1 - \frac{1}{\deg P}\right). \quad (9)$$

If  $\alpha = 0$ , we have  $\text{cl}_{\omega(x_0)} J(f, E) = k(x_0).U_1^{\omega(x_0)}$  as required. If  $\alpha > 0$ , we have  $\deg P = 1$  by (9), since  $\epsilon(x') = \epsilon(x_0) - 1$  and  $i_0 - 1 > 0$ . We now choose  $(X, u_1, u_2, v := P(u_2, u_3))$  as r.s.p. of  $R$ , where  $X$  has been chosen in such a way that the polyhedron  $\Delta(h; u_1, u_2, v; X)$  is minimal. The point  $x'$  has parameters  $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, v' = \frac{v}{u_2})$  and  $\Delta(h'; u'_1, u'_2, v'; X')$  is thus minimal (see the argument at the beginning of the proof of **II.5.4.2**(iii)). Since  $\epsilon(x') = \epsilon(x_0) - 1$ , we get that in (6),  $\Psi$  is of the form  $\Psi = \Psi_1(U_1, V) + U_1^{\epsilon(x_0)-i_0} U_2 \Psi_2(U_1, V)$  with  $\Psi_2(0, V) \neq 0$ . On the other hand,  $\omega(x') = \epsilon(x')$ , thus  $\Psi_2 \in k(x_0)[U_1, V^p]$ . Therefore,  $E = \text{div}(u_1)$  and

$$\text{VDir}(\text{cl}_{\omega(x_0)} J(f, E)) = \langle U_1, V \rangle,$$

which proves the statement about the directrix.

*Case 1b:*  $\text{ord}_{\eta'(x')} F_{i_0}(1, u'_3) = i_0$ . Then  $(F_{i_0}(1, u'_3)) = (v'^{i_0})$ ,  $x'$  is rational over  $x_0$  and we have  $\epsilon(x') \leq \epsilon(x_0)$ . If inequality is strict, we get  $\Omega(x') \leq \Omega(x_0)$  as in case 1a, and the equality case (only if  $\omega'(x_0) = 2$ ) is dealt with similarly.

If  $\epsilon(x') = \epsilon(x_0)$ , with notations as in case 1a,  $\Psi$  is of the form  $\Psi = U_1^{\epsilon(x_0)-i_0} \Psi_1(U_1, V)$  with  $\Psi_1(0, V) \neq 0$ . Thus

$$\epsilon(x_0) - 1 \leq \omega(x') \leq \text{ord}_{\eta'(x')} (H(x')^{-1} \frac{\partial f'}{\partial v'}) = \epsilon(x_0) - 1 = \omega(x_0) \quad (10)$$

and  $\text{VDir}(\text{cl}_{\omega(x_0)} J(f, E)) = \langle U_1, V \rangle$ . We claim that  $\omega'(x') = 2$ , which concludes the proof of case 1b.

To begin with, the vertex  $w = (\frac{a(1)+\epsilon(x_0)-i_0}{p}, \delta(x_0) - 1, i_0/p)$  of  $\Delta(h'; u'_1, u'_2, v'; X')$  is not solvable, since it has two coordinates which are not integers. Furthermore, if we change our r.s.p.  $(u'_1, u'_2, v')$  of  $S'$  to a new r.s.p. adapted to  $E'$ , the latter is of the form  $(u''_1 := \gamma'_1 u'_1, u''_2 := \gamma'_2 u'_2, v'' := \gamma' v' + \varphi)$ , where  $\varphi \in (u''_1, u''_2) S'$  and  $\gamma'_1 \gamma'_2 \gamma' \in S'$  is a unit. Then  $w$  still appears as a vertex in  $\Delta(h'; u''_1, u''_2, v''; X')$  as well as in the minimal polyhedron  $\Delta(h'; u''_1, u''_2, v''; Z'')$  after performing a translation  $Z'' = X' - \theta''$ . The computation in (10) remains valid with derivatives w.r.t.  $(u''_1, u''_2, v'')$ , so  $x'$  is in case **II.4(v)**,  $\omega'(x') = 2$  and the claim is proved.

*Case 2:*  $i_0 \equiv 0 \pmod{p}$ . Then the series  $\Theta'$  in (8) may be nonzero. Since  $i_0 > 0$ , we have  $i_0 \geq p \geq 2$ . Note that the terms  $H(x_0) u_1^{\epsilon(x_0)-i_0} F_{i_0}(u_2, u_3)$  correspond to a face of the *minimal* polyhedron  $\Delta(h; u_1, u_2, u_3; X)$ , so that in particular  $U_2^{a(2)} F_{i_0}(U_2, U_3)$  is not a  $p^{\text{th}}$ -power. We apply **I.5.3.2(iii)**, (iv) to  $U_2^{a(2)} F_{i_0}(U_2, U_3)$ . Then, in (8), we get

$$u_1^{a(1)} u_2^{p(\delta(x_0)-1)} u_1^{\epsilon(x_0)-i_0} F_{i_0}(1, u'_3) + \Theta'^p \equiv u_1^{a(1)} u_2^{p(\delta(x_0)-1)} u_1^{\epsilon(x_0)-i_0} (\gamma' v'^e + u'_2 \varphi),$$

with  $\gamma'$  invertible,  $\varphi \in S'$  and  $e \leq i_0$ . Then

$$\epsilon(x') \leq e + \epsilon(x_0) - i_0 \leq \epsilon(x_0). \quad (11)$$

If some inequality is strict (for instance if  $e < i_0$ ), we get  $\Omega(x') \leq \Omega(x_0)$  as in case 1a, and the equality case (only if  $\omega'(x_0) = 2$ ) is dealt with similarly.

So from now on, we assume that all inequalities in (11) are equalities. In particular,  $\epsilon(x_0) = \epsilon(x')$ . By **I.5.3.2(iv)**, we have  $v' = \lambda + u'_3$ ;  $\lambda \in k(x_0)$ ,  $U_2 \frac{\partial F_{i_0}}{\partial U_2}, \frac{\partial F_{i_0}}{\partial \lambda_i}$ ,  $4 \leq i \leq s$  are multiples of  $(\lambda U_2 + U_3)^{i_0}$ ;  $\frac{\partial F_{i_0}}{\partial U_3}$  is a multiple of  $(\lambda U_2 + U_3)^{i_0-1}$  (remember that  $p$  divides  $a(2)$ ). Since  $i_0 \equiv 0 \pmod{p}$ , we must have  $\frac{\partial F_{i_0}}{\partial U_3} = 0$ ; since  $F_{i_0}$  is not a  $p^{\text{th}}$ -power, one of the other derivatives is nonzero. So

$$\text{Vect}_{k(x_0)} \langle U_2 \frac{\partial F_{i_0}}{\partial U_2}, \{ \frac{\partial F_{i_0}}{\partial \lambda_i} \}_{4 \leq i \leq s} \rangle = k(x_0) \cdot (\lambda U_2 + U_3)^{i_0}.$$

Therefore, there exists  $0 \neq \mu_{i_0} \in k(x_0)$ ,  $\Theta \in k(x_0)[U_2, U_3]_{\frac{i_0}{p}}$  such that

$$F_{i_0}(U_2, U_3) = \mu_{i_0} (\lambda U_2 + U_3)^{i_0} + \Theta^p.$$

Since  $\Delta(h; u_1, u_2, u_3; X)$  is minimal, the monomial  $\mu_{i_0} u_1^{a(1)+\epsilon(x_0)-i_0} u_2^{a(2)} u_3^{i_0}$  must define a non solvable vertex  $w$  of the initial face of  $\Delta(h; u_1, u_2, u_3; X)$ , hence  $\mu_{i_0} \notin k(x_0)^p$ .

We now choose  $(X, u_1, u_2, v = u_3 + \lambda u_2)$  as r.s.p. of  $R$ , where  $X$  has been chosen in such a way that the polyhedron  $\Delta(h; u_1, u_2, v; X)$  is minimal. The point  $x'$  has parameters  $(X' = \frac{X}{u_2}, u'_1 =$

$\frac{u_1}{u_2}, u_2' = u_2, v' = \frac{v}{u_2}$ ) and  $\Delta(h'; u_1', u_2', v'; X')$  is thus minimal (see the argument at the beginning of the proof of **II.5.4.2**(iii)). Since  $\epsilon(x') = \epsilon(x_0)$ , we have  $F_i(1, u_3') = \mu_i v'^i$ ,  $\mu_i \in k(x_0)$  in (7) for each  $i$ ,  $0 \leq i \leq i_0$ , so  $F_i(U_2, U_3) = \mu_i V^i$ . This means that the initial face of  $\Delta(h; u_1, u_2, v; X)$  is a segment with ends

$$a := \left( \frac{a(1) + \epsilon(x_0) - i_0}{p}, \frac{a(2)}{p}, \frac{i_0}{p} \right), \quad b := \left( \frac{a(1) + \epsilon(x_0)}{p}, \frac{a(2)}{p}, 0 \right), \quad (12)$$

where  $a$  corresponds to the vertex  $w$  and  $b$  to the monomial  $Xg^{p-1}$ . By (7), this also implies that the face of  $\Delta(h'; u_1', u_2', v'; X')$  with minimal second coordinate is the segment with ends

$$a' := \left( \frac{a(1) + \epsilon(x_0) - i_0}{p}, \delta(x_0) - 1, \frac{i_0}{p} \right), \quad b' := \left( \frac{a(1) + \epsilon(x_0)}{p}, \delta(x_0) - 1, 0 \right). \quad (13)$$

Suppose we changed  $X$  into

$$Z := X - \mu u_1^{\frac{\alpha(1)+\epsilon(x_0)-i_0}{p}+i} u_2^{\frac{\alpha(2)}{p}} v^{\frac{i_0}{p}-i},$$

where  $i \in \{0, 1\}$  is such that  $i_0/p - i \not\equiv 0 \pmod{p}$  and  $\mu \frac{i_0}{p} - i - \mu \bar{\gamma}^{p-1} \neq 0$ , where  $\bar{\gamma}$  is the image of  $\gamma$  in  $k(x_0)$ . We would have  $\Delta(h; u_1, u_2, v; Z) = \Delta(h; u_1, u_2, v; X)$  by construction, and the coefficient of the monomial  $u_1^{\frac{\alpha(1)+\epsilon(x_0)-i_0}{p}+i} u_2^{\frac{\alpha(2)}{p}} v^{\frac{i_0}{p}-i}$  in

$$f + \left( u_1^{\frac{\alpha(1)+\epsilon(x_0)-i_0}{p}+i} u_2^{\frac{\alpha(2)}{p}} v^{\frac{i_0}{p}-i} \right)^p - u_1^{\frac{\alpha(1)+\epsilon(x_0)-i_0}{p}+i} u_2^{\frac{\alpha(2)}{p}} v^{\frac{i_0}{p}-i} g^{p-1}$$

would be nonzero. This proves that  $\text{VDir}(\text{cl}_{\omega(x_0)} J(f, E)) = \langle U_1, V \rangle$  and, comparing with (7), that  $\omega(x') = \epsilon(x_0) - 1 = \omega(x_0)$ . If  $\omega'(x_0) = 3$  or if  $\omega'(x') = 2$ , we are done.

Fortunately, we cannot have at the same time  $\omega'(x_0) = 2$  and  $\omega'(x') = 3$ . The idea is that, since  $x'$  is rational over  $x_0$ , if there are translations over  $X'$  which drop the order of  $J(f', E')$  by one, there are corresponding translations on  $X$  which also drop the order of  $J(f, E)$  by one.

More precisely, suppose we have  $\Omega(x') = (\omega(x_0), 3)$ . By definition, there exists a r.s.p.  $(w_1', w_2', w_3')$  of  $S'$ , with  $w_1' = \gamma_1' u_1'$ ,  $w_2' = \gamma_2' u_2'$ ,  $w_3' = \nu_1' u_1' + \nu_2' u_2' + \nu_3' v$ ,  $\gamma_i', \nu_j' \in S'$ ,  $\gamma_1' \gamma_2' \nu_3'$  invertible, and  $\theta' \in \widehat{S}'$  such that the following holds: denoting  $Z' := X' - \theta'$ , the polyhedron  $\Delta(h'; w_1', w_2', w_3'; Z')$  is minimal and the order of  $J(f' + \theta'^p - \theta' g'^{p-1}, E')$  is  $\epsilon(x')$ . Now, by (7), the initial form  $\Phi'$  of  $f'$ , written in the coordinates  $(W_1', W_2', W_3')$  is of the form

$$\Phi' = (\overline{\gamma_1'}^{-1} W_1')^{\alpha(1)} (\overline{\gamma_2'}^{-1} W_2')^{p(\delta(x_0)-1)} \left( \sum_{0 \leq i \leq i_0} (\overline{\gamma_1'}^{-1} W_1')^{\epsilon(x_0)-i} \overline{\mu_i \nu_3'}^{-i} (W_3' - \frac{\overline{\nu_1'}}{\overline{\gamma_1'}} W_1')^i + W_2' \Sigma \right),$$

where  $\overline{\gamma_i'}, \overline{\nu_j'}$  are the images of  $\gamma_i', \nu_j'$  in  $k(x') = k(x_0)$ . Since the Newton polyhedron of  $\theta'$  is a subset of  $\Delta(h'; w_1', w_2', w_3'; X')$ , the segment (13), whose ends are not solvable, is still a face of  $\Delta(h'; w_1', w_2', w_3'; Z')$ :  $b'$  is not solvable by definition, and we have

$$\text{in}_{a'} f' = \mu_{i_0} \left( (\overline{\gamma_1'}^{-1} W_1')^{\frac{\alpha(1)+\epsilon(x_0)-i_0}{p}} (\overline{\gamma_2'}^{-1} W_2')^{\delta(x_0)-1} (\overline{\nu_3'} W_3')^{\frac{i_0}{p}} \right)^p,$$

with  $\mu_{i_0} \notin k(x')^p$ , so  $a'$  is not solvable either.

Since  $\text{ord}_{\eta'(x')} J(f' + \theta'^p - g'^{p-1}\theta', E') = \epsilon(x')$ ,  $\Phi'$  must be of the form

$$\Phi' = W_1'^{a(1)} W_2'^{p(\delta(x_0)-1)} \left( \sum_{0 \leq j \leq \frac{i_0}{p}} \mu'_{jp} W_1'^{\epsilon(x_0)-jp} W_3'^{jp} + W_2' \Sigma(W_1', W_2', W_3'^p) \right) + \Sigma_1^p, \quad (14)$$

for some  $\mu'_{jp} \in k(x')$  and  $\Sigma_1 \in k(x')[W_1', W_2', W_3']$ . We pick  $\gamma_i, \nu_j \in S$  such that  $\gamma_i \equiv \gamma'_i \pmod{m_{S'}}$ ,  $\nu_j \equiv \mu'_j \pmod{m_{S'}}$  and let  $w_i := \gamma_i u_i$ ,  $i = 1, 2$  and  $w_3 := \nu_1 u_1 + \nu_2 u_2^2 + \nu_3 v$ . By construction,  $\Delta(h; w_1, w_2, w_3; X)$  and  $\Delta(h; u_1, u_2, v; X)$  have the same initial face: the segment (12). The vertex  $a$  is given by the monomial  $\mu_{i_0} \lambda_1^{-(a(1)+\epsilon(x_0)-i_0)} \nu_3^{-i_0} w_1^{a(1)+\epsilon(x_0)-i_0} w_2^{a(2)} w_3^{i_0}$  and is not solvable, since  $\mu_{i_0} \notin k(x_0)^p$ . The vertex  $b$  is not solvable either by definition. Consider, if necessary, a change of coordinates  $Z := X - \theta$  making  $\Delta(h; w_1, w_2, w_3; Z)$  minimal. Then  $f$  becomes  $f_Z := f + \theta^p - \theta g^{p-1}$  in these new coordinates. Since neither  $a$  nor  $b$  is solvable, we have  $\theta^p \in H(x_0) m_S^{\epsilon(x_0)+1}$ , so that  $f \equiv f_Z \pmod{H(x_0) m_S^{\epsilon(x_0)+1}}$ . Comparing now (7) and (14), the initial form  $\Phi$  of  $f$  (or  $f_Z$ ), written in the variables  $(W_1, W_2, W_3)$  must be

$$\Phi = W_1^{a(1)} W_2^{a(2)} \left( \sum_{0 \leq j \leq \frac{i_0}{p}} \mu'_{jp} W_1^{\epsilon(x_0)-jp} W_3^{jp} \right).$$

This shows that  $\text{ord}_{\eta(x_0)} J(f_Z, E) = \epsilon(x_0)$ . Since  $\omega(x_0) = \epsilon(x_0) - 1$  and  $\Delta(h; w_1, w_2, w_3; Z)$  is minimal, we have  $\omega'(x_0) = 3$ , thus  $\Omega(x') = \Omega(x_0)$ .

This concludes the study of the first chart of the blowing up  $X_1$ . The last point  $x'$  to look at is the point with coordinates  $(\frac{X}{u_3} =: X', \frac{u_1}{u_3} =: u'_1, \frac{u_2}{u_3} =: u'_2, u_3 =: u'_3)$ .

If  $E = \text{div}(u_1)$ , then  $u_2$  and  $u_3$  have symmetric role and by changing indices, we come back to the origin of the chart studied above. Only note that if there was a point  $x''$  with  $\Omega(x'') = \Omega(x_0)$  and  $\omega'(x_0) = 3$  in the first chart, (11) implies that  $\epsilon(x') \leq \epsilon(x_0) - i_0 \leq \epsilon(x_0) - p$ , so that the uniqueness statement in (iv) is established.

If  $\text{div}(u_1 u_2) \subseteq E$ , then, with notations as in (7), we have  $h' = X'^p - X' g'^{p-1} + f'$  with

$$g'^{p-1} = \gamma^{p-1} u_1'^{\frac{p-1}{p}(a(1)+\epsilon(x_0))} u_2'^{\frac{p-1}{p}a(2)} u_3'^{(p-1)(\delta(x_0)-1)},$$

$$f' = u_1'^{a(1)} u_2'^{a(2)} u_3'^{p(\delta(x_0)-1)} \left( \sum_i u_1'^{\epsilon(x_0)-i} F_i(u'_2, 1) + u_3' \Sigma \right),$$

where  $\Sigma \in S'$ . We are at the origin of a chart, so that by the argument in the proof of **II.5.4.2(iii)**,  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal.

If, for all  $i$ ,  $F_i \in k(x_0)[U_2]$ , we have  $\Omega(x_0) = (\epsilon(x_0), 1)$ . Since  $E' = \text{div}(u'_1 u'_2 u'_3)$  in this case, we have  $\omega(x') = \epsilon(x') \leq \text{ord}_{\eta'(x')} (H(x')^{-1} g'^p) = \epsilon(x_0) = \omega(x_0)$ . We thus have  $\Omega(x') < \Omega(x_0)$  if  $\epsilon(x') < \text{ord}_{\eta'(x')} (H(x')^{-1} g'^p)$ , and  $\Omega(x') = \Omega(x_0)$  if  $\epsilon(x') = \text{ord}_{\eta'(x')} (H(x')^{-1} g'^p)$ .

If, for some  $i$ ,  $F_i \notin k(x_0)[U_2]$ , then  $\epsilon(x') \leq \epsilon(x_0) - 1$ . We have  $\Omega(x') \leq \Omega(x_0)$ , with equality only if  $\Omega(x_0) = (\epsilon(x_0) - 1, 2)$  and  $\Psi$  is of the form  $\Psi = \Psi_1(U_1, U_2) + U_3 \Psi_2(U_1, U_2)$ . Therefore  $E = \text{div}(u_1 u_2)$  and  $\langle U_1 \rangle \subseteq \text{VDir}(\text{cl}_{\omega(x_0)} J(f, E)) \subseteq \langle U_1, U_2 \rangle$  as required. This concludes the proof.

**II.5.4.3 Lemma.** *If  $v_\delta(g) > 1$ ,  $\epsilon(x_0) = \epsilon(y)$  (first kind of permissible blowing-up) and  $x' \in \Sigma_p(X_1)$ , we have  $\Omega(x') \leq \Omega(x_0)$ . If equality holds, then:*

(i)  $\eta'(x') \in \text{PDir}(\text{cl}_{\epsilon(x_0)} J(f, E, x_0))$  if  $\omega(x_0) = \nu(x_0) = \epsilon(x_0)$ ,

(ii)  $\eta'(x') \in \text{PDir}(\text{cl}_{\omega(x_0)}J(f, E))$  if  $\omega(x_0) = \nu(x_0) = \epsilon(x_0) - 1$ .

Proof. Since  $v_\delta(g) > 1$ , we have  $\omega'(x_0) = 2$  and  $x_0$  belongs to case **II.4(ii)**. We keep notations as in the beginning of the proof of **II.5.4**. By **II.5.2**, we have: either  $\text{ord}_{\eta(y)}(H(x_0)^{-1}g^p) > \text{ord}_{\eta(y)}(H(x_0)^{-1}f) = \epsilon(y)$ , and then  $g'^p \in tH(x')S'$  in **II.5.4(1)**, so

$$\theta' t^{(p-1)(\sum_{i \in I_E} \beta_i - 1)} \tilde{g}^{p-1} \in tH(x')S' \quad (1)$$

in **II.5.4(4)**; or  $\text{ord}_{\eta(y)}(H(x_0)^{-1}g^p) = \text{ord}_{\eta(y)}(H(x_0)^{-1}f) = \epsilon(y)$ , in which case some  $u_r$  with  $1 \leq r \leq e$ ,  $r \notin I_E$ , divides  $H(x_0)^{-1}g^p$ , since  $\text{ord}_{\eta(x_0)}(H(x_0)^{-1}g^p) > \text{ord}_{\eta(x_0)}(H(x_0)^{-1}f) = \epsilon(x_0) = \epsilon(y)$ . In this case, we have

$$\theta' t^{(p-1)(\sum_{i \in I_E} \beta_i - 1)} \tilde{g}^{p-1} \in u_r H(x')S' \quad (1')$$

in **II.5.4(4)**. In both cases, **II.5.4(3)** and **II.5.4.1** together imply that

$$\phi \equiv H(x')\Psi(\{u'_i\}_{i \in I}) + \theta'^p \pmod{H(x')(t, u_r)S'} \quad (2)$$

in **II.5.4(4)**. By **II.5.4(2)** and **II.5.4.1**, we have

$$H(x')^{-1}\mathcal{J}(\phi, E') \equiv t^{-\epsilon(y)}H(x_0)^{-1}\mathcal{J}(H(x_0)\Psi(\{u_i\}_{i \in I}), E, Y) \pmod{H(x')(t, u_r)S'} \quad (3)$$

if  $Y = \{x_0\}$  (then  $\{r\} = \emptyset$ ), or if ( $Y$  is a curve and  $\text{div}(u_r) \subseteq E$ ). If ( $Y$  is a curve and  $\text{div}(u_r) \not\subseteq E$ ), then

$$H(x')^{-1}\mathcal{J}(\phi, E') \equiv (t^{-\epsilon(y)}H(x_0)^{-1}\mathcal{J}(H(x_0)\Psi(\{u_i\}_{i \in I}), E, Y), H(x')^{-1}\frac{\partial \phi}{\partial u_r}) \pmod{H(x')(t, u_r)S'}. \quad (3')$$

We consider two case:

*Case 1:*  $\omega(x_0) = \epsilon(x_0)$ . As the blowing up is of the first kind, we have by **II.5.2(i)**

$$\omega(x_0) = \epsilon(x_0) = \epsilon(y) = \text{ord}_{\eta(x_0)}(H(x_0)^{-1}\mathcal{J}(H(x_0)\Psi(\{u_i\}_{i \in I}), E, Y)),$$

so  $\omega(x') \leq \omega(x_0)$  by (3) or (3'). Furthermore,

$$\text{cl}_{\epsilon(x_0)}(H(x_0)^{-1}\mathcal{J}(H(x_0)\Psi(\{u_i\}_{i \in I}), E, Y) = \text{cl}_{\epsilon(x_0)}J(f, E, x_0) = \text{cl}_{\epsilon(x_0)}J(f, E, Y),$$

so if  $\eta'(x') \notin \text{PDir}(\text{cl}_{\epsilon(x_0)}J(f, E, x_0))$ , we have  $\omega(x') < \omega(x_0)$  as required.

Finally, if we change  $(t, \{u'_i\}_{i \in I_i}, \{u_i\}_{i \in I^c})$  to new variables in  $S'$ , then minimize the corresponding polyhedron by a translation over  $Z'$ ,  $Z'' := Z' - \theta''$ , (1) (resp. (1')) implies that

$$\text{ord}_t(\theta'' g'^{p-1}) > \text{ord}_t H(x'),$$

$$(\text{resp. } \text{ord}_{u_r}(\theta'' g'^{p-1}) > \text{ord}_{u_r} H(x')).$$

This translation therefore only adds some  $p^{\text{th}}$ -power to  $H(x')\Psi(\{u'_i\}_{i \in I})$  modulo  $(tH(x'))$  (resp. modulo  $(u_r H(x'))$ ) in (2). So if  $\omega(x') = \omega(x_0) = \epsilon(x_0)$ , we have

$$\omega(x') = \text{ord}_{\eta'(x')}(H(x')^{-1}\mathcal{J}(H(x')\Psi(\{u'_i\}_{i \in I}), E')$$

independently of coordinate changes at  $x'$ , so  $\omega'(x') \leq 2$  and  $\Omega(x') \leq \Omega(x_0)$  as required.

Case 2:  $\omega(x_0) = \epsilon(x_0) - 1$ . As the blowing up is of the first kind, we have by **II.5.2(ii)**

$$\omega(x_0) = \epsilon(x_0) - 1 = \epsilon(y) - 1 = \text{ord}_{\eta(x_0)}(H(x_0)^{-1}\mathcal{J}(H(x_0)\Psi(\{u_i\}_{i \in I}), E, Y)) - 1.$$

It is easily seen that

$$I(Y)J(f, E) \subseteq J(f, E, Y), \quad I(Y)H(x_0)^{-1}\mathcal{J}(H(x_0)\Psi, E) \subseteq H(x_0)^{-1}\mathcal{J}(H(x_0)\Psi, E, Y),$$

so with notations as in **II.5.4(2)**, we have

$$t^{-\omega(x_0)}J(f, E) = t^{-\epsilon(x_0)}I(Y)J(f, E) \subseteq J(f', E').$$

Now (3) or (3') combined with **II.5.4(3)** imply that

$$\omega(x') \leq \text{ord}_{\eta'(x')} (t^{-\omega(x_0)}H(x_0)^{-1}\mathcal{J}(H(x_0)\Psi(\{u_i\}_{i \in I}), E)) \leq \omega(x_0).$$

If  $\eta'(x') \notin \text{PDir}(\text{cl}_{\omega(x_0)}J(f, E))$ , we have  $\text{ord}_{\eta'(x')} (t^{-\omega(x_0)}H(x_0)^{-1}\mathcal{J}(H(x_0)\Psi(\{u_i\}_{i \in I}), E)) < \omega(x_0)$ , so  $\omega(x') < \omega(x_0)$ . If  $\eta'(x') \in \text{PDir}(\text{cl}_{\omega(x_0)}J(f, E))$  and  $\omega(x') = \omega(x_0)$ , we conclude by the same argument as in case 1 that  $\omega'(x') \leq 2$ , and therefore  $\Omega(x') \leq \Omega(x_0)$ . ■

**II.5.4.4 Lemma.** *If  $Y$  is a permissible center of second kind at  $x_0$ , then  $\omega'(x_0) \geq 2$ . There exists a r.s.p.  $(X, u_1, u_2, u_3)$  of  $R$  such that  $\Delta(h; u_1, u_2, u_3; X)$  is minimal and*

(i)  $\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2)$  and  $Y = V(X, u_1, u_2)$ ;

(ii)  $\Psi = \text{cl}_{\epsilon(x_0)}(H(x_0)^{-1}f) = U_3\Psi_3(U_1, U_2) + \Psi_0(U_1, U_2)$ ,  $\Psi_3 \neq 0$ ,  $\Psi_0, \Psi_3 \in k(x_0)[U_1, U_2]$ .

Furthermore, every  $x' \in \Sigma_p(X_1)$  has  $\Omega(x') \leq \Omega(x_0)$ . If  $\Omega(x') = \Omega(x_0)$ , we have  $\omega'(x_0) = 2$ ,  $\Psi_3(U_1, U_2)$  is the power of a linear form, and  $\eta'(x') = \text{PDir}(\Psi_3(U_1, U_2))$ . In particular,  $x'$  is rational over  $x_0$ .

Proof. By **II.5.1.1**,  $Y$  is a curve. We choose the r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  in the definition of permissibility of second kind. Then  $\Delta(h; u_1, u_2, u_3; X)$  is minimal,  $E \subseteq \text{div}(u_1 u_2 u_3)$ ,  $\text{div}(u_1) \subseteq E_0$  and  $Y = V(X, u_1, u_2)$ ; moreover, we have  $H(x_0)^{-1}f \in (u_1, u_2)^{\epsilon(x_0)-1}$  and  $\text{ord}_{\eta(y)}J(f, E, Y) = \epsilon(x_0) - 1$ , so that there exists an expression of  $\Psi$  as in (ii) (see remarks in the beginning of the proof of theorem **II.5.4**). Note that, still by definition of permissibility of second kind, we have

$$\text{ord}_{\eta(x_0)}J(f, E, Y) = \epsilon(x_0) - 1 < \epsilon(x_0) \leq \text{ord}_{x_0} \left( H(x_0)^{-1}(u_1, u_2) \frac{\partial f}{\partial u_i} \right)$$

for  $i = 1, 2, 3$ . Therefore  $H(x_0)^{-1} \frac{\partial f}{\partial u_3} \in J(f, E, Y)$ , so  $\text{div}(u_3) \not\subseteq E$  and this gives (i) and (ii). Also note that

$$0 < \omega(x_0) = \epsilon(x_0) - 1 < \text{ord}_{x_0}(H(x_0)^{-1}g^p), \quad (1)$$

which implies that  $\omega'(x_0) \geq 2$ .

Let us prove that  $\Omega(x') \leq \Omega(x_0)$ . If  $x'$  is in the chart of origin  $(\frac{X}{u_1} = X', u_1 = u'_1, \frac{u_2}{u_1} = u'_2, u_3 = u'_3)$ , then

$$h' = X'^p - X'g'^{p-1} + u'_1{}^{a(1)+a(2)+\omega(x_0)-p}u'_2{}^{a(2)}(\Psi_3(1, u'_2)u'_3 + u'_1\phi'_1 + u'_3{}^2\phi'_2) \quad (2)$$

with  $\phi'_1, \phi'_2 \in S'$ , where  $H(x_0) = u_1^{a(1)}u_2^{a(2)}$ . Since  $(H(x')^{-1}g'^p) = u_1^{-\epsilon(y)}(H(x_0)^{-1}g^p)$  in **II.5.4(1)** and  $\epsilon(y) = \epsilon(x_0) - 1$ , (1) implies that  $u'_1$  divides  $H(x')^{-1}g'^p$ . When performing a translation

$Z' = X' - \theta'$  in order to get  $\Delta(h'; u'_1, v', u'_3)$  minimal (where  $(u'_1, v', u'_3)$  is a r.s.p. of  $S'$ ), we thus get an expression  $h' = Z'^p - g'^{p-1}Z' + f'_{Z'}$  with

$$f'_{Z'} \equiv u'_1{}^{a(1)+a(2)+\omega(x_0)-p} u'_2{}^{a(2)} \Psi_3(1, u'_2) u'_3 + \phi'^p \pmod{H(x')(u'_1, u'_3{}^2)}$$

for some  $\phi' \in \widehat{S}'$  by (2). So if  $\text{ord}_{v'} \Psi_3(1, u'_2) < \epsilon(x_0) - 1 = \deg \Psi_3$ , then  $\nu(x') < \epsilon(x_0) - 1 = \omega(x_0)$  and  $\omega(x') < \omega(x_0)$ . So we are interested in the case where  $\text{ord}_{v'} \Psi_3(1, u'_2) = \epsilon(x_0) - 1$ , that is,  $\Psi_3 = \lambda(U_2 + \mu U_1)^{\epsilon(x_0)-1}$ ,  $0 \neq \lambda \in k(x_0)$ ,  $\mu \in k(x_0)$ , and  $R'$  has r.s.p.  $(X', u'_1, v' := u'_2 + \mu, u'_3)$ . By the previous comments,

$$\text{ord}_{\eta'(x')}(H(x')^{-1} \frac{\partial f'_{Z'}}{\partial u'_3}) \leq \text{ord}_{v'} \Psi_3(1, u'_2) = \epsilon(x_0) - 1.$$

This proves that  $\nu(x') \leq \epsilon(x_0) - 1 = \omega(x_0)$ . If equality holds, note that

$$\Psi' := \text{cl}_{\epsilon x_0}(H(x')^{-1} f'_{Z'}) = \lambda V'^{\epsilon(x_0)-1} U'_3 + U'_1 F'(U'_1, V', U'_3)$$

for some  $F' \in k(x')[U'_1, V', U'_3]_{\epsilon(x_0)-1}$ . We then deduce that  $\omega'(x') \leq 2$ , therefore  $\Omega(x') \leq \Omega(x_0)$ , since  $\text{ord}_{u'_1}(H(x')^{-1} g'^p) > 0$  and  $\Psi'(0, V', U'_3) \notin k(x')[V'^p, U'_3{}^p]$ .

We now have to consider the case where  $R'$  has parameters  $(\frac{X}{u_2} = X', \frac{u_1}{u_2} = u'_1, u_2 = u'_2, u_3 = u'_3)$ . If  $E = \text{div}(u_1 u_2)$ , then  $u_1$  and  $u_2$  play symmetric roles and there is nothing to prove. Otherwise,

$$h' = X'^p - X' g'^{p-1} + H(x')(\Psi_3(u'_1, 1) u'_3 + u'_2 \phi'_1 + u'_3{}^2 \phi'_2)$$

with  $\phi'_1, \phi'_2 \in S'$ . As above we see that  $u'_2$  divides  $H(x')^{-1} g'^p$  and the only case to be considered is when  $\Psi_3 = \lambda U_1^{\epsilon(x_0)-1}$ ,  $0 \neq \lambda \in k(x_0)$ ; the proof ends like in the first chart.

**II.5.5 Definition.** *If  $e : X' \rightarrow X_0$  is a composition of permissible blowing ups, a point  $x' \in e^{-1}(x_0)$  is said to be near (resp. very near)  $x_0$  if  $x' \in \Sigma_p(X')$  (resp.  $x' \in \Sigma_p(X')$ ) and  $\Omega(x') = \Omega(x_0)$ .*

**II.5.6 Theorem.** *Assume  $x_0 \in \Sigma_p$  and  $\omega'(x_0) = 3$ . Let*

$$X_0 \leftarrow X_1 \leftarrow \cdots X_{n-1} \leftarrow X_n \leftarrow \cdots$$

*be the quadratic sequence along  $\mu$ , i.e.  $X_i$  is the blowing up along the center  $x_{i-1}$  of  $\mu$  in  $X_{i-1}$  for  $i \geq 1$ . There exists  $n \geq 1$  such that  $x_n$  is not very near  $x_0$ .*

*Proof.* By **II.5.4.2**(iii), the only case we have to consider is when  $\text{div}(H(x_0)^{-1} g^p)$  has only one component, say  $\text{div}(u_1)$ . Then, by **II.5.4.2**(iv), for  $i \geq 1$ ,  $x_i$  is on the strict transform of  $\text{div}(u_1)$  and  $x_i$  is rational over  $x_0$  if  $x_i$  is very near  $x_0$ . For each  $i \geq 0$ , there is a map  $\eta_i : (X_i, x_i) \rightarrow \text{Spec} S_i$ , where  $S_i$  is an iterated quadratic transform of  $S$ . Let  $\pi_i : \text{Spec} S_i \rightarrow \text{Spec} S$  be the composed map and  $E_i := \pi_i^{-1}(E)_{\text{red}}$  be the exceptional divisor. Then  $E_i$  has at most two components as long as  $\Omega(x_i) = \Omega(x_0)$  since  $\omega'(x_0) = 3$ . By **II.5.4.2**(i),  $E_i$  has exactly two components: an ‘‘old’’ component, the strict transform of  $\text{div}(u_1) \subset \text{Spec} S$ , and a ‘‘new’’ component which is exceptional for  $\text{Spec} S_i \rightarrow \text{Spec} S_{i-1}$  for  $i \geq 1$ .

So we can choose the r.s.p.  $(X, u_1, u_2, u_3)$  of  $R$  such that  $\Delta(h; u_1, u_2, u_3; X)$  is minimal and  $u_2 = 0$  is the equation of the exceptional divisor of  $\text{Spec} S_i \rightarrow \text{Spec} S_{i-1}$  for  $i \geq 1$ . So the equation

of  $X_i$  at  $x_i$  is  $h^{(i)} = u_2^{-ip}h =: X^{(i)p} - X^{(i)}g^{(i)p-1} + f^{(i)}$ , where  $g^{(i)} = u_2^{-i}g$ . Then  $\mu(g^{(i)}) = \mu(g) - i\mu(u_2) > 0$ . Since the value group of  $\mu$  is Archimedean, we have

$$i < \frac{\mu(g)}{\mu(u_2)} < \infty.$$

**II.5.7 Remark.** As a consequence of **II.5.6**, we lower  $\Omega(x_0)$  in the ambiguous case  $\omega'(x_0) = 3$  by performing a finite quadratic sequence along the valuation  $\mu$ .

**We assume from now on that  $\omega'(x_0) \leq 2$ .**

**II.6 Theorem.** *If  $H(x_0) \neq 1$ , the function  $\omega$  is upper-semicontinuous over  $\Sigma_p(X_0)$ .*

Proof. If  $g = 0$ , then  $\omega'(x_0) = 2$  by definition and  $\omega(x_0) = \text{ord}_{\eta(x_0)}J(f, E)$ . Since  $J(f, E)$  is well defined as a sheaf of ideals on  $\text{Spec}S$  (**II.3.1**) and is independent on choices of coordinates,  $\omega$  is upper-semicontinuous on  $\Sigma_p(X_0)$ . So we have only to deal with the case  $g \neq 0$ .

By **II.4.7**, the set  $W := \{x \in \Sigma_p \mid \omega(x) \geq 1\}$  is Zariski closed and of dimension at most one. There only remains to prove the following: if  $Y \subseteq \Sigma_p$  is a curve,  $y$  its generic point, and  $\omega(y) \geq 1$ , then  $\omega(y) \leq \omega(x_0)$ . This is implied by the following very useful lemma (where  $g$  may or may not be nonzero).

**II.6.1 Lemma.** *Assume that  $H(x_0) \neq 1$ . Let  $Y \subseteq \Sigma_p$  be a curve with generic point  $y$  such that  $\omega(y) \geq 1$ . For  $n \geq 0$ , let  $e_n : X_{n+1} \rightarrow X_n$  be the blowing up along some closed point  $x_n \in X_n$ , where  $x_n$  is on the strict transform  $Y_n$  of  $Y$ .*

*Then, for  $n \gg 0$ ,  $Y_n$  is permissible of the first kind at  $x_n$  and*

$$\omega(y) = \omega(y_n) \leq \omega(x_n) \leq \omega(x_0),$$

where  $y_n$  is the generic point of  $Y_n$ .

Proof. First note that the condition  $H(x_0) \neq 1$  is preserved by blowing up closed points lying on the consecutive strict transforms  $Y_n$  of  $Y$  if  $Y \subseteq \Sigma_p$  (see proof of **II.5.3.1**).

For each  $n \geq 0$ , there is a map  $\eta_n : (X_n, x_n) \rightarrow \text{Spec}S_n$ , where  $S_n$  is an iterated quadratic transform of  $S$ . Let  $\pi_n : \text{Spec}S_n \rightarrow \text{Spec}S$  be the composed map and  $E_n := \pi_n^{-1}(E)_{\text{red}}$  be the reduced exceptional divisor. Since  $e_n$  is an isomorphism at  $y_n$ , we have  $Y_n \subseteq \Sigma_p(X_n) \subseteq E_n$  and  $\omega(y_n) = \omega(y)$ . For  $n$  big enough,  $Y_n$  is regular at  $x_n$  and  $\eta_n(Y_n)$  has normal crossing with  $E_n$ . Also note that  $x_{n+1}$  is rational over  $x_n$ , since  $Y_n$  is regular at  $x_n$ , and that  $E_n$  has at least two irreducible components. By **II.5.4**, it can be assumed that  $n = 0$  without loss of generality.

By **II.2.1**, we can choose a r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  such that  $Y = V(X, u_1, u_2)$ , with  $\text{div}(u_1 u_3) \subseteq E$  and the polyhedra  $\Delta(h; u_1, u_2, u_3; X)$  and  $\Delta(h; u_1, u_2; X)$  are both minimal. We denote  $H(x_0) = u_1^{a(1)} u_2^{a(2)} u_3^{a(3)}$ ,  $g^p = \gamma^p H(x_0) u_1^{b_1} u_2^{b_2} u_3^{b_3}$  (with  $a(2) = b_2 = 0$  if  $\text{div}(u_2) \not\subseteq E$ ), and

$$H(x_0)^{-1}f = \sum_{(a_1, a_2, a_3) \in \mathbb{N}^3} \lambda_{a_1 a_2 a_3} u_1^{a_1} u_2^{a_2} u_3^{a_3}, \quad \lambda_{a_1 a_2 a_3} \in k(x_0).$$

We have

$$\epsilon(y) = \min\{b_1 + b_2, \{a_1 + a_2 \mid \exists a_3 \in \mathbb{N} : \lambda_{a_1 a_2 a_3} \neq 0\}\}, \quad (1)$$

and

$$\epsilon(x_0) = \min\{b_1 + b_2 + b_3, \{a_1 + a_2 + a_3 \mid \lambda_{a_1, a_2, a_3} \neq 0\}\}. \quad (2)$$

The point  $x' := x_1$  has coordinates  $(\frac{X}{u_3} =: X', \frac{u_1}{u_3} =: u'_1, \frac{u_2}{u_3} =: u'_2, u_3 =: u'_3)$ . We have  $Y_1 = V(X', u'_1, u'_2)$ , with  $\text{div}(u'_1 u'_3) \subseteq E_1$ . Let  $h' = X'^p - X'g'^{p-1} + f'$  be the strict transform

of  $h$ . Then  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal, since  $x'$  is rational over  $x_0$  (see the argument at the beginning of the proof of **II.5.4.2**(iii)). By **II.3.3**, **II.5.4.1** and elementary computations, we have

$$g'^p = \gamma^p H(x') u_1'^{b_1} u_2'^{b_2} u_3'^{b_1+b_2+b_3-\epsilon(x_0)}, \quad (3)$$

and

$$H(x')^{-1} f' = \sum_{(a_1, a_2, a_3) \in \mathbb{N}^3} \lambda_{a_1 a_2 a_3} u_1'^{a_1} u_2'^{a_2} u_3'^{a_1+a_2+a_3-\epsilon(x_0)}. \quad (3')$$

One deduces from (1), (2), (3) and (3') that  $\epsilon(x_0) \geq \epsilon(x') \geq \epsilon(y)$ , and that  $\epsilon(x_0) = \epsilon(x')$  if and only if  $\epsilon(x_0) = \epsilon(y)$ .

Therefore for  $n \gg 0$ , we get  $\epsilon(x_n) = \epsilon(y) = \epsilon(y_n)$  and this proves that  $Y_n$  is permissible of the first kind for  $n \gg 0$ . As before, it can be assumed that  $n = 0$  without loss of generality. There remains to prove that  $\omega(x_n) \geq \omega(y_n)$ . By theorem **II.5.4**, it can be assumed that  $\Omega(x_n)$  is constant for  $n \geq 0$ . Since  $\epsilon(x_n) - 1 \leq \omega(x_n) \leq \epsilon(x_n)$  and  $\epsilon(y_n) - 1 \leq \omega(y_n) \leq \epsilon(y_n)$ , we must prove that  $\omega(x_n) = \epsilon(x_n) - 1$  and  $\omega(y_n) = \epsilon(y_n)$  cannot hold at the same time for arbitrarily large  $n$ .

The latter is certainly true if  $E = \text{div}(u_1 u_2 u_3)$ , since this implies that  $E_n$  has three components for each  $n \geq 0$ , and therefore  $\epsilon(x_n) = \omega(x_n)$ . So, we now assume that  $E = \text{div}(u_1 u_3)$ , and in particular  $b_2 = 0$  if  $g \neq 0$ . Since  $\epsilon(x_0) = \epsilon(y)$ , each  $(a_1, a_2, a_3) \in \mathbb{N}^3$  achieving the minimum in (2) has  $a_3 = 0$ ; if  $g \neq 0$  and this minimum is achieved by  $(b_1, b_2, b_3)$ , then we have  $b_3 = b_2 = 0$ .

Assume that  $\omega(y) = \epsilon(y)$ . Since  $\Delta(h; u_1, u_2; X)$  is minimal, we now have

$$\text{cl}_{\epsilon(x_0)}(H(x_0)^{-1}(g^p, f)) \in k(x_0)[U_1, U_2]_{\epsilon(x_0)}. \quad (4)$$

This proves that  $\omega(x_0) = \epsilon(x_0)$  if  $\omega'(x_0) \leq 2$ . Suppose that  $\omega'(x_0) = 3$ . Since  $\Delta(h; u_1, u_2, u_3; X)$  is minimal, we must have  $\nu_\delta(g) = 1$ , so  $a(1) + \epsilon(x_0)$  and  $a(3)$  are divisible by  $p$ , and

$$\Psi = \text{cl}_{\epsilon(x_0)}(H(x_0)^{-1} f) \notin k(x_0).U_1^{\epsilon(x_0)}.$$

We expand  $\Psi = \sum_{0 \leq i \leq i_0} U_2^{pi} \Psi_i(U_1)$ , where  $i_0 > 0$  is such that  $\Psi_{i_0} \neq 0$ . Then  $(\frac{a(1)+\epsilon(x_0)}{p} - i_0, i_0, \frac{a(3)}{p}) \in \mathbb{N}^3$  is a vertex of  $\Delta(h; u_1, u_2, u_3; X)$  and thus  $\Psi_{i_0} = \lambda U_1^{\epsilon(x_0)-pi_0}$  with  $\lambda \notin k(x_0)^p$  (in particular,  $k(x_0)$  is infinite). Let  $\mu \in S$  be a unit and let

$$\theta := \mu u_1^{\frac{a(1)+\epsilon(x_0)}{p}-1} u_2 u_3^{\frac{a(3)}{p}}, \quad Z := X - \theta.$$

Since  $k(x_0)$  is infinite, the monomial  $u_1^{a(1)+\epsilon(x_0)-1} u_2 u_3^{a(3)}$  appears with nonzero coefficient in  $f + \theta^p - \theta g^{p-1}$  for a general value of the residue class  $\bar{\mu} \in k(x_0)$ . On the other hand,  $\Delta(h; u_1, u_2, u_3; Z) = \Delta(h; u_1, u_2, u_3; X)$  so that *a fortiori*  $\Delta(h; u_1, u_2; Z) = \Delta(h; u_1, u_2; X)$  and this proves that the polygon  $\Delta(h; u_1, u_2; Z)$  is minimal. Therefore

$$\omega(y) \leq \text{ord}_{\eta(y)} H(x_0)^{-1} \left( \frac{\partial(f + \theta^p - \theta g^{p-1})}{\partial u_2} \right) = \epsilon(y) - 1,$$

and this contradicts the assumption  $\omega(y) = \epsilon(y)$ . The argument extends to the proof of  $\omega(x_n) \geq \omega(y_n)$  for all  $n \geq 0$  and the lemma is proved.

**II.6.2 Remark** (not used elsewhere). Theorem **II.6** can be extended to the following more general situation:  $S$  is a  $k$ -algebra of finite type and dimension three, every point of  $\text{Spec} S$  is regular, and  $k$  is differentially finite over the perfect field  $k_0$ . So we have infinitely many closed points in  $\text{Spec} S$

and in each subvariety  $Y \subseteq X_0 = \text{Spec}(S[X]/(h))$  of positive dimension. The normal crossings divisor  $E$  is chosen as in **I** and  $H(x)$ ,  $\mathcal{J}(f, E)_x$  as well as  $\Omega(x)$  are defined pointwise for  $x \in X_0$ , i.e. w.r.t. the local ring  $S_x$ .

By a well known criterion of Nagata ([Ma] theorem 24.2), theorem **II.6** extends to this more general situation provided we can prove the following: for each integral subscheme  $Y \subseteq \Sigma_p(X_0)$  of positive dimension, there is a nonempty open subset  $U \subseteq Y$  such that  $\omega(x) = \omega(y)$  for all  $x \in U$ , where  $y$  is the generic point of  $Y$ . We give the proof when  $Y$  is a curve, the proof being somewhat simpler when  $Y$  is a component of  $E_0$  (see the argument below when  $\omega(y) = 0$ ).

By theorem **II.5.4**, it can be assumed that  $\eta(Y)$  is regular and has normal crossings with  $E$ . We pick some closed point  $x_0 \in Y$  as origin such that  $E_y$  and  $E_{x_0}$  have the same irreducible components. By **II.2.2**, there exists a system of coordinates  $(X, u_1, u_2, u_3)$  at  $x_0$ , with  $\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2 u_3)$ , such that  $Y = V(X, u_1, u_2)$ ,  $\Delta(h; u_1, u_2; X)$  minimal and we can read the  $d_i$ 's on this polyhedron (in particular,  $H(x_0) = H(y) =: u_1^{a(1)} u_2^{a(2)}$  with  $a(2) = 0$  if  $\text{div}(u_2) \not\subseteq E$ ). Then

$$h = X^p - X \left( \gamma u_1^{\frac{a(1)+b_1}{p}} u_2^{\frac{a(2)+b_2}{p}} \right)^{p-1} + u_1^{a(1)} u_2^{a(2)} \sum_{(a_1, a_2) \in F} \mu_{a_1 a_2} u_1^{a_1} u_2^{a_2},$$

where  $\gamma \in S_y$  is invertible,  $F$  is a finite set such that  $(\frac{a(1)+a_1}{p}, \frac{a(2)+a_2}{p}) \in \Delta(h; u_1, u_2; X)$  for each  $(a_1, a_2) \in F$ , and  $\mu_{a_1 a_2} \in S_y$  is invertible. Note that, in particular, we have  $(a_1, a_2) \in F$  whenever  $(\frac{a(1)+a_1}{p}, \frac{a(2)+a_2}{p})$  is a vertex of  $\Delta(h; u_1, u_2; X)$  and  $(a_1, a_2) \neq (b_1, b_2)$ . It can also be assumed that  $\Omega_{S/k_0}^1$  is a free module. We fix a basis  $\mathcal{B}$  of the dual space  $\text{Der}_{k_0}(S)$  containing  $\frac{\partial}{\partial u_1}$  and  $\frac{\partial}{\partial u_2}$ .

Let  $U$  be the nonempty open set consisting of those  $x \in Y$  where

- (i)  $\gamma$  is defined and invertible at  $x$ ;
- (ii) the  $\mu_{a_1 a_2}$ 's are defined and invertible at  $x$  for each  $(a_1, a_2) \in F$  (so in particular,  $f$  is defined at  $x$ );
- (iii)  $E_y$  and  $E_x$  have the same irreducible components (thus  $H(x) = H(y)$  by **II.2** and  $\mathcal{J}(f, E)_x$  as defined in **II.3** is the stalk at  $x$  of one and the same ideal  $\mathcal{J}(f, E)$ ).

(iv) for each vertex  $w = (\frac{a(1)+a_1}{p}, \frac{a(2)+a_2}{p})$ ,  $w \neq w_0 := (\frac{a(1)+b_1}{p}, \frac{a(2)+b_2}{p})$ , of  $\Delta(h; u_1, u_2; X)$  with integer coordinates, and for each  $D \in \mathcal{B}$ ,  $D\mu_{a_1 a_2}$  either vanishes at  $y$  or is invertible at  $x$ .

First note the following consequence of conditions (i) and (ii): if  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is the projection  $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ , then  $\Delta(h; u_1, u_2, v; X) = \pi^{-1}(\Delta(h; u_1, u_2; X))$  whenever  $(X, u_1, u_2, v)$  is an adapted system of coordinates at  $x$ .

Also by (ii), a vertex  $(w, 0)$ ,  $w = (\frac{a(1)+a_1}{p}, \frac{a(2)+a_2}{p})$ , of  $\Delta(h; u_1, u_2, v; X)$  is solvable if only if  $w \neq w_0$ ,  $w$  has integer coordinates and the image of  $\mu_{a_1 a_2}$  in  $k(x)$  is a  $p^{\text{th}}$ -power. By (iv), if some  $D \in \mathcal{B}$  such that  $D\mu_{a_1 a_2}$  is invertible at  $x$  is a derivation w.r.t. constants at  $x$ , then  $(w, 0)$  is not solvable. If there is only one  $D_w \in \mathcal{B}$  such that  $D_w \mu_{a_1 a_2}$  is invertible at  $x$  and  $D_w$  is a derivation w.r.t. a local parameter at  $x$ , then it can be assumed that  $D_w = \frac{\partial}{\partial v}$ . In this case,  $(w, 0)$  is solvable. However, after performing a translation  $Z := X - \theta$  in order to resolve  $(w, 0)$ ,  $f$  is changed into  $f' = f + \theta^p - \theta g^{p-1}$  and  $w_x := (w, \frac{1}{p})$  is a vertex of  $\Delta(h; u_1, u_2, v; Z)$  (remember that  $w \neq w_0$ ). Hence

$$\text{ord}_{\eta(x)}(D_w(H(x)\mu_{a_1 a_2} u_1^{a_1} u_2^{a_2})) = \text{ord}_{\eta(y)}(D_w(H(x)\mu_{a_1 a_2} u_1^{a_1} u_2^{a_2})).$$

This proves that there exists  $X_x \in \widehat{S}_x$  such that  $\Delta(h; u_1, u_2, v; X_x)$  is minimal,  $h = X_x^p - X_x g^{p-1} + f_x$  and

$$\text{ord}_{\eta(x)} \mathcal{J}(f_x, E_x) = \text{ord}_{\eta(y)} \mathcal{J}(f, E_y)$$

whenever  $x \in U$ . Therefore  $\omega(x) \leq \omega(y)$  and the inequality can possibly be strict only if  $\omega'(x) = 3$ .

In this last case, one proceeds as in the end of the proof of lemma **II.6.1**: we must have  $\epsilon(x) = \epsilon(y)$  and there exists a coordinate change of the form  $Z_x := X_x - \theta$ ,  $\theta := \mu u_1^{\frac{\alpha(1)+\epsilon(x)-1}{p}} u_2$ ,  $\mu \in S_x$  invertible, such that  $\Delta(h; u_1, u_2, v; X_x) = \Delta(h; u_1, u_2, v; Z_x)$ . We then have

$$\omega(y) \leq \text{ord}_{\eta(y)}(H(x)^{-1} \frac{\partial f_x}{\partial u_2}) \leq \text{ord}_{\eta(x)}(H(x)^{-1} \frac{\partial f_x}{\partial u_2}) = \epsilon(x) - 1 = \omega(x)$$

as required.

**We assume that  $\omega'(x_0) \leq 2$  from this point on.**

### III Nonpermissible blowing ups.

In certain situations (see chapter 3, section **II** below), we will perform some blowing ups of a particular type which are not permissible: blowing ups of prime ideals  $I \subset S$  such that  $V(I)$  has normal crossing with  $E$  (blowing up the base  $\text{Spec}S$ ).

If  $\pi_0 : Z \rightarrow \text{Spec}S$  is such a blowing up, and  $z' \in \pi_0^{-1}(\eta(x_0))$ , there is an induced map

$$\pi : X' := \text{Spec}R' \rightarrow \text{Spec}R,$$

where  $S' := \mathcal{O}_{Z, z'}$ ,  $R' := S'[X]_{x'}$  and  $x' := (m_{S'}, X)$ . We have a projection  $\eta' : \text{Spec}R' \rightarrow \text{Spec}S'$  and a normal crossings divisor  $E' := \pi_0^{-1}(E)_{\text{red}}$  which satisfies the requirements in **I**. Also note that  $H(x_0)$  divides  $H(x')$  in  $S'$ , so  $H(x') \neq 1$  if  $H(x_0) \neq 1$ .

Then the invariants  $\omega(x')$  and  $\Omega(x')$  are defined as in **II.4** and we point out that **II.4.7** remains valid for the germ  $(X', x')$ . We also keep on using the terminology of “near” and “very near” in definition **II.5.5** whenever  $e : X' \rightarrow X_0$  is a composition of permissible and nonpermissible blowing ups of the above type and  $x' \in e^{-1}(x_0)$ .

## CHAPTER 2: a few easy cases.

In this section, we consider some cases where our main invariant  $\Omega(x) = (\omega(x), \omega'(x))$ ,  $\omega(x) \geq 1$  (chapter 1, definition **II.4**) can be decreased by permissible blowing ups (chapter 1, definition **II.5.1**). Section I of this chapter contains resolution when  $\omega'(x) = 1$ . Section II contains some cases when  $\omega'(x) = 2$ , which mainly rely on the directrix and its associated invariant  $\tau(x)$  (chapter 1, definition **II.4**).

From now on,  $x$  will denote the center of the valuation  $\mu$  in some blowing up  $X$  of  $X_0$  obtained by a composition of permissible or of nonpermissible blowing ups of the type described in chapter 1, **III**. In particular, the local equation of  $X$  at  $x$  is of the form  $h = X^p - Xg^{p-1} + f$ ,  $\text{ord}_x h = p$  and  $H(x)$ ,  $\Omega(x)$  are always defined.

### I Resolution of the case $\omega'(x) = 1$ .

By chapter **I**, **II.5.6**, the uniformization problem is reduced to the case  $\omega'(x) \leq 2$ . As usual, we suppose that the r.s.p.  $(u_1, u_2, u_3)$  of  $S$  and  $X \in \widehat{R}$  are such that  $E = \text{div}(u_1 \cdots u_j)$  and the polyhedron  $\Delta(h; u_1, u_2, u_3; X)$  is minimal. If  $\omega'(x) = 1$ , then by chapter 1, definition **II.4**, we have  $g \neq 0$  and

$$\text{ord}_{\eta(x)}(H(x)^{-1}g^p) = \omega(x) = \epsilon(x).$$

**I.1 Theorem.** *Assume that  $x \in \Sigma_p$ ,  $\omega'(x) = 1$  and  $\text{div}(H(x)^{-1}g^p)$  has at least two irreducible components. Let*

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_{n-1} \leftarrow X_n \leftarrow \cdots$$

*be the quadratic sequence along  $\mu$ , i.e.  $X_i$  is the blowing up along the center  $x_{i-1}$  of  $\mu$  in  $X_{i-1}$  for  $i \geq 1$ . There exists  $n \geq 1$  such that  $x_n \notin \Sigma_p(X_n)$  or  $\Omega(x_n) < \Omega(x)$ .*

*Proof.* By chapter 1 **II.5.4.2(i)**, at most one point  $x'$  in  $X_1$  is very near  $x = x_0$ . If  $x' = x_1$  is very near  $x$ ,  $\text{div}(H(x)^{-1}g^p)$  has exactly two irreducible components, say  $H(x)^{-1}g^p = \gamma^p u_1^{a_1} u_2^{a_2}$ ,  $a_1, a_2 > 0$ ,  $\gamma$  invertible, and  $x' \in X_1$  has r.s.p.  $(X' := \frac{X}{u_3}, u'_1 := \frac{u_1}{u_3}, u'_2 := \frac{u_2}{u_3}, u'_3 := u_3)$ . We have

$$H(x')^{-1}g'^p = \gamma^p u_1'^{a_1} u_2'^{a_2} = u_3^{-p} H(x)^{-1}g^p.$$

Hence  $\mu(H(x')^{-1}g'^p) = \mu(H(x)^{-1}g^p) - p\mu(u_3)$ , where  $\mu$  is the given valuation. As the group of values of  $\mu$  is Archimedean, we have

$$n < \frac{\mu(g)}{\mu(u_3)} < \infty$$

provided  $x_n \in \Sigma_p(X_n)$  and  $\Omega(x_n) = \Omega(x)$ , which concludes the proof.

**I.2.** If  $\omega'(x) = 1$ , it can furthermore be assumed that  $H(x)^{-1}g^p = \gamma^p u_1^{\epsilon(x)}$  by **I.1**. Let  $m(x)$  be the number of irreducible components of  $E$ . By chapter 1 **II.5.4.2(i)**, if  $e : X' \rightarrow X$  is a permissible blowing up of the first kind and  $x' \in e^{-1}(x)$  is very near  $x$ , then  $x'$  is on the strict transform of  $\text{div}(u_1)$ . Therefore it can be assumed that  $m(x) \geq 2$ , i.e.  $\text{div}(u_1 u_2) \subseteq E$ , after blowing up once along  $x$ . Let us denote:

$$H(x) =: u_1^{a(1)} u_2^{a(2)} u_3^{a(3)}, \quad a(3) = 0 \text{ if } E = \text{div}(u_1 u_2);$$

$$g^p =: \gamma^p u_1^{p\beta_1} u_2^{p\beta_2} u_3^{p\beta_3}, \quad \gamma \text{ invertible, } \beta_i = 0 \text{ if } \text{div}(u_i) \not\subseteq E_0;$$

$$H(x)^{-1}f =: \sum_{0 \leq i \leq \epsilon(x)} u_1^{\epsilon(x)-i} \phi_i, \quad \phi_0 \in \widehat{S} \text{ and } \phi_i \in k(x)[[u_2, u_3]], \quad 1 \leq i \leq \epsilon(x).$$

**I.2.1 Definition.** *With notations as above, let us denote*

$$A_j := \inf \left\{ \frac{\text{ord}_{u_j} \phi_i}{i}, 1 \leq i \leq \epsilon(x) \right\}, j = 2, 3;$$

$$B := \inf \left\{ \frac{\text{ord}_{(u_2, u_3)} \phi_i}{i}, 1 \leq i \leq \epsilon(x) \right\}, C := B - A_2 - A_3 \geq 0;$$

$$\beta := \inf \left\{ \frac{\text{ord}_{u_3}(\phi_i/u_2^{iA_2} \bmod(u_2))}{i}, 1 \leq i \leq \epsilon(x) \right\}, \text{ where } \text{ord}_{u_3} \text{ denotes the natural valuation of the discrete valuation ring } k(x)[[u_3]];$$

$$\gamma := 1 + \lfloor C \rfloor \text{ (resp. } \gamma := \sup\{1, \lfloor \beta \rfloor\} \text{) if } E = \text{div}(u_1 u_2 u_3) \text{ (resp. } E = \text{div}(u_1 u_2)).$$

Obviously, these definitions may depend on  $(u_1, u_2, u_3)$ , but not on  $X$ , since  $\Delta(h; u_1, u_2, u_3; X)$  is minimal and  $A_j, B$  and  $C$  (resp.  $\beta$ ) are computed by evaluating the minimum of a linear function on some (linear) projection of this polyhedron. When there is a risk of confusion, we will make explicit this dependence on  $(u_1, u_2, u_3)$  by writing  $A_j(u_1, u_2, u_3)$ , etc... We also use the notation  $A_j(x), A_j(x')$ , etc... when dealing with a blowing up  $e : X' \rightarrow X$  and  $x' \in e^{-1}(x)$ . In this case, we always compute invariants w.r.t.  $E' := (e^{-1}E)_{\text{red}}$ .

Finally note that  $B(x) \geq 1$ , since  $\text{ord}_{\eta(x)} f \geq \epsilon(x)$ , and that all of these invariants are finite (i.e.  $\phi_i \neq 0$  for some  $i, 1 \leq i \leq \epsilon(x)$ ) by definition of  $H(x)$ , since  $\omega(x) \geq 1$ .

**I.2.2 Theorem.** *Assume that the following condition holds:*

$$(*) \ x \in \Sigma_p, \omega'(x) = 1, H(x)^{-1}g^p = \gamma^p u_1^{\epsilon(x)} \text{ and } m(x) \geq 2.$$

*Let  $e : X' \rightarrow X$  be the blowing up at  $x$  and  $x' \in X'$  be very near  $x$ . Then  $x'$  also satisfies  $(*)$  and there exists a r.s.p.  $(Z, v_1, v_2, v_3)$  at  $x'$  such that  $\Delta(h'; v_1, v_2, v_3; Z)$  is minimal and the following holds:*

$$(i) \ \gamma(v_1, v_2, v_3) \leq \gamma(u_1, u_2, u_3);$$

$$(ii) \ \text{if } m(x) = m(x') = 2, \text{ then } \beta(v_1, v_2, v_3) \leq \beta(u_1, u_2, u_3);$$

$$(iii) \ \text{if } x' \text{ is in the chart with origin } \left(\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, \frac{u_3}{u_2}\right) \text{ then } A_2(v_1, v_2, v_3) = B(u_1, u_2, u_3) - 1 \text{ and}$$

$$\beta(v_1, v_2, v_3) < 1 + \lfloor \beta(u_1, u_2, u_3) \rfloor.$$

*If moreover  $x'$  is distinct from the origin of the chart, we also have*

$$\beta(v_1, v_2, v_3) < 1 + \lfloor C(u_1, u_2, u_3) \rfloor;$$

*(iv) if  $(x'$  is not rational over  $x$  and  $\gamma(u_1, u_2, u_3) \geq 2$ ), then  $\gamma(v_1, v_2, v_3) < \gamma(u_1, u_2, u_3)$ , except possibly if  $(m(x) = 2$  and  $\beta(u_1, u_2, u_3) = 2)$  in which case  $\beta(v_1, v_2, v_3) < 2$ ;*

*(v) if  $x' = \left(\frac{X}{u_3}, \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3\right)$  is the point at infinity, then  $A_2(v_1, v_2, v_3) = A_2(u_1, u_2, u_3)$  and*

$$\beta(v_1, v_2, v_3) = \beta(u_1, u_2, u_3) + A_2(u_1, u_2, u_3) - 1.$$

*If moreover  $(m(x) = 2$  and  $\gamma(u_1, u_2, u_3) \geq 2$ ), then  $\gamma(v_1, v_2, v_3) < \gamma(u_1, u_2, u_3)$  except possibly if  $(\beta(u_1, u_2, u_3) = 2$  and  $C(v_1, v_2, v_3) = 1)$ .*

*Proof.* For  $1 \leq i \leq \epsilon(x)$ , we let  $\Phi_i := \text{in}_x \phi_i \in k(x)[U_2, U_3]$  and write  $\Phi_i =: U_2^{a_i(2)} U_3^{a_i(3)} \Psi_i$  with  $U_j$  not dividing  $\Psi_i, j = 2, 3$ . By definition, we have  $\deg \Phi_i \geq iB(x)$  and  $a_i(j) \geq iA_j(x), j = 2, 3$ . If  $L$  is the linear form on  $\mathbb{R}^4$  given by

$$L(x_1, x_2, x_3, X) = \frac{1}{\epsilon(x) + a(1) + \frac{a(2)+a(3)}{B(x)}} \left( x_1 + \frac{x_2 + x_3}{B(x)} \right) + \frac{X}{p},$$

then

$$\text{in}_L h = X^p - XG^{p-1} + U_1^{a(1)} U_2^{a(2)} U_3^{a(3)} \sum_{\text{ord}_x \phi_i = iB(x)} U_1^{\epsilon(x)-i} \Phi_i, \quad (1)$$

where  $G := \text{in}_x g$ .

By chapter 1, **II.5.4.2**, any  $x'$  very near  $x$  maps to the strict transform of  $\text{div}(u_1)$ . We first look at the chart with origin  $(X' := \frac{X}{u_2}, u'_1 := \frac{u_1}{u_2}, u'_2 := u_2, u'_3 := \frac{u_3}{u_2})$ . We have  $E' = \text{div}(u'_1 u'_2 u'_3)$ . By chapter 1, **II.5.4.1**, we have

$$(H(x')) = (u'_1{}^{a(1)} u'_2{}^{p(\delta(x)-1)} u'_3{}^{a(3)})$$

and  $(H(x')^{-1} g'^p) = (u'_1{}^{\epsilon(x)})$ .

**I.2.2.1** If  $x' = (X', u'_1, u'_2, u'_3)$  is the origin of the chart, then the polyhedron  $\Delta(u_2^{-p} h; u'_1, u'_2, u'_3; X')$  remains minimal. Computing in these coordinates, it is easily seen that

$$A_2(x') = B(x) - 1, \quad \beta(x') \leq C(x) + A_3(x) \leq \beta(x), \quad C(x') \leq C(x),$$

and these inequalities give (i),(ii) and (iii) (note that in this case, the hypotheses of (iv) and (v) do not occur).

**I.2.2.2** If  $u'_3$  is invertible at  $x'$ , let  $P \in k(x)[u_2, u_3]$  be irreducible, homogeneous and unitary in  $u_3$  such that  $(X', u'_1, u'_2, v')$  is a system of coordinates at  $x'$ , where  $v' := P(1, u'_3)$ . We have  $E' = \text{div}(u'_1 u'_2)$  in this case and take  $H(x') := u'_1{}^{a(1)} u'_2{}^{p(\delta(x)-1)}$ . If  $B(x) > 1$  and  $L'$  is the linear form on  $\mathbb{R}^4$  given by

$$L(x'_1, x'_2, x'_3, X') = \frac{1}{\epsilon(x) + a(1) + \frac{p(\delta(x)-1)}{B(x)-1}} \left( x'_1 + \frac{x'_2}{B(x)-1} \right) + \frac{X'}{p},$$

then

$$\text{in}_{L'}(u_2^{-p} h) = X'^p - X' G'^{p-1} + U_1{}^{a(1)} U_2{}^{p(\delta(x)-1)} u_3{}^{a(3)} \sum_{\text{ord}_x \phi_i = iB(x)} U_1{}^{\epsilon(x)-i} U_2{}^{i(B(x)-1)} u_3{}^{a_i(3)} \Psi_i(1, u'_3),$$

where  $G' := \text{in}_{x'} g'$ .

Let  $i_0 := \sup\{i \mid \text{ord}_x \phi_i = iB(x)\} \geq 1$ . Any translation  $Z' := X' - \theta'$  we have to make in order to get the polyhedron  $\Delta(u_2^{-p} h; u'_1, u'_2, v'; Z')$  minimal will verify  $\text{ord}_{u'_2} \theta' \geq \delta(x) - 1$  and

$$\text{ord}_{u'_1} \left( \frac{\theta'^p}{u_2{}^{p(\delta(x)-1)}} \text{ mod}(u'_2) \right) \geq a(1) + \epsilon(x) - i_0.$$

The latter inequality is strict if  $a(1) + \epsilon(x) - i_0 \not\equiv 0 \pmod{p}$ . Since  $\text{ord}_{u_1} g^p = a(1) + \epsilon(x) \equiv 0 \pmod{p}$ ,  $a(1) + \epsilon(x) - i_0 \not\equiv 0 \pmod{p}$  is equivalent to  $i_0 \not\equiv 0 \pmod{p}$ . We consider two cases:

*Case 1:*  $i_0 \not\equiv 0 \pmod{p}$ . The above translation preserves the term

$$H(x') u_3{}^{a(3)} u_1{}^{\epsilon(x)-i_0} u_2{}^{i_0(B(x)-1)} u_3{}^{a_{i_0}(3)} \Psi_{i_0}(1, u'_3)$$

in  $u_2^{-p} h$ . Therefore  $A_2(x') = B(x) - 1 \leq \beta(x) + C(x) - 1$  and

$$\beta(x') \leq \frac{\text{ord}_{v'} \Psi_{i_0}(1, u'_3)}{i_0} \leq \frac{C(x)}{[k(x') : k(x)]} \leq \frac{\beta(x)}{[k(x') : k(x)]}.$$

This gives (i)(ii)(iii)(iv), the exceptional case in (iv) not occurring in this case.

*Case 2:*  $i_0 \equiv 0 \pmod{p}$ . The possible translation we have to make on  $X'$  will at most add to  $H(x')u_3^{a(3)}u_1^{\epsilon(x)-i_0}u_2^{i_0(B(x)-1)}u_3^{a_{i_0}(3)}\Psi_{i_0}(1, u_3)$  some  $p^{th}$ -power of the form

$$A^p := u_1^{a(1)+\epsilon(x)-i_0}u_2^{i_0(B(x)-1)+p(\delta(x)-1)}A'(v')^p,$$

since  $i_0 > 0$ . Then  $U_2^{a_{i_0}(2)+a(2)}U_3^{a_{i_0}(3)+a(3)}\Psi_{i_0}(U_2, U_3)$  is not a  $p^{th}$ -power because it induces an edge (or vertex) of  $\Delta(h; u_1, u_2, u_3; X)$  by (1), and we apply theorem **II.5.3.2**(i) of chapter 1. This gives

$$e := \text{ord}_{v'} \left( u_3^{a_{i_0}(3)+a(3)}\Psi_{i_0}(1, u_3) + A'(v')^p \right) \leq \frac{\deg \Psi_{i_0}}{[k(x') : k(x)]} + 1, \quad (2)$$

which implies  $A_2(x') = B(x) - 1$  and

$$i_0\beta(u'_1, u'_2, v') \leq e \leq \left\lfloor \frac{\deg \Psi_{i_0}}{[k(x') : k(x)]} \right\rfloor + 1. \quad (3)$$

By definition **I.2.1**, we get

$$\deg \Psi_{i_0} \leq i_0 C(u_1, u_2, u_3) \leq i_0 \beta(u_1, u_2, u_3). \quad (4)$$

First assume that  $e \leq \left\lfloor \frac{\deg \Psi_{i_0}}{[k(x') : k(x)]} \right\rfloor$  in (2). Then (3) is strengthened to

$$i_0\beta(u'_1, u'_2, v') \leq \left\lfloor \frac{\deg \Psi_{i_0}}{[k(x') : k(x)]} \right\rfloor,$$

and then (4) implies (i), (ii), (iii) and (iv).

Up to the end of the proof of case 2, we assume that  $e = \left\lfloor \frac{\deg \Psi_{i_0}}{[k(x') : k(x)]} \right\rfloor + 1$  in (2). Then the integer  $e$  in (2) is not divisible by  $p$  by **II.5.3.2**(ii) of chapter 1. Therefore

$$\beta(u'_1, u'_2, v') < 1 + \left\lfloor \frac{e}{i_0} \right\rfloor \leq 1 + \left\lfloor \frac{\deg \Psi_{i_0}}{i_0 [k(x') : k(x)]} \right\rfloor. \quad (5)$$

Comparing with (4), this completes the proof of (iii).

If  $m(x) = 3$ , then (i) is a consequence of (iii) and (iv) is a direct consequence of (4) and (5).

If  $m(x) = 2$ , we have  $a(3) = 0$  and  $i_0 A_3(x) \leq a_{i_0}(3)$ . We now apply theorem **II.5.3.2**(iii) of chapter 1 to  $U_2^{a_{i_0}(2)+a(2)}F(U_2, U_3)$ , where  $F(U_2, U_3) := U_3^{a_{i_0}(3)}\Psi_{i_0}(U_2, U_3)$ . We get (note that  $i_0 \geq p \geq 2$ )

$$i_0\beta(u'_1, u'_2, v') \leq e \leq \deg \Psi_{i_0} + a_{i_0}(3). \quad (6)$$

By definition **I.2.1**, we actually have the following refinement of (4):

$$\deg \Psi_{i_0} + \sum_{j=2}^3 (a_{i_0}(j) - i_0 A_j(x)) = i_0 C(u_1, u_2, u_3) \leq i_0 (\beta(u_1, u_2, u_3) - A_3(x)).$$

Comparison with (6) gives  $\beta(u'_1, u'_2, v') \leq \beta(u_1, u_2, u_3)$  and this proves (ii), hence (i). We finally prove (iv): by (4) and (5), we have

$$\beta(u'_1, u'_2, v') < 1 + \left\lfloor \frac{\deg \Psi_{i_0}}{i_0 [k(x') : k(x)]} \right\rfloor \leq 1 + \left\lfloor \frac{\beta(u_1, u_2, u_3)}{[k(x') : k(x)]} \right\rfloor \leq \lceil \beta(u_1, u_2, u_3) \rceil,$$

since  $[k(x') : k(x)] \geq 2$  and  $\gamma(u_1, u_2, u_3) = \lceil \beta(u_1, u_2, u_3) \rceil \geq 2$ . If  $\beta(u_1, u_2, u_3) > 2$ , the right-hand side inequality is strict and (iv) is proved. If  $\beta(u_1, u_2, u_3) = \gamma(u'_1, u'_2, v') = 2$ , then  $[k(x') : k(x)] = 2$  and  $\beta(u'_1, u'_2, v') \leq 1 + \frac{1}{i_0} < 2$  by (3) (remark: this last case does actually occur when  $p = 2$ ).

**I.2.2.3** If  $x'$  is the point with parameters  $(\frac{X}{u_3}, \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3)$ , the polyhedron  $\Delta(u_3^{-p}h; \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3; \frac{X}{u_3})$  is minimal. We have  $E' = \text{div}(u'_1 u'_2 u'_3)$  and it is easily seen that  $A_2(x') = A_2(x)$ ,  $A_3(x') = A_2(x) + A_3(x) + C(x) - 1$ ,  $\beta(x') = \beta(x) + A_2(x) - 1$ ,  $C(x') \leq C(x)$  and

$$C(x') \leq \beta(x) - A_3(x) - C(x). \quad (7)$$

By symmetry, (i) has already been proved if  $m(x) = 3$ . So assume that  $m(x) = 2$ , and let us prove (i) and (v). Since  $C(x') \leq C(x)$ , (7) implies

$$C(x') \leq \frac{\beta(x) - A_3(x)}{2} \leq \frac{\beta(x)}{2},$$

and this concludes the proof.

We now begin the first two steps in the definition of our secondary invariant  $\kappa(x)$ . The function  $\kappa$  is a multiform function which is defined recursively, and takes values in the set  $\{0, 1, 2, 3, 4, 5\}$ . The statement “ $\kappa(x) \leq i$ ” means “some value of  $\kappa(x)$  is not greater than  $i$ ”. The statement “ $\kappa(x) > i$ ” is the set-theoretic complement of “ $\kappa(x) \leq i$ ”.

**I.2.3 Definition.** Let  $x \in \Sigma_p(X)$ . We say that  $\kappa(x) = 0$  if the following algorithm is finite.

1. Let  $X' \rightarrow X$  be the blowing up of  $X$  along  $x$  and  $x'$  be the center of  $\mu$  in  $X'$ . If  $x' \notin \Sigma_p(X')$ , or if  $\Omega(x') < \Omega(x)$  then STOP. Otherwise, go to 2;
2. Replace  $(X, x)$  with  $(X', x')$  and go to 1.

Note that theorem **II.5.4** of chapter 1 implies that  $\Omega(x') = \Omega(x)$  whenever the algorithm passes through step 2.

**I.2.4 Proposition.** With notations and hypotheses of **I.2.1** and **I.2.2**, if  $(A_2(x) < 1$  and  $\beta(x) < 1)$ , then  $\kappa(x) = 0$ .

Proof. Assume that  $x'$  in step 1 of the above algorithm is very near  $x$ .

By **I.2.2**(iii), if  $x'$  is in the chart with origin  $(\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, \frac{u_3}{u_2})$  then  $\beta(x') < 1$  and  $A_2(x') = B(x) - 1 \leq A_2(x) + \beta(x) - 1 < A_2(x)$ .

By **I.2.2**(v), if  $x'$  is the point at infinity  $(\frac{X}{u_3}, \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3)$ , then  $A_2(x') = A_2(x)$  and  $\beta(x') = \beta(x) + A_2(x) - 1 < \beta(x)$ .

In both cases,  $(A_2(x'), \beta(x')) < (A_2(x), \beta(x))$  for the lexicographical ordering. An induction ends the proof.

**I.2.5 Definition.** Let  $x \in \Sigma_p(X)$ . We say that  $\kappa(x) = 1$  if there exist consecutive choices of  $Y$ 's in step 1 of the following algorithm for which it is finite.

1. Choose  $Y \subseteq \Sigma_p(X)$  to be permissible at  $x$ . Let  $X' \rightarrow X$  be the blowing up of  $X$  along  $Y$  and  $x'$  be the center of  $\mu$  in  $X'$ . If  $x' \notin \Sigma_p(X')$  or if  $\Omega(x') < \Omega(x)$ , then STOP. Otherwise, go to 2;
2. Replace  $(X, x)$  with  $(X', x')$  and go to 1.

Note that necessarily  $x' \in \Sigma_p(X')$  and  $\Omega(x') = \Omega(x)$  whenever the algorithm passes through step 2. Also  $\kappa(x) = 0$  implies  $\kappa(x) = 1$ , and in particular  $\kappa(x) = 1$  whenever  $(\Omega(x') = \Omega(x)$  and  $\kappa(x') = 0)$  in step 1. We also point out that  $Y$  will not be uniquely determined in general when step 1 results in a stop. Finally, note that we can achieve a reduction in  $(\text{ord}_x h, \Omega(x))$  for the lexicographical ordering if  $\kappa(x) = 1$ .

**I.2.6 Proposition.** *With notations and hypotheses of I.2.1 and I.2.2, if  $(m(x) = 2$  and  $\beta(x) < 1$ ), or if  $(m(x) = 3$  and  $C(x) = 0$ ), then  $\kappa(x) \leq 1$ .*

*Proof.* If  $A_2(x) < 1$  and  $\beta(x) < 1$ , then  $\kappa(x) = 0$  by I.2.4. If  $(A_2(x) < 1, A_3(x) < 1$  and  $C(x) = 0$ ), then  $\beta(x) = A_3(x) < 1$ , so  $\kappa(x) = 0$  as well.

Otherwise, it can be assumed that  $\text{div}(u_1 u_2) \subseteq E$  and  $A_2(x) \geq 1$ . Let  $\mathcal{P} := (X, u_1, u_2) \subset \widehat{R}$ . By lemma I.2.6.1 below,  $Y := V(X, u_1, u_2)$  is actually a regular curve on  $X$ . Since  $A_2(x) \geq 1$ , we have

$$\text{ord}_{(u_1, u_2)}(g^p, f) \geq a(1) + \epsilon(x) = \text{ord}_{u_1} g^p \geq p,$$

therefore  $\text{ord}_{\mathcal{P}} h = p$  (recall that  $\epsilon(x) = \omega(x) > 0$ ), so  $Y \subseteq \Sigma_p(X)$ . Since  $A_2(x) \geq 1$ , we have  $\epsilon(y) = \epsilon(x)$ , where  $y$  is the generic point of  $Y$ , i.e.  $Y$  is permissible of the first kind. This gives our choice of  $Y$  in step 1 of the algorithm in I.2.5.

By II.5.4.2 (i), the only point  $x'$  in the blowing up  $X'$  of  $X$  along  $Y$  which may be very near  $x$  has parameters  $(\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, u_3)$ . This is the origin of a chart, there is no translation to do, and  $m(x') = m(x)$ . We have  $A_2(x') = A_2(x) - 1$ ,  $A_3(x') = A_3(x)$ , and  $\beta(x') = \beta(x)$  (resp.  $C(x') = C(x) = 0$ ) if  $m(x) = 2$  (resp.  $m(x) = 3$ ). An induction on  $A_2(x) + A_3(x)$  ends the proof.

**I.2.6.1 Lemma.** *Assume that the r.s.p.  $(u_1, u_2, u_3)$  of  $S$  and  $\widehat{X} \in \widehat{R}$  are such that*

(i)  $\text{div}(u_1) \subseteq E_0 \subseteq E \subseteq \text{div}(u_1 u_2 u_3)$ , and

(ii)  $h \in \mathcal{P}$ , where  $\mathcal{P} := (\widehat{X}, u_1, u_2) \subset \widehat{R}$ .

*Then there exists  $X \in R$  such that  $\mathcal{P} = (X, u_1, u_2)$ .*

*Proof.* Pick any  $Z \in R$  such that  $(Z, u_1, u_2, u_3)$  is a r.s.p. of  $R$ . then there exists a series  $\varphi(u_3) \in k(x)[[u_3]]$  such that  $(Z - \varphi(u_3), u_1, u_2) = (\widehat{X}, u_1, u_2)$ . Let  $h = Z^p - Zg^{p-1} + f_Z$  be the expansion of  $h$ , with  $f_Z, g \in S$ . Since  $h \in \mathcal{P}$ , we have

$$f_Z + \varphi(u_3)^p - \varphi(u_3)g^{p-1} \in (u_1, u_2)\widehat{S}.$$

Since  $u_1$  divides  $g$  by assumption (i), we must have  $f + \varphi(u_3)^p \in (u_1, u_2)\widehat{S}$ . But  $f \in S$  and  $S$  is regular, so  $\varphi(u_3) \in S$ . Let  $X := Z - \varphi(u_3)$ .

**I.2.7 Theorem.** *Assume that  $x \in \Sigma_p(X)$ ,  $\omega'(x) = 1$  and  $H(x)^{-1}g^p = \gamma^p u_1^{\epsilon(x)}$ . Then  $\kappa(x) \leq 1$ .*

*Proof.* Let

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_{n-1} \leftarrow X_n \leftarrow \cdots$$

be the quadratic sequence along  $\mu$ , i.e.  $X_n$  is the blowing up along the center  $x_{n-1}$  of  $\mu$  in  $X_{n-1}$  for  $n \geq 1$ . Then  $(\Omega(x_n))_{n \geq 0}$  is a non-increasing sequence, and we have  $\kappa(x) = 0$  unless it is constant. We thus assume that  $\Omega(x_n) = \Omega(x)$  for  $n \geq 0$ . We have  $m(x_n) \geq 2$  for  $n \geq 2$ .

**I.2.7.1** If there exists  $n_1 \geq 0$  such that  $m(x_n) = 3$  for  $n \geq n_1$ ,  $x_n$  is always at the origin of a chart in the blowing up  $X_n \rightarrow X_{n-1}$  and there is no translation to do to minimize polyhedra.

By standard arguments, the ideal  $(\{\phi_i^{\frac{\epsilon(x)!}{i}}\}_{1 \leq i \leq \epsilon(x)})$  gets principal and monomial in  $u_{2,n}, u_{3,n}$  for  $n \gg 0$ , where  $h(n) := X(n)^p - X(n)g(n)^{p-1} + f(n)$  is a local equation of  $X_n$  at  $x_n$  and  $\Delta(h(n); u_{1,n}, u_{2,n}, u_{3,n}; X(n))$  is minimal, i.e. we have  $C(x_n) = 0$ . Therefore  $\kappa(x) \leq 1$  by I.2.6.

**I.2.7.2** If there exists  $n_1 \geq 0$  such that  $m(x_n) = 2$  for  $n \geq n_1$ , then it can be assumed that  $n_1 = 0$  without loss of generality. Each  $x_n$  is on the strict transform of  $\text{div}(u_1)$  and we can choose a r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  such that the exceptional divisor of  $X_n \rightarrow X_{n-1}$  is  $\text{div}(u_2)$  for all  $n \geq 0$ , the strict transform of  $h$  being  $h(n) := \frac{h}{u_2^n}$ . As seen in I.1, or in II.5.6 of chapter 1, we have  $\frac{g}{u_2^n} \in \mathcal{O}_{X_n, x_n}$ ,  $\mu(\frac{g}{u_2^n}) > 0$ , so

$$n < \frac{\mu(g)}{\mu(u_2)} < \infty$$

since the value group of  $\mu$  is Archimedean. Therefore  $\kappa(x) = 0$  in this case.

**I.2.7.3** None of the above. By **I.2.2(i)**, there exists  $n_1$  such that  $\gamma(x_n) = \gamma(x_{n_1})$  for  $n \geq n_1$ . By **I.2.2(v)**, we have  $\gamma(x_{n_1}) \leq 2$ . Pick  $n_2 > n_1$  such that  $m(x_{n_2-1}) = 3$  and  $m(x_{n_2}) = 2$ . By **I.2.2(iii)**,

$$\beta(x_{n_2}) < \gamma(x_{n_2-1}) = \gamma(x_{n_1}) \leq 2. \quad (1)$$

Let  $n_3$  be the least integer  $n > n_2$  such that  $m(x_{n-1}) = 2$  and either  $x_n$  is not rational over  $x_{n-1}$  or  $m(x_n) = 3$ . By definition of  $n_3$ , we have  $m(x_n) = 2$  for  $n_2 \leq n \leq n_3 - 1$ . Therefore (1) and **I.2.2(iii)** imply that

$$\beta(x_{n_3-1}) \leq \beta(x_{n_2}) < 2.$$

By **I.2.2(iv)** or (v), we have  $\gamma(x_{n_3}) = \gamma(x_{n_1}) \leq 1$ . Going back to (1), we now get the sharper upper bound  $\beta(x_{n_2}) < 1$ , so  $\kappa(x) = \kappa(x_{n_2}) \leq 1$  by **I.2.6**.

**We assume that  $\omega'(x) = 2$  from this point on.**

## II A few cases where $\kappa(x) \leq 1$ .

In view of theorems **I.1**, **I.2.7** and chapter 1 **II.5.6**, we may restrict our attention to the case  $\omega'(x) = 2$ . From now on and up to the end of this article, we thus assume that  $\omega'(x) = 2$ .

All along this section, we suppose that the r.s.p.  $(u_1, u_2, u_3)$  of  $\widehat{S}$  and  $X \in \widehat{R}$  are such that  $E \subseteq \text{div}(u_1 u_2 u_3)$  and the polyhedron  $\Delta(h; u_1, u_2, u_3; X)$  is minimal. We gather in here some criteria to have  $\kappa(x) \leq 1$  that will be used in the next chapters, as well as some auxiliary lemmas.

**II.1 Proposition.** *Assume that  $\text{div}(u_1) \subseteq E_0$  and  $H(x)^{-1}f = u_1\phi + \gamma_1 u_2^{\omega(x)} u_3$ , with  $\text{ord}_{\eta(x)}(u_1\phi) = \omega(x)$  and  $\gamma_1$  invertible. We have  $\kappa(x) \leq 1$  provided one of the following properties holds:*

- (i)  $\omega(x) \geq 2$  and  $E \subseteq \text{div}(u_1 u_2)$ ;
- (ii)  $\omega(x) \geq 3$ ;
- (iii)  $\omega(x) = 2$ ,  $\text{div}(u_1 u_3) \subseteq E$  and  $\text{cl}_1 \phi \notin k(x).U_3$ ;
- (iv)  $\omega(x) = 1$  and  $E = \text{div}(u_1 u_2)$ .

We first recollect from chapter 1 a lemma which leads to important corollaries.

**II.1.1 Lemma.** *We suppose that  $\omega'(x) = 2$  and that the r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  is such that  $E \subseteq \text{div}(u_1 u_2 u_3)$  and the polyhedron  $\Delta(h; u_1, u_2, u_3; X)$  is minimal. Let  $Y$  be a permissible center of first kind, and  $X' \rightarrow X$  be the blowing up of  $X$  along  $Y$ .*

*If  $x' \in X'$  is very near  $x$ , then  $x'$  maps to  $\text{PDir}(\text{cl}_{\epsilon(x)} J(f, E, x))$  (resp.  $\text{PDir}(\text{cl}_{\omega(x)} J(f, E))$ ) if  $\epsilon(x) = \omega(x)$  (resp. if  $\epsilon(x) = 1 + \omega(x)$ ).*

*Proof.* When  $\epsilon(x) = \omega(x)$ , as  $\omega'(x) = 2$ , we have  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \epsilon(x)$  and chapter 1 **II.5.4.3(i)** gives the conclusion. When  $\epsilon(x) = 1 + \omega(x)$ , chapter 1 **II.5.4.2(ii)**, (iii), (iv) or **II.5.4.3(ii)** give the conclusion.

This lemma leads to the next statements. Recall the definition of  $\tau(x)$  in chapter 1, **II.4**.

**II.1.2 Corollary.** *If  $\omega(x) = \epsilon(x)$  and  $\tau(x) = 3$ , then  $\kappa(x) = 0$ .*

*Proof.* Here,  $\tau(x) = 3$  means that  $\text{PDir}(\text{cl}_{\epsilon(x)} J(f, E, x)) = \emptyset$ .

We now introduce some remarks and definitions about the directrix.

If  $\epsilon(x) = \omega(x)$ , we have  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \omega(x)$  since  $\omega'(x) = 2$ . More generally, if  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \epsilon(x)$  then  $\text{cl}_{\epsilon(x)} J(f, E, x)$  (resp.  $\text{cl}_{\omega(x)} J(f, E)$ ) does not depend on any choice

of parameters  $(X, u_1, u_2, u_3)$  with  $E \subseteq \text{div}(u_1 u_2 u_3)$  and  $\Delta(h; u_1, u_2, u_3; X)$  minimal if  $\epsilon(x) = \omega(x)$  (resp.  $\epsilon(x) = 1 + \omega(x)$ ), as noticed in chapter 1 **II.3.2**(ii).

If  $\epsilon(x) = 1 + \omega(x)$  and  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) = \epsilon(x)$ , then  $\text{cl}_{\omega(x)}J(f, E)$  may depend on the parameters  $(X, u_1, u_2, u_3)$  with  $E \subseteq \text{div}(u_1 u_2 u_3)$  and  $\Delta(h; u_1, u_2, u_3; X)$  minimal. When making a translation on  $X$ , say  $Z := X - \theta$  is the new variable,  $f$  is changed into  $f_Z := f + \theta^p - \theta g^{p-1}$ . If  $\Delta(h; u_1, u_2, u_3; Z)$  is minimal, the Newton polyhedron of  $\theta$  is a subset of  $\Delta(h; u_1, u_2, u_3; X)$ ,  $H(x)$  divides  $\theta g^{p-1}$ , so we have the following congruence of vector spaces:

$$\text{cl}_{\omega(x)}J(f_Z, E) \equiv \text{cl}_{\omega(x)}J(f, E) \pmod{k(x)[\{U_i : u_i \mid H(x)^{-1}g^p\}]_{\omega(x)}}.$$

Furthermore, by chapter 1 **II.5.4.2**(i), if we blow up  $x$ , every point  $x'$  very near  $x$  maps to the strict transform of  $H(x)^{-1}g^p$ . By **II.1.1** above,  $x'$  maps to  $\mathbf{Proj}(W)$  where

$$I(W) = \text{IDir}(\text{cl}_{\omega(x)}J(f, E) + (\{U_i : u_i \mid H(x)^{-1}g^p\})),$$

and  $W$  does not depend on any choice of parameters  $(X, u_1, u_2, u_3)$  with  $E \subseteq \text{div}(u_1 u_2 u_3)$  and  $\Delta(h; u_1, u_2, u_3; X)$  minimal. This leads to the following definitions.

**II.1.3 Definition.** *Let*

$\text{VDir}(x) := \text{VDir}(\text{cl}_{\epsilon(x)}J(f, E, x))$  if  $\epsilon(x) = \omega(x)$ ;

$\text{VDir}(x) := \text{VDir}(\text{cl}_{\omega(x)}J(f, E) + (\{U_i : u_i \mid H(x)^{-1}g^p\}))$  if  $\epsilon(x) = 1 + \omega(x)$  and  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) = \epsilon(x)$ ;

$\text{VDir}(x) := \text{VDir}(\text{cl}_{\omega(x)}J(f, E))$  if  $\epsilon(x) = 1 + \omega(x)$  and  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \epsilon(x)$ .

We denote by  $\tau'(x)$  the codimension of the  $k(x)$ -vector space  $\text{VDir}(x)$  and by  $\text{IDir}(x)$  the ideal generated by  $\text{VDir}(x)$ .

The previous considerations give

**II.1.4 Corollary.** *The vector space  $\text{VDir}(x)$  and the integer  $\tau'(x)$  do not depend on choices of parameters with  $E \subseteq \text{div}(u_1 u_2 u_3)$  and  $\Delta(h; u_1, u_2, u_3; X)$  minimal. Furthermore, if  $\tau'(x) = 3$ , then  $\kappa(x) = 0$ .*

**Proof of II.1.** The assumption on the expansion of  $f$  implies that

$$\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \text{ord}_{\eta(x)}(H(x)^{-1}f) = \epsilon(x) = \omega(x). \quad (1)$$

Moreover,  $U_1$  divides  $\text{cl}_{\omega(x)}J(f, E, x)$ , so by **II.1.1**, if  $e : X' \rightarrow X$  is a permissible blowing up of the first kind, any  $x' \in X'$  very near  $X$  maps to the strict transform of  $\text{div}(u_1)$ . Let  $Y := V(X, u_1, u_2) \subseteq \text{Spec}(\widehat{R}/(h))$  and  $y$  be the generic point of  $Y$ .

If  $\text{div}(u_1 u_2) \subseteq E$ , then  $Y = V(X, v_1, v_2)$  if  $(v_1, v_2, v_3)$  is any r.s.p. of  $S$  such that  $\text{div}(v_i) = \text{div}(u_i)$  for  $i = 1, 2$ . Therefore lemma **I.2.6.1** applies and  $Y$  is a regular curve on  $X$  such that  $\eta(Y)$  has normal crossings with  $E$ .

If  $\omega(x) = 1$ , assumption (iv) holds, so  $E = \text{div}(u_1 u_2)$ . Then

$$\text{ord}_{\eta(x)}J(f, E, x) = \text{ord}_{\eta(y)}J(f, E, Y) = 1, \quad \text{ord}_{\eta(x)}(H(x)^{-1}g^p) = \text{ord}_{\eta(y)}(H(x)^{-1}g^p),$$

and  $\text{ord}_{\eta(x)}f = \text{ord}_{\eta(y)}f = \text{ord}_{\eta(x)}H(x) + \omega(x)$ , so  $Y \subseteq \Sigma_p(X)$  and  $\epsilon(x) = \epsilon(y)$ :  $Y$  is permissible of the first kind.

More generally, if  $\text{div}(u_1 u_2) \subseteq E$  and  $Y$  is permissible of the first kind, Take  $e : X' \rightarrow X$  to be the blowing up along  $Y$ ; by **II.1.1**, the only point which can possibly be very near  $x$  is

$x' = (X' := \frac{X}{u_2}, u'_1 := \frac{u_1}{u_2}, u'_2 := u_2, u_3)$ . Then  $\text{div}(u'_1 u'_2) \subseteq E' := (e^{-1}E)_{\text{red}}$  and  $h' := u_2^{-p}h = X'^p - X'g'^{p-1} + f'$ , where

$$f' = H(x')(u'_1 u'_2^{-(\omega(x)-1)}\phi + \gamma_1 u_3).$$

We are at the origin of a chart, so  $\Delta(h'; u'_1, u'_2, u_3; X')$  is minimal. Hence  $\omega(x') \leq 1$  and  $\omega(x') = 0$  if (iv) holds, so  $\kappa(x) = 1$ .

From now on, we assume that  $\omega(x) \geq 2$ . Let  $H(x) = u_1^{a(1)}u_2^{a(2)}u_3^{a(3)}$ , with  $a(i) = 0$  if  $\text{div}(u_i) \not\subseteq E$ . Let  $e : X' \rightarrow X$  be the blowing up at  $x$ ,  $E' := (e^{-1}E)_{\text{red}}$  and  $x' \in X'$  be very near  $x$ . Then  $x'$  maps to the strict transform of  $\text{div}(u_1)$  (cf. comments in the beginning of the proof of this proposition). We claim that  $x'$  is on the strict transform of  $Y$  if  $\kappa(x') > 0$ .

If  $x'$  is in the chart of origin  $(X' := \frac{X}{u_2}, u'_1 := \frac{u_1}{u_2}, u'_2 := u_2, u'_3 := \frac{u_3}{u_2})$ , then  $(X', u'_1, u'_2, v')$  is a system of coordinates at  $x'$ , where  $v' := P(1, u'_3)$  for some irreducible homogeneous polynomial  $P \in k(x)[U_2, U_3]$ , unitary in  $U_3$ . Let  $h' := u_2^{-p}h = X'^p - X'g'^{p-1} + f'$ .

If  $P \neq U_3$ , then  $E' = \text{div}(u'_1 u'_2)$ ,  $H(x') := u_1^{a(1)}u_2^{a(2)+\omega(x)-p}$ , and we have

$$H(x')^{-1}f' = u_3^{a(3)}(u'_1 u'_2^{-(\omega(x)-1)}\phi + \gamma_1 u'_2 u'_3). \quad (2)$$

By (1),  $u'_2$  divides  $H(x')^{-1}g'^p$ . We may have to make a translation  $Z' := X' - \theta'$  on  $X'$  to get the polyhedron  $\Delta(h'; u'_1, u'_2, v'; Z')$  minimal.

If  $\text{ord}_{\eta'(x')}(H(x')^{-1}g'^p) \leq 2$ , we get  $\Omega(x') \leq (2, 1)$ :  $x'$  is not very near to  $x$ , since  $\Omega(x) \geq (2, 2)$ .

If  $\text{ord}_{\eta'(x')}(H(x')^{-1}g'^p) \geq 3$ , as  $\text{ord}_{(u'_1, u'_2)}(H(x')^{-1}\theta'^p) \geq 1$  by (2), we get

$$\text{ord}_{(u'_1, u'_2)}(H(x')^{-1}\theta'g'^{p-1}) \geq 3\frac{p-1}{p} + \frac{1}{p} = 3 - \frac{2}{p} \geq 2.$$

If  $p$  does not divide  $a(1)$ , we have

$$\text{ord}_{\eta'(x')}(H(x')^{-1}u'_1 \frac{\partial(f' + \theta'^p - \theta'g'^{p-1})}{\partial u'_1}) = 1$$

by (2), since  $u'_3$  is invertible at  $x'$ . If  $p$  divides  $a(1)$ , we apply chapter 1, **II.5.3.2(i)** to the monomial  $\overline{\gamma}_1 U_2^{a(2)+\omega(x)} U_3^{a(3)+1}$  (which is not a  $p^{\text{th}}$ -power because  $\Delta(h; u_1, u_2, u_3; X)$  is minimal), where  $\overline{\gamma}_1$  denotes the image of  $\gamma_1$  in  $k(x)$ . We then get

$$\text{ord}_{\eta'(x')}(H(x')^{-1} \frac{\partial(f' + \theta'^p - \theta'g'^{p-1})}{\partial \lambda'_i}) = 1$$

for some  $i$ ,  $2 \leq i \leq s$ , with conventions on derivations as in chapter 1 **II.3**. Therefore  $\omega(x') \leq 1 < \omega(x)$ .

If  $P = U_3$ ,  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal and we get

$$H(x')^{-1}f' = u'_1 u'_2^{-(\omega(x)-1)}\phi + \gamma_1 u'_2 u'_3. \quad (3)$$

Therefore  $\omega(x') \leq \text{ord}_{\eta'(x')}(u'_2 u'_3) = 2$ . We are done unless possibly if  $\omega(x) = \omega(x') = 2$  (so  $\text{div}(u_3) \subseteq E$ ), in which case assumption (iii) holds. In particular, we now have  $E' = \text{div}(u'_1 u'_2 u'_3)$ .

Let then  $\text{cl}_1 \phi =: \lambda_1 U_1 + \nu_3 U_2 + \mu_2 U_3$ ,  $\lambda_1, \mu_2, \nu_3 \in k(x)$ , and  $(\lambda_1, \nu_3) \neq (0, 0)$  by assumption. Since  $\omega(x') = 2$ , we have  $\nu_3 = 0$  by (3), so  $\lambda_1 \neq 0$ . We claim that  $\tau(x') = 3$ , whence  $\kappa(x) = \kappa(x') = 0$

by **II.1.2**. Note that  $x'$  then satisfies the assumptions of lemma **II.1.5** below with  $\mu_1 := \overline{\gamma_1}$  and  $\lambda_2 := 0$  for some  $\mu_3 \in k(x')$ , and the conclusion follows.

The last point we have to consider is thus the point  $x' = (X' := \frac{X}{u_3}, u'_1 := \frac{u_1}{u_3}, u'_2 := \frac{u_2}{u_3}, u'_3 := u_3)$ , i.e.  $x'$  is on the strict transform of  $Y$ . This is the origin of a chart, so  $\Delta(u_3^{-p}h; u'_1, u'_2, u'_3; X')$  is minimal. Let  $h' := u_3^{-p}h = X'^p - X'g'^{p-1} + f'$ , where

$$H(x')^{-1}f' = u'_1 u'_3^{-\omega(x)-1} \phi + \gamma_1 u'_2 \omega(x) u'_3.$$

Let us see that  $x'$ ,  $(X', u'_1, u'_2, u'_3)$  verify the hypotheses **II.1(ii)** or **(iii)**. This is obvious if  $\omega(x) \geq 3$ . If  $\omega(x) = \omega(x') = 2$ , then  $\overline{\phi} := \text{cl}_1 \phi \in \langle U_1, U_2 \rangle$ ,

$$\text{cl}_1(u'_3^{-1} \phi) = \overline{\phi}(U'_1, U'_2) \notin k(x').U'_3$$

and  $\text{div}(u'_1 u'_3) \subseteq E'$ , so **(iii)** holds.

Let

$$X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_{n-1} \leftarrow X_n \leftarrow \cdots$$

be the quadratic sequence along  $\mu$ , i.e.  $X_n$  is the blowing up along the center  $x_{n-1}$  of  $\mu$  in  $X_{n-1}$  for  $n \geq 1$ . Let  $h(n)$  be a local equation of  $X_n$ .

By well known results,  $\text{ord}_{x_n} h(n) = \text{ord}_y h$  for  $n \gg 0$ , so  $x_n \notin \Sigma_p(X_n)$  if  $Y \not\subseteq \Sigma_p(X)$ . Since  $\omega(x) \geq 2$ , we have  $Y = V(X, u_1, u_2) \subseteq W := \{z \in X \mid \omega(z) > 0\}$ . If  $Y \subseteq \Sigma_p(X)$ , the strict transform  $Y_n$  of  $Y$  in  $X_n$  is permissible of the first kind at  $x_n$  for  $n \gg 0$  by chapter 1 **II.6.1**. As noticed in the beginning of the proof, this implies  $\kappa(x) \leq 1$ .

**II.1.5 Lemma.** *Assume that  $E = \text{div}(u_1 u_2 u_3)$ ,  $\Delta(h; u_1, u_2, u_3; X)$  is minimal and*

$$\Psi := \text{cl}_2(H(x)^{-1}f) = \lambda_1 U_1^2 + \mu_2 U_1 U_3 + \mu_3 U_1 U_2 + \lambda_2 U_2^2 + \mu_1 U_2 U_3,$$

with  $\lambda_2 \mu_2 = 0$  and  $\lambda_1 \mu_1 \neq 0$ . Then  $\tau(x) = 3$ .

Proof. For each  $F = \sum_{a_1 a_2 a_3} \lambda_{a_1 a_2 a_3} U_1^{a_1} U_2^{a_2} U_3^{a_3} \in k(x)[U_1, U_2, U_3]_2$ , we denote

$$S(F) := \{(a_1, a_2, a_3) \mid \lambda_{a_1 a_2 a_3} \neq 0\}.$$

Let  $K \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 2\}$  be the convex hull of  $S(\Psi)$ . Since  $E = \text{div}(u_1 u_2 u_3)$ , we have  $S(F) \subseteq K$  for every  $F \in \text{cl}_2 J(f, E, x)$ ; in particular, the monomial  $U_3^2$  and either  $U_2^2$  or  $U_1 U_3$  appears in no  $F \in \text{cl}_2 J(f, E, x)$ . Furthermore, the monomial  $U_1^2$  (resp.  $U_2 U_3$ ) appears with nonzero coefficient in some element  $G_1$  (resp.  $G_2$ ) of  $\text{cl}_2 J(f, E, x)$ , since  $H(x)u_1^2$  and  $H(x)u_2 u_3$  induce vertices of  $\Delta(h; u_1, u_2, u_3; X)$  which is minimal.

Suppose that  $\tau(x) \leq 2$ . Then  $\text{cl}_2 J(f, E, x) \subseteq k(x)[V_1, V_2]_2$ , where  $V_1 = U_1 + \alpha_2 U_2 + \alpha_3 U_3$  and  $V_2 = \alpha'_2 U_2 + \alpha'_3 U_3$ , since  $U_1^2$  appears in  $G_1$ . The rank of the matrix

$$\begin{pmatrix} \alpha_2 & \alpha_3 \\ \alpha'_2 & \alpha'_3 \end{pmatrix}$$

is two, since  $U_2 U_3$  but not  $U_3^2$  appears in  $G_2$ . Without loss of generality, it can thus be assumed that  $\alpha_2 \alpha_3 = 0$ . Let  $\{i, j\} = \{2, 3\}$  be such that  $\alpha_i = 0$  and  $\alpha_j \neq 0$ . Without loss of generality, it can also be assumed that  $\alpha'_i = 1$ .

Since  $U_3^2$  and either  $U_2^2$  or  $U_1 U_3$  appears in no element of  $\text{cl}_2 J(f, E, x)$ , elementary considerations show that  $\alpha'_j \neq 0$  and that every  $F \in \text{cl}_2 J(f, E, x)$  must be a scalar multiple of one and

the same polynomial  $G := V_1^2 + c_1 V_1 V_2 + c_2 V_2^2$  for some  $c_1, c_2 \in k(x)$ . We get the following list of possible cases:

*Case 1:*  $i = 3$ ,  $\mu_2 \neq 0$ ,  $\lambda_2 = 0$ ,  $c_2 = 0$ ,  $c_1 = -\frac{\alpha_2}{\alpha_2'}$ ;

*Case 2:*  $i = 2$ ,  $\lambda_2 = 0$ ,  $c_2 = 0$ ,  $c_1 = -\frac{\alpha_3}{\alpha_3'}$ ;

*Case 3:*  $i = 2$ ,  $\mu_2 = 0$ ,  $c_1 = -2\frac{\alpha_3}{\alpha_3'}$ ,  $c_2 = (\frac{\alpha_3}{\alpha_3'})^2$ .

Case 3 is easily discarded from the assumption  $\mu_1 \neq 0$ . We give the proof in case 2, the proof of case 1 being similar. Let us denote  $H(x) = u_1^{a(1)} u_2^{a(2)} u_3^{a(3)}$  and  $F_i := \text{cl}_2(H(x)^{-1} u_i \frac{\partial f}{\partial u_i})$ ,  $i = 1, 2, 3$ . All of these forms should be proportional:

$$G = U_1^2 + c_1 U_1 U_2 + \alpha_3 U_1 U_3 + c_1 \alpha_3 U_2 U_3;$$

$$F_1 = a(1)\mu_1 U_2 U_3 + (a(1) + 1)(\mu_3 U_1 U_2 + \mu_2 U_1 U_3) + (a(1) + 2)\lambda_1 U_1^2;$$

$$F_2 = a(2)(\lambda_1 U_1^2 + \mu_2 U_1 U_3) + (a(2) + 1)(\mu_3 U_1 U_2 + \mu_1 U_2 U_3);$$

$$F_3 = a(3)(\lambda_1 U_1^2 + \mu_3 U_1 U_2) + (a(3) + 1)(\mu_2 U_1 U_3 + \mu_1 U_2 U_3).$$

So the matrix

$$M := \begin{pmatrix} 1 & c_1 & \alpha_3 & c_1 \alpha_3 \\ (a(1) + 2)\lambda_1 & (a(1) + 1)\mu_3 & (a(1) + 1)\mu_2 & a(1)\mu_1 \\ a(2)\lambda_1 & (a(2) + 1)\mu_3 & a(2)\mu_2 & (a(2) + 1)\mu_1 \\ a(3)\lambda_1 & a(3)\mu_3 & (a(3) + 1)\mu_2 & (a(3) + 1)\mu_1 \end{pmatrix}$$

must have rank one. The last two rows are not both zero, and are linearly dependent if and only if  $(\mu_2 = \mu_3 = 0$  and  $a(2) = a(3))$ . But then  $M$  has rank two, since  $c_1 \neq 0$ : a contradiction which proves that  $\tau(x) = 3$ .

**II.2 Proposition.** *Assume that  $E = \text{div}(u_1)$  and  $\epsilon(x) = \omega(x)$ . If*

$$\text{VDir}(x) \equiv \langle U_2, U_3 \rangle \pmod{U_1},$$

*then  $\kappa(x) = 0$ .*

*Proof.* We have  $\text{ord}_{\eta(x)}(H(x)^{-1} g^p) > \text{ord}_{\eta(x)}(H(x)^{-1} f) = \epsilon(x) = \omega(x)$  since  $\omega'(x) = 2$ . Let

$$F := \text{cl}_{\omega(x)}(H(x)^{-1} f).$$

Since  $\epsilon(x) = \omega(x)$ , we have  $F \in k(x)[U_1, U_2^p, U_3^p]$ . If  $\tau(x) = 3$ , then  $\kappa(x) = 0$  by **II.1.2**, so we assume that  $\tau(x) = 2$ .

We can now pick a r.s.p.  $(u_1, v_2, v_3)$  of  $S$ , where  $v_i := u_i + \alpha_i u_1$ ,  $\alpha_i \in S$  invertible,  $i = 2, 3$ , and get an expression  $H(x)^{-1} f = F(u_1, v_2, v_3) + \phi$ ,  $\text{ord}_{\eta(x)} \phi > \epsilon(x)$  and  $\text{VDir}(x) = \langle V_2, V_3 \rangle$ . Of course, we may have to do a translation  $Z := X - \theta$  on  $X$  in order to get  $\Delta(h; u_1, v_2, v_3; Z)$  minimal. Since  $\text{ord}_{\eta(x)}(H(x)^{-1} g^p) > \omega(x)$ , we have

$$\text{cl}_{\epsilon(x)} J(f, E, x) = \text{cl}_{\epsilon(x)} J(f_Z, E, x) \in k(x)[V_1, V_2]_{\omega(x)},$$

where  $h = Z^p - g^{p-1} Z + f_Z$ .

By **II.1.1**, if  $e : X' \rightarrow X$  is the blowing up of  $x$  and  $x' \in X'$  is very near  $X$ , then  $x' = (Z' := \frac{Z}{u_1}, u_1' := u_1, v_2' := \frac{v_2}{u_1}, v_3' := \frac{v_3}{u_1})$ . This is the origin of a chart, so  $\Delta(h'; u_1', v_2', v_3'; Z')$  is minimal,  $E' := (e^{-1} E)_{\text{red}} = \text{div}(u_1')$  and we get an expression

$$H(x')^{-1} f' = F(1, v_2', v_3') + u_1' \phi'.$$

If  $x'$  is very near  $x$ , we have  $\text{ord}_{\eta'(x')} \phi' \geq \omega(x) - 1$  and therefore  $\epsilon(x') = \omega(x)$ . Also note that  $\text{ord}_{u'_1}(H(x')^{-1}g'^p) > \omega(x)$ , since otherwise  $\Omega(x') \leq (\omega(x), 1) < \Omega(x)$ . By construction,

$$\text{cl}_{\omega(x)} J(f', E', x') \equiv \text{cl}_{\omega(x)} J(H(x')F(1, v'_2, v'_3), E', x') \pmod{U'_1 k(x') [U'_1, V'_2, V'_3]_{\omega(x)-1}},$$

hence  $\text{VDir}(x') \equiv \langle V'_2, V'_3 \rangle \pmod{U'_1}$  and  $x'$  satisfies again the assumptions of **II.1** w.r.t. the r.s.p.  $(Z', u'_1, v'_2, v'_3)$ .

Let

$$X = X_0 \leftarrow X_1 \leftarrow \cdots X_{n-1} \leftarrow X_n \leftarrow \cdots$$

be the quadratic sequence along  $\mu$ , i.e.  $X_n$  is the blowing up along the center  $x_{n-1}$  of  $\mu$  in  $X_{n-1}$  for  $n \geq 1$ . We iterate the previous argument: if  $\kappa(x) > 0$ , there exists series  $w_i := u_i + \sum_{j \geq 1} \alpha_{ij} u_1^j$ ,  $\alpha_{ij} \in k(x)$ ,  $i = 2, 3$  and  $Z \in \widehat{R}$  with the following property:

- (a) the polyhedron  $\Delta(h; u_1, w_2, w_3; Z)$  is minimal;
- (b) if  $x_n$  is very near  $x$ , then  $x_n$  is on the strict transform  $Y_n$  in  $X_n$  of  $Y := V(Z, w_2, w_3) \subseteq \text{Spec}(\widehat{R}/(h))$ .

As in the proof of proposition **II.1**, we have  $x_n \notin \Sigma_p(X_n)$  for  $n \gg 0$  unless  $Y \subseteq \Sigma_p(X)$ . But this is a contradiction, since  $\eta(Y) \not\subseteq E$ .

**II.3 Proposition.** *Assume that  $\omega(x) \geq 2$ ,  $E = \text{div}(u_1)$  and  $H(x)^{-1}f = F(u_2, u_3) + u_1\phi$  with  $\text{ord}_{\eta(x)}(u_1\phi) = \omega(x)$  and  $F \in k(x)[u_2, u_3]_{1+\omega(x)}$ . If moreover*

$$\text{VDir}(\text{cl}_{\omega(x)} J(f, E, x) + \text{cl}_{\omega(x)} J(F, E)) = \langle U_1, U_2, U_3 \rangle, \quad (1)$$

then  $\kappa(x) \leq 1$ .

Proof. We have  $\text{ord}_{u_1}(H(x)^{-1}g^p) > \text{ord}_{\eta(x)}(H(x)^{-1}f)$  since  $\epsilon(x) = \omega(x)$ . In particular, we have  $F \neq 0$  by definition of  $H(x)$ . Moreover,  $\text{cl}_{\omega(x)}(u_1\phi) \in k(x)[U_1, U_2^p, U_3^p]$ , since  $\epsilon(x) = \omega(x)$ . We discuss according to the value of  $\tau(x)$ .

**II.3.1** If  $\tau(x) = 3$ , then  $\kappa(x) = 0$  by **II.1.1**.

**II.3.2** If  $\tau(x) = 2$ , as  $\text{ord}_{\eta(x)}(u_1\phi) = \omega(x)$ ,  $U_1$  divides  $\text{cl}_{\omega(x)} J(f, E, x)$ , so  $U_1 \in \text{VDir}(x)$ . Let  $\text{VDir}(x) = \langle U_1, \lambda U_2 + \mu U_3 \rangle$ ,  $\lambda, \mu \in k(x)$ ,  $(\lambda, \mu) \neq (0, 0)$ . In particular,  $\text{cl}_{\omega(x)-1}\phi \notin k(x)[U_1]$ , and thus  $\omega(x) \geq 1 + p$ .

By symmetry, it can be assumed that  $\mu \neq 0$  and we replace  $u_2$  by  $v := u_2 + \frac{\mu}{\lambda} u_2$ . We may have to do a translation  $Z := X - \theta$ ,  $\theta \in k(x)[[u_1, u_2, u_3]]$  in order to get  $\Delta(h; u_1, v, u_3; Z)$  minimal. Then  $f$  is changed to  $f_Z := f + \theta^p - \theta g^{p-1}$ . As  $\omega'(x) = 2$ ,  $E = \text{div}(u_1)$  and  $\omega(x) = \epsilon(x)$ ,  $u_1$  divides  $H(x)^{-1}g^{p-1}\theta$  and  $\text{ord}_{\eta(x)}(H(x)^{-1}g^{p-1}\theta) > \omega(x)$ , so  $H(x)^{-1}f_Z = F(v, u_3) + u_1\phi_Z$  and the vector space  $\text{cl}_{\omega(x)} J(f, E, x)$  is unchanged. Hypothesis (1) is equivalent to:

$$\left\langle \frac{\partial F}{\partial U_2}, \frac{\partial F}{\partial U_3} \right\rangle \not\subseteq k(x)[V].$$

This condition is independent of the choice of the variables  $(u_2, u_3)$ ; in other terms, it can be assumed without loss of generality that  $\text{VDir}(x) = \langle U_1, U_2 \rangle$ , hypothesis (1) then reading

$$F(U_2, U_3) \notin k(x)[U_2] + U_3^p k(x)[U_2^p, U_3^p]. \quad (2)$$

Let  $e : X' \rightarrow X$  be the blowing up along  $x$ . If  $x' \in e^{-1}(x)$  is very near  $x$ , we have  $x' = (X' := \frac{X}{u_3}, u'_1 := \frac{u_1}{u_3}, u'_2 := \frac{u_2}{u_3}, u'_3 := u_3)$  by **II.1.1**, so  $E' := e^{-1}(x) = \text{div}(u'_1 u'_3)$ . This is the origin of a chart, so  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal. Let  $\Phi(U_1, U_2) := \text{cl}_{\omega(x)-1}\phi$ . We get

$$H(x')^{-1}f' = u'_3 F(u'_2, 1) + u'_1 \phi',$$

where  $\phi' = \Phi(u'_1, u'_2) + u'_3\psi'$  and  $\psi' \in k(x)[[u'_1, u'_2, u'_3]]$ . We have  $\text{ord}_{(u'_1, u'_3)}(H(x')^{-1}g'^p) > \epsilon(x)$ . If  $x'$  is very near  $x$ , we get  $\text{ord}_{\eta'(x')}(u'_1\phi') = \omega(x') = \omega(x)$  and

$$\omega(x) = \omega(x') = \epsilon(x') \leq 1 + \text{ord}_{u'_2}F(u'_2, 1) \leq 1 + \omega(x),$$

where the right-hand side inequality holds because  $F \notin k(x)[U_2]$ . So  $x'$  is not very near  $x$  unless possibly if  $\text{ord}_{u'_2}F(u'_2, 1) \geq \omega(x) - 1$ . We consider two cases:

*Case 1:*  $\text{ord}_{u'_2}F(u'_2, 1) = \omega(x)$ . Then  $H(x')^{-1}f' = \gamma'u'_3u'_2{}^{\omega(x)} + u'_1\phi'$  with  $\gamma'$  invertible. Since  $\omega(x) \geq 1 + p \geq 3$ , we have  $\kappa(x') \leq 1$  by **II.1**(ii).

*Case 2:*  $\text{ord}_{u'_2}F(u'_2, 1) = \omega(x) - 1$ , Then  $H(x')^{-1}f' = \gamma'u'_3u'_2{}^{\omega(x)-1} + u'_1\phi'$  with  $\gamma'$  invertible. If  $x'$  is very near  $x$ , then  $\omega(x) - 1 \equiv 0 \pmod{p}$ , otherwise

$$\omega(x') \leq \text{ord}_{\eta'(x')}(H(x')^{-1}\frac{\partial f'}{\partial u'_2}) < \omega(x).$$

We have  $H(x') = u'_1{}^{a(1)}u'_3{}^{a(1)+\omega(x)-p}$  and

$$F' := \text{cl}_{\omega(x)}(H(x')^{-1}f') = U'_1\Phi(U'_1, U'_2) + U'_1U'_3K(U'_1, U'_2, U'_3) + \overline{\gamma'}U'_3U'_2{}^{\omega(x)-1}.$$

By the lemma below (with indices 2 and 3 exchanged and  $\Psi = 0$ ), we get  $\tau(x') = 3$  so  $\kappa(x') = 0$ .

**II.3.3 Lemma.** *Assume that  $\omega(x) \geq 2$ ,  $E = \text{div}(u_1u_2)$  and the polyhedron  $\Delta(h; u_1, u_2, u_3; X)$  is minimal. We have  $\tau(x) = 3$  provided*

- (i)  $H(x) = u_1{}^{a(1)}u_2{}^{a(2)}$ , with  $a(2) \equiv a(1) + 1 \pmod{p}$ , and
- (ii)  $F := \text{cl}_{\omega(x)}(H(x)^{-1}f)$  has an expansion

$$F = U_1\Phi(U_1, U_2, U_3) + U_2^2\Psi(U_2, U_3) + \lambda U_2U_3{}^{\omega(x)-1}, \quad (1)$$

with  $\Phi \neq 0$  and  $\lambda \neq 0$ .

Proof. Since  $F \neq 0$ , we have  $\epsilon(x) = \omega(x)$ , so  $\Phi \in k(x)[U_1, U_2, U_3^p]$ ,  $\Psi \in k(x)[U_2, U_3^p]$  and  $\omega(x) - 1 \equiv 0 \pmod{p}$ . With notations about derivations as in chapter 1 **II.3**, the vector space  $\text{cl}_{\epsilon(x)}J(f, E, x)$  is generated by forms

$$F_i = U_1\Phi_i(U_1, U_2, U_3) + U_2^2\Psi_i(U_2, U_3) + \lambda_i U_2U_3{}^{\omega(x)-1}, \quad 1 \leq i \leq s.$$

Since  $\Delta(h; u_1, u_2, u_3; X)$  is minimal, there exists  $i_0, i_1$  such that  $\lambda_{i_0} \neq 0$ ,  $\Phi_{i_1} \neq 0$  by (ii). We have  $0 < \deg_{U_3}F_{i_0} = \omega(x) - 1 < \deg F_{i_0} = \omega(x)$ , so  $\tau(x) \geq 2$ .

If  $U_1 \in \text{VDir}(x)$ , then  $U_2^2\Psi_{i_0}(U_2, U_3) + \lambda_{i_0}U_2U_3{}^{\omega(x)-1} \in \text{IDir}(x)^{\omega(x)}$ , so  $\tau(x) = 3$ . If  $U_2 \in \text{VDir}(x)$ , then  $U_1\Phi_{i_1}(U_1, U_2, U_3) \in \text{IDir}(x)^{\omega(x)}$ , so  $U_1 \in \text{VDir}(x)$ . Hence  $\tau(x) = 3$  as well in this case.

Suppose that  $\tau(x) = 2$ . The previous discussion shows that

$$\text{VDir}(x) = \langle U := \mu_1U_1 + \mu_2U_2, V := U_3 + \nu_1U_1 + \nu_2U_2 \rangle,$$

with  $\mu_1, \mu_2, \nu_1, \nu_2 \in k(x)$  and  $\mu_1\mu_2 \neq 0$ . It can be assumed that  $\nu_1 = 0$  in  $V$  by replacing  $V$  with  $V - \frac{\nu_1}{\mu_1}U$ . Let  $F' := F(U_1, U_2, V - \nu_2U_2)$ . Then  $F'$  has an expansion as in (1) with  $\lambda$  and the property  $\Phi \neq 0$  unchanged. The vector space  $\text{cl}_{\epsilon(x)}J(f, E, x)$  is also unchanged. In other terms, it can be assumed that  $\text{VDir}(x) = \langle \mu_1U_1 + \mu_2U_2, U_3 \rangle$ ,  $\mu_1\mu_2 \neq 0$ .

There is an expansion

$$F = U_3^{\omega(x)-1}(\alpha U_1 + \lambda U_2) + \sum_{1 \leq a \leq \frac{\omega(x)-1}{p}} U_3^{\omega(x)-1-ap} F_a(U_1, U_2),$$

with  $F_a \in k(x)[U_1, U_2]_{ap+1}$ ,  $\alpha \in k(x)$ . Then

$$F_1 = U_3^{\omega(x)-1}((a(1) + 1)\alpha U_1 + a(1)\lambda U_2) + \sum_{1 \leq a \leq \frac{\omega(x)-1}{p}} U_3^{\omega(x)-1-ap} F_{a,1}(U_1, U_2),$$

and

$$F_2 = U_3^{\omega(x)-1}(a(2)\alpha U_1 + (a(2) + 1)\lambda U_2) + \sum_{1 \leq a \leq \frac{\omega(x)-1}{p}} U_3^{\omega(x)-1-ap} F_{a,2}(U_1, U_2).$$

Since  $\text{VDir}(x) = \langle \mu_1 U_1 + \mu_2 U_2, U_3 \rangle$ , the linear forms  $L_0 := \mu_1 U_1 + \mu_2 U_2$ ,  $L_1 := (a(1) + 1)\alpha U_1 + a(1)\lambda U_2$  and  $L_2 := a(2)\alpha U_1 + (a(2) + 1)\lambda U_2$  must be proportional. The linear forms  $L_1$  and  $L_2$  are linearly dependent if and only if  $(a(1) + a(2) + 1)\alpha \equiv 0 \pmod{p}$ , so  $2(a(1) + 1)\alpha \equiv 0 \pmod{p}$  by (i). Hence  $p = 2$ , or  $a(1) + 1 \equiv 0 \pmod{p}$  or  $\alpha = 0$ .

If  $a(1) + 1 \equiv 0 \pmod{p}$ , then  $L_2 = \lambda U_2$  is not proportional to  $L_0$  since  $\lambda \mu_1 \mu_2 \neq 0$ .

If  $p = 2$  and  $a(1) + 1 \not\equiv 0 \pmod{p}$ , then  $L_1 = \alpha U_1$ , so we must have  $\alpha = 0$ , since  $\mu_1 \mu_2 \neq 0$ .

If  $\alpha = 0$  and  $a(1) + 1 \not\equiv 0 \pmod{p}$ , then  $L_1 = a(1)\lambda U_2$ ,  $L_2 = (a(1) + 2)\lambda U_2$  so we must have  $p = 2$ . Then, since the monomial  $\lambda u_1^{a(1)} u_2^{a(2)+1} u_3^{\omega(x)-1}$  induces a vertex of the minimal polyhedron  $\Delta(h; u_1, u_2, u_3; X)$ , we have  $\lambda \notin k(x)^2$ . Hence

$$F_\lambda := \frac{\partial F}{\partial \lambda} = U_2 U_3^{\omega(x)-1} + \sum_{1 \leq a \leq \frac{\omega(x)-1}{p}} U_3^{\omega(x)-1-ap} F_{a,\lambda}(U_1, U_2) \notin k(x)[U_3, \mu_1 U_1 + \mu_2 U_2],$$

thus contradicting at last the assumption  $\tau(x) = 2$ . This concludes the proof.

**II.3.4** (end of the proof of **II.3**) If  $\tau(x) = 1$ , as  $\text{ord}_{\eta(x)}(u_1 \phi) = \omega(x)$ , we have  $\text{cl}_{\omega(x)} J(f, E, x) = k(x).U_1^{\omega(x)}$  and  $\text{VDir}(x) = k(x).U_1$ . We thus have

$$H(x)^{-1} f = F(u_2, u_3) + \lambda u_1^{\omega(x)} + u_1 \psi,$$

with  $\lambda \in k(x)$  nonzero and  $\text{ord}_{\eta(x)} \psi \geq \omega(x)$ . Assumption (1) in the proposition is then equivalent to

$$\text{VDir}\left(\frac{\partial F}{\partial U_2}, \frac{\partial F}{\partial U_3}\right) = \langle U_2, U_3 \rangle. \quad (1)$$

Let  $e : X' \rightarrow X$  be the blowing up along  $x$ . By **II.1.1**, a point  $x' \in e^{-1}(x)$  is very near  $x$  only if it maps to the strict transform of  $\text{div}(u_1)$ . As  $u_2$  and  $u_3$  play symmetric roles, we can assume  $x'$  is in the chart with origin  $(X' := \frac{X}{u_2}, u'_1 := \frac{u_1}{u_2}, u'_2 := u_2, u'_3 := \frac{u_3}{u_2})$ . Then  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1 u'_2)$ ,  $H(x') = u'_1{}^{a(1)} u'_2{}^{a(1)+\omega(x)-p}$  and

$$H(x')^{-1} f' = u'_2 F(1, u'_3) + \lambda u'_1{}^{\omega(x)} + u'_1 u'_2 \psi'$$

for some  $\psi' \in \widehat{S}'$ . We pick local coordinates  $(u'_1, u'_2, v' := P(1, u'_3))$  at  $x'$ , with  $P \in k(x)[U_2, U_3]$  irreducible, homogeneous and unitary in  $U_3$ .

If  $P \neq U_3$ , we may have to perform a translation  $Z' := X' - \theta'$ ,  $\theta' \in k(x')[[u'_1, u'_2, v']]$  to get  $\Delta(u_2^{-p}h; u'_1, u'_2, v'; Z')$  minimal. Then  $f'$  is changed into  $f'_{Z'} := f' + \theta'^p - \theta'g'^{p-1}$  and get an expansion

$$H(x')^{-1}f'_{Z'} = u'_2F_{Z'}(v') + \lambda_{Z'}u'_1{}^{\omega(x)} + u'_1u'_2\psi'_{Z'}, \quad (2)$$

with  $\lambda_{Z'}, \psi'_{Z'} \in k(x')[[u'_1, u'_2, v']]$ ,  $F_{Z'}(v) \in k(x')[[v']]$ . As  $u'_1$  divides  $H(x')^{-1}g'^p$ , the following holds:

(a)  $H(x')u'_2(F_{Z'}(v') - F(1, u'_3)) \in (k(x')[[u'_1, u'_2, v']])^p$ . In particular,

$$J(f_{Z'}, E') \equiv J(H(x')u'_2F(1, u'_3), E') \pmod{u'_1}.$$

(b)  $\text{ord}_{v'}F_{Z'}(v') > \text{ord}_{v'}F(1, u'_3)$  only if  $(a(1), \omega(x) + 1) \equiv (0, 0) \pmod{p}$ .

(c)  $\lambda_{Z'}$  is not invertible if and only if  $(a(1) + \omega(x)) \equiv 0 \pmod{p}$  and  $\bar{\lambda} \in k(x')^p$ . In this case, since  $\bar{\lambda} \notin k(x)^p$ ,  $P$  is inseparable, i.e.  $\frac{\partial P}{\partial U_3} = 0$ .

Note that the conditions on  $a(1)$  and  $\omega(x)$  in (b) and (c) are mutually exclusive. By (1), we have

$$\text{ord}_{v'}J(H(x')u'_2F(1, u'_3), E') \leq 1 + (\omega(x) - 1) = \omega(x). \quad (3)$$

Assume that  $x'$  is very near  $x$ . By (a), equality holds in (3) and this implies  $\omega(x) - 1 \leq \text{ord}_{v'}F_{Z'}(v') \leq \omega(x)$ . Moreover, we must have  $\text{ord}_{\eta'(x')}(u'_1u'_2\psi'_{Z'}) \geq \omega(x)$  in (2).

Let us first look at the case where  $(a(1), \omega(x) + 1) \not\equiv (0, 0) \pmod{p}$ . By (b), we have  $\text{ord}_{v'}F_{Z'}(v') = \text{ord}_{v'}F(1, u'_3)$ .

*Case 1:*  $\text{ord}_{v'}F_{Z'}(v') = \omega(x)$ . We have  $F = P^{\omega(x)}Q$ , with  $Q \in k(x)[U_2, U_3]$  homogeneous. Counting degrees, we get

$$\omega(x) + 1 = [k(x') : k(x)]\omega(x) + \deg Q.$$

Since  $\omega(x) \geq 2$  by assumption in the proposition, we must have  $k(x') = k(x)$ . The last statement in (c) then implies that  $\lambda_{Z'} = \lambda$  is a unit. Then  $\kappa(x') \leq 1$  by **II.1**. Note that, if  $\omega(x) = 2$ , the extra assumption in **II.1** (iii) holds since

$$U_1'^{-1}\text{cl}_2(H(x')^{-1}f'_{Z'}) = \bar{\lambda}U_1' + U_2'\psi'_{Z'}(0, 0, 0) \notin k(x').U_2'.$$

*Case 2:*  $\text{ord}_{v'}F_{Z'}(v') = \omega(x) - 1$ . Since equality holds in (3), we have  $\omega(x) - 1 \equiv 0 \pmod{p}$ . By (b),  $F = P^{\omega(x)-1}Q$ , with  $Q \in k(x)[U_2, U_3]$  homogeneous. Counting degrees, we get

$$\omega(x) + 1 = [k(x') : k(x)](\omega(x) - 1) + \deg Q. \quad (4)$$

Suppose that  $x'$  is not algebraic over  $x$ , i.e.  $[k(x') : k(x)] \geq 2$ . Since  $\omega(x) \geq 2$  by assumption in the proposition, counting degrees in (4) we get:

$\deg Q = 0$ ,  $[k(x') : k(x)] = 2$  and  $\omega(x) = 3$ , or

$\deg Q = 0$ ,  $[k(x') : k(x)] = 3$  and  $\omega(x) = 2$ , or

$\deg Q = 1$ ,  $[k(x') : k(x)] = 2$  and  $\omega(x) = 2$ .

By (c),  $\lambda_{Z'}$  is a unit unless  $k(x')/k(x)$  is inseparable. Since  $\omega(x) - 1 \equiv 0 \pmod{p}$ ,  $\lambda_{Z'}$  is a unit unless possibly if  $(p = 2, \deg Q = 0, [k(x') : k(x)] = 2 \text{ and } \omega(x) = 3)$ . In this case, we have  $F = \mu P^2$ ,  $\mu \in k(x)$  which contradicts (1), since  $p = 2$ .

It has thus been proved that  $\lambda_{Z'}$  is a unit in all cases (by (c) if  $x'$  is rational over  $x$  and by the above argument otherwise). Thus the initial form  $F' := \text{cl}_{\omega(x)}(H(x')^{-1}f'_{Z'})$  has an expansion

$$F' = U_1'\Phi(U_1', U_2', V') + \lambda'U_2'V'^{\omega(x)-1}$$

with  $\Phi \neq 0$  (since  $\overline{\lambda_{Z'}} \neq 0$ ) and  $\lambda' \neq 0$ . By lemma **II.3.3**, we have  $\tau(x') = 3$ , so  $\kappa(x') \leq 1$  by **II.1.2**.

Finally, if  $(a(1), \omega(x) + 1) \equiv (0, 0) \pmod{p}$ , then  $\lambda_{Z'} = \lambda$  is a unit by (c) and we conclude that  $\kappa(x') \leq 1$  using **II.1** (resp. **II.3.3**) as in case 1 (resp. case 2) above.

**II.4 Proposition.** *Assume that  $E = \text{div}(u_1)$  and  $H(x) - 1f = F(u_2, u_3) + u_1\phi + \psi$ , where  $\text{ord}_{\eta(x)}(u_1\phi) = \omega(x)$ ,  $F \in k(x)[u_2, u_3^p]$  is homogeneous of degree  $1 + \omega(x)$  (so  $F \neq 0$ ) and  $\psi \in (u_2, u_3)^{\omega(x)+2}$ . If moreover*

$$\text{VDir}(\text{cl}_{\omega(x)}J(F, E)) = \langle U_2, U_3 \rangle, \quad (1)$$

then  $\kappa(x) = 0$ .

*Proof.* We have  $\text{ord}_{u_1}(H(x)^{-1}g^p) > \text{ord}_{\eta(x)}(H(x)^{-1}f)$  and  $\text{cl}_{\omega(x)}(u_1\phi) \in k(x)[U_1, U_2^p, U_3^p]$ , since  $\epsilon(x) = \omega(x)$ . As  $\text{ord}_{\eta(x)}(u_1\phi) = \omega(x)$ ,  $U_1$  divides  $\text{cl}_{\omega(x)}J(f, E, x)$ , so  $U_1 \in \text{VDir}(x)$ . As  $F \in k(x)[u_2, u_3^p]$ ,  $\text{cl}_{\omega(x)}J(F, E) = k(x) \cdot \frac{\partial F}{\partial U_2} \neq 0$  by (1).

By the Euler identity,  $(1 + \omega(x))F = U_2 \frac{\partial F}{\partial U_2} \neq 0$ , whence  $1 + \omega(x) \not\equiv 0 \pmod{p}$ . We get  $F = U_2 F'(U_2, U_3)$ , and claim that  $\text{VDir}(F') = \langle U_2, U_3 \rangle$ . If not, then  $F' \in k(x) \cdot (\lambda_2 U_2 + \lambda_3 U_3)^{\omega(x)}$  for some  $\lambda_2, \lambda_3 \in k(x)$ . Since  $F' \in k(x)[U_2, U_3^p]$ , we would have  $\lambda_3 = 0$  or  $\omega(x) \equiv 0 \pmod{p}$ , which both contradict (1).

Let  $e : X' \rightarrow X$  be the blowing up along  $x$ . By **II.1.1**, a point  $x' \in X'$  is very near  $x$  only if it maps to the strict transform of  $\text{div}(u_1)$ .

We first look at the chart with origin  $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, u'_3 = \frac{u_3}{u_2})$ . Then  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1 u'_2)$ ,  $H(x') = u'_1{}^{a(1)} u'_2{}^{a(1)+\omega(x)-p}$  and there is an expression

$$H(x')^{-1}f' = u'_2 F(1, u'_3) + u'_1 \Phi(u'_1, 1, u'_3) + u'_1 u'_2 \phi' + u'_2{}^2 \psi', \quad (2)$$

with  $\phi', \psi' \in \widehat{S}'$ ,  $\Phi := \text{cl}_{\omega(x)-1}\phi$ . We pick local coordinates  $(u'_1, u'_2, v' := P(1, u'_3))$  at  $x'$ , with  $P \in k(x)[U_2, U_3]$  homogeneous, irreducible and unitary in  $U_3$ .

If  $P \neq U_3$ , we may have to perform a translation  $Z' := X' - \theta'$ ,  $\theta' \in k(x')[[u'_1, u'_2, v']]$  to get  $\Delta(h'; u'_1, u'_2, v'; Z')$  minimal. Then  $f'$  is changed into  $f_{Z'} := f' + \theta'^p - \theta' g'^{p-1}$  and there is an expansion

$$H(x')^{-1}f_{Z'} = u'_2 F_{Z'}(v') + u'_1 \Phi_{Z'}(u'_1, v') + u'_1 u'_2 \phi'_{Z'} + u'_2{}^2 \psi'_{Z'},$$

with  $F_{Z'}(v) \in k(x')[[v]]$ ,  $\Phi_{Z'}(u'_1, v') \in k(x')[[v]][[u'_1]]$ .

As  $\omega'(x) = 2$ ,  $u'_1 u'_2$  divides  $H(x')^{-1}g'^p$ . Moreover, we have

$$(a(1), a(1) + 1 + \omega(x)) \not\equiv (0, 0) \pmod{p} \quad (3)$$

since  $1 + \omega(x) \not\equiv 0 \pmod{p}$ , so the monomial  $H(x')u'_2$  is not a  $p^{\text{th}}$ -power. Hence

$$F_{Z'}(v') = F(1, u'_3). \quad (4)$$

By (2) and (4), we have

$$H(x')^{-1}u'_1 \frac{\partial f_{Z'}}{\partial u'_1} \equiv H(x')^{-1}u'_1 \frac{\partial H(x')u'_2 F(1, u'_3)}{\partial u'_1} \equiv a(1)u'_2 F(1, u'_3) \pmod{(u'_1, u'_2{}^2)}.$$

Note that

$$H(x')^{-1}u'_1 \frac{\partial H(x')u'_2 F(1, u'_3)}{\partial u'_1} = u_2^{-\omega(x)} H(x)^{-1}u_1 \frac{\partial H(x)F(u_2, u_3)}{\partial u_1} = a(1)u_2^{-\omega(x)} F(u_2, u_3).$$

Similarly,

$$H(x')^{-1}u'_2 \frac{\partial f_{Z'}}{\partial u'_2} \equiv H(x')^{-1}u'_2 \frac{\partial H(x')u'_2 F(1, u'_3)}{\partial u'_2} \equiv (a(1) + 1 + \omega(x))u'_2 F(1, u'_3) \pmod{(u'_1, u'_2{}^2)},$$

and

$$H(x')^{-1}u'_2 \frac{\partial H(x')u'_2 F(1, u'_3)}{\partial u'_2} = (a(1) + 1 + \omega(x))u_2^{-\omega(x)} F(u_2, u_3),$$

since  $H(x')^{-1}u'_2 \frac{\partial H(x')u'_2 F(1, u'_3)}{\partial u'_2} = u_2^{-\omega(x)} H(x)^{-1} \sum_{1 \leq i \leq 3} u_i \frac{\partial H(x)F(u_2, u_3)}{\partial u_i}$  and  $F \in k(x)[u_2, u_3^p]$ . By (3) and the above computations,

$$u_2^{-\omega(x)} F(u_2, u_3) = u'_2 F(1, u'_3) \in J(f', E', x') + (u'_1, u'_2{}^2). \quad (5)$$

As  $F = u_2 F'(u_2, u_3)$  and  $\text{VDir}(F') = \langle U_2, U_3 \rangle$ , we have  $\text{ord}_v F(1, u'_3) \leq \omega(x) - 1$ . If  $x'$  is very near  $x$ , we get from (5) that

$$\text{ord}_v F(1, u'_3) = \omega(x) - 1, \quad (6)$$

thus  $\epsilon(x') = \omega(x')$  and therefore  $\omega(x) - 1 \equiv 0 \pmod{p}$ . Necessarily  $\omega(x) \geq p + 1$ , since  $U_3^p$  explicitly appears in the form  $F' \in k(x)[U_2, U_3]_{\omega(x)}$ .

As  $\omega(x) + 1 \not\equiv 0 \pmod{p}$ , we have  $p \geq 3$ . We resume the argument at the beginning of case 2, end of the proof of proposition **II.3**: equation (6) implies that  $x'$  is rational over  $x$  unless possibly if  $\omega(x) \leq 3$ . But here  $\omega(x) \geq p + 1 \geq 4$ , since  $p \geq 3$ .

The argument at the beginning of **II.3.2** then shows that, after possibly changing  $u_3$  with  $u_3 + \lambda u_2$ ,  $\lambda \in S$  invertible, we may assume that  $x' = (\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, \frac{u_3}{u_2})$  is the origin of the chart (i.e.  $P = U_3$  with notations as above). Hence  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is already minimal and we turn back to equation (2): since  $x'$  is very near  $x$ , we have

$$\text{ord}_{\eta'(x')}(H(x')^{-1}f') = \text{ord}_{(u'_1, u'_3)}(u'_1 \Phi(u'_1, 1, u'_3)) = \omega(x)$$

and **II.3.3** gives  $\tau(x') = 3$ , so  $\kappa(x') = 0$ .

We finally turn to the point at infinity  $x' = (X' = \frac{X}{u_3}, u'_1 = \frac{X}{u_3}, u'_2 = \frac{u_2}{u_3}, u'_3 = u_3)$ . We get

$$H(x')^{-1}f' = u'_3 F(u'_2, 1) + u'_1 \Phi(u'_1, u'_2, 1) + u'_1 u'_3 \phi' + u'_3{}^2 \psi',$$

with  $\phi', \psi' \in \widehat{S}'$ ,  $\Phi := \text{cl}_{\omega(x)-1} \phi$ . This is the origin of a chart, so  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal. As  $F \in k(x)[u_2, u_3^p]$  and  $F \notin k(x)[u_2]$ , we have  $\text{ord}_{(u'_2, u'_3)}(u'_3 F(u'_2, 1)) \leq 1 + \omega(x) - p < \omega(x)$ , so  $x'$  is not very near  $x$ .

**II.5 Proposition.** *Let  $(a(1), a(2), a(3)) \in \mathbb{N}^3$  and  $H := u_1^{a(1)} u_2^{a(2)} u_3^{a(3)} \in k(x)[u_1, u_2, u_3]$ . Let  $(a, \omega) \in \mathbb{N}^2$  satisfy  $0 < a < \omega$  and  $F \in k(x)[u_2, u_3]$  be homogeneous of degree  $a$  (so  $F \neq 0$ ). We assume that*

$$\psi := H u_1^{\omega-a} F(u_2, u_3) \notin (k(x)[u_1, u_2, u_3])^p. \quad (1)$$

*With conventions on derivations as in chapter 1 **II.3**, we have  $\tau(W) \geq 2$ , where*

$$W := \langle \{H^{-1} \lambda_i \frac{\partial \psi}{\partial \lambda_i}\}_{1 \leq i \leq s} \rangle \subseteq k(x)[u_1, u_2, u_3]_{\omega}.$$

*Assume moreover that*

$$H u_1^{\omega-a} (F(u_2, u_3) - G_i(u_i)) \notin (k(x)[u_1, u_2, u_3])^p \quad (2)$$

for all  $G_i \in k(x)[u_i]$  and each  $i = 2, 3$ . Then  $\tau(W) = 3$  except possibly if one of the following conditions is satisfied:

(i)  $a \not\equiv 0 \pmod{p}$ ,  $a(1) + \omega - a \equiv 0 \pmod{p}$ ,  $a(2)a(3) \not\equiv 0 \pmod{p}$  and  $\widehat{a(2)} + \widehat{a(3)} + \widehat{a} = p$  where  $\widehat{\alpha} \in \{0, \dots, p-1\}$  denotes the remainder of the division of the integer  $\alpha$  by  $p$ . In particular  $p \geq 3$ ;

(ii)  $a \equiv 0 \pmod{p}$  and

$$\psi - cHu_1^{\omega-a}(u_2 + \mu u_3)^a \in (k(x)[u_1, u_2, u_3])^p$$

for some  $c, \mu \in k(x) - \{0\}$ . In particular,  $\text{VDir}(W) = \langle u_1, u_2 + \mu u_3 \rangle$ .

Proof. Let

$$H^{-1}\lambda_i \frac{\partial \psi}{\partial \lambda_i} =: u_1^{\omega-a} F_i(u_2, u_3) \in W,$$

where  $F_i \in k(x)[u_2, u_3]_a$ . By (1) and the assumption  $a > 0$ ,  $k(x).u_1 \subset \text{VDir}(W)$  (inclusion is necessarily strict), so  $\tau(x) \geq 2$  and the first part of the proposition is proved.

Assume now that  $\tau(W) = 2$ . Then all  $F_i$ 's are proportional to some  $(u_2 + \mu u_3)^a$ ,  $\mu \neq 0$  by (2). Let us denote

$$F_i =: d_i(u_2 + \mu u_3)^a, \quad (3)$$

where  $d_i \neq 0$  for some  $i$ ,  $1 \leq i \leq s$ .

**II.5.1** Assume that  $a \equiv 0 \pmod{p}$ . Let  $c \in k(x)$  be such that  $\deg_{u_2}(F - c(u_2 + \mu u_3)^a) < a$ . Since  $a \equiv 0 \pmod{p}$ , we have

$$H^{-1}\lambda_i \frac{\partial Hc(u_2 + \mu u_3)^a u_1^{\omega-a}}{\partial \lambda_i} =: c_i u_1^{\omega-a} (u_2 + \mu u_3)^a$$

for some  $c_i \in k(x)$ . As  $\deg_{u_2}(H(x)^{-1}\lambda_i \frac{\partial H(F - c(u_2 + \mu u_3)^a)}{\partial \lambda_i}) < a$ , we have  $c_i = d_i$  and therefore

$$Hu_1^{\omega-a}(F(u_2, u_3) - c(u_2 + \mu u_3)^a) \in (k(x)[u_1, u_2, u_3])^p,$$

which proves (ii).

**II.5.2** Assume now that  $a \not\equiv 0 \pmod{p}$ . Let us denote  $\alpha(j) \in k(x)$  the coefficient of  $u_j^a$  in  $F$ ,  $j = 2, 3$ . By (3), we have  $\alpha(2)\alpha(3) \neq 0$ . Computing the coefficient of  $u_j^a$  in  $F_1, F_2, F_3$  for  $j = 2, 3$ , the following couples must be proportional:

$$v_1 := ((a(1) + \omega - a)\alpha(2), (a(1) + \omega - a)\alpha(3)),$$

$$v_2 := ((a(2) + a)\alpha(2), a(2)\alpha(3)), \text{ and}$$

$$v_3 := (a(3)\alpha(2), (a(3) + a)\alpha(3)).$$

Since  $\det(v_1, v_2) = 0$  and  $a \not\equiv 0 \pmod{p}$ , we have  $a(1) + \omega - a \equiv 0 \pmod{p}$  and we are thus reduced to a question on  $u_2^{a(2)} u_3^{a(3)} F(u_2, u_3)$ . Since  $\det(v_2, v_3) = 0$ , we have

$$a(2) + a(3) + a \equiv 0 \pmod{p}. \quad (4)$$

Suppose  $a(2) + a \equiv 0 \pmod{p}$ . Then  $a(3) \equiv 0 \pmod{p}$  by (4). The Euler identity applied to  $u_2^{a(2)} F(u_2, u_3)$  gives

$$F_2 + u_3 \frac{\partial F}{\partial u_3} = 0.$$

On the other hand, we have  $\deg_{u_2} F_2 < a$ , so  $F_2 = 0$ , and we get  $\frac{\partial F}{\partial u_3} = 0$ , so  $a = \deg_{U_3} F \equiv 0 \pmod{p}$ , since  $\alpha(3) \neq 0$ : a contradiction. Hence  $a(2) + a \not\equiv 0 \pmod{p}$ . *Mutatis mutandis*,  $a(3) + a \not\equiv 0 \pmod{p}$ . All this, together with (4), leads to  $a(2)a(3) \not\equiv 0 \pmod{p}$ .

Let  $a =: a_0 + a_1p + \dots + a_m p^m$  be the  $p$ -adic expansion of  $a$ ,  $a_i \in \{0, \dots, p-1\}$ ,  $a_0 \neq 0$ . For each  $j \in \mathbb{N}$ ,  $0 \leq j \leq a$ , let  $j =: j_0 + j_1p + \dots + j_m p^m$  be the  $p$ -adic expansion of  $j$ . As

$$(X + Y)^a = (X + Y)^{a_0} (X^p + Y^p)^{a_1} \dots (X^{p^m} + Y^{p^m})^{a_m}$$

in  $\mathbb{Z}/p$ , we have  $\binom{a}{j} \equiv \binom{a_0}{j_0} \binom{a_1}{j_1} \dots \binom{a_m}{j_m} \pmod{p}$ . Then, for each  $j$ ,  $0 \leq j \leq \widehat{a} < p$ , we have  $\binom{a}{j} \equiv \binom{\widehat{a}}{j} \not\equiv 0 \pmod{p}$ .

Let  $F =: \sum_{0 \leq j \leq a} \gamma_j u_2^j u_3^{a-j}$ . Since  $a(2) + a \not\equiv 0 \pmod{p}$ , we have  $d_2 \neq 0$  in  $F_2 = d_2(u_2 + \mu u_3)^a$ . Computing explicitly  $F_2$ , we get

$$\gamma_j(a(2) + j) = d_2 \binom{a}{j} \neq 0$$

whenever  $0 \leq j \leq \widehat{a} < p$ , hence  $a(2) + j \not\equiv 0 \pmod{p}$ . We deduce  $\widehat{a(2)} + \widehat{a} < p$ , since  $a(2) + a \not\equiv 0 \pmod{p}$ . By (4),  $\widehat{a(2)} + \widehat{a(3)} + \widehat{a} = p$ .

### CHAPTER 3: Resolution when there is transverseness

In all this chapter, we assume that  $x \in \Sigma_p$ ,  $\Omega(x) = (\omega(x), 2)$  and that the r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  is such that  $E \subseteq \text{div}(u_1 u_2 u_3)$  and the polyhedron  $\Delta(h; u_1, u_2, u_3; X)$  is minimal.

We naturally define “transverseness” for  $J(f, E)$  by the following property:

$$J(f, E) \text{ is transverse} \Leftrightarrow \text{cl}_{\omega(x)} J(f, E) \not\subseteq (\{U_i \mid \text{div}(u_i) \subseteq E\}). \quad (1)$$

This is our definition of  $\kappa(x) = 4$  in **III.2** below. A slightly more general definition can be given adjoining to (1) those cases when  $\epsilon(x) = \omega(x)$  and

$$\text{cl}_{\omega(x)} J(f, E, x) \not\subseteq k(x)[\{U_i \mid \text{div}(u_i) \subseteq E\}]. \quad (2)$$

This is our definition of  $\kappa(x) = 2$  (i) in **I.1** below. Under each assumption (1) or (2), we have  $E \subseteq \text{div}(u_1 u_2)$  up to renumbering components of  $E$ .

Two main difficulties arise: on the one hand, transverseness as defined above is not preserved by permissible blowing ups at very near points, and we will have to use those nonpermissible blowing ups specified in chapter 1, **III** in resolving the  $\kappa(x) = 3$  case (section **II** below); on the other hand, some easy reductions (corollary **I.3** and section **III**) focus our attention on those cases where  $E \subseteq \text{div}(u_1 u_2)$ ,  $J(f, E) \equiv (u_3^{\omega(x)}) \text{ mod}(u_1, u_2)$  and  $\tau(x) = 1$ . These in turn contain two very different subcases:

$\kappa(x) = 2$ , which contains again two subcases:

*Case (\*1) or (\*2):*  $(H(x)^{-1} f) \equiv (u_3^{\omega(x)}) \text{ mod}(u_1, u_2)$ ,  $E = \text{div}(u_1)$  or  $E = \text{div}(u_1 u_2)$ , and

*Case (\*3):*  $(H(x)^{-1} \frac{\partial f}{\partial u_2}) \equiv (u_3^{\omega(x)}) \text{ mod}(u_1)$ ,  $E = \text{div}(u_1)$ ,  $\omega(x) \equiv 0 \text{ mod } p$ ;

$\kappa(x) = 3$ :  $(H(x)^{-1} f) \equiv (u_3^{1+\omega(x)}) \text{ mod}(u_1, u_2)$ ,  $1 + \omega(x) \not\equiv 0 \text{ mod } p$ .

Unfortunately, these assumptions are not stable by blowing up and we still have to introduce more general definitions of  $\kappa(x) = 2, 3$ . We recall that,  $\kappa$  being a multiform function, one may have at the same time  $\kappa(x) \in \{2, 3, 4\}$  and  $\kappa(x) \leq 1$ .

#### I. Resolution of the case $\kappa(x) = 2$ .

**I.1 Definition of  $\kappa(x) = 2$ .** *We say that  $\kappa(x) = 2$  if one of the following (mutually exclusive) conditions is satisfied:*

(i)  $\epsilon(x) = \omega(x)$ ,  $E \subseteq \text{div}(u_1 u_2)$ ,  $\text{VDir}(x) \not\subseteq \langle \{U_i \mid \text{div}(u_i) \subseteq E\} \rangle$ ;

(ii)  $\epsilon(x) = 1 + \omega(x)$ ,  $E = \text{div}(u_1)$  and, for a suitable r.s.p.  $(u_1, u_2, u_3)$  of  $\widehat{S}$  with  $\Delta(h; u_1, u_2, u_3; X)$  minimal,

$$\text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f}{\partial u_2}) = \Phi(U_2, U_3) + U_1 \Psi(U_1, U_2, U_3),$$

where  $\Phi \in k(x)[U_2, U_3^p] \setminus k(x)[U_2]$ , and  $U_3 \in \text{VDir}(\Phi)$ .

The reader verifies that this definition includes the two subcases of  $\kappa(x) = 2$  stated above in the introduction of this chapter, and that  $\kappa(x) \neq 2$  in the example  $f = u_1^q((u_2 + \lambda u_3)^{1+\omega(x)} + \{\text{higher order terms}\})$ ,  $\lambda \neq 0$ . We first recollect from definition **II.1.3** in chapter 2 and related comments:

##### **I.1.1 Remarks.**

(i) *If  $\kappa(x) = 2$  and  $\epsilon(x) = \omega(x)$  then:  $\text{ord}_{\eta(x)}(H(x)^{-1} g^p) > \omega(x)$ ;  $\text{VDir}(x) := \text{VDir}(\text{cl}_{\omega(x)} J(f, E, x))$  is independent of any choice of parameters with  $\Delta(h; u_1, u_2, u_3; X)$  minimal and  $E \subseteq \text{div}(u_1 u_2 u_3)$ .*

(ii) If  $\kappa(x) = 2$ ,  $\epsilon(x) = 1 + \omega(x)$  and  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) = \epsilon(x)$  (resp.  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \epsilon(x)$ ), then

$$\text{VDir}(x) := \text{VDir}(\text{cl}_{\omega(x)}J(f, E)) + k(x) \cdot U_1 \quad (\text{resp. } \text{VDir}(x) := \text{VDir}(\text{cl}_{\omega(x)}J(f, E)))$$

is independent of any choice of the parameters satisfying the conditions in **I.1(ii)** above.

**I.1.2 Definition.** Assume that  $\kappa(x) = 2$ . We say that condition (\*) holds if there exists a r.s.p.  $(u_1, u_2, u_3)$  of  $\widehat{S}$  (with  $\Delta(h; u_1, u_2, u_3; X)$  minimal) such that one of the following conditions is satisfied:

- (i)  $\epsilon(x) = \omega(x)$ ,  $(H(x)^{-1}f) \equiv (u_3^{\omega(x)}) \pmod{(u_1, u_2)}$ ;
- (ii)  $\epsilon(x) = 1 + \omega(x)$  and  $\deg_{U_3}\Phi = \omega(x)$  in **I.1(ii)**.

**I.2 Proposition.** Assume that  $\kappa(x) = 2$  and condition (\*) does not hold. We have  $\kappa(x) \leq 1$  or the following holds:

- (i)  $\omega(x) > p$ ,  $\tau(x) = 2$  and  $U_3 \in \text{VDir}(x)$  for a suitable choice of parameters  $(u_1, u_2, u_3)$  satisfying **I.1**. Moreover, we have  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \epsilon(x)$ ;
- (ii) there is no permissible curve  $\mathcal{C}$  containing  $x$ .

Let  $e : X' \rightarrow X$  be the blowing up along  $x$  and  $x' \in e^{-1}(x)$  be very near  $x$ .

- (iii) if  $(E = \text{div}(u_1)$  and  $U_1 \notin \text{VDir}(x)$ ), then  $\epsilon(x) = 1 + \omega(x)$  and  $\text{VDir}(x) = \langle U_2, U_3 \rangle$  for a suitable choice of the parameters satisfying **I.1(ii)**. We have  $x' = (X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = \frac{u_2}{u_1}, u'_3 = \frac{u_3}{u_1})$ ,  $\kappa(x') = 2$ , and either condition (\*) holds at  $x'$  or  $\text{VDir}(x') \equiv \langle U'_2, U'_3 \rangle \pmod{(U'_1)}$ ;
- (iv) if  $(E = \text{div}(u_1 u_2)$  and  $\text{VDir}(x) = \langle U_2 + \lambda U_1, U_3 \rangle$ ,  $\lambda \neq 0$ ), then  $x' = (X' = \frac{X}{u_1}, u'_1 = u_1, v'_2 = \frac{u_2}{u_1} + \lambda, u'_3 = \frac{u_3}{u_1})$ ,  $\kappa(x') = 2$ , and either condition (\*) holds at  $x'$  or  $x'$  is in case (iii) above;
- (v) if  $(\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2)$  and  $\text{VDir}(x) = \langle U_1, U_3 \rangle$ ), then  $x' = (X' = \frac{X}{u_1}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, u'_3 = \frac{u_3}{u_1})$ ,  $\kappa(x') = 2$ , and either condition (\*) holds at  $x'$  or  $\text{VDir}(x') \equiv \langle U'_1, U'_3 \rangle \pmod{(U'_2)}$ .

**I.2.1 Proof of (i).** If  $\epsilon(x) = 1 + \omega(x)$ , then  $\omega(x) \geq p$  by definition **I.1(ii)**, and  $\omega(x) = p$  implies  $\deg_{U_3}\Phi = p$ , i.e. condition (\*) holds. If  $\epsilon(x) = \omega(x) \leq p$ , we have

$$\Phi_0 := \text{cl}_{\omega(x)}(H(x)^{-1}f) \notin k(x)[\{U_i \mid \text{div}(u_i) \subseteq E\}]$$

by definition **I.1(i)**, say  $U_3$  occurs in the expansion of  $\Phi_0$ . Since  $\epsilon(x) = \omega(x)$ , we must have  $\frac{\partial \Phi_0}{\partial U_3} = 0$ , so  $\deg_{U_3}\Phi_0 \geq p$ , i.e. we have  $\omega(x) = p$  and condition (\*) holds.

In all cases, we have  $U_3 + \lambda_1 U_1 + \lambda_2 U_2 \in \text{VDir}(x)$  for some  $\lambda_1, \lambda_2 \in k(x)$  (after possibly renaming variables if  $\epsilon(x) = \omega(x)$  and  $E = \text{div}(u_1)$ ). If  $\epsilon(x) = \omega(x)$  (resp.  $\epsilon(x) = 1 + \omega(x)$ ), then  $\text{cl}_{\omega(x)}J(f, E, x)$  (resp.  $\text{cl}_{\omega(x)}J(f, E)$ ) is not generated by an  $\omega(x)^{\text{th}}$ -power since condition (\*) does not hold by assumption, so  $\tau(x) \geq 2$ . Therefore  $\tau(x) = 2$  if  $\kappa(x) > 0$ . Furthermore, if we are in case **I.1(ii)**, we must have  $\text{VDir}(\Phi) = \langle U_2, U_3 \rangle$  since  $U_3 \in \text{VDir}(\Phi)$  but  $\deg_{U_3}\Phi < \omega(x)$  by assumption.

If  $\epsilon(x) = \omega(x)$ , then after replacing  $u_3$  with  $v := u_3 + \lambda_1 u_1 + \lambda_2 u_2$ , then picking  $Z := X - \theta$  in order to have  $\Delta(h; u_1, u_2, v; Z)$  minimal, we get  $V \in \text{VDir}(x)$ . If  $\epsilon(x) = 1 + \omega(x)$  and  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) = \epsilon(x)$ , then  $\tau'(x) = 3$ , so  $\kappa(x) = 0$  by corollary **II.1.4** in chapter 2. Therefore  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \epsilon(x)$  and the end of the proof goes as in the case  $\epsilon(x) = \omega(x)$ .

**I.2.2 Proof of (ii).** If  $\mathcal{C}$  is permissible of the first kind (for example if  $\epsilon(x) = \omega(x)$ , cf. chapter 1 **II.5.1**), we have

$$\text{cl}_{\epsilon(x)}(H(x)^{-1}f) \in k(x)[\{U \in m_S \setminus m_S^2 \mid \mathcal{C} \subseteq \text{div}(u)\}].$$

By chapter 1 **II.5.4.3**, there is no point very near  $x$  on the blowing up of  $X$  along  $\mathcal{C}$  because  $\tau(x) \geq 2$ , so  $\kappa(x) = 1$ .

If  $\mathcal{C}$  is permissible of the second kind (then  $\epsilon(x) = 1 + \omega(x)$ , cf. chapter 1 **II.5.1**), we have

$$\text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f}{\partial u_2}) \in k(x)[\{U \in m_S \setminus m_S^2 \mid \mathcal{C} \subseteq \text{div}(u)\}],$$

where partial must be w.r.t.  $u_2$  because  $\Phi \in k(x)[U_2, U_3^p]$ . Hence

$$\text{cl}_{\epsilon(x)}(H(x)^{-1} f) \in k(x)[U_1, U_3]_{1+\omega(x)} + U_2 k(x)[U_1, U_3]_{\omega(x)}$$

by definition of permissibility of the second kind, so  $\Phi \in k(x)[U_3]$ . This proves that condition (\*) holds.

**I.2.3** Proof of (iii). It can be assumed that  $\text{VDir}(x) = \langle U_2, U_3 \rangle$  after possibly changing coordinates, the assumption being unchanged. By chapter 2, **II.2**, we have  $\kappa(x) = 0$  if  $\epsilon(x) = \omega(x)$ , so  $\epsilon(x) = 1 + \omega(x)$ .

By chapter 1 **II.5.4.3**,  $x' = (X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = \frac{u_2}{u_1}, u'_3 = \frac{u_3}{u_1})$  if  $x'$  is very near  $x$ . We have  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1)$ . As we are at the origin of a chart,  $\Delta(u_1^{-p}h; u'_1, u'_2, u'_3; X')$  is minimal. Let us denote  $f = u_1^{a(1)}(F(u_1, u_2, u_3) + \phi)$ , where  $F \in k(x)[u_1, u_2, u_3]$  is homogeneous of degree  $1 + \omega(x)$ ,  $\text{ord}_{\eta(x)}\phi \geq 2 + \omega(x)$ . Then  $f' = u_1^{a(1)+1+\omega(x)-p}(F(1, u'_2, u'_3) + u'_1\phi')$ , and we have

$$\min\{\text{ord}_{(u'_2, u'_3)}F(1, u'_2, u'_3), \text{ord}_{\eta'(x')} (u'_1\phi')\} \geq \omega(x),$$

since  $x'$  is very near  $x$ .

If  $\text{ord}_{(u'_2, u'_3)}F(1, u'_2, u'_3) = \omega(x)$ , then  $\epsilon(x') = \omega(x) = \omega(x')$  and  $\kappa(x') = 2$ . If condition (\*) holds for  $x'$  neither w.r.t.  $u'_2$  nor w.r.t.  $u'_3$ , then  $U'_2 U'_3$  divides the nonzero form  $\text{cl}_{\omega(x)}F(1, u'_2, u'_3)$ , so  $\text{VDir}(x') \equiv \langle U'_2, U'_3 \rangle \pmod{U'_1}$ .

If  $\text{ord}_{\eta'(x')}F(1, u'_2, u'_3) = 1 + \omega(x)$ , then  $F \in k(x)[u_2, u_3]$  and  $\text{VDir}(x) = \langle U_2, U_3 \rangle$  implies that  $\text{VDir}(\text{cl}_{\omega(x)}J(F(u'_2, u'_3), E')) = \langle U'_2, U'_3 \rangle$ . Also note that  $U'_1 \in \text{VDir}(x')$  if  $\text{ord}_{\eta'(x')} (u'_1\phi') = \omega(x)$ , so  $\kappa(x') \leq 1$  in this case by chapter 2, **II.3**. From now on,  $\text{ord}_{\eta'(x')} (u'_1\phi') \geq 1 + \omega(x)$ , so  $\epsilon(x') = 1 + \omega(x)$ . We then have

$$\text{cl}_{\epsilon(x')} (H(x')^{-1} \frac{\partial f'}{\partial u'_2}) = \Phi(U'_2, U'_3) + U'_1 \Psi'(U'_1, U'_2, U'_3)$$

for some  $\Psi' \in k(x)[U'_1, U'_2, U'_3]$ , so  $\kappa(x') = 2$  and  $\text{VDir}(x') \equiv \langle U'_2, U'_3 \rangle \pmod{U'_1}$  as required.

**I.2.4** Proof of (iv). By definition of  $\kappa(x) = 2$ , we have  $\epsilon(x) = \omega(x)$ . Since  $\text{VDir}(x) = \langle U_2 + \lambda U_1, U_3 \rangle$ ,  $\lambda \in k(x) - \{0\}$ ,  $x' = (X' = \frac{X}{u_1}, u'_1, v'_2 = \frac{u_2}{u_1} + \lambda, u'_3 = \frac{u_3}{u_1})$  and  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1)$ . Let us denote  $f = u_1^{a(1)} u_2^{a(2)}(F(u_1, u_2, u_3) + \phi)$  where  $F \in k(x)[u_1, u_2, u_3]_{\omega(x)}$  and  $\text{ord}_{\eta(x)}\phi \geq 1 + \omega(x)$ . We have  $F = \sum_{0 \leq i \leq \lfloor \frac{\omega(x)}{p} \rfloor} U_3^{pi} F_i(U_1, U_2)$ , where  $F_i \in k(x)[U_1, U_2]_{\omega(x)-pi}$ . We get

$$f' = u_1^{a(1)+a(2)+\omega(x)-p} (v'_2 - \lambda)^{a(2)} \left( \sum_{0 \leq i \leq \lfloor \frac{\omega(x)}{p} \rfloor} u_3'^{pi} F_i(1, v'_2 - \lambda) + u'_1 \phi' \right). \quad (1)$$

Let  $Z' := X' - \theta'$  be such that  $\Delta(h'; u'_1, v'_2, u'_3; Z')$  is minimal, and we let  $f'_{Z'} = f' + \theta'^p - \theta' g'^{p-1}$  as usual. We consider two cases:

Case 1:  $a(1) + a(2) + \omega(x) \not\equiv 0 \pmod{p}$ . The term

$$u_1^{a(1)+a(2)+\omega(x)-p}(v_2' - \lambda)^{a(2)} \sum_{0 \leq i \leq \lfloor \frac{\omega(x)}{p} \rfloor} u_3'^{pi} F_i(1, v_2' - \lambda)$$

in (1) induces unsolvable vertices of  $\Delta(h'; u_1', v_2', u_3'; Z')$ , so we have  $\omega(x') = \epsilon(x')$  since  $x'$  is very near  $x$ . Since  $F_i \neq 0$  for at least one index  $i \geq 1$  and condition (\*) does not hold,  $F$  is not the power of a linear form, hence  $\text{VDir}(x') \equiv \langle V_2', U_3' \rangle \pmod{U_1'}$ . By **II.2** of chapter 2, we have  $\kappa(x) = 0$ .

Case 2:  $a(1) + a(2) + \omega(x) \equiv 0 \pmod{p}$ . We have  $f'_{Z'} \equiv f' + \theta'^p \pmod{u_1'}$ , since  $u_1'$  divides  $H(x')^{-1}g'^p$ . Therefore  $J(f'_{Z'}, E') \equiv J(f', E') \pmod{u_1'}$ . Given  $D \in \mathcal{D}(x)$ , we denote  $F_D := \text{cl}_{\omega(x)}(H(x)^{-1}Df) \in k(x)[U_1, U_2, U_3]_{\omega(x)}$ . By assumption, we may pick  $D \in \mathcal{D}(x)$  such that  $F_D \notin k(x)[U_1, U_2]$ . If  $e^*D \in \mathcal{D}(x')$ , then

$$\bar{F}'_D := \text{cl}_{\omega(x')}(H(x')^{-1}(e^*D)f'_{Z'}) \pmod{U_1'} \in k(x)[V_2', U_3'^p] \setminus k(x)[V_2'],$$

so  $\omega(x') = \epsilon(x')$ ,  $\kappa(x') = 2$  and  $\tau(\bar{F}'_D) = 2$  if condition (\*) does not hold at  $x'$ . Then  $\kappa(x') = 0$  by chapter 2, **II.2**.

Assume now that  $e^*D \notin \mathcal{D}(x')$  for each such choice of  $D$ . We can pick  $D \in \{u_1 \frac{\partial}{\partial u_1}, u_2 \frac{\partial}{\partial u_2}\}$ : since  $\epsilon(x) = \omega(x)$ , we have  $F_{\frac{\partial}{\partial u_3}} = 0$ , moreover,  $x'$  being rational over  $x$ , we have (with notations as in chapter 1, **II.3**)

$$e^* \frac{\partial}{\partial \lambda_4}, \dots, e^* \frac{\partial}{\partial \lambda_s} \in \mathcal{D}(x').$$

We thus get

$$\text{cl}_{\omega(x')}(H(x')^{-1}(e^*D)f'_{Z'}) \in k(x)[U_1', V_2', U_3'^p] \setminus k(x)[U_1', V_2']. \quad (2)$$

If  $\epsilon(x) = 1 + \omega(x)$ , this proves that  $\kappa(x') = 2$ . Since condition (\*) does not hold at  $x$ , we have  $F_{\frac{\omega(x)}{p}} = 0$  if  $\omega(x) \equiv 0 \pmod{p}$ . Therefore

$$\text{cl}_{\omega(x')}(H(x')^{-1}(e^*D)f'_{Z'}) \in V_2' k(x)[U_1', V_2', U_3'^p], \quad (3)$$

by (2), which proves that  $\text{VDir}(x') \equiv \langle V_2', U_3' \rangle \pmod{U_1'}$ . Hence  $x'$  is in case (iii) of the proposition or  $\kappa(x) = 0$ .

If  $\epsilon(x) = \omega(x)$ , we expand

$$H(x')^{-1}f'_{Z'} = \phi'(v_2', u_3') + u_1'\psi',$$

where  $\phi' \in k(x)[[v_2', u_3']]$  and  $\text{ord}_{\eta'(x')}\phi' \geq 1 + \omega(x')$ . Since condition (\*) does not hold at  $x$ , equation (3) above implies that  $x'$  satisfies the assumptions of chapter 2, **II.4**, so  $\kappa(x) = 0$  in this case.

**I.2.5 Proof of (v).** By **I.1** (ii), we have  $\tau(x) = 3$  (so  $\kappa(x) = 0$ ) if  $\epsilon(x) = 1 + \omega(x)$ . Assume now that  $\epsilon(x) = \omega(x)$ . Since  $\text{VDir}(x) = \langle U_1, U_3 \rangle$ , we have  $x' = (X' = \frac{X}{u_2}, u_1' = \frac{u_1}{u_2}, u_2' = u_2, u_3' = \frac{u_3}{u_2})$  and  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u_1'u_2')$ . We are at the origin of a chart, so  $\Delta(h; u_1', u_2', u_3'; X')$  is minimal. Note that  $\text{cl}_{\omega(x)}(H(x)^{-1}f) \in k(x)[U_1, U_3]$ , so we get  $\epsilon(x') = \omega(x')$ ,  $\text{VDir}(x') \equiv \langle U_1', U_3' \rangle \pmod{U_2'}$ , and the conclusion follows.

**I.3 Corollary.** Let  $X = X_0 \leftarrow X_1 \leftarrow \dots \leftarrow X_i \leftarrow \dots$  be the quadratic sequence along  $\mu$ . If for each  $i \geq 1$ , the center  $x_i$  of  $\mu$  in  $X_i$  satisfies ( $x_i \in \Sigma_p(X_i)$ ,  $\Omega(x_i) = (\omega(x_i), 2)$  and  $\kappa(x_i) = 2$ ), then  $\kappa(x) \leq 1$  or condition (\*) holds for some  $i \geq 1$ .

Proof. Otherwise, there exists  $i_0 \geq 0$  such that either all  $x_i$ 's are in case (iii) or are in case (v) for  $i \geq i_0$  by **I.2**. Without loss of generality, it can be assumed that  $i_0 = 0$ . In case (iii), all  $x_i$ 's are on the strict transform of a formal curve  $\mathcal{C} := V(\widehat{X}, \widehat{u}_2, \widehat{u}_3)$ , where  $\widehat{X} = X + \sum \lambda_i u_1^i$  and  $\widehat{u}_j = u_j + \sum \lambda_{ij} u_1^i$  for  $j = 2, 3$ . We have  $\mathcal{C} \not\subseteq E$ , so  $x_i$  is a regular point of  $X_i$  for  $i \gg 0$ : a contradiction. In case (v), we have  $E_i = \text{div}(u_{1,i} u_2)$ , where  $E_i$  is the exceptional divisor at  $x_i$  and  $u_{1,i} = u_2^{-i} u_1$ , so

$$i < \frac{\mu(u_1)}{\mu(u_2)} < \infty,$$

since the value group of  $\mu$  is Archimedean: a contradiction.

**I.4 Remark.** *From now on till the end of section I, we may therefore assume that  $(\kappa(x) = 2$  and condition  $(*)$  holds at  $x$ ). In particular, we have  $\omega(x) \equiv 0 \pmod{p}$  by **I.1** and **I.1.2**. Namely, by **I.3**, local uniformization in this special case implies local uniformization whenever  $\kappa(x) = 2$ .*

### I.5 Well prepared variables, invariants.

Let  $m(x)$  be the number of irreducible components of  $E$  at  $x$ , where  $\kappa(x) = 2$  and condition  $(*)$  holds at  $x$ . There are three different cases:

(\*1)  $m(x) = 1$ ,  $\epsilon(x) = \omega(x)$ ;

(\*2)  $m(x) = 2$ ,  $\epsilon(x) = \omega(x)$ ;

(\*3)  $m(x) = 1$ ,  $\epsilon(x) = 1 + \omega(x)$ . In the following definition, the r.s.p.  $(u_1, u_2, u_3)$  of  $\widehat{S}$  is such that  $\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2)$ . Remember that  $X \in \widehat{R}$  is such that  $\Delta(h; u_1, u_2, u_3; X)$  is minimal. We expand

$$f =: H(x) \sum_{0 \leq j \leq \omega(x)} u_3^{\omega(x)-j} \phi_j,$$

where  $\phi_j \in k[[u_1, u_2]]$ ,  $1 \leq j \leq \omega(x)$ ,  $\phi_0 \in \widehat{S}$ . We have  $\phi_0$  invertible in cases (\*1) or (\*2), and  $(u_1, \phi_0, u_3)$  is a r.s.p. of  $\widehat{S}$  in case (\*3).

**I.5.1 Preparation of the variables.** We now define well-preparedness of variables in cases (\*1) and (\*2), then in case (\*3). Existence of well prepared variables is proved in **I.6** below.

**I.5.1.1 Definition.** *In cases (\*1) or (\*2), we say that  $(X, u_1, u_2, u_3)$  is well prepared if the following conditions are satisfied:*

(wp1)  $\Delta(h; u_1, u_2, u_3; X)$  is minimal;

(wp2) no vertex  $w = (w_1, w_2)$  of  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)$  is solvable (definition of this polyhedron and of vertex solvability below).

The polygon in (wp2) is defined in this way: given an ideal  $I \subseteq \widehat{S}$ , let  $NP(I) \subset \mathbb{R}_{\geq 0}^3$  be its Newton polyhedron, i.e.  $NP(I)$  is the convex hull of  $\{\mathbf{s} + \mathbb{R}_{\geq 0}^3\}$ , where  $\mathbf{s} =: (s_1, s_2, s_3) \in \mathbb{N}^3$  ranges over all monomials  $u_1^{s_1} u_2^{s_2} u_3^{s_3}$  appearing with nonzero coefficient in the expansion of some  $\phi \in I$ .

Assume that moreover  $\text{ord}_{u_3}(I \bmod(u_1, u_2)) = \omega(x)$ . Then  $\Delta(I; u_1, u_2; u_3)$  is defined to be  $1/\omega(x)$  times the projection of  $NP(I)$  on the first two coordinates plane from the point  $(0, 0, \omega(x))$ . Note: each vertex  $w$  of  $\Delta(I; u_1, u_2; u_3)$  has coordinates in  $(\frac{1}{\omega(x)!} \mathbb{N})^2$ .

It is easily checked that

$$\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3) = \Delta(H(x)^{-1}(\{\lambda_i \frac{\partial f}{\partial \lambda_i}\}_{1 \leq i \leq s}, g^p); u_1, u_2; u_3).$$

A vertex  $w = (w_1, w_2) \in \mathbb{N}^2$  of this polygon is then called solvable if there exists  $\lambda \in k(x)$ ,  $\lambda \neq 0$ , such that following condition is satisfied:

$$\forall i, 1 \leq i \leq s, \text{in}_w(H(x)^{-1}\lambda_i \frac{\partial f}{\partial \lambda_i}) = a_i(U_3 - \lambda U_1^{w_1} U_2^{w_2})^{\omega(x)}, \quad a_i \in k(x), \quad (1)$$

and if

$$(w_1, w_2) \neq \frac{1}{\omega(x)}(\text{ord}_{u_1}(H(x)^{-1}g^p), \text{ord}_{u_2}(H(x)^{-1}g^p)). \quad (2)$$

**I.5.1.2 Definition.** In case (\*3), we say that  $(X, u_1, u_2, u_3)$  is well prepared if the following conditions are satisfied:

(wp1)  $\Delta(h; u_1, u_2, u_3; X)$  is minimal;

(wp2) no “left” vertex  $w = (w_1, w_2)$  of  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$  is solvable (definition of this polyhedron, of “left” vertices and of vertex solvability below).

We define  $NP(u_2^{-1}I)$  as follows for an ideal  $I \subseteq \widehat{S}$  such that

- (a)  $\text{ord}_x I = \text{ord}_{(u_2, u_3)}(I \bmod(u_1)) = \omega(x) + 1$ , and
- (b)  $\text{ord}_{u_3}(\frac{\partial \phi}{\partial u_2} \bmod(u_1, u_2)) = \omega(x)$  for some  $\phi \in I$ .

The polyhedron  $NP(u_2^{-1}I)$  is the convex hull of

$$\{(0, 0, \omega(x)) + \mathbb{R}_{\geq 0}^3\} \cup \{\mathbf{s} + \mathbb{R}_{\geq 0}^3\},$$

where  $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{N} \times (\mathbb{N} \cup \{-1\}) \times \mathbb{N}$  ranges over all monomials  $u_1^{s_1} u_2^{s_2} u_3^{s_3}$  appearing with nonzero coefficient in the expansion of some  $u_2^{-1}\phi \in u_2^{-1}I$  with  $0 \leq s_3 \leq \omega(x) - 1$  (i.e. we disregard all monomials with  $s_3 \geq \omega(x)$  distinct from  $u_3^{\omega(x)}$ ). Then  $\Delta(u_2^{-1}I; u_1, u_2; u_3)$  is defined to be  $1/\omega(x)$  times the projection of  $NP(I)$  on the first two coordinates plane from the point  $(0, 0, \omega(x))$  as before, thus allowing vertices with *negative second coordinate*. If  $w = (w_1, w_2)$  is a vertex of  $\Delta(u_2^{-1}I; u_1, u_2; u_3)$ , we thus have

$$w_2 \in \bigcup_{1 \leq i \leq \omega(x)} \left( \frac{1}{i} \mathbb{N} \cup \left\{ -\frac{1}{i} \right\} \right).$$

Note that, since  $\Phi \in k(x)[U_2, U_3^p]$  in definition **I.1** (ii) and condition (\*3) holds, (a) and (b) are verified for  $I = H(x)^{-1}(f, g^p)$ .

In case (\*3) a vertex  $w = (w_1, w_2) \in \mathbb{N} \times \mathbb{N}$  is then called solvable if there exists  $\lambda \in k(x)$ ,  $\lambda \neq 0$ , such that following conditions are satisfied:

$$\forall i, 1 \leq i \leq s, \text{in}_w(H(x)^{-1}u_2^{-1}\lambda_i \frac{\partial f}{\partial \lambda_i}) = a_i(U_3 - \lambda U_1^{w_1} U_2^{w_2})^{\omega(x)}, \quad a_i \in k(x), \quad (1)$$

and

$$(w_1, w_2) \neq \frac{1}{\omega(x)}(\text{ord}_{u_1}(H(x)^{-1}g^p), -1). \quad (2)$$

A “left” vertex is a vertex  $w$  such that there exists a linear form  $L \in (\mathbb{R}^2)^\vee$ ,  $L(w_1, w_2) = mw_1 + nw_2$ , ( $m, n \in \mathbb{Q}_+$  and  $m > n$ ) with

$$\{w\} = \Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3) \cap \{v \mid L(v) = 1\}.$$

All other vertices are called “right” vertices. We have

$$\text{in}_w(H(x)^{-1}u_2^{-1}\lambda_i \frac{\partial H(x)u_3^{\omega(x)}\phi_0}{\partial \lambda_i}) \in U_2^{-1}k(x)[U_2, U_3]_{1+\omega(x)} \quad (3)$$

for each “left” vertex  $w$ , since  $\epsilon(x) = 1 + \omega(x)$  and  $L(1,0) > L(0,1)$  for linear forms as in the definition of “left” vertices. Actually, the left hand side of (5) is  $a_i U_3^{\omega(x)}$  for some  $a_i \in k(x)$  if the “left” vertex  $w$  is distinct from  $(1,0)$ . However for “right” vertices, we emphasize the following fact: we only have  $\text{in}_w(H(x)^{-1}u_2^{-1}\lambda_i \frac{\partial f}{\partial \lambda_i}) \in U_2^{-1}k(x)[U_1, U_2, U_3]$ . For example, assume  $F := \text{cl}_{1+\omega(x)}(H(x)^{-1}f)$  has an expansion

$$F = U_3^{\omega(x)}(aU_1 + bU_2 + cU_3) + \{\text{terms of smaller degree in } U_3\},$$

with  $a \neq 0$ . Then for any “right” vertex, we have

$$\text{in}_w(H(x)^{-1}u_2^{-1}\lambda_i \frac{\partial f}{\partial \lambda_i}) = (a_i \frac{u_1}{u_2} + b_i)(u_3^{\omega(x)}) + \{\text{terms of smaller degree in } u_3\},$$

with  $a_i \neq 0, b_i \in k(x)$  for some  $i, 1 \leq i \leq s$ . Fortunately, only left vertices are important w.r.t. the invariants that we define now. A similar fact occurs when  $x$  is in case (\*1), although right vertices are better behaved in this case.

**I.5.2 Notations and invariants.** We define the resolution invariants in cases (\*1) and (\*2) (resp. (\*3)) from the polygon  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)$  (resp.  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)$ ), once well prepared coordinates have been chosen.

**I.5.2.1 Definition.** In cases (\*1) and (\*2), let us denote

$$\begin{aligned} A_j &:= \inf \left\{ \frac{\text{ord}_{u_j}(H(x)^{-1}g^p)}{\omega(x)}, \frac{\text{ord}_{u_j}\phi_i}{i}, 1 \leq i \leq \omega(x) \right\}, j = 1, 2; \\ B &= \inf \left\{ \frac{\text{ord}_{(u_1, u_2)}(H(x)^{-1}g^p)}{\omega(x)}, \frac{\text{ord}_{(u_1, u_2)}\phi_i}{i}, 1 \leq i \leq \omega(x) \right\}, C := B - A_1 - A_2 \geq 0; \\ \beta &:= \inf \left\{ \frac{\text{ord}_{u_2}(u_1^{-\omega(x)A_1}H(x)^{-1}g^p \bmod(u_1))}{\omega(x)}, \frac{\text{ord}_{u_2}(u_1^{-iA_1}\phi_i \bmod(u_1))}{i}, 1 \leq i \leq \omega(x) \right\}, \text{ where } \text{ord}_{u_2} \text{ de-} \\ &\text{notes the natural valuation of the discrete valuation ring } k(x)[[u_2]]. \end{aligned}$$

Note that

$$A_j = \inf \{w_j : (w_1, w_2) \in \Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)\} \geq 0, j = 1, 2,$$

$$B = \inf \{w_1 + w_2 \mid (w_1, w_2) \in \Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)\} \geq 1,$$

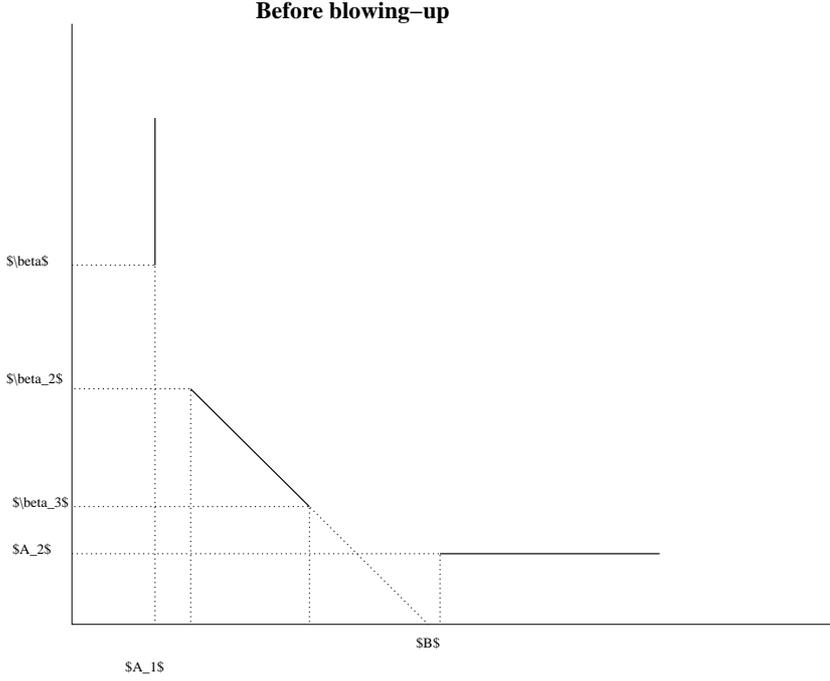
and that

$$\beta = \inf \{w_2 \mid (A_1, w_2) \in \Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)\} \geq 0.$$

We will denote those vertices  $w = (w_1, w_2)$  of  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)$  with  $x_1 + x_2 = B$  by  $(\alpha_2, \beta_2)$  and  $(\alpha_3, \beta_3)$  with  $\alpha_2 \leq \alpha_3$ .

We point out that, if  $x$  is in case (\*1),

$$A_1 = \frac{\text{ord}_{u_1}(H(x)^{-1}g^p)}{\omega(x)} \implies \beta = 0.$$



**I.5.2.2 Definition.** In case  $(\ast 3)$ , let us denote

$$A_1 = \inf \left\{ \frac{\text{ord}_{u_1}(H(x)^{-1}g^p)}{\omega(x)}, \frac{\text{ord}_{u_1}\phi_i}{i}, 1 \leq i \leq \omega(x) \right\};$$

$$B_3 = \inf \left\{ \frac{\text{ord}_{(u_1, u_2)}(H(x)^{-1}u_2^{-1}g^p)}{\omega(x)}, \frac{\text{ord}_{(u_1, u_2)}(u_2^{-1}\phi_i)}{i}, 1 \leq i \leq \omega(x) \right\};$$

$$\beta_3 := \inf \left\{ \frac{\text{ord}_{u_2}(u_1^{-\omega(x)A_1}u_2^{-1}H(x)^{-1}g^p \bmod(u_1))}{\omega(x)}, \frac{\text{ord}_{u_2}(u_1^{-iA_1}u_2^{-1}\phi_i \bmod(u_1))}{i}, 1 \leq i \leq \omega(x) \right\}, \text{ where}$$

$\text{ord}_{u_2}$  denotes the natural valuation of the discrete valuation ring  $k(x)[[u_2]]$ .

$$\beta := \inf \left\{ \frac{\text{ord}_{u_2}(u_1^{-\omega(x)A_1}H(x)^{-1}g^p \bmod(u_1))}{\omega(x)}, \frac{\text{ord}_{u_2}(u_1^{-iA_1}\phi_i \bmod(u_1))}{i}, 1 \leq i \leq \omega(x) \right\}.$$

Note that

$$A_1 = \inf \{w_1 : (w_1, w_2) \in \Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)\} \geq 0,$$

$$B_3 = \inf \{w_1 + w_2 \mid (w_1, w_2) \in \Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)\} \geq 1,$$

and that

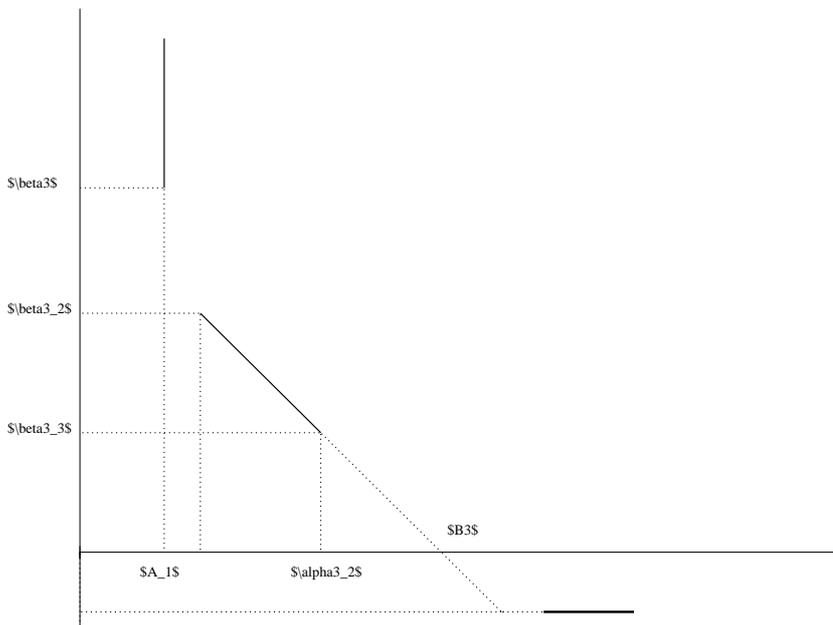
$$\beta_3 = \inf \{w_2 \mid (A_1, w_2) \in \Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)\} \geq -1.$$

We will denote those vertices  $w = (w_1, w_2)$  of  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$  with  $x_1 + x_2 = B_3$  by  $(\alpha_3_2, \beta_3_2)$  and  $(\alpha_3_3, \beta_3_3)$  with  $\alpha_3_2 \leq \alpha_3_3$ . Note that a vertex  $w = (w_1, w_2)$  is a “left” vertex of  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$  if and only if  $w_1 \leq \alpha_3_2$ .

We point out the following implications, if  $x$  is in case  $(\ast 3)$ :

$$A_1(x) = \frac{\text{ord}_{u_1}(H(x)^{-1}g^p)}{\omega(x)} \implies -1 \leq \beta_3(x) < 0 \Leftrightarrow \beta(x) = 0. \quad (1)$$

**Before blowing up case (\*3)**



Obviously, the above definitions may depend on the choice of well prepared coordinates  $(X, u_1, u_2, u_3)$ . When there is a risk of confusion, we will make explicit this dependence on  $(u_1, u_2, u_3)$  by writing  $A_j(u_1, u_2, u_3)$ , etc... We also use the notation  $A_j(x)$ ,  $A_j(x')$ , etc... when dealing with a blowing up  $e : X' \rightarrow X$  and  $x' \in e^{-1}(x)$  if  $\kappa(x') = 2$  and  $x'$  satisfies condition (\*). In this case, we always compute invariants w.r.t.  $E' := (e^{-1}E)_{\text{red}}$ .

**I.5.2.3 Definition.** Assume that  $(X, u_1, u_2, u_3)$  is well prepared. We denote

$\gamma(u_1, u_2, u_3) := \lceil \beta(u_1, u_2, u_3) \rceil \geq 0$  if  $x$  is in case (\*1);

$\gamma(u_1, u_2, u_3) := 1 + \lfloor C(u_1, u_2, u_3) \rfloor \geq 1$  if  $x$  is in case (\*2);

$\gamma(u_1, u_2, u_3) := 1 + \lfloor \beta_3(u_1, u_2, u_3) \rfloor \geq 0$  if  $x$  is in case (\*3).

We will also use the notation  $\gamma(x) = \gamma(u_1, u_2, u_3)$  for short. About existence of well prepared coordinates, we have:

**I.6 Proposition.** Assume that  $\kappa(x) = 2$  and condition (\*) holds for the r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$ . There exists  $\psi \in (u_1, u_2)k(x)[[u_1, u_2]]$ ,  $\theta \in \widehat{S} = k(x)[[u_1, u_2, u_3]]$  such that  $(X - \theta, u_1, u_2, v := u_3 - \psi)$  is well prepared.

*Proof.* By assumption, (wp1) holds for  $(X, u_1, u_2, u_3)$ . In cases (\*1) and (\*2) (resp. (\*3)), suppose there exists a vertex (resp. a left vertex)  $w = (w_1, w_2)$  of  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)$  (resp.  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$ ) which is solvable. By definition of solvability, if  $L$  is a linear form on  $\mathbb{R}^3$  with coefficients in  $\mathbb{R}_{>0}$  (resp. and with  $L(1, 0, 0) > L(0, 1, 0)$ ) such that  $\text{in}_w(H(x)^{-1}f) = \text{in}_L(H(x)^{-1}f)$ , then in  $\text{gr}_L(S)$  (resp. in  $\text{gr}_L(u_2^{-1}S)$ ), we have equality of ideals (resp. submodules):

$$\text{in}_L(H(x)^{-1}(\{\lambda_i \frac{\partial f}{\partial \lambda_i}\}_{1 \leq i \leq s}, g^p)) = k(x).(U_3 - \lambda U_1^{w_1} U_2^{w_2})^{\omega(x)}$$

$$\text{(resp. } \text{in}_L(H(x)^{-1}u_2^{-1}(\{\lambda_i \frac{\partial f}{\partial \lambda_i}\}_{1 \leq i \leq s}, g^p)) = k(x).(U_3 - \lambda U_1^{w_1} U_2^{w_2})^{\omega(x)}).$$

**I.6.1** We perform the dissolution of  $w$  in cases (\*1) and (\*2) (resp. in case (\*3)). Let us denote  $v_w := u_3 - \psi_w$ ,  $\psi_w := \lambda u_1^{w_1} u_2^{w_2}$ . The basis

$$(du_1, du_2, du_3, d\lambda_4, \dots, d\lambda_s)$$

of  $\Omega_{S/k_0}^1$  gets changed to

$$(du_1, du_2, dv_w, d\lambda_4, \dots, d\lambda_s),$$

so  $H(x)^{-1} \frac{\partial f}{\partial u_3} = H(x)^{-1} \frac{\partial f}{\partial v_w}$  and, for  $1 \leq i \leq s$ ,  $H(x)^{-1} \lambda_i \frac{\partial f}{\partial \lambda_i}$  is unchanged modulo the ideal  $(u_1^{w_1} u_2^{w_2} H(x)^{-1} \frac{\partial f}{\partial u_3})$ .

**I.6.2** If  $L$  is a linear form as above, we have

$$\text{in}_L(H(x)^{-1} \lambda_i \frac{\partial f}{\partial \lambda_i}) = \gamma_i V_w^{\omega(x)} \quad (\text{resp. } \text{in}_L(H(x)^{-1} u_2^{-1} \lambda_i \frac{\partial f}{\partial \lambda_i}) = \gamma_i V_w^{\omega(x)}).$$

So  $L(w', 0) > L(0, 0, \omega(x))$  for all vertices  $w'$  of

$$\Delta(H(x)^{-1}(f, g^p); u_1, u_2; v_w) = \Delta(H(x)^{-1}(\{\lambda_i \frac{\partial f}{\partial \lambda_i}\}_{1 \leq i \leq s}, g^p); u_1, u_2; v_w)$$

(resp. of  $\Delta(H(x)^{-1} u_2^{-1}(f, g^p); u_1, u_2; v_w)$ ). Let  $w' \neq w$  be a vertex (resp. a “left” vertex) of

$$\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3) \quad (\text{resp. of } \Delta(H(x)^{-1} u_2^{-1}(f, g^p); u_1, u_2; u_3)),$$

and  $L'$  be a linear form on  $\mathbb{R}^3$  with coefficients in  $\mathbb{R}_{>0}$  (resp. and with  $L'(1, 0, 0) > L'(0, 1, 0)$ ) such that

$$\text{in}_{w'}(H(x)^{-1} f) = \text{in}_{L'}(H(x)^{-1} f) \quad (\text{resp. } \text{in}_{w'}(H(x)^{-1} u_2^{-1} f) = \text{in}_{L'}(H(x)^{-1} u_2^{-1} f)).$$

In particular, we have  $L'(w, 0) > L'(w', 0)$ . By **I.6.1** above,

$$\text{in}_{w'}(H(x)^{-1}(\{\lambda_i \frac{\partial f}{\partial \lambda_i}\}_{1 \leq i \leq s}, g^p)) \quad (\text{resp. } \text{in}_{w'}(H(x)^{-1} u_2^{-1}(\{\lambda_i \frac{\partial f}{\partial \lambda_i}\}_{1 \leq i \leq s}, g^p)))$$

is unaffected by the change of differential basis. So  $w'$  is still a vertex of

$$\Delta(H(x)^{-1}(f, g^p); u_1, u_2; v_w) = \Delta(H(x)^{-1} u_2^{-1}(\{\lambda_i \frac{\partial f}{\partial \lambda_i}\}_{1 \leq i \leq s}, g^p); u_1, u_2; v_w)$$

(resp. of  $\Delta(H(x)^{-1} u_2^{-1}(f, g^p); u_1, u_2; v_w)$ ).

We may have lost (wp1) in this process, in which case we make a change of variable  $X_w := X - \theta_w$ , to get (wp1) anew. This translation necessarily makes not larger the polygon

$$\Delta(H(x)^{-1}(f_{X_w}, g^p); u_1, u_2; v_w) \quad (\text{resp. } \Delta(H(x)^{-1} u_2^{-1}(f_{X_w}, g^p); u_1, u_2; v_w))$$

which is a projection (resp. the projection of a translate) of  $\Delta(h; u_1, u_2, v_w; X_w)$ , where  $f_{X_w} := f + \theta_w^p - g^{p-1} \theta_w$ . After iterating (possibly infinitely many times) this vertex dissolution and minimizing step, one gets a non-increasing sequence of polyhedra of the form

$$\Delta(H(x)^{-1} u_2^{-1}(f_j, g^p); u_1, u_2; v_j), \quad j \geq 1,$$

where  $h = (X - \theta_j)^p - g^{p-1}(X - \theta_j) + f_j$  and  $v_j = u_3 - \psi_j$ . The series  $\psi_j$  and  $\theta_j$  respectively converge in  $k(x)[[u_1 u_2]]$  and in  $\widehat{S}$ , since  $(w_1 + w_2, w_1)$  increases for the lexicographical ordering in each step **I.6.1**.

**I.6.3 Proposition.** *Assume that  $\kappa(x) = 2$  and  $x$  is in case (\*3) for the r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$ . If  $w := (0, 1)$  is a vertex of  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$ , then  $w$  is not solvable. As a consequence, we have*

$$v = u_3 + u_1\phi_1 + \phi,$$

with  $\phi_1, \phi \in \widehat{S}$ ,  $\text{ord}_{\eta(x)}\phi \geq 2$  in the dissolution **I.6**. Therefore  $V \in \text{VDir}(H(x)^{-1}\frac{\partial f_Z}{\partial u_2})$ , where  $h = Z^p - g^{p-1}Z + f_Z$ .

Proof. Suppose that  $w := (0, 1)$  is a vertex of  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$ . By **I.1** (ii), we have  $\text{VDir}(\Phi) = \langle U_3, U_2 \rangle$ , so  $w$  does not satisfy equation (1) in **I.5.1.2** and the proposition follows.

**I.7 Proposition.** *Assume that  $\kappa(x) = 2$ , condition (\*) holds and  $(X, u_1, u_2, u_3)$  is well prepared. We have  $\kappa(x) \leq 1$  or the following holds:*

- (i) in cases (\*1) and (\*2), either  $U_3 \in \text{VDir}(x)$  or  $(m(x) = 2, B(x) = C(x) = 1)$ , and there exists a r.s.p.  $(Z, u_1, u_2, v)$  such that condition (\*2) holds,  $(Z, u_1, u_2, v)$  is well prepared,  $B(u_1, u_2, v) = C(u_1, u_2, v) = 1$ , and we have  $\text{VDir}(x) = \langle V, U_1 + bU_2 \rangle$  for some  $b \neq 0$ ;
- (ii) in case (\*3), either

$$U_3 \in \text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1}\frac{\partial f}{\partial u_2})),$$

or  $(B3(x) = 1)$  and there exists a r.s.p.  $(Z, u_1, u_2, v)$  with  $v \equiv u_3 \pmod{(u_1)}$  such that condition (\*3) holds,  $(Z, u_1, u_2, v)$  is well prepared,

$$V \in \text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1}\frac{\partial f_Z}{\partial u_2})),$$

and the following holds:  $\Delta(H(x)^{-1}u_2^{-1}(f_Z, g^p); u_1, u_2; v)$  and  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$  have the same "left" vertices, one of which is  $(\alpha3_2, \beta3_2)$ , where  $0 \leq \alpha3_2 < 1$  and  $\alpha3_2 + \beta3_2 = 1$  (in particular  $B3(u_1, u_2, v) = 1$  and  $\beta3(u_1, u_2, v) = \beta3(u_1, u_2, u_3)$ ).

**I.7.1** Proof of (i). If we are in cases (\*1) or (\*2), then

$$\text{cl}_{\omega(x)}(H(x)^{-1}f) = \lambda(U_3^{\omega(x)} + \sum_{1 \leq i \leq \omega(x)} U_3^{\omega(x)-i} P_i(U_1, U_2)) =: \lambda F,$$

where  $\lambda \neq 0$ ,  $P_i \in k(x)[U_1, U_2]_i$ , and  $P_i = 0$  if  $B(x) > 1$  or if  $i \not\equiv 0 \pmod{p}$ . The case  $B(x) > 1$  is proved (take  $v := u_3$ ), so assume  $B(x) = 1$ . By well-preparedness, we have  $\tau(x) \geq 2$ , so  $\tau(x) = 2$  if  $\kappa(x) > 1$ .

Assume that  $U_3 \notin \text{VDir}(x)$ . Then  $\text{VDir}(x) = \langle U_3 + aU_i, U_j + bU_i \rangle$ , where  $a \neq 0$ ,  $b \in k(x)$  and  $\{i, j\} = \{1, 2\}$ . By chapter 2 **II.2**, we have  $\kappa(x) = 0$  if  $\text{div}(u_j) \not\subseteq E$ , so it can be assumed that  $j = 1, i = 2$ . We claim that  $b \neq 0$ . Indeed, if  $b = 0$ , then for  $1 \leq j \leq s$ , we have

$$\text{cl}_{\omega(x)}(H(x)^{-1}\lambda_j \frac{\partial f}{\partial \lambda_j}) = \mu_j(U_3 + aU_2)^{\omega(x)} + \sum_{1 \leq i \leq \omega(x)} \mu_{ji}(U_3 + aU_2)^{\omega(x)-i} U_1^i.$$

By equation (1) of **I.5.1.1**, the vertex  $(0, 1)$  of  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)$  is solvable: a contradiction, so  $b \neq 0$ . Once again, this implies  $E = \text{div}(u_1 u_2)$  by chapter 2 **II.2**. If  $C(x) < 1$ , we have  $A_1(x) > 0$  or  $A_2(x) > 0$ . Therefore there exists  $i_0 \in \{1, 2\}$  such that for each  $j, 1 \leq j \leq s$ , we have

$$\text{cl}_{\omega(x)}(H(x)^{-1}\lambda_j \frac{\partial f}{\partial \lambda_j}) - \mu_j U_3^{\omega(x)} \in U_{i_0} k(x)[U_1, U_2, U_3], \mu_j \in k(x),$$

so  $U_{i_0} \in \text{VDir}(x)$ : a contradiction. Hence  $C(x) = B(x) = 1$ . Then we replace  $u_3$  by  $v := u_3 + au_2$  and get  $\text{VDir}(x) = \langle V, U_1 + bU_2 \rangle$ . The r.s.p.  $(Z, u_1, u_2, v)$  is obtained after applying the algorithm **I.6**. The same argument as above shows that  $B(x)$  and  $C(x)$  are unchanged.

**I.7.2** Proof of (ii). Assume that  $U_3 \notin \text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f}{\partial u_2}))$ . In definition **I.1**(ii), we have  $\Psi \neq 0$  necessarily. If  $(0, 1)$  is not a vertex of  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$ , then  $\Phi \in k(x)[U_3]$ , so we must have

$$\text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f}{\partial u_2}) \in k(x) \cdot [U_3 + aU_1]$$

for some  $a \in k(x)$ ,  $a \neq 0$ . In particular  $B\mathfrak{3}(x) = 1$ . The vertex  $(\alpha\mathfrak{3}_2, \beta\mathfrak{3}_2)$  (defined at the end of **I.5.2.2**) of the initial face of  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$  is unsolvable by definition of well-preparedness and has  $0 < \alpha\mathfrak{3}_2 \leq 1$  by assumption. Suppose that  $\alpha\mathfrak{3}_2 = 1$ . Then

$$\text{cl}_{\omega(x)}(H(x)^{-1}f) = \mu U_2(U_3 + aU_1)^{\omega(x)} + P,$$

where  $\mu \neq 0$  and  $P \in k(x)[U_1, U_3]_{1+\omega(x)}$ . This is impossible, since  $(1, 0)$  is not solvable.

We have proved that  $0 \leq \alpha\mathfrak{3}_2 < 1$ . Then we replace  $u_3$  by  $v := u_3 + au_1$ , the r.s.p.  $(Z, u_1, u_2, v)$  being obtained after applying the algorithm **I.6**. We get

$$V \in \text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f_Z}{\partial u_2})).$$

Note that  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$  and  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; v)$  have the same ‘‘left’’ vertices.

**I.8 Theorem.** *Assume that  $\kappa(x) = 2$ . Then  $\kappa(x) \leq 1$ .*

Note that theorem **I.8** proves local uniformization when  $\kappa(x) = 2$ . By **I.3**, it can be assumed that condition (\*) holds. By **I.6**, there exists then a well prepared r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$ . We maintain these assumptions up to the end of this chapter. The proof needs long computations. We start by an easy lemma.

**I.8.1 Lemma.** *If  $\beta(x) = 0$ , then  $\kappa(x) \leq 1$ . In particular, theorem **I.8** holds when  $\gamma(x) = 0$ .*

Proof. We have  $\epsilon(x) \leq \omega(x)(A_1(x) + \beta(x))$ . So  $A_1(x) \geq 1$  which implies  $\epsilon(\mathcal{C}) = \epsilon(x) \geq p$  where  $\mathcal{C} := V(X, u_1, u_3)$ . By chapter 1 **II.4.7**,  $\mathcal{C}$  is an analytic branch of an algebraic curve on  $X$ . On the other hand, we have

$$J(f, E, x) \equiv (u_3^{\omega(x)}) \pmod{(u_1)} \quad (\text{resp. } J(f, E) \equiv (u_3^{\omega(x)}) \pmod{(u_1)})$$

if  $\epsilon(x) = \omega(x)$  (resp.  $\epsilon(x) = 1 + \omega(x)$ ). Therefore  $\mathcal{C}$  is the unique analytic curve on  $E$  (resp. on  $E \setminus \text{div}(u_2)$ ) in the set

$$W := \{y \in \Sigma_p(x) \mid \omega(y) \geq 1\}$$

if  $m(x) = 1$  (resp. if  $m(x) = 2$ ). Hence  $\mathcal{C}$  is a regular curve on  $X$ .

If  $\epsilon(x) = \omega(x)$ , i.e.  $x$  is in case (\*1) or (\*2),  $\mathcal{C}$  is thus permissible of the first kind. By **I.7**(i), we have  $U_3 \in \text{VDir}(x)$  since  $B(x) > C(x)$ . Let  $e : X' \rightarrow X$  be the blowing up along  $\mathcal{C}$ . If  $x' \in e^{-1}(x)$  is very near  $x$ , then  $x' = (X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = u_2, u'_3 = \frac{u_3}{u_1})$ ,  $x'$  remains in case (\*1) (resp. (\*2)) if  $x$  is in case (\*1) (resp. (\*2)), and  $(X', u'_1, u'_2, u'_3)$  remains well prepared at  $x'$  w.r.t.  $h' = u_1^{-p}h$ . We thus have  $\beta(x') = 0$ ,  $A_1(x') = A_1(x) - 1$  and  $\text{ord}_{\eta'(x')}(H(x')^{-1}f') \leq A_1(x')\omega(x)$ . The conclusion follows by descending induction on  $A_1(x)$ .

If  $\epsilon(x) = 1 + \omega(x)$ , i.e.  $x$  is in case (\*3), then there appears in the expansion of  $H(x)^{-1}f$  or of  $H(x)^{-1}g^p$  the monomial  $u_3^{\omega(x)-i}u_1^{iA_1(x)}$  with nonzero coefficient for some  $i$ ,  $1 \leq i \leq \omega(x)$ . In particular,  $A_1(x) > 1$ , so

$$J(f, E, \mathcal{C}) \in (u_3^{\omega(x)}) + (u_1, u_3)^{1+\omega(x)},$$

i.e.  $\mathcal{C}$  is permissible of the second kind. Let  $e : X' \rightarrow X$  be the blowing up along  $\mathcal{C}$ . We have  $\text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1}\frac{\partial f}{\partial u_2})) = k(x).U_3$ , since  $A_1(x) > 1$ . If  $x' \in e^{-1}(x)$  is very near  $x$ , then  $x' = (X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = u_2, u'_3 = \frac{u_3}{u_1})$  by chapter 1, **II.5.4.4**. We consider two cases.

If  $\epsilon(x') = 1 + \omega(x)$ , then  $x'$  remains in case (\*3) and  $(X', u'_1, u'_2, u'_3)$  remains well prepared. We have  $\beta(x') = 0$  and  $A_1(x') = A_1(x) - 1$ .

If  $\epsilon(x') = \omega(x') = \omega(x)$ , then  $A_1(x) = 2$  and there is an expansion

$$f' = u'_1{}^{a(1)+\omega(x)-p}(\gamma' u'_2 u'_3{}^{\omega(x)} + u'_1 \phi'),$$

where  $\gamma'$  is a unit and  $\text{ord}_{\eta'(x')}(u'_1 \phi') = \omega(x)$ . But this in turn implies  $\kappa(x') \leq 1$  by chapter 2, **II.1** (note that  $\omega(x) \geq p \geq 2$ ).

We conclude by descending induction on  $A_1(x)$ . The last statement in the lemma follows from equation (1) in **I.5.2.2**.

Lemma **I.8.1** settles cases (\*1) and (\*3) when  $\gamma(x) = 0$ . Therefore from now on, we assume that  $\gamma(x) \geq 1$ . We now control the behavior of  $\gamma(x)$  by blowing up a closed point.

**I.8.2 Notations.** Assume that  $\kappa(x) = 2$ , condition (\*1) or (\*2) holds and  $(X, u_1, u_2, u_3)$  is well prepared. Let us denote  $H(x) =: u_1^{a(1)}u_2^{a(2)}$ ,  $g^p =: \gamma^p u_1^{b(1)}u_2^{b(2)}$ ,  $\gamma$  invertible (with  $a(2) = b(2) = 0$  if  $x$  is in case (\*1)), and:

$$H(x)^{-1}f =: \sum_{0 \leq j \leq \omega(x)} u_3^{\omega(x)-j} \phi_j,$$

with  $\phi_0 \in k(x)[[u_1, u_2, u_3]]$  invertible,  $\phi_j \in k(x)[[u_1, u_2]]$  for  $1 \leq j \leq \omega(x)$ . Let  $\phi_j =: u_1^{a_j(1)}u_2^{a_j(2)}\psi_j$ , where  $\psi_j = 0$  or  $\psi_j$  is not divisible by  $u_1$ , nor by  $u_2$  if  $x$  is in case (\*2). As usual, we take  $a_j(2) = 0$  and  $\psi_j = 0$  or  $\psi_j$  is not divisible by  $u_1$  if  $x$  is in case (\*1).

**I.8.2.1 Definition.** For  $1 \leq j \leq \omega(x)$  such that  $\phi_j \neq 0$ , we denote  $\Phi_j := \text{in}_x \phi_j = U_1^{a_j(1)}U_2^{a_j(2)}\Psi_j$  and  $d_j := \deg \Psi_j \in \mathbb{N}$ . Let  $J_0 := \{j, 1 \leq j \leq \omega(x) \mid B(x) = \frac{\text{ord} \phi_j}{j}\}$ .

The following lemma is obvious from the definitions.

**I.8.2.2 Lemma.** We have

$$\sup_{j_0 \in J_0} \left\{ \frac{d_{j_0}}{j_0} \right\} \leq \beta_2(x) - A_2(x) \leq C(x) \leq \beta(x) - A_2(x).$$

**I.8.2.3 Definition.** Let  $\mu_0$  be the monomial valuation on  $\widehat{S}$  given by

$$\mu_0\left(\sum_{abc} \lambda_{abc} u_1^a u_2^b u_3^c\right) = \inf\left\{c + \frac{a+b}{B(x)} \mid \lambda_{abc} \neq 0\right\}.$$

We denote by  $\text{cl}_{\mu_0, \omega(x)} J$  the  $k(x)$ -vector space  $\text{in}_{\mu_0} J$ , where  $J$  is the ideal

$$J := \left( \left\{ H(x)^{-1} \lambda_j \frac{\partial f}{\partial \lambda_j} \right\}_{1 \leq j \leq s} \right) + (H(x)^{-1} g^p).$$

By definition of  $B(x)$  and well-preparedness, we have  $\dim_{k(x)}(\text{cl}_{\mu_0, \omega(x)} J) \geq 1$ , and for every  $\lambda \in k(x)$ ,

$$\text{cl}_{\mu_0, \omega(x)} J \neq k(x) \cdot (U_3 - \lambda U_1^{\alpha_2(x)} U_2^{\beta_2(x)})^{\omega(x)}.$$

Let  $e : X' \rightarrow X$  be the blowing up along  $x$ . We call “first chart” the chart with origin  $(X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = \frac{u_2}{u_1}, u'_3 = \frac{u_3}{u_1})$ . Let  $E' := (e^{-1}(x))_{\text{red}}$ . We have  $E' = \text{div}(u'_1)$  if  $x$  is in case (\*1), or  $E' = \text{div}(u'_1 u'_2)$  if  $x$  is in case (\*2).

**I.8.3 Lemma.** *With hypotheses and notations of I.8.2 (in particular  $x$  is in case (\*1) or (\*2)). Assume that the center  $x'$  of  $\mu$  in  $X'$  belongs to the first chart. Let  $d := [k(x') : k(x)]$ . We have  $\kappa(x) \leq 1$  or the following holds:*

- (a)  $\kappa(x') = 2$  and  $x'$  maps to the strict transform of  $\text{div}(u_3)$  except possibly if  $(m(x) = 2$  and  $B(x) = C(x) = 1)$ ;
- (b) if  $x' = (X', u'_1, u'_2, u'_3)$  is the origin of the chart, then  $x'$  is in case (\*1) (resp. (\*2)) if  $x$  is in case (\*1) (resp. (\*2)) and we have  $C(x') \leq C(x)$ ,  $\beta(x') \leq \beta(x)$ ,  $A_1(x') = B(x) - 1$ ;
- (c) if  $x' \neq (X', u'_1, u'_2, u'_3)$ , then  $x'$  is in case (\*1) or (\*3). Moreover,  $x'$  is in case (\*1) if  $a(1) + a(2) \not\equiv 0 \pmod{p}$  or if ( $x$  is in case (\*1) and  $x'$  is separable over  $x$ );
- (d) we have  $\gamma(x') \leq \gamma(x)$ ;
- (e) assume that  $x' \neq (X', u'_1, u'_2, u'_3)$ .

If  $(m(x) = 2$  and  $B(x) = C(x) = 1)$ , then  $\beta(x') \leq 2$ . Equality holds only if  $p = \omega(x) = 2$ , and there exist well prepared variables  $(Z', u'_1, u'_3, v')$  at  $x'$  such that  $h' = Z'^2 - g'Z' + f'_{Z'}$ , where

$$f'_{Z'} = u'_1{}^{a(1)'} (\mu'_2 v'^2 + \mu'_1 v' u'_3{}^2 + u'_1 \varphi'), \quad (\text{Dis})$$

with  $a(1)' \equiv 0 \pmod{2}$  and  $\mu'_1 \mu'_2 \in k(x')[[u'_1, v', u'_3]]$  invertible.

If we do not have  $(m(x) = 2$  and  $B(x) = C(x) = 1)$ , then

$$\beta(x') < 1 + \left\lfloor \frac{C(x)}{d} \right\rfloor, \quad (1)$$

$$\beta(x') \leq \sup\{\beta(x), \frac{1}{p}\} \text{ if } x \text{ is in case } (*1), \quad (2)$$

and all properties below are satisfied:

- (i) we have  $B(x) < \frac{\text{ord}_{(u_1, u_2)}(H(x)^{-1} g^p)}{\omega(x)}$ ;
- (ii) we have  $A_1(x') = B(x) - 1$ ;
- (iii) if  $J_0 \not\subset p\mathbf{N}$ , then

$$\beta(x') \leq \sup_{j_0 \in J_0} \left\{ \frac{d_{j_0}}{j_0 d} \right\} \leq \frac{C(x)}{d} \leq \frac{\beta(x)}{d}.$$

If moreover  $x'$  is in case (\*3), then

$$\beta(x') \leq \sup_{j_0 \in J_0} \left\{ \frac{d_{j_0}}{j_0 d} - \frac{1}{j_0} \right\} < \frac{C(x)}{d} \leq \frac{\beta(x)}{d}.$$

(iv) if  $(J_0 \subset p\mathbf{N}$  and  $B(x) \notin \mathbf{N})$ , then

$$\beta(x') \leq \sup_{j_0 \in J_0} \left\{ \frac{d_{j_0}}{j_0 d} + \frac{1}{j_0} \right\} \leq \frac{C(x)}{d} + \frac{1}{p}.$$

If moreover  $x'$  is in case (\*3), then

$$\beta\mathfrak{B}(x') \leq \sup_{j_0 \in J_0} \left\{ \frac{d_{j_0}}{j_0 d} \right\} \leq \frac{C(x)}{d}.$$

(v) if  $(J_0 \subset p\mathbf{N}, B(x) \in \mathbb{N} \text{ and } \dim_{k(x)}(\text{cl}_{\mu_0, \omega(x)} J) \geq 2)$ , then

$$\beta(x') \leq \sup_{j_0 \in J_0} \left\{ \frac{d_{j_0}}{j_0 d} + \frac{1}{j_0} \right\} \leq \frac{C(x)}{d} + \frac{1}{p}.$$

If moreover  $x'$  is in case (\*3), then

$$\beta\mathfrak{B}(x') \leq \sup_{j_0 \in J_0} \left\{ \frac{d_0}{j_0 d} \right\} \leq \frac{C(x)}{d}.$$

(vi) if  $(J_0 \subset p\mathbf{N}, B(x) \in \mathbb{N} \text{ and } \dim_{k(x)}(\text{cl}_{\mu_0, \omega(x)} J) = 1)$ , let

$$G := U_3^{\omega(x)} + \sum_{j_0 \in J_0} U_3^{\omega(x)-j} G_j(U_1, U_2)$$

be such that  $\text{cl}_{\mu_0, \omega(x)} J = k(x).G$  and define

$$C(G) := B(x) - \inf_{j_0 \in J_0} \left\{ \frac{\text{ord}_{U_1} G^{j_0}}{j_0} \right\} - \inf_{j_0 \in J_0} \left\{ \frac{\text{ord}_{U_2} G^{j_0}}{j_0} \right\} \leq C(x).$$

If  $x'$  is in case (\*1), we have

$$\beta(x') \leq \frac{C(G)}{d} + \frac{1}{p} \text{ and } \beta(x') < 1 + \left\lfloor \frac{C(G)}{d} \right\rfloor.$$

If  $x'$  is in case (\*3), then

$$\beta\mathfrak{B}(x') \leq \frac{C(G)}{d} + \frac{1}{p} \text{ and } \beta\mathfrak{B}(x') < 1 + \left\lfloor \frac{C(G)}{d} \right\rfloor.$$

(vii) if  $(m(x) = 2, a(1) \equiv a(2) \pmod{p}, C(x) = 1 \text{ and } x' \text{ is in case } (*3))$ , then:

if  $p \geq 3$ , we have  $\beta\mathfrak{B}(x') < 1$  and  $x'$  is not rational over  $x$ ;

if  $p = 2$ , we have  $\beta\mathfrak{B}(x') \leq \frac{3}{2}$ , and if equality holds, then  $a(1) \not\equiv 0 \pmod{2}$ ,  $x'$  is rational over  $x$  and the monomial

$$u_3'^{\omega(x)-j_1} u_1'^{j_1 A_1(x')} v'^{\frac{3}{2}j_1+1}$$

appears with nonzero coefficient in the expansion of  $H(x')^{-1} v' \frac{\partial f_{Z'}}{\partial v'}$ , where  $1 \leq j_1 \leq \omega(x)$ ,  $j_1 \equiv 0 \pmod{4}$ , and  $(Z', u_1', v', u_3')$  is a suitable well prepared r.s.p. at  $x'$ ;

(viii) if  $x'$  is not rational over  $x$ , then:

if  $\gamma(x) \geq 3$ , we have  $\gamma(x') < \gamma(x)$ ;

if  $\gamma(x) = \gamma(x') = 2$ , then  $(x \text{ is in case } (*1), J_0 \subset p\mathbf{N} \text{ and } \beta(x) = C(x) = 2)$ . If  $x'$  is in case (\*1), then  $\beta(x') < 2$ . If  $x'$  is in case (\*3), then  $\beta\mathfrak{B}(x') \leq 1$  or  $(p = 2 \text{ and } \beta\mathfrak{B}(x') \leq \frac{3}{2})$ , in which case equality holds only if the monomial

$$u_3'^{\omega(x)-j_1} u_1'^{j_1 A_1(x')} v'^{\frac{3}{2}j_1+1}$$

appears with nonzero coefficient in the expansion of  $H(x')^{-1}v' \frac{\partial f_{Z'}}{\partial v'}$ , where  $1 \leq j_1 \leq \omega(x)$ ,  $j_1 \equiv 0 \pmod{4}$ , and  $(Z', u'_1, v', u'_3)$  is a suitable well prepared r.s.p. at  $x'$ ; (ix) if  $C(x) < 1$  or if  $(\beta(x) < 2$  and  $x'$  is not rational over  $x$ ), then  $\beta(x') < 1$ .

Proof. We assume all along the proof that  $x'$  is very near  $x$ . The last statement in (a) is a direct consequence of **I.7**(i). If  $(m(x) = 2$  and  $B(x) = C(x) = 1)$ , we apply **I.7**(i) and replace  $(X, u_1, u_2, u_3)$  with  $(Z, u_1, u_2, v)$ . All computations will henceforth be performed with the variables  $(Z, u_1, u_2, v)$ . Note that  $\gamma(u_1, u_2, v) = \gamma(u_1, u_2, u_3) = 2$  and that  $\beta(u_1, u_2, v) > 1$ .

In general, we denote  $h' = u_1^{-p}h = X'^p - X'g'^{p-1} + f'$  the strict transform of  $h$ , with

$$f' = u_1'^{a(1)+a(2)+\omega(x)-p} u_2'^{a(2)} \left( u_3'^{\omega(x)} \phi_0 + \sum_{1 \leq j \leq \omega(x)} u_3'^{\omega(x)-j} u_1'^{a_j(1)+a_j(2)+d_j-j} u_2'^{a_j(2)} \psi_j' \right), \quad (1)$$

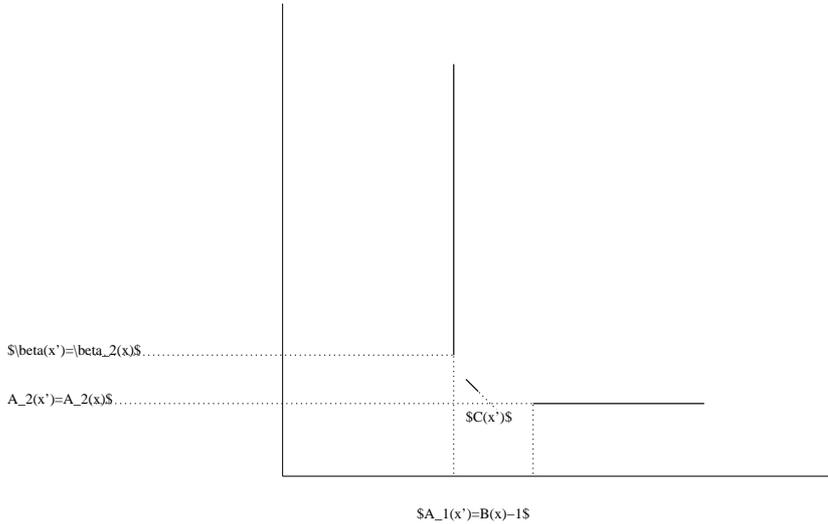
where  $\psi_j'$  is the strict transform of  $\psi_j$ ,  $g'^p = (u_1'^{-1}g)^p = \gamma^p u_1'^{b(1)+b(2)-p} u_2'^{b(2)}$ .

Assume that  $x' = (X', u'_1, u'_2, u'_3)$  is the origin of the chart. Then  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal, from which follows that  $\kappa(x') = 2$ ,  $x'$  is in case (\*1) (resp. (\*2)) if  $x$  is in case (\*1) (resp. (\*2)) and  $(X', u'_1, u'_2, u'_3)$  is well prepared. Looking at the figure below after blowing up and comparing with that in **I.5.2.1**, the reader sees that

$$C(x') \leq \beta_2(x) - A_2(x) \leq B(x) - \alpha_2(x) - A_2(x) \leq C(x),$$

and that  $\beta(x') \leq \beta_2(x) \leq \beta(x)$ ,  $A_1(x') = B(x) - 1$ . This proves (b) and the corresponding part of (a) and (d).

After blowing up, origin of the first chart



We assume from now on that  $x' \neq (X', u'_1, u'_2, u'_3)$  and choose  $v' := P(1, u'_2)$ , where  $P \in k(x)[u_1, u_2]$  is irreducible, unitary and of degree  $d$  in  $u_2$  such that  $(X', u'_1, v', u'_3)$  is a r.s.p. at  $x'$ . We have  $E' = \text{div}(u'_1)$  and denote  $S' := (S[u'_2, u'_3])_{\eta'(x')}$ . We may take  $H(x') = u_1'^{a(1)+a(2)+\omega(x)-p}$  by **II.5.4.1** of chapter 1. The point is that, in general,  $(X', u'_1, v', u'_3)$  is not well prepared.

Let  $M'$  be the monomial  $M' := u_1'^{a(1)+a(2)+\omega(x)-p} u_2'^{a(2)} u_3'^{\omega(x)} \overline{\phi_0}$  in expression (1), where  $\overline{\phi_0}$  denotes the image of  $\phi_0$  in  $k(x')$ . When making a translation on  $X'$ ,  $Z' := X' - u_1'^a \theta'$ ,  $\theta' \in \widehat{S}'$ ,  $a \geq \frac{a(1)+a(2)+\omega(x)-p}{p}$  in order to get  $\Delta(h'; u_1', v', u_3'; Z')$  minimal, we consider three cases:

*Case 1:*  $a(1) + a(2) \not\equiv 0 \pmod{p}$ . The monomial  $M'$  is preserved by this translation.

*Case 2:*  $a(1) \equiv a(2) \equiv 0 \pmod{p}$  and  $x'$  separable over  $x$ . Then  $\overline{\phi_0}$  is not a  $p^{\text{th}}$ -power in  $k(x)$ , therefore neither in  $k(x')$  (recall that  $\omega(x) \equiv 0 \pmod{p}$ ). As  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \epsilon(x)$ , the monomial  $M'$  is preserved by this translation.

In cases 1 and 2, we get  $\epsilon(x') \leq \omega(x) = \epsilon(x)$ . Since  $x'$  is very near  $x$ , we have  $\kappa(x') = 2$  and condition (\*1) holds at  $x'$ .

*Case 3:* all other cases. In particular, we have  $a(1) + a(2) \equiv 0 \pmod{p}$ . Then

$$f'_{Z'} := f' + (u_1'^a \theta')^p - u_1'^a \theta' g'^{p-1}$$

satisfies

$$H(x')^{-1} f'_{Z'} \equiv (u_2'^{a(2)} \phi_0 + \psi'^p) u_3'^{\omega(x)} + \sum_{1 \leq j \leq \frac{\omega(x)}{p}} u_3'^{\omega(x)-pj} (u_2'^{a(2)} \Phi'_{pj}(u_2') + \psi'^p_{pj}(v')^p) \pmod{u_1'}, \quad (2)$$

where  $\deg_{u_2'} \Phi'_{pj} \leq pj$ , and  $\text{ord}_{v'} \psi'^p_{pj}(v')^p \geq \text{ord}_{v'} \Phi'_{pj}(u_2')$ ,  $1 \leq j \leq \frac{\omega(x)}{p}$ .

Let  $\lambda' := u_2'^{a(2)} \phi_0 + \psi'^p$ . There exists  $D' \in \mathcal{D}' := \{D \in \text{Der}_{S'/k_0} \mid D(I(E')) \subseteq I(E')\}$  such that  $D'\lambda'$  is a unit, since  $J(f', E')$  is the weak transform of  $J(f, E, x)$ . Hence either ( $\lambda'$  is a unit and  $D' \in \mathcal{D}'(x')$ ), or ( $D' = \frac{\partial}{\partial v'}$  and  $D'\lambda'$  is a unit).

*Case 3a:* in the former case, we have  $\omega(x') = \omega(x)$ ,  $\kappa(x') = 2$  and  $x'$  verifies (\*1).

*Case 3b:* in the latter one, we claim that  $\kappa(x') \leq 1$  or ( $\kappa(x') = 2$  and  $x'$  is in case (\*1) or (\*3)). To prove the claim, we consider two more subcases:

*Case 3ba:* assume that  $\Phi'_{pj} = 0$  for  $1 \leq j \leq \frac{\omega(x)}{p}$ . Then  $\kappa(x') = 1$  by chapter 2 **II.1** if  $\text{ord}_{\eta'(x')}(H(x')^{-1}f'_{Z'}) = \omega(x)$ . Assume now that  $\text{ord}_{\eta'(x')}(H(x')^{-1}f'_{Z'}) = 1 + \omega(x)$ , so

$$H(x')^{-1} \frac{\partial f'_{Z'}}{\partial v'} \equiv \frac{\partial \lambda'}{\partial v'} u_3'^{\omega(x)} \pmod{u_1'},$$

so  $\kappa(x') = 2$  and  $x'$  is in case (\*3).

*Case 3bb:* assume now that not all  $\Phi'_{pj}$ 's are zero. Then  $B(x) = 1$  and  $U_i \notin \text{VDir}(x)$  for  $i = 1, 2$  since  $x' \neq (X', u_1', u_2', u_3')$ , so  $C(x) = B(x) = 1$ . Since  $x'$  is very near  $x$ , we have  $\text{VDir}(x) = \langle U_3, bU_1 + U_2 \rangle$ ,  $b \neq 0$  and  $v' = b + u_2'$ . Moreover,  $m(x) = 2$  and  $a(2) \not\equiv 0 \pmod{p}$  since we are dealing with case 3. By chapter 2 **II.5(ii)** (with variables  $u_1$  and  $u_3$  exchanged), this implies that

$$u_2'^{a(2)} \Phi'_{pj}(u_2') = \mu_{pj}(v' - b)^{a(2)} v'^{pj} + \varphi'_{pj}(v')^p,$$

for some  $\mu_{pj} \in k(x)$  and  $\varphi'_{pj}(v') \in k(x')[[v']]$  for each  $j$ ,  $1 \leq j \leq \frac{\omega(x)}{p}$ .

If  $\mu_{pj} b^{a(2)} \notin k(x)^p$  for some  $j$ , then  $\epsilon(x') = \omega(x)$  and  $\kappa(x') = 2$ . By chapter 2 **II.2**, we have  $\kappa(x) = 0$  unless  $\mu_{\omega(x)} b^{a(2)} \notin k(x)^p$  and  $\mu_{pj} b^{a(2)} \in k(x)^p$  for  $pj \neq \omega(x)$ , in which case (2) leads to

$$H(x')^{-1} f'_{Z'} \equiv \mu'_1(v') v' u_3'^{\omega(x)} + \sum_{1 \leq j \leq \frac{\omega(x)}{p} - 1} \mu'_{pj}(v') u_3'^{\omega(x)-j} v'^{pj+1} + \mu'_{\omega(x)}(v') v'^{\omega(x)} \pmod{u_1'},$$

with  $\mu'_1(v')\mu'_{\omega(x)}(v')$  invertible and  $\mu'_{pj}(v')$  invertible or zero for  $1 \leq j \leq \frac{\omega(x)}{p} - 1$ . Hence  $\kappa(x') = 2$  and  $x'$  is in case (\*1). Since  $\mu'_1(0) \neq 0$  and  $\omega(x) \equiv 0 \pmod{p}$ , there exists a vertex

$$w := (0, w_2) \in \Delta(H(x')^{-1}(f'_{Z'}, g'^p); u'_1, u'_3; v')$$

which is not solvable, so  $A_1(x') = 0$ . We have  $\beta(x') = \frac{\omega(x)}{\omega(x)-1}$  (resp.  $\beta(x') = 1 + \frac{1}{pj_0}$ ) if  $\mu'_{pj}(v')$  is zero for each  $j$ ,  $1 \leq j \leq \frac{\omega(x)}{p} - 1$  (resp. if  $j_0 = \inf\{j, 1 \leq j \leq \frac{\omega(x)}{p} - 1 \mid \mu'_{pj}(v') \text{ invertible}\}$ ). So  $\beta(x') \leq \frac{\omega(x)}{\omega(x)-1} \leq 2$ , with  $\beta(x') = 2$  only if  $p = \omega(x) = 2$  and

$$H(x')^{-1}f'_{Z'} \equiv \mu'_1 v' u'_3{}^2 + \mu'_2 v'^2 \pmod{u'_1}.$$

This is the special case announced in the statement of (e), where “Dis” stands for “Disaster” (change of directrix).

If  $\mu_{pj}b^{a(2)} \in k(x)^p$  for each  $j$ ,  $1 \leq j \leq \frac{\omega(x)}{p}$ , note that

$$\text{cl}_{\mu_0, \omega(x)} J = \text{cl}_{\omega(x)} J(f, E, x) = \text{cl}_{\omega(x)} (H(x)^{-1} u_1 \frac{\partial f}{\partial u_1}) =: k(x).G,$$

the first equality since  $B(x) = 1$ , so  $\mu_0 = \text{ord}_{\eta(x)}$  (definition **I.8.2.3**). Therefore we get

$$J(f', E') \equiv (G(1, v' - b, u'_3)) \pmod{u'_1}.$$

Explicitly, we have (up to multiplication by a constant)

$$G(1, v' - b, u'_3) = \mu_1 u'_3{}^{\omega(x)} + \sum_{1 \leq j \leq \frac{\omega(x)}{p}} \mu_{pj} u'_3{}^{\omega(x)-pj} v'^{pj}, \quad (3)$$

where  $\mu_1 = \overline{\phi_0}$ . Now, (2) leads to

$$H(x')^{-1}f'_{Z'} = \mu'_1(v')v'u'_3{}^{\omega(x)} + \sum_{1 \leq j \leq \frac{\omega(x)}{p}} \mu'_{pj}(v')u'_3{}^{\omega(x)-pj} v'^{pj+1} + u'_1 \varphi',$$

with  $\mu'_1(v')$  invertible,  $\mu'_{pj}(v')$  invertible or zero if  $1 \leq j \leq \frac{\omega(x)}{p}$  and  $\varphi' \in \widehat{S}'$ , so we get by identification with (3)

$$(H(x')^{-1}f'_{Z'}) \equiv (v'G(1, v' - b, u'_3)) \pmod{u'_1, (v', u'_3)^{\omega(x)+2}}. \quad (4)$$

Now note that, since  $(X, u_1, u_2, u_3)$  is well prepared,  $G$  is not a scalar multiple of an  $\omega(x)^{th}$ -power.

If  $\text{ord}_{\eta'(x')}(u'_1 \varphi') \geq 1 + \omega(x)$ , then  $\kappa(x') = 2$  and  $x'$  is in case (\*3). By (3) and (4),  $(0, 1)$  is an unsolvable vertex of  $\Delta(H(x')^{-1}v'^{-1}(f'_{Z'}, g'^p); u'_1, v'; u'_3)$ , since  $G$  is not an  $\omega(x)^{th}$ -power. We get  $A_1(x') = 0$  and  $\beta_3(x') = 1$ .

If  $\text{ord}_{\eta'(x')}(u'_1 \varphi') = \omega(x)$ , it is easily seen along the following lines that  $\kappa(x') = 0$ : we have  $U'_1 \in \text{VDir}(x')$ ; by (3) and (4), every  $x''$  near  $x'$  has

$$\epsilon(x'') \leq 1 + \omega(x) - 1 = \omega(x),$$

since  $G$  is not a scalar multiple of an  $\omega(x)^{th}$ -power; equality holds implies  $\omega(x'') = \epsilon(x) - 1 < \omega(x)$ , so  $x''$  is not very near  $x'$ .

This concludes the proof of the claim, hence of (c) and of the remaining part of (a).

We now prove **I.8.3** in the special case  $B(x) = 1$ .

Assume first that  $(m(x) = 2 \text{ and } B(x) = C(x) = 1)$ . In the above case 1, we have  $\kappa(x') = 0$  by chapter 2, **II.2**. In case 3, we are already done by the above computations except in case 3a. We argue along the lines and notations of the proof in case 3bb above: if  $\mu_{pj}b^{a(2)} \notin k(x)^p$  for some  $j$ ,  $1 \leq j \leq \frac{\omega(x)}{p}$ , then  $A_1(x') = 0$  and  $\text{VDir}(x') \equiv \langle U_3', V' \rangle \pmod{U_1'}$  so  $\kappa(x') = 0$  by chapter 2, **II.2**; if  $\mu_{pj}b^{a(2)} \in k(x)^p$  for each  $j$ ,  $1 \leq j \leq \frac{\omega(x)}{p}$ , then  $A_1(x') = 0$  and

$$\beta(x') = 1 + \inf\left\{\frac{1}{pj} \mid \mu_{pj} \neq 0\right\} < 2.$$

Assume now that  $B(x) = 1$  and either  $C(x) < 1$  or  $m(x) = 1$ . Since  $x' \neq (X', u_1', u_2', u_3')$  and we are in the first chart, it can be assumed as well that  $\langle U_3, bU_1 + U_2 \rangle \subseteq \text{VDir}(x)$ ,  $b \neq 0$ . If  $m(x) = 1$ , we have  $\kappa(x) = 0$  by chapter 2, **II.2**. If  $m(x) = 2$ , then  $U_i \in \text{VDir}(x)$  for some  $i \in \{1, 2\}$  because  $C(x) = 1$  (see argument at the end of **I.7.1**), so  $\tau(x) = 3$  and  $\kappa(x) = 0$ .

All statements have been proved for  $B(x) = 1$ , so we assume that  $B(x) > 1$  from now on.

Before proving (d), (e) and statements (i)-(ix), recall that  $a_j(1) + a_j(2) + d_j - j \geq j(B(x) - 1)$  in (1), with equality if and only if  $j \in J_0$  (definition **I.8.2.1**). As the well preparation will replace  $\Delta(H(x')^{-1}(f', g'^p); u_1', v'; u_3')$  by a smaller polyhedron, we get

$$A_1(x') \geq B(x) - 1.$$

**I.8.3.1** Proof of (i). If  $b(1) + b(2) - (a(1) + a(2)) = \omega(x)B(x)$ , then  $A_1(x') = B(x) - 1$  and  $(A_1(x'), 0)$  is a vertex of  $\Delta(H(x')^{-1}(f', g'^p); u_1', v'; u_3')$  which is not solvable by definition, so  $\beta(x') = 0$ . We have  $\kappa(x') \leq 1$  by **I.8.1**.

**I.8.3.2** Preliminary remarks and proof of (ii). Let  $\mu_1$  be the monomial valuation on  $\widehat{S}'$  defined by

$$\mu_1\left(\sum_{abc} \lambda_{abc} u_1'^a v'^b u_3'^c\right) = \min\left\{c + \frac{a}{B(x) - 1} \mid \lambda_{abc} \neq 0\right\}.$$

Note that  $\mu_1(H(x')^{-1}f') = \omega(x)$ . In the well preparation algorithm at  $x'$ , we replace  $u_3'$  by

$$w' := u_3' - u_1'^a s', \quad s' \in k(x')[[u_1', v']], \quad a \geq B(x) - 1, \quad (1)$$

and  $X'$  by

$$Z' := X' - u_1'^{a'} \theta', \quad \theta' \in \widehat{S}', \quad a' \geq \frac{a(1) + a(2) + \omega(x) - p}{p}. \quad (2)$$

The Newton polyhedron of  $u_1'^{a'} \theta'$  is a subset of  $\Delta(h'; u_1', u_2', u_3'; X')$ , so

$$\mu_1(u_1'^{a'} \theta') \geq \frac{\mu_1(H(x')) + \omega(x)}{p}.$$

Let

$$F' := u_2'^{a(2)} \left( \frac{\phi_0}{\phi_0} U_3'^{\omega(x)} + \sum_{j_0 \in J_0} U_3'^{\omega(x) - j_0} U_1'^{j_0(B(x) - 1)} u_2'^{a_j(2)} \text{cl}_0 \psi_j' \right) \quad (3)$$

be the initial form of  $H(x')^{-1}f'$  in the graded algebra  $G_{\mu_1}(\widehat{S}') = k(x')[[v']][U'_1, U'_3]$  of  $\widehat{S}'$  w.r.t.  $\mu_1$ , where  $\text{cl}_0$  denotes residue w.r.t.  $\mu_1$ , i.e. the image in  $\widehat{S}'/(u'_1, u'_3) \simeq k(x')[[v']]$ . Note that  $F'$  is the weak transform of

$$U_1^{a(1)}U_2^{a(2)}\text{in}_{\mu_0} \left( \sum_{0 \leq j \leq \omega(x)} u_3^{\omega(x)-j} \phi_j \right) =: U_1^{a(1)}U_2^{a(2)}F$$

(see definition **I.8.2.3** for the definition of  $\mu_0$ ).

Substitution (1) changes  $F'$  by an automorphism  $U'_3 \mapsto W' + \lambda(v')U_1^{B(x)-1}$ ,  $\lambda \in k(x')[[v']]$  (with  $\lambda = 0$  if  $B(x) \notin \mathbb{N}$ ), while substitution (2) only changes it by some  $H(x')^{-1}\Theta^p$ ,  $\Theta \in G_{\mu_1}(\widehat{S}')$  by (i).

Therefore we have  $\mu_1(J') = \omega(x)$  and

$$\text{in}_{\mu_1} J' = \langle (\text{in}_{\mu_1} H(x'))^{-1} \cdot (\text{in}_{\mu_0} H(x)) \cdot \text{cl}_{\mu_0, \omega(x)} J \rangle, \quad (4)$$

where  $J' := J(U_1^{a(1)+a(2)+\omega(x)-p}F', E')$ . By well-preparedness, the right hand side is not generated by an  $\omega(x)^{th}$ -power, so the left hand side is not generated by  $W'^{\omega(x)}$ : letting  $f'_{Z'} := f' + (u_1^{a'}\theta')^p - u_1^{a'}\theta'g'^{p-1}$ , there exists a vertex  $\mathbf{v}' = (B(x) - 1, v'_2)$  in  $\Delta(H(x')^{-1}(f'_{Z'}, g'^p); u'_1, v'; w')$ . This proves that  $A_1(x') = B(x) - 1$ , so (ii) holds.

We now proceed to prove (d) and (e) in each of cases (iii)-(vi). Note that (d) is a trivial consequence of equations (1) and (2) in (e) if  $x'$  is in case (\*1) and of (iii)-(vi) if ( $x$  is in case (\*2) and  $x'$  is in case (\*3)). So (d) only needs to be proved when ( $x$  is in case (\*1) and  $x'$  is in case (\*3)). By (c), we may then assume furthermore that  $x'$  is inseparable over  $x$ .

**I.8.3.3** Proof in case (iii). Let  $j_1 := \inf\{j_0 \in J_0 | j_0 \not\equiv 0 \pmod{p}\}$ . The comments below equation (3) in **I.8.3.2** show that the monomial

$$H(x')u_2^{a(2)}u_1^{a_{j_1}(1)+a_{j_1}(2)+d_{j_1}-j_1}u_2^{a_{j_1}(2)}u_3^{\omega(x)-j_1}\Psi_{j_1}(1, u'_2)$$

in  $f'$  is preserved by the well preparation algorithm at  $x'$ , so  $\beta(x') \leq \frac{d_{j_1}}{j_1 d}$  and  $\beta_3(x') \leq \frac{d_{j_1}}{j_1 d} - \frac{1}{j_1}$  if  $x'$  is in case (\*3). The conclusion of (iii) follows from this fact and lemma **I.8.2.2**. The corresponding parts of (d) and (e) are trivial consequences of (iii) (note the trivial fact

$$\forall d \geq 2, \forall y > 0, 1 + \left\lfloor \frac{y}{d} \right\rfloor \leq [y], \quad (1)$$

where equality holds only if  $0 < y \leq 1$  or if  $d = y = 2$ ).

**I.8.3.4** Proof in case (iv). Equation (1) and subsequent comments in **I.8.3.2** imply that the monomial

$$H(x')u_2^{a(2)}u_1^{a_{j_0}(1)+a_{j_0}(2)+d_{j_0}-j_0}u_2^{a_{j_0}(2)}u_3^{\omega(x)-j_0}\Psi_{j_0}(1, u'_2)$$

in  $f'$  is preserved by the translation  $w' = u'_3 - u_1^a s'$  for each  $j_0 \in J_0$ , since  $B(x) \notin \mathbb{N}$ . Let

$$j_1 := \inf\{j_0 \in J_0 | U_1^{a(1)+a_{j_0}(1)}U_2^{a(2)+a_{j_0}(2)}\Psi_{j_0}(U_1, U_2) \notin (k(x)[U_1, U_2])^p\}.$$

Since  $\Delta(h; u_1, u_2, u_3; X)$  is minimal,  $j_1$  exists. By equation (2) and subsequent comments in **I.8.3.2**, the translation  $Z' = X' - u_1^{a'}\theta'$  plugs into  $f'_{Z'}$  a term of the form

$$H(x')u_3^{\omega(x)-j_1}(u_2^{a(2)+a_{j_1}(2)}\Psi_{j_1}(1, u'_2) + A_{j_1}(v')^p),$$

with  $A_{j_1}(v') \in k(x')[[v']]$ . We apply **II.5.3.2**(i) of chapter 1 to the above form and get:

$$\text{ord}_{v'}(u_2'^{a(2)+a_{j_1}(2)})\Psi_{j_1}(1, u_2') + A_{j_1}(v')^p \leq \frac{d_{j_1}}{d} + 1. \quad (1)$$

This implies  $\beta(x') \leq \frac{d_{j_1}}{j_1 d} + \frac{1}{j_1}$ . If  $x'$  is in case (\*3),  $\beta_3(x') \leq \frac{d_{j_1}}{j_1 d}$ . From this and **I.8.2.2**, (iv) is established.

We now prove (e). By **II.5.3.2**(ii) of chapter 1, we have

$$\text{ord}_{v'}(u_2'^{a(2)+a_{j_1}(2)})\Psi_{j_1}(1, u_2') + A_{j_1}(v')^p < p(1 + \left\lfloor \frac{d_{j_1}}{pd} \right\rfloor).$$

Therefore the right hand side in (1) above is not in  $p\mathbb{N}$  if equality holds in (1). Since  $j_1 \equiv 0 \pmod{p}$ , we have

$$\beta(x') \leq \frac{\text{ord}_{v'}(u_2'^{a(2)+a_{j_1}(2)})\Psi_{j_1}(1, u_2') + A_{j_1}(v')^p}{j_1} < 1 + \left\lfloor \frac{d_{j_1}}{j_1 d} \right\rfloor.$$

If  $E = \text{div}(u_1)$ , then  $a(2) = a_{j_1}(2) = 0$  by definition, so **II.5.3.2**(iii) of chapter 1 now yields

$$j_1\beta(x') \leq \text{ord}_{v'}(\Psi_{j_1}(1, u_2') + A_{j_1}(v')^p) \leq \deg\Psi_{j_1} \leq j_1\beta(x) \quad (2)$$

provided  $\deg\Psi_{j_1} \geq 1$ . If  $\deg\Psi_{j_1} = 0$ , then  $j_1\beta(x') \leq 1$ . This concludes the proof of (e).

To prove (d), we may assume (last paragraph before **I.8.3.3**) that ( $x$  is in case (\*1),  $x'$  is in case (\*3) and  $d \geq 2$ ). So  $\gamma(x') \leq 1 + \lfloor \frac{\beta(x)}{d} \rfloor \leq \gamma(x)$  by lemma **I.8.2.2** and **I.8.3.3**(1).

**I.8.3.5** Proof of (v). Since  $\dim_{k(x)}(\text{cl}_{\mu_0, \omega(x)}J) \geq 2$ , there exists  $0 \neq G \in \text{cl}_{\mu_0, \omega(x)}J$  of the form

$$G = \sum_{j_0 \in J_0} U_3^{\omega(x)-j_0} U_1^{a_{j_0}(1)} U_2^{a_{j_0}(2)} G_{j_0}(U_1, U_2),$$

with  $G_{j_0}$  homogeneous of degree  $d_{j_0}$ . Let

$$G' := U_1^{-\omega(x)}G = \sum_{j_0 \in J_0} U_3^{\omega(x)-j_0} U_1^{j_0(B(x)-1)} u_2'^{a_{j_0}(2)} G_{j_0}(1, u_2').$$

By **I.8.3.2**(4), we have  $G' \in \text{in}_{\mu_1}J'$ .

Let  $j_1 := \inf\{j_0 \in J_0 \mid G_{j_0} \neq 0\}$ . Then  $U_3^{\omega(x)-j_1} U_1^{j_1(B(x)-1)} u_2'^{a_{j_1}(2)} G_{j_1}(1, u_2')$  is preserved by any translation on  $u_3'$  or on  $X'$  in the well preparation algorithm at  $x'$ . In general, we can only insure that

$$v'G' \in \text{in}_{\mu_1}J(f'_{Z'}, E', x'),$$

so

$$j_1\beta(x') \leq \frac{d_{j_1}}{d} + 1 \quad (1)$$

and we get  $\beta(x') \leq \frac{d_{j_1}}{d j_1} + \frac{1}{j_1}$ ,  $\beta_3(x') \leq \frac{d_{j_1}}{j_1 d}$  if  $x'$  is in case (\*3), so (v) is established.

Now note that equality holds in (1) only if the monomial  $u_3'^{\omega(x)-j_1} v'^{j_1\beta(x')-1}$  appears with nonzero coefficient in the expansion of  $H(x')^{-1} \frac{\partial f'_{Z'}}{\partial v'}$ . Since  $j_1 \in p\mathbb{N}$ , this implies that  $\beta(x') \notin \mathbb{N}$ . The first statement in (e) follows easily from this remark. For the proof of the second part of (e)

and of (d),  $x$  is now in case (\*1). We get the same upper bound as in **I.8.3.4**(2) from which the conclusion follows.

**I.8.3.6** Proof of (vi). Write

$$G = U_3^{\omega(x)} + \sum_{j_0 \in J_0} U_3^{\omega(x)-j_0} U_1^{a_{j_0}(1)} U_2^{a_{j_0}(2)} G'_{j_0}(U_1, U_2),$$

with  $G'_{j_0} \in k(x)[U_1, U_2]_{d_{j_0}}$ . Let  $j_1 := \inf\{j_0 \in J_0 | G'_{j_0} \neq 0\}$ . We denote  $\omega(x) = p^\alpha l$  where  $l$  is prime to  $p$ .

If  $j_1 = p^\alpha$  and  $U_1^{a_{p^\alpha}(1)} U_2^{a_{p^\alpha}(2)} G'_{p^\alpha} \in (k(x)[U_1, U_2])^{p^\alpha}$ , say  $U_1^{a_{p^\alpha}(1)} U_2^{a_{p^\alpha}(2)} G'_{p^\alpha} =: K(U_1, U_2)^{p^\alpha}$ , we replace  $u_3$  by  $w := u_3 + l^{-\frac{1}{p^\alpha}} K(u_1, u_2)$ , so

$$G = W^{\omega(x)} + \sum_{j_0 \in J_0} W^{\omega(x)-j_0} U_1^{a'_{j_0}(1)} U_2^{a'_{j_0}(2)} G''_{j_0}(U_1, U_2).$$

Since  $\Delta(H(x)^{-1}f; u_1, u_2; u_3)$  has no solvable vertex by well-preparedness, its initial side is unchanged by the above translation. Therefore not all  $G''_{j_0}$ 's are zero and we have

$$\deg G''_{j_0} \leq j_0 C(G). \quad (1)$$

Note that we do not mean that  $h$  is well prepared for  $(X, u_1, u_2, w)$ , only that the derivative  $G$  is unchanged by further translations  $Z := X - \theta$  in order to get  $\Delta(f; u_1, u_2, w; Z)$  minimal. We assume from now on that the above preparation has been performed and denote  $j'_1 := \inf\{j_0 | G''_{j_0} \neq 0\} > p^\alpha$ .

If no preparation has been performed, we let  $w = u_3$ ,  $j'_1 = j_1 \geq p$  in what follows.

Let  $w' := \frac{w}{u_2}$ . By **I.8.3.2**(4), we have

$$k(x').G' = \text{in}_{\mu_1} J' = \text{in}_{\mu_1} (H(x')^{-1} (u'_1 \frac{\partial f'}{\partial u'_1}, \frac{\partial f'}{\partial v'}, w' \frac{\partial f'}{\partial w'}, \lambda_i \frac{\partial f'}{\partial \lambda_i})), \quad (2)$$

where  $G' = U_1^{-\omega(x)} \text{in}_{\mu_0} G(U_1, U_2, W')$ . By (2) above, either  $G' \in \text{in}_{\mu_1} J(f', E', x')$  or  $\langle G' \rangle = \text{in}_{\mu_1} (H(x')^{-1} \frac{\partial f'}{\partial v'})$ . Note that  $G'$  is unchanged by any translation on  $X'$  in the well-preparation algorithm at  $x'$ , since  $G'$  is (the initial form of) some derivative of  $f'$ .

Assume that  $x'$  is in case (\*1). Then  $G' \in \text{in}_{\mu_1} J(f', E', x')$  and we consider two cases:

*Case 1:*  $j'_1 \neq p^\alpha$ . The vertex of first coordinate  $A_1(x')$  of  $\Delta(H(x')^{-1}f'; u'_1, v'; w')$  is not solvable: any translation on  $w'$  in the well preparation algorithm is of the form  $w' \mapsto w' - u'_1{}^a s'$  with  $a > A_1(x')$ , so preserves the monomial  $W'^{\omega(x)-j'_1} U_1^{j'_1(B(x)-1)} u'_2{}^{a'_{j'_1}(2)} G''_{j'_1}(1, u'_2)$  in  $\text{in}_{\mu_1} (H(x')^{-1}f')$ . We get

$$j'_1 \beta(x') \leq \frac{d_{j'_1}}{d} \leq j'_1 \frac{C(G)}{d},$$

from which (vi) follows.

*Case 2:*  $j'_1 = p^\alpha$ . The translation on  $w'$  in the well-preparation algorithm produces some term of the form

$$W'^{p^\alpha(l-1)} U_1^{p^\alpha(B(x)-1)} (u'_2{}^{a_{p^\alpha}(2)} G'_{p^\alpha}(1, u'_2) + \varphi'(v')^{p^\alpha})$$

in  $H(x')^{-1}f'_{Z'}$ . Let  $\alpha' < \alpha$  be the largest integer such that  $U_1^{a_{p^\alpha}(1)} U_2^{a_{p^\alpha}(2)} G'_{p^\alpha} \in (k(x)[U_1, U_2])^{p^{\alpha'}}$ .

By **II.5.3.2**(i) of chapter 1 applied to the form  $(U_1^{a_{p^\alpha}(1)} U_2^{a_{p^\alpha}(2)} G'_{p^\alpha})^{\frac{1}{p^{\alpha'}}}$ , we get

$$p^\alpha \beta(x') \leq \text{ord}_{v'}(u'_2{}^{a_{p^\alpha}(2)} G'_{p^\alpha}(1, u'_2) + \varphi'(v')^{p^\alpha}) \leq \frac{p^\alpha C(G)}{d} + p^{\alpha'} \leq p^\alpha \left( \frac{C(G)}{d} + \frac{1}{p} \right).$$

Furthermore, by **II.5.3.2**(ii) of chapter 1,

$$\text{ord}_{v'}(u_2^{a_{p^\alpha}(2)} G'_{p^\alpha}(1, u_2) + \varphi'(v')^{p^\alpha}) < p^\alpha \left(1 + \left\lfloor \frac{C(x)}{d} \right\rfloor\right),$$

and this completes the proof of (vi) when  $x'$  is in case (\*1).

Assume that  $x'$  is in case (\*3). Then  $G' \in \text{in}_{\mu_1}(H(x')^{-1} \frac{\partial f'}{\partial v'})$ . The proof runs along the same lines as above, with  $\beta(x')$  replaced by  $\beta_3(x')$ . The worst upper bound we can get is

$$p^\alpha \beta_3(x') \leq \frac{d_{p^\alpha}}{d} + p^{\alpha'} \leq p^\alpha \left(C(G) + \frac{1}{p}\right).$$

The remaining statements are proved along the same lines as in **I.8.3.4** or **I.8.3.5**.

**I.8.3.7** Proof of (vii). We include this statement here to deal with some extra difficulty when  $p = 2$  (see **I.11.1** below). If  $x'$  is not rational over  $x$ , i.e.  $d \geq 2$ , the result follows from (iii)-(vi). From now on,  $x'$  is rational over  $x$ . We have

$$f' = u_1^{a(1)+a(2)+\omega(x)-p} u_2^{a(2)} (\phi_0 u_3^{\omega(x)} + u_1' \varphi'),$$

with  $\varphi' \in k(x)[[u_1, u_2, u_3]][u_2', u_3']$ . If  $2a(1) \not\equiv 0 \pmod{p}$  or if  $\bar{\phi}_0 \notin k(x)^p$  (for example if  $a(1) \equiv 0 \pmod{p}$ ), then  $x'$  is in case (\*1). So we have  $p = 2$ ,  $a(1) \not\equiv 0 \pmod{2}$  and  $\bar{\phi}_0 \in k(x)^2$ .

In cases (iii)-(v) above, we get  $\beta_3(x') \leq 1$ . In case (vi), we get  $\beta_3(x') \leq 1 + \frac{1}{p} = \frac{3}{2}$ . With notations as in the end of the proof of (vi),  $\beta_3(x') = \frac{3}{2}$  implies that the monomial

$$u_3^{\omega(x)-p^\alpha} u_1^{p^\alpha A_1(x')} v'^{p^\alpha \beta_3(x')}$$

appears with nonzero coefficient in the expansion of  $H(x')^{-1} \frac{\partial f'}{\partial v'}$ . Necessarily,  $j_1 \equiv 0 \pmod{4}$ .

**I.8.3.8** Proof of (viii). Assume that  $\gamma(x) \geq 2$  and  $d \geq 2$ . By (e) and (iii)-(vi),  $\gamma(x') < \gamma(x)$  except possibly if ( $d = 2$ ,  $\gamma(x) = 2$ ,  $\beta(x) = C(x) = 2$  and  $x$  is in case (\*1)). In this case, we only get  $\beta(x') < 2$  if  $x'$  is in case (\*1). If  $x'$  is in case (\*3), then we only get  $\beta_3(x') \leq 1 + \frac{1}{p} \leq \frac{3}{2}$ . Equality implies  $p = 2$ , the end of the proof of **I.8.3.6** giving the required statement.

**I.8.3.9** Proof of (ix). This follows from (e).

We now deal with when  $x$  is in case (\*3).

**I.8.4 Notations.** Assume that  $\kappa(x) = 2$ ,  $x$  is in case (\*3) and  $(X, u_1, u_2, u_3)$  is well prepared. We denote:

$$H(x)^{-1} u_2^{-1} f = \sum_{0 \leq j \leq \omega(x)} u_3^{\omega(x)-j} \phi_j,$$

with  $\psi_0 := u_2 \phi_0 \in (u_1, u_2, u_3)$ ,  $\psi_j := u_2 \phi_j \in k(x)[[u_1, u_2]]$  for  $1 \leq j \leq \omega(x)$  and  $\frac{\partial \psi_0}{\partial u_2}$  invertible. We let  $a_j(1) := \text{ord}_{u_1} \psi_j \geq j A_1(x)$  for  $1 \leq j \leq \omega(x)$ .

**I.8.5 Lemma.** With assumptions and notations of **I.8.4**, assume furthermore that  $B_3(x) \leq A_1(x)$ . Then  $\kappa(x) \leq 1$ .

Proof. We argue by induction on  $[A_1(x)]$ . If  $A_1(x) \leq 1$ , then we have  $A_1(x) = B_3(x) = 1$ , since  $\text{ord}_{\eta(x)}(u_2^{-1} H(x)^{-1} f) = \omega(x)$ . Hence for any value of  $A_1(x) \geq 1$ , **I.7**(ii) yields  $U_3 \in \text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f}{\partial u_2})$ .

The first step of the induction  $[A_1(x)] = 1$  is performed in (i) of the next lemma for  $1 < A_1(x) < 2$  (which yields  $\kappa(x) = 1$ , since  $1 \leq B3(x') = B3(x) - 1$  if  $\kappa(x) > 1$ ) and in (ii) of the next lemma for  $A_1(x) = 1$ . The induction step is performed in (i) of the next lemma, and thus completes the proof.

**I.8.6 Lemma.** *With assumptions and notations of I.8.4, assume furthermore that  $A_1(x) \geq 1$ . The curve  $\mathcal{C} = V(X, u_1, u_3)$  is permissible of the second kind. Let  $e : X' \rightarrow X$  be the blowing up along  $\mathcal{C}$ . There is at most one point  $x' \in e^{-1}(x)$  very near  $x$ , with r.s.p.  $(\frac{X}{u_1}, u_1, u_2, \frac{u_3}{u_1})$ . Moreover, (i) if  $A_1(x) > 1$ , then  $\kappa(x) \leq 1$  or the following holds:  $\kappa(x') = 2$ ,  $x'$  is in case (\*3), and we have  $A_1(x') = A_1(x) - 1$ ,  $B3(x') = B3(x) - 1$  and  $\beta3(x') = \beta3(x)$ ; (ii) if  $A_1(x) = B3(x) = 1$  or if  $(A_1(x) = 1 \text{ and } \beta3(x) < 1 - \frac{1}{\omega(x)})$ , then  $\kappa(x) \leq 1$ .*

Proof. Since  $A_1(x) \geq 1$ , we have  $\epsilon(\mathcal{C}) = \omega(x) > 0$ . Since  $A_1(x) > 0$ ,  $u_1$  divides  $H(x)^{-1}g^p$  and  $J(f, E) \equiv (u_3^{\omega(x)}) \pmod{u_1}$  so  $\mathcal{C} = \{y \in \eta^{-1}(E) \mid \omega(y) > 0\}$ . By II.4.7 of chapter 1,  $\mathcal{C}$  is a curve on  $X$ . Therefore  $\mathcal{C}$  is permissible of the second kind by chapter 1, II.5.1(ii).

By II.5.4.4 of chapter 1, we have  $\kappa(x) \leq 1$  unless  $\Psi := \text{cl}_{\epsilon(x)}(H(x)^{-1}f)$  is of the form

$$\Psi = \lambda U_2 U_3^{\omega(x)} + P(U_1, U_3) \quad (1)$$

with  $\lambda \in k(x)$ ,  $P \in k(x)[U_1, U_3]_{\epsilon(x)}$ , since  $U_3 \in \text{VDir}(x)$  (comments in the proof of I.8.5).

Let  $x' := (X' := \frac{X}{u_1}, u'_1 := u_1, u_2, u'_3 := \frac{u_3}{u_1}) \in e^{-1}(x)$  and assume that  $x'$  is very near  $x$ . We have  $E' := e^{-1}(E) = \text{div}(u'_1)$  and  $H(x') = u_1^{a(1)+\omega(x)-p}$ . This is the origin of a chart, so  $\Delta(f'; u'_1, u_2, u'_3, X')$  is minimal (where  $f' = u_1^{-p}f$ ). We get

$$H(x')^{-1}u_2^{-1}f' = \sum_{0 \leq j \leq \omega(x)} u_3^{\omega(x)-j} u_1^{-j} \phi_j(u'_1, u_2), \quad (2)$$

with  $u_2 \phi_0 \in (u'_1, u_2)$ ,  $u_2 \phi_j \in (u'_1)^{jA_1(x)} k(x)[[u'_1, u_2]]$  for  $1 \leq j \leq \omega(x)$  and  $\frac{\partial u_2 \phi_0}{\partial u_2}$  invertible.

First assume that  $B3(x) < 2$  (in particular  $1 \leq A_1(x) \leq 2$ ). If  $A_1(x) = \frac{\text{ord}_{u_1}(H(x)^{-1}g^p)}{\omega(x)}$ , then  $\Omega(x') \leq (\omega(x), 1)$ : a contradiction, since  $x'$  is very near  $x$ . By (2), there appears in  $H(x')^{-1}f'$  some term of the form  $u_3^{\omega(x)-j} u_2 u_1^{-j} \phi_j(u'_1, u_2)$ , where  $1 \leq j \leq \omega(x)$  and either  $\text{ord}_{(u'_1, u_2)} u_1^{-j} \phi_j = j(B3(x) - 1)$  or  $(A_1(x) = 1 \text{ and } \text{ord}_{u_2} u_1^{-j} \phi_j(0, u_2) = j\beta3(x) < 1 - \frac{1}{\omega(x)})$ .

In the former case, note that  $\omega(x) - j + \text{ord}_{(u'_1, u_2)} u_1^{-j} \phi_j < \omega(x)$ , since  $B3(x) < 2$ . Since  $x'$  is very near  $x$ , we have  $\epsilon(x') = \omega(x)$  and  $\text{ord}_{(u'_1, u_2)} u_1^{-j} \phi_j = j - 1$ , so  $2 - B3(x) = \frac{1}{j}$ . If  $A_1(x) = 1$ , we must have  $j = 1$  and  $\text{ord}_{u_2}(u_1^{-1}(u_2 \phi_1) \pmod{u'_1}) = 1$ : a contradiction by (1). Hence  $A_1(x) > 1$  and  $u_1^{-j} \phi_j(u'_1, u_2) \in (u'_1)$ , so  $\kappa(x) \leq 1$  by chapter 2, II.1.

In the latter case, note that  $\omega(x) - j + j\beta3(x) < \omega(x) - \frac{j}{\omega(x)} < \omega(x)$  by assumption. Since  $x'$  is very near  $x$ , we have  $\epsilon(x') = \omega(x)$  and  $j < \omega(x)$  whenever  $\text{ord}_{u_2} \phi(0, u_2) = j\beta3(x)$ . So there is an expression

$$\text{cl}_{\omega(x)}(H(x')^{-1}f') = \sum_{1 \leq j \leq \frac{\omega(x)}{p} - 1} \mu_j U_3^{\omega(x)-pj} U_2^{pj} + U_1 P'(U_1, U_2, U_3),$$

with  $\mu_j \in k(x)$  and  $\mu_j \neq 0$  for some  $j$ . Hence  $\text{VDir}(x') \equiv \langle U_2, U_3 \rangle \pmod{U_1}$  and  $\kappa(x') \leq 1$  by II.2 of chapter 2.

Assume now that  $B3(x) \geq 2$  (in particular  $A_1(x) > 1$ ). Then  $\epsilon(x') = 1 + \omega(x)$  and it is easily seen from (2) that  $x'$  is in case (\*3). Moreover,  $(X', u'_1, u_2, u'_3)$  remains well prepared since we are

at the origin of a chart. We get  $A_1(x') = A_1(x) - 1$ ,  $B3(x') = B3(x) - 1$  and  $\beta3(x') = \beta3(x)$ . This concludes the proof.

We now turn to the general case  $B3(x) > A_1(x)$ . With assumptions and notations of **I.8.4**, we define the analogues of **I.8.2** when  $x$  is in case (\*3).

**I.8.6.1 Definition.** Let  $J_0 := \{j, 1 \leq j \leq \omega(x) \mid B3(x) = \frac{\text{ord}\phi_j}{j}\}$ . For  $j_0 \in J_0$ , we denote  $\Phi_{j_0} := \text{in}_x \phi_{j_0} = U_1^{a_{j_0}(1)} U_2^{-1} \Psi_{j_0}$  and  $d_{j_0} := \deg \Psi_{j_0} - 1 \in \mathbb{N} \cup \{-1\}$ . We also denote  $\Psi_0 := \text{in}_x(u_2 \phi_0) \in k(x)[U_1, U_2, U_3]_1$ .

The definition of  $J_0$  and  $d_{j_0}$  for  $j_0 \in J_0$  is motivated by the following obvious fact, where only preparation of “left” vertices is needed (definition **I.5.1.2**):

**I.8.6.2 Lemma.** We have

$$\sup_{j_0 \in J_0} \left\{ \frac{d_{j_0}}{j_0} \right\} \leq \beta3_2(x) \leq B3(x) - A_1(x) \leq \beta3(x),$$

and

$$\sup_{j_0 \in J_0} \left\{ \frac{1 + d_{j_0}}{j_0} \right\} \leq \beta(x) \leq \gamma(x).$$

**I.8.6.3 Definition.** Let  $\mu_0$  be the monomial valuation on  $u_2^{-1} \widehat{S}$  given by

$$\mu_0 \left( \sum_{abc} \lambda_{abc} u_1^a u_2^b u_3^c \right) = \inf \left\{ c + \frac{a+b}{B3(x)} \mid \lambda_{abc} \neq 0 \right\}.$$

We denote by  $\text{cl}_{\mu_0, \omega(x)} J$  the  $k(x)$ -vector space  $U_2^{-1} \text{in}_{\mu_0} J$ , where  $J$  is the ideal

$$J := \left( \left\{ H(x)^{-1} \lambda_j \frac{\partial f}{\partial \lambda_j} \right\}_{1 \leq j \leq s} \right) + (H(x)^{-1} g^p).$$

By definition  $B3(x)$  and well-preparedness, we have  $\dim_{k(x)}(\text{cl}_{\mu_0, \omega(x)} J) \geq 1$ , and for every  $\lambda \in k(x)$ , we have

$$\text{cl}_{\mu_0, \omega(x)} J \neq k(x) \cdot (U_3 - \lambda U_1^{\alpha3_2(x)} U_2^{\beta3_2(x)})^{\omega(x)}.$$

Let  $e : X' \rightarrow X$  be the blowing up along  $x$ . We call “first chart” the chart with origin  $(X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = \frac{u_2}{u_1}, u'_3 = \frac{u_3}{u_1})$ . Let  $E' := (e^{-1}(x))_{\text{red}} = \text{div}(u'_1)$ . For  $x' \in e^{-1}(x)$  in the first chart, we pick  $P(u_1, u_2)$  homogeneous of degree  $d \geq 1$ , irreducible and unitary in  $u_2$  such that  $v' := P(1, u'_2) \in m_{\eta'(x')}$  as usual.

**I.8.7 Lemma.** With hypotheses and notations of **I.8.4** (in particular  $x$  is in case (\*3)), assume that the center  $x'$  of  $\mu$  belongs to the first chart. Let  $d := [k(x') : k(x)]$ . We have  $\kappa(x) \leq 1$  or the following holds:

- (a)  $\kappa(x') \leq 2$  and  $x'$  satisfies condition (\*1) or (\*3);
- (b) if  $B3(x) = 1$ , then  $\beta3(x) \geq 1 - \frac{1}{\omega(x)}$  and  $x'$  is rational over  $x$ . We have  $A_1(x') = 0$  and  $\gamma(x') \leq \sup\{\gamma(x), 2\}$ . If  $(\gamma(x) = 1 \text{ and } \gamma(x') = 2)$ , then  $x'$  is in case (\*1) and either  $\beta(x') < 2$ , or  $(p = \omega(x) = 2 \text{ and } x' \text{ satisfies equation (Dis) in I.8.3(e))$ ;
- (b') if  $B3(x) > 1$ , then the point  $x'$  maps to the strict transform of  $\text{div}(u_3)$  (in particular,  $(X', u'_1, v', u'_3)$  is a r.s.p. at  $x'$ ) and  $A_1(x') = B3(x) - 1$ .

$$(c) \ B3(x) < \frac{\text{ord}_{u_1}(H(x)^{-1}g^p)-1}{\omega(x)};$$

From now on, we assume that either  $B3(x) > 1$  or ( $B3(x) = 1$  and  $x'$  is in case (\*3)). The following holds:

$$(d) \ \text{we have } \gamma(x') \leq \gamma(x);$$

More precisely, the following holds:

(i) if  $J_0 \not\subset p\mathbb{N}$ , then

$$\beta(x') \leq \sup_{j_0 \in J_0} \left\{ \frac{1 + d_{j_0}}{j_0 d} \right\},$$

and

$$\beta3(x') \leq \sup_{j_0 \in J_0} \left\{ \frac{d_{j_0}}{j_0 d} \right\}$$

if  $x'$  is in case (\*3);

(ii) if  $J_0 \subset p\mathbb{N}$ , then  $\beta(x') \leq \frac{1}{p}$  or

$$\beta(x') \leq \inf \left\{ \frac{\beta(x)}{d} + \frac{1}{p}, \sup_{j_0 \in J_0} \left\{ \frac{1 + d_{j_0}}{j_0} \right\} \right\}.$$

(iii) if ( $J_0 \subset p\mathbb{N}$  and  $x'$  is in case (\*3)), then  $\beta3(x') < \frac{1}{p}$  or

$$\beta3(x') \leq \sup_{j_0 \in J_0} \left\{ \frac{d_{j_0}}{j_0} \right\} \text{ and } \beta3(x') < \frac{\beta(x)}{d} + \frac{1}{p}.$$

(iv) if  $x'$  is not rational over  $x$ , then:

if  $\gamma(x) > 2$ , we have  $\gamma(x') < \gamma(x)$ ;

if  $\gamma(x) = \gamma(x') = 2$ , we have: if  $x'$  is in case (\*1), then  $\beta(x') \leq \frac{3}{2}$ ; if  $x'$  is in case (\*3), then ( $p = 2$  and  $\beta3(x') < \frac{3}{2}$ );

(v) if ( $\gamma(x) = \beta(x) = 1$ ,  $x'$  is not rational over  $x$  and  $x'$  is in case (\*3)), then  $\beta(x') < 1$ .

**Proof.** We assume all along the proof that  $x'$  is very near  $x$ . If  $\kappa(x) > 1$ ,  $x'$  maps to the strict transform of  $\text{div}(u_3)$  unless possibly if  $B3(x) = 1$  by **I.7(ii)**. In this case, without loss of generality, it can be assumed that  $(u_1, u_2, u_3)$  is the r.s.p.  $(u_1, u_2, v)$  given in **I.7(ii)**, since  $B3(u_1, u_2, v) = 1$ ,  $\beta3(u_1, u_2, v) = \beta3(u_1, u_2, u_3)$  and the vertex  $(\alpha3_2(x), \beta3_2(x))$  is unaffected by this coordinate change.

**I.8.7.1** We first prove the theorem when  $B3(x) = 1$ . In particular,  $U_3 \in \text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f}{\partial u_2}))$  by the previous comments. Note that cases 1 and 2 below are unaffected by the above coordinate change. Also note that  $\omega(x') = 1 < \omega(x)$  if  $\text{ord}_{u_1}(H(x)^{-1}g^p) = \omega(x) + 1$ , so (c) holds.

*Case 1:*  $\text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f}{\partial u_2})) \equiv \langle U_3, U_2 \rangle \pmod{U_1}$ . We have  $\tau(x) = 2$ , since  $x'$  is very near  $x$ . Then  $(0, 1)$  is an unsolvable vertex of  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$ , so  $A_1(x) = 0$ ,  $\beta3(x) = 1$  and  $\gamma(x) = 2$ . After possibly changing  $u_2$  to  $u_2 + au_1$ ,  $a \in k(x)$  and picking again well prepared coordinates, it can be assumed that

$$\text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f}{\partial u_2})) = \langle U_2, U_3 \rangle,$$

all assumptions remaining unchanged. Since  $x$  is in case (\*3) and  $\text{VDir}(x) = \langle U_2, U_3 \rangle$ , we have

$$\text{cl}_{1+\omega(x)}(H(x)^{-1}f) \in U_2 k(x)[U_2^p, U_3^p] \oplus U_3 k(x)[U_2^p, U_3^p] \oplus U_1 k(x)[U_1^p, U_2^p, U_3^p]. \quad (1)$$

Since  $x'$  is very near  $x$ , we have  $x' = (X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = \frac{u_2}{u_1}, u'_3 = \frac{u_3}{u_1})$ , so  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal. We get

$$H(x')^{-1}f' \equiv u'_3{}^{\omega(x)}\Psi_0(1, u'_2, u'_3) + \sum_{j_0 \in J_0} u'_3{}^{\omega(x)-j_0}\Psi_{j_0}(1, u'_2) \pmod{u'_1}, \quad (2)$$

with notations as in **I.8.6.1**. Note that  $\omega(x') \leq \epsilon(x') \leq \omega(x) - p$  unless we have

$$\text{cl}_{1+\omega(x)}(H(x)^{-1}f) = U_2F_2(U_2^p, U_3^p) + U_3F_3(U_2^p, U_3^p) + U_1F_1(U_2^p, U_3^p), \quad (3)$$

which we assume from now on. Remember that by assumption, we have  $\text{VDir}(F_2(U_2^p, U_3^p)) = \langle U_2, U_3 \rangle$ . Finally, we have  $\kappa(x') = 0$  by **II.2** of chapter 2 if

$$\tau(J(U_1^{a(1)+1}F_1(U_2^p, U_3^p), E, x)) = 2.$$

This yields to the following subcases:

*Case 1a:*  $F_1 = 0$ . If  $\epsilon(x') = \omega(x)$ , then  $\kappa(x') \leq 1$  by (2) and **II.3** of chapter 2. Otherwise, we have  $\epsilon(x') = 1 + \omega(x)$ , so  $x'$  is again in case (\*3). Note that  $(0, 1)$  is an unsolvable vertex of  $\Delta(H(x')^{-1}u'_2{}^{-1}(f', g'^p); u'_1, u'_2; u'_3)$  by **I.6.3**, so we have  $A_1(x') = 0$ ,  $\beta_3(x') = 1$ ,  $\gamma(x') = 2$  and (ii),(iii) hold.

*Case 1b:*  $F_1 \neq 0$ . Then  $\epsilon(x') = \omega(x)$ . Let  $k(x).W := \text{VDir}(J(U_1^{a(1)+1}F_1(U_2^p, U_3^p), E, x))$ , so

$$F_1(U_2^p, U_3^p) =: \mu W^{\omega(x)} + F'_1(V, W)^p, \quad (4)$$

where  $W$  is picked in such a way that  $\langle V, W \rangle = \langle U_2, U_3 \rangle$ . Note that  $\mu \neq 0$ , and that  $\mu \notin k(x)^p$  (resp.  $F'_1 = 0$ ) if  $a(1) + 1 \equiv 0 \pmod{p}$  (resp.  $a(1) + 1 \not\equiv 0 \pmod{p}$ ). Let  $F(U_2, U_3) := U_2F_2(U_2^p, U_3^p) + U_3F_3(U_2^p, U_3^p)$ . We expand

$$F(U_2, U_3) = \mu_{-1}W^{1+\omega(x)} + \mu_0VW^{\omega(x)} + \sum_{p-1 \leq j \leq \omega(x)} \mu_j V^{1+j}W^{\omega(x)-j} =: F'(V, W).$$

Since  $\text{VDir}(\frac{\partial F}{\partial U_2}, \frac{\partial F}{\partial U_3}) = \text{VDir}(\frac{\partial F'}{\partial V}, \frac{\partial F'}{\partial W}) = \langle V, W \rangle$ , there exists  $j_0 \geq p - 1$  such that  $\mu_{j_0} \neq 0$ . Equation (2) then reads

$$H(x')^{-1}f' \equiv F'(v, w) + \mu w^{\omega(x)} + F'_1(v, w)^p \pmod{u'_1},$$

where  $\mu$  and  $F'_1$  are defined in (4). In particular,  $x'$  is in case (\*1). After picking well prepared coordinates  $(Z', u'_1, v' := v, w')$  at  $x'$ , the vertex  $(0, 1 + \frac{1}{j_0})$  of the polygon  $\Delta(H(x')^{-1}(f', g'^p); u'_1, v'; w')$  is not solvable. Therefore  $\beta(x') \leq 2$ , so  $\gamma(x') \leq 2$ .

*Case 2:*  $\text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1}\frac{\partial f}{\partial u_2})) = \langle U_3, U_1 \rangle$ . The only possible very near point has r.s.p.  $(\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, \frac{u_3}{u_2})$  which does not belong to the first chart.

*Case 3:*  $\text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1}\frac{\partial f}{\partial u_2})) = k(x).U_3$ . By **I.7**(ii),  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$  has an unsolvable vertex of the form  $(\alpha_3_2, \beta_3_2)$ , where  $0 \leq \alpha_3_2 < 1$  and  $\alpha_3_2 + \beta_3_2 = 1$ . Note that we now have

$$\text{cl}_{1+\omega(x)}(H(x)^{-1}f) \in U_2k(x)[U_3^p] \oplus U_3k(x)[U_2^p, U_3^p] \oplus U_1k(x)[U_1, U_2^p, U_3].$$

First assume that  $\alpha_3_2 = 0$ . This implies that  $\text{VDir}(x) = \langle U_2, U_3 \rangle \pmod{U_1}$ ,  $A_1(x) = 0$ ,  $\beta_3(x) = 1$  and  $\gamma(x) = 2$ . After possibly changing  $u_2$  to  $u_2 + au_1$ ,  $a \in k(x)$  and picking again well

prepared coordinates, it can be assumed that  $\text{VDir}(x) = \langle U_3, U_2 \rangle$ , all assumptions remaining unchanged. Since  $x'$  is very near  $x$ , we have  $x' = (X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = \frac{u_2}{u_1}, u'_3 = \frac{u_3}{u_1})$ , so  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal. The proof now runs parallel to that of case 1; to begin with, we have

$$\text{cl}_{1+\omega(x)}(H(x)^{-1}f) = U_2 F_2(U_3^p) + U_3 F_3(U_2^p, U_3^p) + U_1 F_1(U_2^p, U_3^p),$$

since  $x'$  is very near  $x$ . Now, we split case 3 into case 3a ( $F_1 = 0$ ) and case 3b ( $F_1 \neq 0$ ), and have the same conclusion as in cases 1a and 1b.

Assume now that  $\alpha 3_2 > 0$ . We now have

$$\text{cl}_{1+\omega(x)}(H(x)^{-1}f) \in U_2 k(x)[U_3^p] \oplus U_3 k(x)[U_3^p] \oplus U_1 k(x)[U_1, U_2^p, U_3].$$

Since  $x'$  is very near  $x$  and belongs to the first chart, we have  $U_1 \notin \text{VDir}(x)$ . Therefore

$$\text{cl}_{1+\omega(x)}(H(x)^{-1}f) = \mu_2 U_2 U_3^{\omega(x)} + \mu_3 U_3^{1+\omega(x)} + \sum_{0 \leq j \leq \frac{\omega(x)}{p}} U_1^{1+pj} F_{\omega(x)-pj}(U_2^p, U_3^p),$$

where  $F_{\omega(x)-pj}(U_2^p, U_3^p) \notin k(x)[U_3^p]$  for some  $j \geq 0$ , since  $B3(x) = 1$  and  $\alpha 3_2 < 1$ . Moreover, we have either  $a(1) + 1 \not\equiv 0 \pmod{p}$  or  $F_{\omega(x)-pj}(U_2^p, U_3^p) \notin (k(x)[U_2, U_3])^p$  for some  $j \geq 0$ , since  $\Delta(h; u_1, u_2, u_3; X)$  is minimal.

Note that  $\text{VDir}(x) = k(x).U_3$ . By chapter 1, **II.5.3.2(iv)**,  $x'$  is rational over  $x$  since it is very near  $x$ . After possibly changing  $u_2$  to  $u_2 + au_1$ ,  $a \in k(x)$  and picking again well prepared coordinates, it can be assumed that  $x' = (X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = \frac{u_2}{u_1}, u'_3 = \frac{u_3}{u_1})$ , all assumptions remaining unchanged. In particular,  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal. We finally get

$$\text{cl}_{1+\omega(x)}(H(x)^{-1}f) = \mu_2 U_2 U_3^{\omega(x)} + \mu_3 U_3^{1+\omega(x)} + U_1 F_{\omega(x)}(U_2^p, U_3^p),$$

where  $\mu_2 \neq 0$ ,  $F_{\omega(x)}(U_2^p, U_3^p) \notin k(x)[U_3^p]$  and either  $a(1) + 1 \not\equiv 0 \pmod{p}$  or  $F_{\omega(x)}(U_2^p, U_3^p) \notin (k(x)[U_2, U_3])^p$ . This proves in particular that  $\epsilon(x') = \omega(x)$ . We have  $\kappa(x') = 0$  by chapter 2, **II.2** if

$$\tau(J(U_1^{a(1)+1} F_{\omega(x)}(U_2^p, U_3^p), E, x)) = 2.$$

Assume finally that  $\text{VDir}(J(U_1^{a(1)+1} F_1(U_2^p, U_3^p), E, x))$  has dimension one, i.e.

$$F_{\omega(x)}(U_2^p, U_3^p) =: \mu(U_2 + \lambda U_3)^{\omega(x)} + F'_1(U_2, U_3)^p,$$

where  $\mu \neq 0$  and  $\mu \notin k(x)^p$  (resp.  $F'_1 = 0$ ) if  $a(1) + 1 \equiv 0 \pmod{p}$  (resp.  $a(1) + 1 \not\equiv 0 \pmod{p}$ ). In particular,  $(\alpha 3_2(x), \beta 3_2(x)) = (\frac{1}{\omega(x)}, 1 - \frac{1}{\omega(x)})$ , so  $\beta 3(x) \geq 1 - \frac{1}{\omega(x)}$ . Moreover, this proves that  $x'$  is in case (\*1). Since  $\mu_2 \neq 0$ , after picking well prepared coordinates  $(Z', u'_1, u'_3, v')$  at  $x'$ ,  $(0, 1 + \frac{1}{\omega(x)-1})$  is a vertex of the polygon  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_3; v')$  and is not solvable. Therefore  $A_1(x') = 0$ ,  $\beta(x') \leq 1 + \frac{1}{\omega(x)-1}$  and  $\gamma(x') \leq 2$ .

In all cases, we have  $\gamma(x') \leq 2$ . Suppose that  $(\gamma(x) = 1$  and  $\gamma(x') = 2)$ . The above analysis shows that we are in the situation of the previous paragraph, with  $\beta 3(x) = 1 - \frac{1}{\omega(x)}$ , and the conclusion follows. Otherwise, we may assume that  $x'$  is in case (\*3) (cases 1a and 3a above) and (ii) holds, so all statements have been proved when  $B3(x) = 1$ .

**I.8.7.2** From now on, we assume that  $B3(x) > 1$ . We have

$$\text{cl}_{1+\omega(x)}(H(x)^{-1}f) = (\mu_1 U_1 + \mu_2 U_2 + \mu_3 U_3) U_3^{\omega(x)},$$

with  $\mu_2 \neq 0$ . In particular  $\text{VDir}(x) = k(x).U_3$  and the first part of (b') is proved.

**I.8.7.3** Proof of (c). Assume that (a) and the second part of (b') have been proved. If  $(H(x)^{-1}g^p) = (u_1)^{1+\omega(x)B3(x)}$ , then  $\kappa(x) \leq 1$  or  $(A_1(x'), 0)$  (resp.  $(A_1(x'), -\frac{1}{\omega(x)})$ ) is a vertex of  $\Delta(H(x')^{-1}(f', g'^p); u'_1, v'; u'_3)$  (resp. of  $\Delta(H(x')^{-1}v'^{-1}(f', g'^p); u'_1, v'; u'_3)$ ) if  $x'$  is in case (\*1) (resp. in case (\*3)) and is not solvable by definition. By **I.8.1** (resp. **I.8.5**), we get  $\kappa(x) \leq 1$ .

**I.8.7.4** From now on,  $B3(x) > 1$  and  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > 1 + B3(x)\omega(x)$ . According to **I.8.4** and **I.8.6.1**, let us denote  $h' = u_1^{-p}h = X'^p - g'^{p-1}X' + f'$ , where

$$H(x')^{-1}f' = u_3^{\omega(x)}\psi'_0 + \sum_{1 \leq j \leq \omega(x)} u_3^{\omega(x)-j} u_1^{a_j(1)+d_j-j} \psi'_j(u'_1, u'_2), \quad (1)$$

where  $\psi'_j$  is the strict transform of  $\psi_j$ , i.e.  $1 + d_j = \text{ord}_{(u_1, u_2)}\psi_j$ ,  $\psi'_0 \in S'$  with  $\frac{\partial \psi'_0}{\partial u_2}$  invertible,  $H(x') = u_1^{a(1)+\omega(x)+1-p}$ , and

$$g'^p = u_1^{-p}g^p = \gamma^p u_1^{b(1)-p}.$$

**I.8.7.5** We first consider the origin  $x' = (X', u'_1, u'_2, u'_3)$  of the first chart. Since  $x'$  is the origin of a chart,  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal. In this case, the module

$$M' := u_2'^{-1} \sum_{1 \leq i \leq s} H(x')^{-1} \lambda_i \frac{\partial f'}{\partial \lambda_i} S' + u_2'^{-1} H(x')^{-1} g'^p S',$$

where  $S' = S[u'_2, u'_3]_{(u'_1, u'_2, u'_3)}$  is equal to  $u_1'^{-\omega(x)}M$ , where

$$M := u_2^{-1} \sum_{1 \leq i \leq s} H(x)^{-1} \lambda_i \frac{\partial f}{\partial \lambda_i} S + u_2^{-1} H(x)^{-1} g^p S.$$

Since  $x'$  is very near  $x$ , we have  $\epsilon(x') \geq \epsilon(x) - 1$ . We consider three cases:

*Case 1:*  $\psi'_0$  is invertible (i.e.  $\mu_1 \neq 0$  in **I.8.7.2**). Then  $\epsilon(x') = \epsilon(x) - 1$ ,  $\kappa(x') = 2$  and  $x'$  is in the case (\*1). Since the vertex  $(\alpha 3_2, \beta 3_2)$  of  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_1, u_2; u_3)$  is not solvable and  $B3(x) > 1$ , there exists an unsolvable vertex of  $\Delta(H(x')^{-1}(f'_{Z'}, g'^p); u'_1, u'_2; w')$  of the form  $(B3(x) - 1, \beta')$ , where

$$\beta' \leq \frac{1 + d_{j_0}}{j_0} \leq \frac{1 + j_0 \beta 3_2(x)}{j_0},$$

for some  $j_0 \in J_0$  and  $(Z', u'_1, u'_2, w')$  well prepared at  $x'$ . We get  $A_1(x') = B3(x) - 1$  and

$$\beta(x') \leq \sup_{j_0 \in J_0} \left\{ \frac{1 + d_{j_0}}{j_0} \right\} \leq \beta(x),$$

by **I.8.6.2**.

*Case 2:*  $\psi'_0$  is not invertible and  $\epsilon(x') = \omega(x)$ . Then  $(u'_1, \psi'_0, u'_3)$  is a r.s.p. of  $S'$ . After picking coordinates  $(Z', u'_1, \psi'_0, u'_3)$  at  $x'$  such that  $\Delta(h'; u'_1, \psi'_0, u'_3; Z')$  is minimal, we get an expression

$$H(x')^{-1}f' = u_3^{\omega(x)}\psi'_0 + u'_1\phi',$$

where  $\text{ord}_{\eta'(x')}(u'_1\phi') = \omega(x)$ , so  $\kappa(x') \leq 1$  by **II.1** of chapter 2.

Case 3:  $\epsilon(x') = 1 + \omega(x)$ . Then  $(u'_1, u'_2, u'_3)$  is a r.s.p. of  $S'$  and there is an expression

$$H(x')^{-1}f' = \mu_2 u'_3{}^{\omega(x)} u'_2 + u'_1 \phi',$$

where  $\text{ord}_{\eta'(x')}(u'_1 \phi') = 1 + \omega(x)$ . In particular  $\kappa(x') = 2$  and  $x'$  is in case (\*3). After picking well prepared coordinates  $(Z', u'_1, u'_2, w')$  at  $x'$  as in case 1 above, we get  $A_1(u'_1, u'_2, w') = B3(u_1, u_2, u_3) - 1$  and

$$\beta 3(x') \leq \frac{d_{j_0}}{j_0} \leq \beta 3(x)$$

for some  $j_0 \in J_0$ . This concludes the proof of **I.8.7** when  $x'$  is the origin of the first chart.

**I.8.7.6** From now on,  $x' \neq (X', u'_1, u'_2, u'_3)$ , i.e.  $u'_2$  is a unit in  $S'$ . There remains to prove (a), the second statement in (b') and all statements from (d) on. Note that (d) is a direct consequence of (i), (ii) and (iii) and **I.8.6.2**. The proof will be parallel to that of **I.8.3** (iii)-(vi).

Recall equation (1) in **I.8.7.4**. Note that

$$H(x')^{-1}f' = u'_3{}^{\omega(x)}(\mu_1 + \mu_2 u'_2 + \mu_3 u'_3 + u'_1 \theta'_0) + \sum_{1 \leq j \leq \omega(x)} u'_3{}^{\omega(x)-j} u'_1{}^{\lfloor j(B3(x)-1) \rfloor} (\Psi'_j(1, u'_2) + u'_1 \theta'_j), \quad (1)$$

where  $\Psi'_j(U_1, U_2) := \Psi_j(U_1, U_2)$  (resp.  $\Psi'_j(U_1, U_2) := 0$ ) if  $j \in J_0$  (resp.  $j \notin J_0$ ) and  $\theta'_j \in \widehat{S}'$  for  $0 \leq j \leq \omega(x)$ . Remember that  $\mu_2 \neq 0$ ,  $\Psi_j(U_1, U_2) \in k(x)[U_1, U_2]_{1+d_j}$  and  $j(B3(x) - 1) \in \mathbb{N}$  if  $j \in J_0$ . Let  $\bar{\mu}$  denote the image of  $\mu := \mu_1 + \mu_2 u'_2$  in  $k(x')$ .

If  $\bar{\mu} \notin k(x')^p$  or if  $(a(1) + 1 \not\equiv 0 \pmod{p}$  and  $\bar{\mu} \neq 0)$ , then  $x'$  is in case (\*1) since  $x'$  is very near  $x$  and we have

$$H(x')^{-1}f'_{Z'} \equiv (\mu + \mu_3 u'_3) u'_3{}^{\omega(x)} \pmod{(u'_1)},$$

where  $Z' := X' - \theta'$ ,  $f'_{Z'} := f + \theta'^p - \theta' g'^{p-1}$  and  $\Delta(h'; u'_1, v', u'_3; Z')$  is minimal.

If  $(\bar{\mu} \in k(x')^p$  and either  $a(1) + 1 \equiv 0 \pmod{p}$  or  $\bar{\mu} = 0)$ , we have similarly

$$H(x')^{-1}f'_{Z'} \equiv \mu' u'_3{}^{\omega(x)} v' \pmod{(u'_1)},$$

with  $\mu' \in \widehat{S}'$  invertible and  $\Delta(h'; u'_1, v', u'_3; Z')$  minimal. If  $\epsilon(x') = \omega(x)$ , then  $\kappa(x) \leq 1$  by **II.1** of chapter 2. Otherwise  $\epsilon(x') = 1 + \omega(x)$  and  $x'$  is again in case (\*3).

This ends the proof of (a).

**I.8.7.7** Proof when  $J_0 \not\subset p\mathbb{N}$ . Let  $j_1 := \inf\{j \in J_0 \mid j \not\equiv 0 \pmod{p}\}$ . Since  $\omega(x) - j_1 \not\equiv 0 \pmod{p}$ , the term  $H(x') u'_3{}^{\omega(x)-j_1} u'_1{}^{j_1(B3(x)-1)} \Psi_{j_1}(1, u'_2)$  in  $f'$  is unaffected by any translation on  $u'_3$  or on  $X'$  in the well-preparation algorithm **I.6**.

If  $x'$  is in case (\*1), we get  $A_1(x') = B3(x) - 1$  and  $\beta(x') \leq \frac{1+d_{j_1}}{j_1 d}$ .

If  $x'$  is in case (\*3), we get  $A_1(x') = B3(x) - 1$ ,  $\beta(x') \leq \frac{1+d_{j_1}}{j_1 d}$  and  $\beta 3(x') \leq \frac{1+d_{j_1}}{d_{j_1}} - \frac{1}{j_1} \leq \frac{\beta 3(x)}{d}$ .

This completes the proof of (b') and (i) from which all other statements in the theorem easily follow in the case  $J_0 \not\subset p\mathbb{N}$  (with  $\gamma(x') < \gamma(x)$  in (iv)).

**I.8.7.8** Proof when  $J_0 \subset p\mathbb{N}$ . We prove together (b'), (ii), (iii) and (v). We consider three cases, exactly like in the proof of **I.8.3**, see **I.8.3.4**, **I.8.3.5** and **I.8.3.6**.

Case 1:  $B3(x) \notin \mathbb{N}$ . The translation  $w' := u'_3 - u'_1{}^a s'$  in the well preparation algorithm will affect none of the terms

$$u'_3{}^{\omega(x)-j_0} u'_1{}^{j_0(B3(x)-1)} \Psi_{j_0}(1, u'_2)$$

in  $H(x')^{-1}f'$  for  $j_0 \in J_0$ . The translation  $Z' = X' - u_1^{a'}\theta'$  plugs into  $f'_{Z'}$  some term of the form

$$H(x')u_3^{\omega(x)-j_0}u_1^{j_0(B3(x)-1)}(\Psi_{j_0}(1, u_2') + A_{j_0}(v')^p)$$

with  $A_{j_0} = 0$  unless  $a(1) + 1 + j_0(B3(x) - 1) \equiv 0 \pmod{p}$ . If  $a(1) + 1 + j_0(B3(x) - 1) \equiv 0 \pmod{p}$  for each  $j_0 \in J_0$ , then  $U_1^{a(1)+a_{j_1}(1)}\Psi_{j_1}(U_1, U_2) \notin (k(x)[U_1, U_2])^p$  for at least one index  $j_1 \in J_0$ , since  $\Delta(h; u_1, u_2, u_3; X)$  is minimal. By chapter 1, **II.5.3.2**(i), (ii) we get

$$\text{ord}_{v'}(\Psi_{j_1}(1, u_2') + A_{j_1}(v')^p) \leq \frac{1 + d_{j_1}}{d} + 1 \quad (1)$$

and  $\frac{1+d_{j_1}}{d} + 1 \notin p\mathbb{N}$  if equality holds. Also note that

$$\text{ord}_{v'}(\Psi_{j_1}(1, u_2') + A_{j_1}(v')^p) \leq \sup\{1 + d_{j_1}, 1\} \quad (2)$$

by chapter 1, **II.5.3.2**(iii). This proves that  $A_1(x') = B3(x) - 1$  (so (b') holds),

$$\beta(x') \leq \frac{1 + d_{j_1}}{dj_1} + \frac{1}{j_1}, \quad (3)$$

and either  $\beta(x') \leq \frac{1}{j_1} \leq \frac{1}{p}$  or

$$\beta(x') \leq \frac{1 + d_{j_1}}{j_1}; \quad (3')$$

if  $x'$  is in case (\*3), then

$$\beta 3(x') \leq \frac{1 + d_{j_1}}{dj_1} \quad (4)$$

and either  $\beta 3(x') \leq 0$  or

$$\beta 3(x') \leq \frac{d_{j_1}}{j_1}. \quad (4')$$

Using lemma **I.8.6.2**, this completes the proof of (ii) and (iii) in case 1. Statement (v) follows from (ii) except possibly if  $d = p = 2$ . In this case, we have  $\beta(x') < 1$  unless equality holds in (1) above. By (3),  $\beta(x') < 1$  except possibly if  $j_1 = 2$ , so  $\frac{1+d_{j_1}}{d} + 1 = 2$ : a contradiction since  $\frac{1+d_{j_1}}{d} + 1 \notin p\mathbb{N}$  if equality holds in (1).

*Case 2:*  $B3(x) \in \mathbb{N}$  and  $\dim_{k(x)}(\text{cl}_{\mu_0, \omega(x)}J) \geq 2$ . Recall the definition of the vector space  $\text{cl}_{\mu_0, \omega(x)}J$  in **I.8.6.3**. Following the lines of **I.8.3.2**, there is a formula

$$\text{in}_{\mu_1}J' = \langle (\text{in}_{\mu_1}H(x'))^{-1} \cdot (\text{in}_{\mu_0}H(x))U_2 \text{cl}_{\mu_0, \omega(x)}J, \rangle$$

where the valuation  $\mu_1$  on  $\widehat{S}' \simeq k(x')[[u_1', v', u_3']]$  is now defined by

$$\mu_1\left(\sum_{abc} \lambda_{abc} u_1'^a v'^b u_3'^c\right) = \min\left\{c + \frac{a}{B3(x) - 1} \mid \lambda_{abc} \neq 0\right\},$$

and  $J' := J(U_1^{a(1)+1+\omega(x)-p}F', E')$ , where

$$F' := \text{in}_{\mu_1}(H(x')^{-1}f') = U_3^{\omega(x)}(\mu_1 + \mu_2 u_2') + \sum_{j_0 \in J_0} U_3^{\omega(x)-j_0} U_1^{j_0(B3(x)-1)} \Psi_{j_0}(1, u_2').$$

The proof goes along the same lines as **I.8.3.5**; following these lines, we pick  $0 \neq G \in \text{cl}_{\mu_0, \omega(x)} J$  of the form

$$G = U_2^{-1} \sum_{j_0 \in J_0} U_3^{\omega(x) - j_0} U_1^{a_{j_0}(1)} G_{j_0}(U_1, U_2),$$

with  $G_{j_0}$  homogeneous of degree  $1 + d_{j_0}$  and  $j_1 := \inf\{j_0 \in J_0 \mid G_{j_0} \neq 0\}$ . We get the same upper bounds (3), (3'), (4) and (4') as in case 1, and the conclusion follows in the same way.

*Case 3:*  $B3(x) \in \mathbb{N}$  and  $\text{cl}_{\mu_0, \omega(x)} J =: k(x) \cdot G$ . We may take  $G = \mu_2^{-1} \frac{\partial F}{\partial U_2}$ , where

$$F := \text{in}_{\mu_0}(H(x)^{-1} f) = U_3^{\omega(x)} (\mu_1 U_1 + \mu_2 U_2) + \sum_{p \leq j_0 \leq \omega(x)} U_3^{\omega(x) - j_0} U_1^{a_{j_0}(1)} \Psi_{j_0}(U_1, U_2).$$

Let  $j_1 := \inf\{j_0 \in J_0 \mid \frac{\partial \Psi_{j_0}}{\partial U_2} \neq 0\}$ . As seen in **I.8.3.6**, we may suppose that  $\mu_2^{-1} U_1^{a_{j_1}(1)} \frac{\partial \Psi_{j_1}}{\partial U_2}$  is not a  $(p^\alpha)^{\text{th}}$ -power if  $j_1 = p^\alpha$ , where  $\omega(x) = p^\alpha l$ ,  $l$  prime to  $p$ . The argument in **I.8.3.6** produces the upper bounds

$$\beta(x') \leq \frac{1 + d_{j_1}}{j_1 d} + \frac{1}{p}, \quad (5)$$

with  $(j_1 = p^\alpha \text{ and } \frac{1 + d_{j_1}}{p^{\alpha-1} d} + 1 \notin p\mathbb{N})$  if equality holds, and

$$\beta(x') \leq \sup\left\{\frac{1}{p}, \frac{1 + d_{j_1}}{j_1}\right\}.$$

If  $x'$  is in case (\*3), we get similarly

$$\beta 3(x') \leq \frac{1 + d_{j_1}}{j_1 d} + \frac{1}{p} - \frac{1}{j_1}$$

and

$$\beta 3(x') \leq \sup\left\{\frac{1}{p} - \frac{1}{j_1}, \frac{d_{j_1}}{j_1}\right\}.$$

This completes the proof of (ii) and (iii) in case 3. Finally, (v) follows from (ii) except possibly if  $d = p = 2$ . Equation (5) above yields  $j_1 = p^\alpha$  and  $1 + d_{j_1} = p^\alpha$ : a contradiction, since  $\frac{1 + d_{j_1}}{p^{\alpha-1} d} + 1 = 2 \notin p\mathbb{N}$  if equality holds in (5).

**I.8.7.9** Proof of (iv). See end of **I.8.7.7** when  $J_0 \not\subset p\mathbb{N}$  and thus assume  $J_0 \subset p\mathbb{N}$ .

If  $\gamma(x) \geq 3$ , (ii) and **I.8.6.2** give

$$\beta(x') \leq \frac{\gamma(x)}{d} + \frac{1}{p} \leq \gamma(x) - 1.$$

When  $x'$  is in case (\*3), we have  $\beta 3(x') < \beta(x')$  in any case, so (iv) holds if  $\gamma(x) \geq 3$ .

If  $\gamma(x) = 2$ , (ii) yields  $\beta(x') \leq \frac{3}{2}$ , so  $\beta 3(x') < \frac{3}{2}$  if  $x'$  is in case (\*3). In this case, when  $d \geq 3$  and  $p \geq 3$ , we get  $\beta 3(x') < \beta(x') \leq 1$ , let us see the case  $d = 2$  and  $p \geq 3$ . Then  $x'$  is separable over  $x$ , we will prove that  $\beta 3(x') < \beta(x') \leq 1$ , this will end the proof of (iv).

*Case 1:*  $\text{in}_{\mu_0}(H(x)^{-1} \frac{\partial f}{\partial u_2})$  is not proportional to an  $\omega(x)$ -power.

$$\text{in}_{\mu_0}(H(x)^{-1} \frac{\partial f}{\partial u_2}) = \overline{\mu_2} U_3^{\omega(x)} + \sum_{p \leq j_0 \leq \omega(x)} U_3^{\omega(x) - j_0} U_1^{a_{j_0}(1)} \frac{\partial \Psi_{j_0}}{\partial U_2}.$$

Let  $j_1$  be the smallest  $j_0$  with  $\frac{\partial \Psi_{j_0}}{\partial U_2} \neq 0$ , as in **I.8.3.6**, we may suppose  $j_1 \neq p^\alpha$  or  $\overline{\mu_2}^{-1} \frac{\partial \Psi_{j_1}}{\partial U_2}$  not a  $p^\alpha$ -power. In the first case, we get

$$\beta_3(x') \leq \beta(u_1^{-\omega(x)} H(x)^{-1} \frac{\partial f}{\partial u_2}; u'_1, v'; w') \leq \frac{d_{j_1}}{j_1 d} < 1,$$

where  $\beta(u_1^{-\omega(x)} H(x)^{-1} \frac{\partial f}{\partial u_2}; u'_1, v'; w')$  is the  $\beta$  of the polyhedron  $\Delta(u_1^{-\omega(x)} H(x)^{-1} \frac{\partial f}{\partial u_2}; u'_1, v'; w')$ .

In the second case, we get, with the notations of **I.8.3.6 Case 2**

$$p^\alpha \beta(u_1^{-\omega(x)} H(x)^{-1} \frac{\partial f}{\partial u_2}; u'_1, v'; w') \leq \text{ord}_{x'} \left( \frac{\partial \Psi_{j_1}}{\partial U_2}(1, u'_2) + \varphi'(v')^{p^\alpha} \right) \leq \frac{d_{j_1}}{d} + p^{\alpha'},$$

as  $\text{ord}_{x'} \left( \frac{\partial \Psi_{j_1}}{\partial U_2}(1, u'_2) + \varphi'(v')^{p^\alpha} \right) \in p^{\alpha'} \mathbb{N}$  and as

$$\text{ord}_{x'} \left( \frac{\partial \Psi_{j_1}}{\partial U_2}(1, u'_2) + \varphi'(v')^{p^\alpha} \right) > \frac{d_{j_1}}{d} \Rightarrow \text{ord}_{x'} \left( \frac{\partial \Psi_{j_1}}{\partial U_2}(1, u'_2) + \varphi'(v')^{p^\alpha} \right) \notin p^{1+\alpha'} \mathbb{N},$$

this leads to:

$$\beta_3(x') \leq \beta(u_1^{-\omega(x)} H(x)^{-1} \frac{\partial f}{\partial u_2}; u'_1, v'; w') < 1 + \lfloor \frac{d_{j_1}}{j_1 d} \rfloor = 1.$$

*Case 2:*  $\text{in}_{\mu_0}(H(x)^{-1} \frac{\partial f}{\partial u_2})$  is proportional to an  $\omega(x)$ -power.

Say  $\text{in}_{\mu_0}(H(x)^{-1} \frac{\partial f}{\partial u_2}) = \overline{\mu_2}(U_3 + Q(U_1, U_2))^{\omega(x)}$ , then we replace  $u_3$  by  $v = u_3 + Q(u_1, u_2)$  and eventually  $X$  by  $Y = X + \theta$  to get  $\Delta(h; u_1, u_2, v; Y)$  minimal, by the preparation of  $u_1, u_2, u_3, X$ , the left vertices of  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)$  and  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; v)$  are the same and well prepared. So we reach the next case.

*Case 3:*  $\text{in}_{\mu_0}(H(x)^{-1} \frac{\partial f}{\partial u_2}) = \overline{\mu_2} U_3^{\omega(x)}$ .

Then, in the translation  $w' = u'_3 - u'_1{}^a s'$ , we get  $a > B_3(x) - 1$ . Let  $j_1$  the smallest  $j_0$  such that  $U_1^{a(1)} U_1^{a j_0} \Psi_{j_0}$  is not a  $p$ -power. During the preparation at  $x'$ , we will only add a  $p$ -power  $K^p$  to  $u'_1{}^{a(1)+a j_1+d_{j_1}-j_1} \Psi_{j_1}(1, u'_2)$ . When  $a(1) + a_{j_1} + d_{j_1} \not\equiv 0 \pmod{p}$ ,  $\beta(x') \leq \frac{1+d_{j_1}}{j_1 d} \leq 1$ . When  $a(1) + a_{j_1} + d_{j_1} \equiv 0 \pmod{p}$ , let us denote

$$F_i := \frac{\partial \Psi_{j_1}(U_1, U_2)}{\partial \lambda_i}, \quad 4 \leq i \leq s.$$

The  $F_i$  are not all 0. Let us denote  $p_i := v_{P(U_1, U_2)}(F_i)$ . i.e.  $F_i = P(U_1, U_2)^{p_i} F'_i$ ,  $F'_i$  prime to  $P$  when  $F_i \neq 0$ . Let  $q = \inf\{p_i | 4 \leq i \leq s\}$ , then  $\beta_3(x') \leq \frac{q}{j_1}$ , as  $\deg(P) = 2 = 2$ ,  $q \leq \frac{1+d_{j_1}}{d} \leq j_1$ , we get the announced result except if  $q = j_1$ , this means that

$$F_i = \gamma_i P(U_1, U_2)^{j_1}, \quad \gamma_i \in k(x), \quad 4 \leq i \leq s.$$

This implies  $\Psi_{j_1}(U_1, U_2) = \mu P(U_1, U_2)^{j_1} + G^p$ ,  $\mu \in k(x) - k(x)^p$ . As  $k(x')/k(x)$  is separable, we get  $\text{ord}_{v'}(\Psi_{j_1}(1, u'_2) + K^p) \leq j_1$  which leads to  $\beta(x') \leq 1$ .

We now turn to the study and control of the invariants for points  $x'$  away from the first chart. This is done in **I.8.8** (resp. **I.8.9**) when  $x$  is in cases (\*1) and (\*2) (resp. in case (\*3)).

When  $x$  is in cases (\*1) and (\*2) and  $\kappa(x) > 1$ , we have  $U_3 \in \text{VDir}(x)$  unless we are in the special case specified in **I.7**(i). But then we have  $\tau(x) = 2$  and  $\text{VDir}(x) \oplus k(x).U_1 = \langle U_1, U_2, U_3 \rangle$ , so every  $x'$  which is very near  $x$  belongs to the first chart. This proves that if  $x' \in e^{-1}(x)$  is very

near  $x$  and does not belong to the first chart, then  $x' = (X' := \frac{X}{u_2}, u'_1 := \frac{u_1}{u_2}, u'_2 := u_2, u'_3 := \frac{u_3}{u_2})$  and we have  $U_3 \in \text{VDir}(x)$ .

**I.8.8 Lemma.** *Assume that  $x$  is in case (\*1) or (\*2) and the center of  $\mu$  in  $X'$  is the point  $x' = (X', u'_1, u'_2, u'_3)$  defined above. With hypotheses and notations as in I.8.2, we have  $\kappa(x) \leq 1$  or all following statements hold:  $x'$  is in case (\*2),  $H(x') = u_1'^{a(1)} u_2'^{a(1)+a(2)+\omega(x)-p}$  and  $(X', u'_1, u'_2, u'_3)$  is well prepared;  $\mathbf{w}'_1 := (A_1(x'), \beta(x')) = (A_1(x), A_1(x) + \beta(x) - 1)$  and  $\mathbf{w}'_2 := (\alpha_2(x), B(x) - 1)$  is the vertex of smaller second coordinate of  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2; u'_3)$ ; moreover,*

(i) *if  $x$  is in case (\*1), then  $C(x') \leq \frac{\beta(x)}{2}$ . Equality holds only if  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2; u'_3)$  has only two vertices:  $\mathbf{w}'_1$  and  $\mathbf{w}'_2$ , which are the ends of its initial side  $x'_1 + x'_2 = B(x')$ ;*

(ii) *if  $x$  is in case (\*1) and  $\gamma(x) \geq 2$ , then  $\gamma(x') < \gamma(x)$  except if  $(\beta(x) = 2, C(x) = 1)$  and  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2; u'_3)$  has only two vertices:  $(A_1(x), A_1(x) + 1)$  and  $(A_1(x) + 1, A_1(x))$ , which are the ends of its initial side;*

(iii) *if  $x$  is in case (\*2), we have  $C(x') \leq C(x)$ ,  $\gamma(x') \leq \gamma(x)$ .*

*Proof.* We have  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1 u'_2)$  and  $H(x') = u_1'^{a(1)} u_2'^{a(1)+a(2)+\omega(x)-p}$ . Let  $h' := u_2'^{-p} h = X'^p - g'^{p-1} X' + f'$ , where

$$H(x')^{-1} f' = u_3'^{\omega(x)} \phi_0 + \sum_{1 \leq j \leq \omega(x)} u_3'^{\omega(x)-j} u_2'^{-j} \phi_j(u'_1 u'_2, u'_2) \quad (1)$$

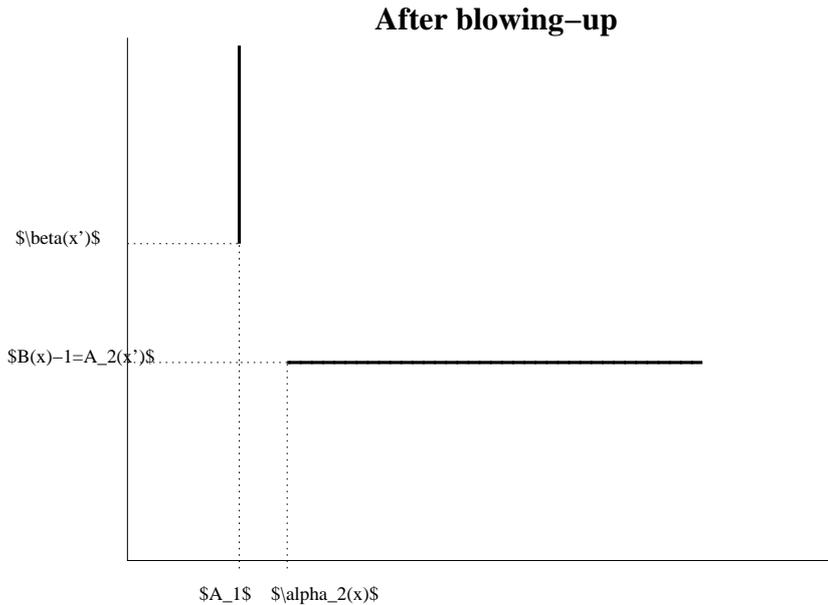
with notations as in I.8.2. Since  $x'$  is the origin of a chart,  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal, hence  $\kappa(x') = 2$  and  $x'$  is in case (\*2) by (1) if  $x'$  is very near  $x$ . The correspondence between vertices of  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)$  and  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2; u'_3)$  is given by

$$(\alpha, \beta) \mapsto (\alpha, \alpha + \beta - 1),$$

so  $(X', u'_1, u'_2, u'_3)$  remains well prepared. The vertex with smaller first (resp. second) coordinate of  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2; u'_3)$  is therefore  $(A_1(x), \beta(x) + A_1(x) - 1)$  (resp.  $(\alpha_2(x), B(x) - 1)$ ). We get

$$C(x') \leq \inf\{\beta(x) + A_1(x) - B(x), \alpha_2(x) - A_1(x)\}. \quad (2)$$

All statements before (i) have been proved. This is visualized in the following figure.



Proof of (i). Since  $\alpha_2(x) \leq B(x)$ , (2) implies that  $C(x') \leq \frac{\beta(x)}{2}$ . Equality holds only if  $C(x') = \beta(x) + A_1(x) - B(x) = \alpha_2(x) - A_1(x)$  and this proves (i).

Proof of (ii). By assumption,  $\beta(x) > 1$ , so  $1 + \lfloor \frac{\beta(x)}{2} \rfloor \leq \lceil \beta(x) \rceil$ , and equality holds only if  $\beta(x) = 2$ . The statement follows from (i) and (2), where  $B(x) - A_1(x) = \alpha_2(x) - A_1(x) = 1$ .

Proof of (iii). This is a consequence of **I.8.3** (b) by symmetry on  $u_1$  and  $u_2$ .

When  $x$  is in case (\*3) and  $\kappa(x) > 1$ , we have  $U_3 \in \text{VDir}(x)$  unless we are in the special case specified in **I.7**(ii). But then we can replace the r.s.p.  $(X, u_1, u_2, u_3)$  by  $(Z, u_1, u_2, v)$  of *loc.cit.* (where  $v \equiv u_3 \pmod{u_1}$ ) and get  $V \in \text{VDir}(x)$ . In other terms, we may assume without loss of generality in the lemma below that, if  $x' \in e^{-1}(x)$  does not belong to the first chart, then  $x' = (X' := \frac{X}{u_2}, u'_1 := \frac{u_1}{u_2}, u'_2 := u_2, u'_3 := \frac{u_3}{u_2})$  and we have  $U_3 \in \text{VDir}(\text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f}{\partial u_2}))$ .

**I.8.9 Lemma.** *Assume that  $x$  is in case (\*3) and the center of  $\mu$  in  $X'$  is the point  $x' = (X', u'_1, u'_2, u'_3)$  defined above. With hypotheses and notations as in **I.8.4**, we have  $\kappa(x) \leq 1$  or all following statements hold:  $x'$  is in case (\*2),  $H(x') = u'_1{}^{\alpha(1)} u'_2{}^{\alpha(1)+1+\omega(x)-p}$  and  $(X', u'_1, u'_2, u'_3)$  is well prepared;  $\mathbf{w}'_1 := (A_1(x'), \beta(x')) = (A_1(x), A_1(x) + \beta 3(x) - 1)$  and  $\mathbf{w}'_2 := (\alpha 3_2(x), B 3(x) - 1)$  is the vertex with smaller second coordinate of  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2, u'_3)$ ;  $C(x') \leq \beta 3(x)$  and  $\gamma(x') \leq \gamma(x)$ ; moreover,*

(i) *if  $B 3(x) - A_1(x) \geq 1$  or if  $\beta 3(x) = 1$ , then  $\gamma(x') < \gamma(x)$ ;*

(ii) *if  $(B 3(x) - A_1(x) < 1$  and  $\beta 3(x) \neq 1$ ), then  $\gamma(x') \leq 2$ ;*

(iii) *if  $(B 3(x) - A_1(x) < 1, \beta 3(x) \neq 1$  and  $\gamma(x) = \gamma(x') = 2$ ), the following holds: either  $(x''$  is in case (\*1) and  $\beta(x'') < 2$ ) or  $(x''$  is in case (\*3) and  $\beta 3(x'') \leq 1)$ , where  $x''$  is the center of  $\mu$  in the blowing up  $X''$  of  $X'$  along  $x'$ .*

Proof. We have  $E' = e^{-1}E = \text{div}(u'_1 u'_2)$  and  $H(x') = u'_1{}^{\alpha(1)} u'_2{}^{\alpha(1)+1+\omega(x)-p}$ . Let  $h' := u_2^{-p} h = X'^p - g'^{p-1} X' + f'$ , where

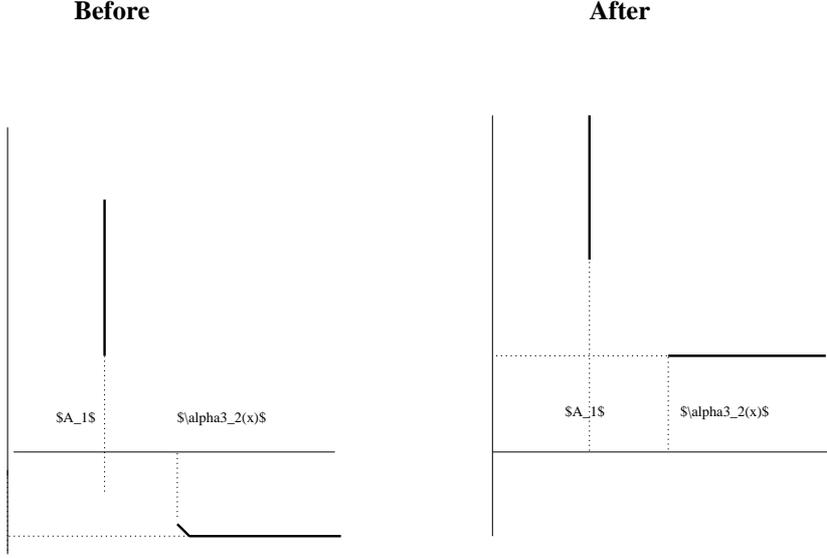
$$H(x')^{-1} f' = u'_3{}^{\omega(x)} \phi_0(u'_1 u'_2, u'_2, u'_3 u'_2) + \sum_{1 \leq j \leq \omega(x)} u'_3{}^{\omega(x)-j} u'_2{}^{-j} \phi_j(u'_1 u'_2, u'_2) \quad (1)$$

with notations as in **I.8.4**. Note that  $\phi_0(u'_1 u'_2, u'_2, u'_3 u'_2) \in \widehat{S}'$  is a unit. Since  $x'$  is the origin of a chart,  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal, hence  $\kappa(x') = 2$  if  $x'$  is very near  $x$  and  $x'$  is in case (\*2) by (1).

Vertices of  $\Delta(H(x)^{-1} u_2^{-1}(f, g^p); u_1, u_2, u_3)$  and  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2, u'_3)$  are in correspondence now given by  $(\alpha, \beta) \mapsto (\alpha, \alpha + \beta - 1)$ , so  $(X', u'_1, u'_2, u'_3)$  is well prepared (it is only used here that no “left” vertex of the former polygon is solvable). The vertex with smaller first (resp. second) coordinate of  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2, u'_3)$  is therefore  $(A_1(x), \beta 3(x) + A_1(x) - 1)$  (resp.  $(\alpha 3_2(x), B 3(x) - 1)$ ). We get

$$C(x') \leq \inf\{\beta 3(x) + A_1(x) - B 3(x), \alpha 3_2(x) - A_1(x)\}. \quad (2)$$

We have  $\kappa(x) \leq 1$  by **I.8.5** if  $A_1(x) - B 3(x) \geq 0$ , so  $C(x') \leq \beta 3(x)$  otherwise by (2), from which  $\gamma(x') \leq \gamma(x)$  immediately follows. This proves all statements before (i). For all remaining statements, it can be assumed that  $A_1(x) - B 3(x) < 0$



**I.8.9.1** Proof of (i). This is an obvious consequence of (2) above.

**I.8.9.2** Proof of (ii). We have  $0 < B3(x) - A_1(x) < 1$ ; in particular,  $\alpha3_2(x) - A_1(x) < 2$  since  $\beta3_2(x) \geq -1$ . By (2) above, this implies  $C(x') < 2$ , so  $\gamma(x') \leq 2$ .

**I.8.9.3** Proof of (iii). Assume that  $\gamma(x') = 2$ , i.e.  $1 \leq C(x') < 2$ . In particular we must have  $\beta3_2(x) < 0$  by the above argument. We discuss according to  $x''$ .

If  $x''$  does not belong to the first chart, equation (2) in the proof of **I.8.8** gives

$$C(x'') \leq \beta(x') - A_2(x') - C(x').$$

We have  $A_2(x') = A_1(x) + (B3(x) - A_1(x) - 1) > A_1(x) - 1$  and  $\beta(x') = A_1(x) + \beta3_2(x) - 1 < A_1(x) + 1$  (since  $\gamma(x) = 2$ ), so  $C(x'') < 2 - C(x') \leq 1$ .

Similarly, if  $x''$  is the origin of the first chart, we have by symmetry on  $u_1$  and  $u_2$ :

$$C(x'') \leq \alpha3_2(x) - A_1(x') - C(x') = (B3(x) - A_1(x)) - \beta3_2(x) - C(x') < 1,$$

since  $\beta3_2(x) > -1$  and  $C(x') \geq 1$ .

Finally, let  $x''$  belong to the first chart and be distinct from the origin. First note that  $A_1(x') > 0$  if  $C(x') = 1$ , since  $C(x') \leq A_1(x') + \beta3_2(x) - 1$ , so  $x''$  is not the special case specified in **I.8.3**(e).

By **I.8.3**(i),  $B(x') < \frac{\text{ord}_{(u'_1, u'_2)}(H(x')^{-1}g'^p)}{\omega(x)}$  if  $\kappa(x') > 1$ . Consistently with **I.8.2.1** and (1) above, we denote

$$J'_0 := \{j, 1 \leq j \leq \omega(x) \mid \frac{\text{ord}_{(u'_1, u'_2)}\phi'_j}{j} = B(x')\},$$

where  $\phi'_j := u'_2{}^{-j}\psi_j(u'_1u'_2, u'_2)$ , and  $\Phi'_j := \text{cl}_{jB(x')}\phi'_j = U_1{}^{a'_j(1)}U_2{}^{a'_j(2)}\Psi'_j$  for  $j \in J'_0$ .

We claim that for all  $j \in J'_0$ ,

$$d'_j := \deg \Psi'_j \leq j. \quad (1)$$

Namely, if  $d'_j > 0$ , there exist two monomials  $M_1 := u_1^a u_2^{b+2d'_j}$ ,  $M_2 := u_1^{a+d'_j} u_2^b$  appearing with nonzero coefficient in the expansion of  $\phi_j(u_1, u_2)$ , where  $2a + b + 2d'_j - j = jB(x')$ . Since  $(A_1(x), A_1(x) + \beta 3(x) - 1)$  is the vertex with smaller first coordinate of  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2; u'_3)$ , we have  $a \geq jA_1(x)$  and

$$2a + b + 2d'_j - j \leq j(2A_1(x) + \beta 3(x) - 1),$$

so

$$b + 2d'_j \leq j\beta 3(x) < 2j. \quad (2)$$

Then  $2(d'_j - j) < -b$  which proves (1), since  $b \geq -1$ .

We apply **I.8.3** to the well-prepared r.s.p.  $(X', u'_1, u'_2, u'_3)$  at  $x'$ . Since  $C(x') < 2$ , **I.8.3**(e) gives  $\beta(x'') < 2$  if  $x''$  is in case (\*1).

Assume from now on that  $x''$  is in case (\*3). By **I.8.3**(iii)-(vi), we have  $\beta 3(x'') \leq 1$  unless  $x'$  is in case **I.8.3**(vi) by (1). If  $x'$  is in case **I.8.3**(vi), an explicit computation gives

$$f'_2 := u_2^{-\omega(x)} H(x)^{-1} \frac{\partial f}{\partial u_2} = H(x')^{-1} (u'_2 \frac{\partial f'}{\partial u'_2} - u'_1 \frac{\partial f'}{\partial u'_1} - u'_3 \frac{\partial f'}{\partial u'_3}).$$

Since  $\text{cl}_{\omega(x)} f'_2 \equiv \frac{\partial \psi_0}{\partial u_2} U_3^{\omega(x)} \pmod{(U'_1, U'_2)}$ , we may take  $G' := \text{cl}_{\mu_0, \omega(x)} f'_2$  in **I.8.3**(vi). We claim that  $C(G') < 1$ , which implies  $\gamma(x'') = 1$  by **I.8.3**(vi).

To prove the claim, let  $(a_1, b_1)$  and  $(a_2, b_2)$ ,  $a_1 \leq a_2$ , be those two vertices of the polygon  $\Delta(H(x)^{-1} \frac{\partial f}{\partial u_2}; u_1, u_2; u_3)$  (hence  $b_2 \geq 0$ ) whose transforms  $(a_1, a_1 + b_1 - 1)$  and  $(a_2, a_2 + b_2 - 1)$  give the ends of the initial side  $x'_1 + x'_2 = B(x')$  of  $\Delta(f'_2; u'_1, u'_2; u'_3)$ . By definition and with notations as in **I.8.3**(vi),  $C(G') = a_2 - a_1$ . We do similar computations as in the above claim ( $d'_j \leq j$ ) w.r.t.  $f'_2$ . This time, no division by  $u_2$  occurs in the computation and we get  $a_1 \geq A_1(x)$ ,  $B(x') = 2a_1 + b_1 - 1 = 2a_2 + b_2 - 1 \leq 2A_1(x) + \beta 3(x) - 1$ , so  $b_1 \leq \beta 3(x) < 2$ ; this yields  $2(a_2 - a_1) = b_1 - b_2 < 2$  and  $C(G') = a_2 - a_1 < 1$ .

We now turn to proving that  $\kappa(x) \leq 1$  in some cases when  $\gamma(x) = 1$ .

**I.9 Theorem.** *Assume that  $\kappa(x) = 2$ , condition (\*) holds and  $(X, u_1, u_2, u_3)$  is well prepared. We have  $\kappa(x) \leq 1$  provided one of the following conditions is satisfied:*

- (i)  $x$  is in case (\*1) or (\*2) and  $\gamma(x) = 1$ ;
- (ii)  $x$  is in case (\*3) and  $\beta 3(u_1, u_2, u_3) < 1 - \frac{1}{\omega(x)}$ .

*Proof.* First assume that  $A_1(x) \geq 1$  and let  $\mathcal{C} := V(X, u_1, u_3)$ . If  $x$  is in case (\*3), then  $\mathcal{C}$  is permissible of the second kind by **I.8.6**, which also reduces theorem **I.9** to  $0 < A_1(x) < 1$ ,  $\beta 3(x)$  being unchanged. If  $x$  is in case (\*1) or (\*2), then similarly  $\mathcal{C}$  is permissible of the first kind and we get reduced to  $0 \leq A_1(x) < 1$ ,  $\beta(x)$  being unchanged (see argument in the beginning of the proof of **I.8.1**).

If  $(A_1(x) = 0$  and  $x$  is in case (\*1)), then  $\beta(x) = 1$ , so we have  $\text{VDir}(x) \equiv \langle U_2, U_3 \rangle \pmod{(U_1)}$ , whence  $\kappa(x) = 0$  by chapter 2, **II.2**. In particular, it can actually be assumed that  $0 < A_1(x) < 1$  as in case (\*3). Finally if  $x$  is in case (\*2), it can be assumed that  $0 \leq A_2(x) < 1$  by symmetry on  $u_1$  and  $u_2$ . Moreover,  $(A_1(x), A_2(x)) \neq (0, 0)$  since  $B(x) \geq 1$ .

Let

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_{n-1} \leftarrow X_n \leftarrow \cdots$$

be the quadratic sequence along  $\mu$ , i.e.  $X_n$  is the blowing up along the center  $x_{n-1}$  of  $\mu$  in  $X_{n-1}$  for  $n \geq 0$ . First assume that  $x_0 := x$  satisfies furthermore the following extra two assumptions:

- (a)  $0 \leq A_1(x_0) < 1$ , and
- (b)  $\beta(x_0) < 1$  if  $x_0$  is in case (\*2).

We prove by induction on  $n \geq 0$  that  $\kappa(x_n) \leq 1$  or  $x_n$  also satisfies the assumptions of **I.9**, together with (a) and (b). Moreover, the invariant  $i_n := (A_1(x_n), d(x_n))$  satisfies  $i_n \leq i_{n-1}$  for  $n \geq 1$ , where  $d(x_n) := \beta(x_n)$  (resp.  $d(x_n) := \beta\mathfrak{B}(x_n)$ ) if  $x$  is in case (\*1) or (\*2) (resp. in case (\*3)), and the ordering is lexicographical.

To prove this claim, first note that  $\kappa(x) \leq 1$ , or  $\gamma(x_n) = \gamma(x_{n-1}) = 1$  by **I.8.1** and **I.8.3(d)**, **I.8.8** (resp. and **I.8.7(b)**, (d), **I.8.9**) if  $x_{n-1}$  is in case (\*1) or (\*2) (resp. in case (\*3)). If  $x_n$  does not belong to the first chart,  $\kappa(x_n) \leq 1$  or  $x_n$  is in case (\*2) by **I.8.8** or **I.8.9** which give

$$i_n = (A_1(x_{n-1}), A_1(x_{n-1}) + \beta(x_{n-1}) - 1) \text{ (resp. } = (A_1(x_{n-1}), A_1(x_{n-1}) + \beta\mathfrak{B}(x_{n-1}) - 1))$$

and the claim is proved (with  $i_n < i_{n-1}$ ), since  $A_1(x_n) < 1$ . Assume now that  $x_n$  belongs to the first chart. By **I.8.3** (resp. **I.8.7**), we have

$$A_1(x_n) = B(x_{n-1}) - 1 \leq A_1(x_{n-1}) + \beta(x_{n-1}) - 1 \text{ (resp. } = B\mathfrak{B}(x_{n-1}) - 1 \leq A_1(x_{n-1}) + \beta\mathfrak{B}(x_{n-1}) - 1),$$

so the claim is proved provided we show that  $\beta(x_n) < 1$  (resp.  $\beta\mathfrak{B}(x_n) < 1 - \frac{1}{\omega(x)}$ ) if  $x_n$  is in case (\*2) (resp. in case (\*3)). Note that moreover  $i_n < i_{n-1}$  except if  $(x_n$  and  $x_{n-1}$  are in case (\*1) and  $\beta(x_{n-1}) = \beta(x_n) = 1$  and  $x_n$  is rational over  $x_{n-1}$ ).

The claim follows directly from **I.8.3(b)** if  $x_n$  is in case (\*2). Assume that  $x_n$  is in case (\*3). We may assume that  $B\mathfrak{B}(x_{n-1}) > 1$  if  $x_{n-1}$  is in case (\*3) by **I.8.7(b)**, in which case the result follows from **I.8.7(i)**, (iii). If  $x_{n-1}$  is in case (\*1) or (\*2), we have  $\beta(x_n) < 1$  by **I.8.3(e)** except possibly if  $(x_{n-1}$  is in case (\*1) and  $\beta(x_{n-1}) = C(x_{n-1}))$ . By **I.8.3(c)**,  $x_n$  must then be inseparable over  $x_{n-1}$  so **I.8.3(e)** also yields  $\beta(x_n) < 1$ . By definition of  $\beta(x_n)$ , there exists some index  $j$ ,  $1 \leq j \leq \omega(x)$ , such that the monomial  $u_{3,n}^{\omega(x)-j} u_{1,n}^{jA_1(x_n)} u_{2,n}^{j\beta(x_n)}$  appears with nonzero coefficient in the expansion of  $H(x_n)^{-1} f_n$ , where  $(X_n, u_{1,n}, u_{2,n}, u_{3,n})$  is some r.s.p. at  $x_n$  which is well prepared. So we have  $j\beta(x_n) \leq j - 1$  and

$$\beta\mathfrak{B}(x_n) \leq \frac{j-2}{j} = 1 - \frac{2}{j} < 1 - \frac{1}{\omega(x)} \quad (1)$$

and the claim is proved.

Remember that, if  $\kappa(x) > 1$ , then  $i_n = i_{n-1}$  only if both  $x_{n-1}$  and  $x_n$  are in case (\*1),  $\beta(x_{n-1}) = \beta(x_n) = 1$  and  $x_n$  is rational over  $x_{n-1}$ . If this happens for all  $n \geq n_0$  for some  $n_0 \geq 0$ , some formal curve  $\mathcal{C}' = V(\widehat{X}_{n_0}, \widehat{u}_{2,n_0}, \widehat{u}_{3,n_0})$  is contained in  $\Sigma_p(X_{n_0})$ , a contradiction since  $\mathcal{C}' \not\subset E_{n_0}$ . Therefore  $i_n$  eventually drops so that  $\kappa(x) \leq 1$ .

We now turn to the general case, so  $x$  is now in case (\*2) and we have  $C(x) < 1$ ,  $\beta(x) \geq 1$ . If  $x_n$  is in case (\*2) for all  $n \geq 0$ , all points  $x_n$  are either at the origin of the first chart, or the unique point away from the first chart. By standard arguments, there exists  $n_0 \geq 0$  such that  $C(x_{n_0}) = 0$ . Otherwise, there exists a smaller  $n_0 \geq 1$  such that  $x_{n_0-1}$  is in case (\*2) and  $x_{n_0}$  is either in case (\*1) or in case (\*3). By **I.8.3(b)**, (ix) and **I.8.8(iii)**, we have  $\beta(x_{n_0}) < 1$  in the latter case. The argument in (1) above shows that  $\beta\mathfrak{B}(x_{n_0}) < 1 - \frac{1}{\omega(x)}$  if  $x_{n_0}$  is in case (\*3), so  $x_{n_0}$  satisfies the assumption of **I.9** in any case.

Summing up, we have  $\kappa(x) \leq 1$  or the following holds: by blowing up permissible curves (argument at the beginning of the proof of this theorem), there exists a composition of blowing ups of permissible curves  $e' : X' \rightarrow X_{n_0}$  such that either  $(x'$  is in case (\*2),  $C(x') = 0$  and

$0 < \max\{A_1(x'), A_2(x')\} < 1$ ) or ( $x'$  is in case (\*1) or (\*3),  $0 < A_1(x') < 1$  and  $x'$  satisfies the assumptions of **I.9**). In the former case, note that  $\beta(x') = A_2(x') < 1$ . Hence in every case,  $x'$  verifies (a) and (b) above, so  $\kappa(x) \leq 1$  and the proof is complete.

We now prove that  $\kappa(x) \leq 1$  in some special cases when ( $x$  is in case (\*1) and  $\gamma(x) = 2$ ).

**I.10 Theorem.** *Assume that  $\kappa(x) = 2$ ,  $x$  is in case (\*1) and  $(X, u_1, u_2, u_3)$  is well prepared. We have  $\kappa(x) \leq 1$  provided one of the following conditions is satisfied:*

- (i)  $\beta(u_1, u_2, u_3) < 2$ ;
- (ii)  $x$  is a “disaster” (as specified in **I.8.3**(e)), i.e.  $(\omega(x) = p = 2$  and

$$H(x)^{-1}f = \mu_2 u_3^2 + \mu_1 u_3 u_2^2 + u_1 \varphi, \quad (\text{Dis})$$

with  $\mu_1 \mu_2 \in \widehat{S}$  invertible and  $\text{ord}_{\eta(x)} \varphi \geq 1$ ).

Proof of (i). Let  $e : X' \rightarrow X$  be the blowing up along  $x$  and  $x'$  be the center of  $\mu$  in  $X'$ .

If  $x'$  does not belong to the first chart, then  $\kappa(x) \leq 1$  by **I.8.8**(i) and **I.9**(i).

If  $x'$  is not rational over  $x$ , then  $\beta(x') < 1$  by **I.8.3**(ix), so  $\kappa(x) \leq 1$  by **I.9**(i) or **I.9**(ii) (using equation (1), proof of **I.9**, if  $x'$  is in case (\*3)).

If  $x'$  is rational over  $x$  and belongs to the first chart, then  $x'$  verifies assumption **I.10**(i) by **I.8.3**(b), (c) and (e), so we iterate the process. An argument already used several times -e.g. in the proof of **I.9** above, right after equation (1)- shows that either  $\kappa(x) \leq 1$ , or there exists some formal curve  $\mathcal{C} = V(\widehat{X}, \widehat{u}_2, \widehat{u}_3)$  contained in  $\Sigma_p(X)$ , a contradiction since  $\mathcal{C} \not\subseteq E$ .

Proof of (ii). We have  $A_1(x) = 0$  and  $\beta(x) = 2$ . By **I.7**,  $U_3 \in \text{VDir}(x)$ . Let  $e : X' \rightarrow X$  be the blowing up along  $x$  and  $x'$  be the center of  $\mu$  in  $X'$ .

If  $x'$  does not belong to the first chart, then  $\omega(x') = 1$  by (Dis), so  $\kappa(x) = 0$  in this case. In particular, if  $\text{ord}_{\eta(x)}(u_1 \varphi) = 2$ , we have  $\langle U_1, U_3 \rangle \subseteq \text{VDir}(x)$ , so  $\kappa(x) = 0$ . From now on, we assume that  $\text{ord}_{\eta(x)}(u_1 \varphi) \geq 3$  and  $x'$  belongs to the first chart.

Suppose that  $x'$  is not the origin of the first chart. If  $x'$  is in case (\*1), then  $x'$  satisfies **I.10**(i) except possibly if  $C(x) = 2$  and  $x'$  is rational over  $x$  by **I.8.3**(e). But  $C(x) = B(x) = 2$  in this case, so  $\text{cl}_2 \varphi \in k(x)[U_1, U_2]_2$  and we get  $\omega(x') = 1$  from (Dis). If  $x'$  is in case (\*3), then  $x'$  is inseparable over  $x$  by **I.8.3**(c); then **I.8.3**(e) yields  $\beta(x') < 1$  (so  $\beta 3(x') < 1 - \frac{1}{\omega(x)}$  and  $\kappa(x) \leq 1$  by **I.9**(ii)) except possibly if  $C(x) = 2$  and  $d = 2$ . Hence  $\text{cl}_2 \varphi \in k(x)[U_1, U_2]_2$  and we also get  $\omega(x') = 1$  from (Dis).

From now on,  $x'$  is the origin of the first chart. In particular,  $(X' := \frac{X}{u_1}, u'_1 := u_1, u'_2 := \frac{u_2}{u_1}, u'_3 := \frac{u_3}{u_1})$  is well prepared and  $x'$  is in case (\*1) if it is very near  $x$ . We get

$$f' = u'_1{}^{a(1)} (\mu_2 u'_3{}^2 + \mu_1 u'_1 u'_3 u'_2{}^2 + u'_1 \Phi'(u'_2, u'_3) + u'_1{}^2 \varphi'),$$

where  $\Phi' \in k(x)[u'_2, u'_3]_{\leq 2}$  and  $\varphi' \in k(x)[[u'_1, u'_2, u'_3]]$ . We have  $\omega(x') = 1$  unless  $\Phi' \in k(x)[u'_2, u'_3]_2$ , which we assume now. We have  $\beta(x') = 1$ , so  $\kappa(x) \leq 1$  by **I.9**, unless  $\Phi' \in k(x)[u'_3]$ , which we also assume from now on. Then the curve  $\mathcal{C}' := V(X', u'_3, u'_1)$  is permissible of the first kind and we perform the blowing up  $e' : X'' \rightarrow X'$  along  $\mathcal{C}'$ . Let  $x''$  be the center of  $\mu$  in  $X''$ . Since  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2; u'_3)$  has no solvable vertex, we have  $\text{VDir}(x') = \langle U'_1, U'_3 \rangle$  if  $\varphi'$  is a unit, so  $\kappa(x) \leq 1$  in this case. Otherwise,  $\text{VDir}(x') = k(x').U'_3$ , so we need only consider the case when  $x''$  is the origin of the first chart. If  $\omega(x'') = 2$ , then tracing back to  $X$ , we had

$$\varphi = \Phi'(u_3) + u_3 \Phi_1(u_1, u_2, u_3) + u_2^2 \Phi_2(u_1, u_2) + \psi,$$

where  $\Phi_1 \in k(x)[u_1, u_2, u_3]_3$ ,  $\Phi_1(u_1, 0, 0) = 0$ ,  $\Phi_2 \in k(x)[u_1, u_2]_1$  and  $\text{ord}_{\eta(x)}\psi \geq 4$ . Hence

$$\varphi' \equiv u_2'^2 \Phi_2(1, u_2') \pmod{(u_1', u_3' u_2', u_3'^2)}.$$

Therefore  $x''$  is in case (\*1),  $(X'' := \frac{X'}{u_1'}, u_1'' := u_1', u_2'' := u_2', u_3'' := \frac{u_3'}{u_1'})$  is well prepared and  $A_1(x'') = 0$ . If  $\Phi_2 \neq 0$ , we get  $\beta(x'') = 1$  or  $\beta(x'') = \frac{3}{2}$ , so  $\kappa(x'') \leq 1$  by (i). Finally, if  $\Phi_2 = 0$ ,  $x''$  satisfies again the assumptions of (ii).

The conclusion now follows as in the proof of (i): either  $\kappa(x) \leq 1$ , or the curve  $\mathcal{C} = V(X, u_2, u_3)$  is contained in  $\Sigma_p(X)$ , a contradiction since  $\mathcal{C} \not\subset E$ .

### I.11 Proof of Theorem I.8.

Let

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots X_{n-1} \leftarrow X_n \leftarrow \cdots$$

be the quadratic sequence along  $\mu$ , i.e.  $X_n$  is the blowing up along the center  $x_{n-1}$  of  $\mu$  in  $X_{n-1}$  for  $n \geq 0$ . We assume that  $\kappa(x) = 2$ ,  $\kappa(x) > 1$  and derive a contradiction. By **I.8.3**, **I.8.7**, **I.8.8** and **I.8.9**, we have  $\gamma(x_n) \leq \gamma(x_{n-1})$  for each  $n \geq 1$  unless we are in the special case specified in **I.8.7(b)**:  $x_{n-1}$  is in case (\*3) with  $\gamma(x_{n-1}) = 2$ , and  $x_n$  satisfies the assumptions of **I.10**. By **I.10**, this is a contradiction since  $\kappa(x_n) \leq 1$  in this case. Therefore there exists  $n_0 \geq 1$  such that  $\gamma(x_n) = \gamma(x_{n-1})$  for  $n \geq n_0$ . Let  $\gamma(\mu)$  be this limit value of  $\gamma(x_n)$ . Without loss of generality, it can be assumed that  $n_0 = 0$ .

First assume that  $\gamma(\mu) = 1$ . By **I.9**, we are done unless  $x_n$  is in case (\*3) for all  $n \geq 0$ . Since  $\beta_3(x_n) < 1$  for each  $n \geq 0$ , we have  $\beta(x_n) \leq 1$ . Moreover,  $\beta_3(x_n) < 1 - \frac{1}{\omega(x)}$  if  $\beta(x_n) < 1$  (see proof of **I.9**, equation (1)), so  $\kappa(x) \leq 1$  by **I.9(ii)** unless  $\beta(x_n) = 1$  for all  $n \geq 0$ . By **I.8.7(v)**,  $x_n$  is rational over  $x_{n-1}$  for all  $n \geq 1$ . Therefore there exists some formal curve  $\mathcal{C} = V(\widehat{X}, \widehat{u}_2, \widehat{u}_3)$  contained in  $\Sigma_p(X)$ : a contradiction since  $\mathcal{C} \not\subset E$ .

Assume that  $\gamma(\mu) \geq 3$ . By **I.8.8(ii)** and **I.8.9(i)**, (ii),  $x_n$  always belong to the first chart of the blowing up along  $x_{n-1}$  provided  $m(x_n) = 1$  for some  $n \geq 1$ . Then  $x_n$  is rational over  $x_{n-1}$  for all  $n \geq 1$  by **I.8.3(viii)** and **I.8.7(b)**, (iv). We conclude as in the case  $\gamma(\mu) = 1$  unless  $x_n$  is in case (\*2) for all  $n \geq 0$ . By standard arguments, we then get  $C(x_n) = 0$  for  $n \gg 0$ , a contradiction, since  $\gamma(\mu) = 3$ .

Assume from now on that  $\gamma(\mu) = 2$ . The argument of the previous paragraph settles the case when  $x_n$  is in case (\*2) for all  $n \gg 0$ , or when ( $x_n$  is rational over  $x_{n-1}$  and  $m(x_n) = 1$ ) for all  $n \gg 0$ . From now on, there exists infinitely many values of  $n \geq 1$  such that ( $m(x_{n-1}) = 1$  and either  $m(x_n) = 2$  or  $x_n$  is not rational over  $x_{n-1}$ ).

We first sum up some of the conclusions of **I.8.3**, **I.8.7** and **I.10**; since  $\kappa(x) > 1$ , the following holds when  $x_n$  is not rational over  $x_{n-1}$ :

- (a)  $x_{n-1}$  is in case (\*1) with  $\beta(x_{n-1}) = C(x_{n-1}) = 2$ ,  $x_n$  is in case (\*3) and either  $\beta_3(x_n) = 1$  or ( $p = 2$  and  $\beta_3(x_n) \leq \frac{3}{2}$ );
- (b)  $x_{n-1}$  is in case (\*3),  $x_n$  is in case (\*3),  $p = 2$  and  $\beta_3(x_n) < \frac{3}{2}$ .

The situation we want to reduce to is that in the lemma below. The argument is somewhat more involved when  $p = 2$ , due to the characteristic two version of **I.8.3(vii)**, **I.8.7(iv)** and (a), (b) above.

**I.11.1 Lemma.** *Assume that  $\kappa(x) > 1$ ,  $x$  is in case (\*1) with  $\beta(x) = 2$  and  $x_1$  is in case (\*2). Then  $p = 2$  and  $x_2$  satisfies the assumptions of **I.11.2** below.*

Proof. By **I.8.8**(ii), we have  $\kappa(x) \leq 1$  unless  $x_1$  has  $C(x_1) = 1$  and  $x_2$  is in case (\*1) or (\*3). By **I.8.3**(e), we have  $\beta(x_2) < 2$  if  $x_2$  is in case (\*1): a contradiction by **I.10**(i). Note that  $x_1$  then satisfies all assumptions in **I.8.3**(vii) from which the conclusion follows.

**I.11.2 Lemma.** *Assume that  $\kappa(x) > 1$  and  $x$  is in case (\*3) with either  $\beta_3(x) < \frac{3}{2}$ , or ( $p = 2$ ,  $\beta_3(x) = \frac{3}{2}$  and the monomial  $u_3^{\omega(x)-j_1} u_1^{j_1 A_1(x)} u_2^{1+\frac{3}{2}j_1}$  appears with nonzero coefficient in the expansion of  $H(x)^{-1} u_2 \frac{\partial f}{\partial u_2}$  for some  $j_1$ ,  $1 \leq j_1 \leq \omega(x)$ ,  $j_1 \equiv 0 \pmod{4}$ , and  $(X, u_1, u_2, u_3)$  is well-prepared).*

*Let  $x' := x_1$  be the center of  $\mu$  in the blowing up  $X' := X_1$  along  $x$ . Exactly one of the following properties holds:*

(i)  *$x'$  is in case (\*1) with  $\beta(x') = 2$  and the monomial  $u_3^{\omega(x)-1} u_1^{A_1(x_1)} v'^2$  appears with nonzero coefficient in the expansion of  $H(x')^{-1} f'_{Z'}$ , where  $(Z', u'_1, v', u'_3)$  is well-prepared at  $x'$ ;*

(ii)  *$x'$  satisfies again the assumptions of **I.11.2**;  $\beta(x') \leq \beta(x)$  and equality is strict if  $x'$  is not rational over  $x$ .*

Proof. Recall the definition of  $J_0$  in **I.8.6.1** and lemma **I.8.6.2**. Remark that  $\beta_3(x) \leq \frac{3}{2}$  implies that

$$\forall j_0 \in J_0, \frac{1+d_{j_0}}{j_0} \leq 2. \quad (1)$$

Note that equality possibly holds only if ( $j_0 = 1$  and  $\beta_3(x) = 1$ ): if  $j_0 \geq 3$ , then  $\frac{1+d_{j_0}}{j_0} \leq \frac{3}{2} + \frac{1}{3} < 2$ ; if  $j_0 = 2$ , then  $\frac{1+d_{j_0}}{j_0} = 2$  implies  $\beta_3(x) = \frac{3}{2}$ , so  $p = 2$ ,  $1+d_2 = 4$ , so the monomial  $u_3^{\omega(x)-2} u_1^{2A_1(x)} u_2^4$  cannot appear with nonzero coefficient in the expansion of  $H(x)^{-1} u_2 \frac{\partial f}{\partial u_2}$ , since  $p = 2$ .

**I.11.2.1** First assume that  $x'$  is not rational over  $x$ . We have  $\gamma(x') \leq 1$  if  $J_0 \not\subset p\mathbb{N}$  by **I.8.7**(i): a contradiction. If  $J_0 \subset p\mathbb{N}$ , we have  $\kappa(x) \leq 1$  by **I.8.7**(iv) and **I.10**(i) if  $x'$  is in case (\*1): a contradiction.

Suppose finally that  $J_0 \subset p\mathbb{N}$  and  $x'$  is in case (\*3). Then  $\beta(x') \leq \frac{3}{2}$  by **I.8.7**(ii), so  $\beta_3(x') < \frac{3}{2}$ . By **I.8.7**(ii) and **I.8.6.2**, we have

$$\beta(x') - \beta(x) \leq \frac{1}{p} - \frac{\beta(x)}{2} < 0.$$

**I.11.2.2** Assume that  $x'$  is in case (\*2). By **I.8.9**(i), (iii) and **I.10**(i), we have  $\kappa(x) \leq 1$  unless possibly ( $x''$  is in case (\*3) and  $\beta_3(x'') = 1$ ), where  $x''$  is the center of  $\mu$  in the blowing up  $X'' := X_2 \rightarrow X'$  of  $X'$  along  $x'$ .

This holds only if equality  $d'_j = \deg \Psi'_j = j$  holds for some  $j \in J'_0$  in **I.8.9.3**(1). Since  $\beta_3(x) \leq \frac{3}{2}$ , (2) of *loc.cit.* gives

$$2(d'_j - j) \leq 1 - \frac{j}{2},$$

so  $d'_j = j$  implies  $j \in \{1, 2\}$ . Since  $1 \leq \beta_3(x) < 2$ , we had

$$(j, j\beta_3(x)) \in \{(1, 1), (2, 2), (2, 3)\}.$$

The comments right after (1) above discard the value (2, 3) in the above list, so we had  $\beta_3(x) = 1$ : a contradiction by **I.8.9**(i), since  $\gamma(x') = \gamma(\mu) = 2$ .

**I.11.2.3** Assume that  $x'$  is in case (\*1). By **I.8.7**(ii), we have  $\beta(x') < 2$  (so  $\kappa(x) \leq 1$  by **I.10**(i)) unless equality holds in (1) above (so  $1 \in J_0$ ) or  $\beta_3(x) = 1$ . If  $\beta_3(x) = 1$ , then checking through **I.8.7.1**,  $\beta(x') = 2$  possibly holds only if  $p = 2$  and either  $x$  is in cases 1b or 3b with  $j_0 = 1$  in

*loc.cit.* (so  $x'$  satisfies **I.11.2**(i) with  $A_1(x') = 0$ ), or  $\omega(x) = p = 2$  (end of **I.8.7.1**) in which case  $x'$  also satisfies **I.11.2**(i) with  $A_1(x') = 0$  (since  $\mu_2 \neq 0$  in *loc.cit.*).

**I.11.2.4** Assume finally that  $x'$  is in case (\*3). By **I.8.7**(i), (ii), (iii) and **I.8.6.2**, we have  $\beta_3(x') \leq \beta_3(x)$ ,  $\beta(x') \leq \beta(x)$ , so (ii) will hold except possibly if  $\beta_3(x') = \beta_3(x) = \frac{3}{2}$ .

In this case,  $H(x')^{-1} \frac{\partial f'}{\partial u_2'}$  is the strict transform of  $H(x)^{-1} \frac{\partial f}{\partial u_2}$  since  $x'$  is rational over  $x$ . Then  $(A_1(x'), \frac{3}{2})$  is a vertex of  $(\Delta(H(x')^{-1} \frac{\partial f'}{\partial u_2'}; u_1', v'; u_3'))$ , where  $(Z', u_1', v', u_3')$  is well-prepared at  $x'$ , and the monomial  $u_3'^{\omega(x)-j_1} u_1'^{j_1 A_1(x)} u_2'^{\frac{3}{2} j_1}$  appears with nonzero coefficient in the expansion of  $H(x')^{-1} \frac{\partial f'}{\partial u_2'}$  for some  $j_1$ ,  $1 \leq j_1 \leq \omega(x)$  (see **I.8.7.1**, **I.8.7.8** and **I.8.7.9**). Necessarily, we have  $j_1 \equiv 0 \pmod{4}$ , since  $p = 2$  in this case.

**I.11.3 Lemma.** *Assume that  $\kappa(x) > 1$ ,  $x$  is in case (\*1) with  $\beta(x) = 2$  and the monomial  $u_3^{\omega(x)-1} u_1^{A_1(x)} u_2^2$  appears with nonzero coefficient in the expansion of  $H(x)^{-1} f$ , where  $(X, u_1, u_2, u_3)$  is well prepared.*

*Let  $x' := x_1$  be the center of  $\mu$  in the blowing up  $X' := X_1$  along  $x$ . Then  $x'$  is rational over  $x$  and satisfies again the assumptions of **I.11.3**.*

**Proof.** We have  $C(x) \leq \beta(x) = 2$ . With notations as in **I.8.2.1**, we have  $C(x) < 2$  or  $(C(x) = 2$  and  $1 \in J_0)$ .

If  $x'$  is again in case (\*1), then  $x'$  satisfies equation (Dis) or  $\beta(x') < 2$  whenever  $C(x) < 2$  by **I.8.3**(e). Otherwise  $1 \in J_0$ , so  $\beta(x') \leq 2$  by **I.8.3**(iii) and  $x'$  satisfies again the assumption of the lemma if equality holds (in which case  $x'$  is rational over  $x$ ). The conclusion follows from **I.10**.

If  $x'$  is in case (\*2), then  $\beta(x) = 2$  and  $x'$  satisfies the conclusion of **I.8.8**(ii). In particular,  $C(x') = 1$ . Let  $x''$  be the center of  $\mu$  in the blowing up  $X'' := X_2 \rightarrow X'$  of  $X'$  along  $x'$ . The conclusion of **I.8.8**(ii) implies the following: if  $x''$  is in case (\*2), we have  $C(x'') = 0$ ; if  $x''$  is in case (\*1) or (\*3), then  $x'$  satisfies the assumption in **I.8.3**(iii) whose conclusion gives  $\gamma(x'') = 1$  or  $x''$  satisfies equation (Dis) (so  $\kappa(x) \leq 1$  by **I.10**(ii)). In all cases, this contradicts the assumption  $\gamma(\mu) = 2$ .

If  $x'$  is in case (\*3), then  $x'$  is inseparable over  $x$  by **I.8.3**(c). Then  $\beta_3(x') < 1$  by **I.8.3**(iii), once again a contradiction, since  $\gamma(\mu) = 2$ .

We can now conclude the proof of theorem **I.8**:

**I.11.4** Recall equations (a) and (b) above and reminder right before them of the assumption on  $\mu$ : there exists infinitely many values of  $n \geq 1$  such that  $m(x_{n-1}) = 1$  and either  $m(x_n) = 2$  or  $x_n$  is not rational over  $x_{n-1}$ ).

**I.11.4.1** First assume that  $m(x_n) = 1$  for every  $n \geq 0$ . Pick  $n_2 > 0$  such that  $x_{n_2}$  is not rational over  $x_{n_2-1}$ . By **I.8.3**(viii) or **I.8.7**(iv) and **I.10**(i),  $x_{n_2}$  satisfies the assumptions of **I.11.2**.

**I.11.4.2** Assume that for some  $n_1 > 0$ ,  $x_{n_1-1}$  is in case (\*1) and  $x_{n_1}$  is in case (\*2). Then  $\kappa(x) \leq 1$  by **I.10**(i) unless possibly if  $\beta(x_{n_1-1}) = 2$ , i.e.  $x_{n_1-1}$  satisfies the assumption of **I.11.1**. The conclusion of **I.11.2** produces some integer  $n_2 := n_1 + 1$  such that  $x_{n_2}$  satisfies the assumptions of **I.11.2**.

**I.11.4.3** Assume finally that for some  $n_1 > 0$ ,  $x_{n_1-1}$  is in case (\*3) and  $x_{n_1}$  is in case (\*2). By **I.8.9**(iii) and **I.10**, we have  $\kappa(x) \leq 1$  or  $x_{n_2}$  satisfies the assumptions of **I.11.2**, where  $n_2 := n_1 + 1$ .

**I.11.4.4** If  $\kappa(x) > 1$ , the conclusion of **I.11.2** either produces some integer  $n_3 > n_2$  such that  $x_{n_3}$  satisfies the assumptions of **I.11.3**, or states that  $x_n$  satisfies the assumptions of **I.11.2** for all  $n \geq n_2$ .

In the former case,  $x_n$  satisfies the assumptions of **I.11.3** (in particular  $x_n$  is in case (\*1)) and  $x_{n+1}$  is rational over  $x_n$  for all  $n \geq n_3$ : this contradicts the assumption on  $\mu$ .

In the latter case, we have  $\beta(x_{n+1}) \leq \beta(x_n)$  for all  $n \geq n_2$ , where equality is strict if  $x_{n+1}$  is not rational over  $x_n$ . The assumption on  $\mu$  implies the existence of an increasing sequence of integers  $(n_i)_{i \geq 2}$  such that  $\beta(x_{n_{i+1}}) < \beta(x_{n_i})$  for all  $i \geq 2$ : a contradiction, since  $\beta(x_{n_i}) \in \frac{1}{\omega(x_i)!} \mathbb{N}$ . Therefore we had  $\kappa(x) \leq 1$  and the proof of **I.8** is thus complete.

## II Resolution of the case $\kappa(x) = 3$ .

We will solve this case by a sequence of permissible and *non-permissible* blowing ups.

**II.1 Notations** We are interested in the case where for a suitable r.s.p.  $(X, u_1, u_2, u_3)$  with  $\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2)$ ,  $1 + \omega(x) \not\equiv 0 \pmod{p}$  and  $\Delta(h; u_1, u_2, u_3; X)$  minimal:

$$f := H(x) \sum_{0 \leq i \leq 1 + \omega(x)} u_3^{1 + \omega(x) - i} \phi_i, \quad \phi_i \in k(x)[[u_1, u_2]], 1 \leq i \leq 1 + \omega(x), \phi_0 \in \widehat{S}, \phi_0 \text{ invertible}$$

or equivalently

$$\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2) \text{ and } (H(x)^{-1} \frac{\partial f}{\partial u_3} \equiv (u_3^{\omega(x)}) \pmod{(u_1, u_3)}.$$

Let us note that we may have  $\epsilon(x) = 1 + \omega(x)$  or  $\epsilon(x) = \omega(x)$ . We do not assume that  $u_1, u_2, u_3$  are in  $S$ : we may take them in  $\widehat{S}$ . We will always assume that

$$u_i \times \text{invertible} \in S \text{ if } \text{div}(u_i) \subseteq E.$$

**Well preparedness of variables** We choose  $v \equiv u_3 \pmod{(u_1, u_2)}$ ,  $v \in k(x)[[u_1, u_2, u_3]]$  such that for all  $s = (s_1, s_2)$  vertex of  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; v)$

$$\text{in}_s \left( \frac{\partial H(x)^{-1} f}{\partial u_3} \right) = \bar{\phi}_0 v^{\omega(x)} \text{ or is not a } \omega(x)^{\text{th}} \text{ - power } \times \bar{\phi}_0, \text{ or } H(x)^{-1} g^p = u_1^{(1 + \omega(x))s_1} u_2^{(1 + \omega(x))s_2}. \quad (1)$$

This can be made this way: let  $s$  the smallest vertex for the order  $(| \cdot |, \text{lex})$  where (1) fails. Then  $\text{in}_s \left( \frac{\partial H(x)^{-1} f}{\partial u_3} \right) = \bar{\phi}_0 (u_3 + \lambda u_1^{s_1} u_2^{s_2})^{\omega(x)}$ , with  $\lambda \neq 0$ ,  $\lambda \in k(x)$  and  $H(x)^{-1} g^p = u_1^a u_2^b$  with  $s = (s_1, s_2) \notin \left( \frac{a}{1 + \omega(x)}, \frac{b}{1 + \omega(x)} \right) + \mathbb{Q}_+^2$ . We take  $v_1 = u_3 + \lambda u_1^{s_1} u_2^{s_2}$ . This translation on  $u_3$  does not modify  $\text{in}_{s'}(H(x)^{-1} f)$  and  $\text{in}_{s'} \left( \frac{\partial H(x)^{-1} f}{\partial u_3} \right) = \text{in}_{s'} \left( \frac{\partial H(x)^{-1} f}{\partial v} \right)$  for  $s' \neq s$  vertex of  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)$ . Either  $s$  is dissolved, or we get (1) for  $s$ . The polyhedra  $\Delta(h; u_1, u_2, v_1; X)$  may be not minimal, if not, we make a translation over  $X$ , we get a new variable

$$X_1 = X + \sum_{1 \leq i \leq \omega(x), (s'_1, s'_2) > s} \lambda_{is'_1, is'_2, 1 + \omega(x) - i} v^{1 + \omega(x) - i} u_1^{is'_1} u_2^{is'_2}, \quad \lambda_{is'_1, is'_2, 1 + \omega(x) - i} \in k(x),$$

and  $f$  becomes  $f_1$ . We have the inclusions  $\Delta(H(x)^{-1}(f_1, g^p); u_1, u_2; v_1) \subseteq \Delta(H(x)^{-1}(f, g^p); u_1, u_2; v_1) \subseteq \Delta(H(x)^{-1}(f, g^p); u_1, u_2; u_3)$ , this translation will not spoil the  $\text{in}_{s'}(H(x)^{-1} f)$  for  $s' < s$ , may add a  $p$ -power to  $\text{in}_s(H(x)^{-1} f)$ , so will not spoil  $\text{in}_s \left( \frac{\partial H(x)^{-1} f}{\partial v} \right) = aV^{\omega(x)}$ . Either we get (1) for  $(u_1, u_2; v_1)$ , or we go on with a new vertex strictly greater than  $s$ . We will get  $v$  and  $X$  as limits in  $\widehat{S}$  and in  $\widehat{S}[X]$ . Note that, in an extreme case, it may happen that there is a r.s.p. such that  $f = H(x)\gamma v^{1 + \omega(x)}$ , which implies condition (1). Now, we mimic chapter 2 and we set

$$A_j = \inf \left\{ \frac{\text{ord}_{u_j}(H(x)^{-1} g^p)}{1 + \omega(x)}, \frac{\text{ord}_{u_j}(\phi_i)}{i}, 1 \leq i \leq 1 + \omega(x) \right\}, \quad j = 2, 3,$$

$$B = \inf \left\{ \frac{\text{ord}_{(u_1, u_2)}(H(x)^{-1} g^p)}{1 + \omega(x)}, \frac{\text{ord}_{(u_1, u_2)}(\phi_i)}{i}, 1 \leq i \leq 1 + \omega(x) \right\},$$

$$C := B - \sum_{j, \text{div}(u_j) \subseteq E} A_j,$$

$$\beta := \inf \left\{ \frac{\text{ord}_{u_2}(H(x)^{-1} g^p \text{ mod } u_1^{(1 + \omega(x))A_1})}{1 + \omega(x)}, \frac{\text{ord}_{u_2}(\phi_i \text{ mod } u_1^{iA_1})}{i}, 1 \leq i \leq 1 + \omega(x) \right\},$$

with the convention  $\text{ord}_{u_2}(\phi_i \text{ mod } u_1^{iA_1}) = +\infty$  if  $\frac{\text{ord}_{u_1}(\phi_i)}{i} > A_1$ ,  $\text{ord}_{u_2}(H(x)^{-1} g^p \text{ mod } u_1^{(1 + \omega(x))A_1}) = +\infty$  if  $\text{ord}_{u_1}(H(x)^{-1} g^p) > (1 + \omega(x))A_1$ . Obviously,  $C, A_j, B$  and  $\beta$  depend on  $(u_1, u_2, u_3 = v)$

verifying (1), if there is no possible confusion, we write  $C$  or  $C(x)$  instead  $C(u_1, u_2, v)$ ,  $A_j$  or  $A_j(x)$  instead of  $A_j(u_1, u_2, v)$ ,  $j = 1, 2$ , etc. Let

$$\begin{aligned}\gamma(u_1, u_2, v) &= 1 + \lfloor C \rfloor \text{ if } E = \text{div}(u_1 u_2), \\ \gamma(u_1, u_2, v) &= \sup(\lfloor \beta \rfloor, 1) \text{ if } E = \text{div}(u_1).\end{aligned}$$

We note  $\gamma(x) = \gamma(u_1, u_2, u_3)$  for short.

**II.1.1 Definition** We say that  $\kappa(x) = 3$  if, for a suitable r.s.p.  $(X, u_1, u_2, u_3)$  with  $\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2)$ ,  $1 + \omega(x) \neq 0 \pmod{p}$  and  $\Delta(h; u_1, u_2, u_3; X)$  minimal:

$$f := H(x) \sum_{0 \leq i \leq 1 + \omega(x)} u_3^{1 + \omega(x) - i} \phi_i, \quad \phi_i \in k(x)[[u_1, u_2]], 1 \leq i \leq 1 + \omega(x), \phi_0 \in \widehat{S}, \phi_0 \text{ invertible.}$$

or equivalently

$$\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2) \text{ and } (H(x)^{-1} \frac{\partial f}{\partial u_3} \equiv (u_3^{\omega(x)}) \pmod{(u_1, u_3)}.$$

**II.1.2 Proposition.** If  $\kappa(x) = 3$  and  $\epsilon(x) = 1 + \omega(x)$ , for any r.s.p.  $(X; u_1, u_2, v)$  with **II.1(1)** above, then either  $V \in \text{IDir}(x)$  or  $\tau'(x) \geq 2$ .

If  $\tau'(x) \geq 2$  and  $A_1 > 0$ , then  $V \in \text{IDir}(x)$ .

If  $\tau'(x) \geq 2$  and  $A_1 = 0$ , then, we can change  $v$  in  $w = v - (au_1 + bu_2)$ ,  $a, b \in k(x)$  such that we have **II.1(1)** for  $(X; u_1, u_2, w)$  and  $W \in \text{IDir}(x)$ . If  $\text{ord}_x(H(x)^{-1} g^p) = \epsilon(x)$ , then  $W \in W(x)$  where

$$W(x) = \text{VDir}(x) + \langle \{U_i | u_i \text{ divides } H(x)^{-1} g^p\} \rangle.$$

Proof. In that case the ideal of the directrix of  $H(x)^{-1} \frac{\partial f}{\partial v}$  is contained in  $\text{IDir}(x)$ . It contains  $V$  if  $\text{in}_x(\frac{\partial H(x)^{-1} f}{\partial v}) = \bar{\phi}_0 V^{\omega(x)}$ ,  $\gamma$  invertible. If  $\text{in}_x(\frac{\partial H(x)^{-1} f}{\partial v}) \neq \bar{\phi}_0 V^{\omega(x)}$ , then  $V \in \text{IDir}(x) \pmod{(U_1, U_2)}$ , so there exists  $w = v - (au_1 + bu_2)$ ,  $a, b \in k(x)$  such that  $W \in \text{IDir}(x)$ . The reader verifies that  $\text{in}_x(\frac{\partial H(x)^{-1} f}{\partial v}) = \text{in}_x(\frac{\partial H(x)^{-1} f}{\partial w})$  and that these initial forms are not  $= \bar{\phi}_0 W^{\omega(x)}$  (else **II(1)** is not true for  $(X; u_1, u_2, v)$ ):  $\tau(x) \geq 2$ . If  $\tau'(x) = 3$ ,  $\kappa(x) = 0$ : from now on  $\tau(x) = \tau'(x) = 2$ . If  $A_1 > 0$  then,  $\text{IDir}(x) = V \pmod{(U_1)}$ , if  $\tau(x) = 2$ , then  $\text{IDir}(x) = (V, U_1)$ .

From now on  $A_1 = 0$  and, by symmetry,  $A_2 = 0$  if  $\text{div}(u_2) \subseteq E$ . Either  $\text{ord}_x(H(x)^{-1} g^p) = \epsilon(x)$ , then  $(W, U_i) \subseteq \text{IDir}(x) + (U_i)$ , where  $u_i$  divides  $H(x)^{-1} g^p$ . As  $\tau'(x) = 2$ ,  $H(x)^{-1} g^p = \gamma u_i^{\epsilon(x)}$ , as  $\text{div}(u_i) \subseteq E$ , we make  $i = 1$ . Then if  $b \neq 0$ ,  $s = (0, 1)$  is a vertex of  $\Delta(\frac{\partial H(x)^{-1} f}{\partial v}; u_1, u_2, v)$ , as  $\text{in}_s(\frac{\partial H(x)^{-1} f}{\partial v})$  is not proportional to an  $\omega(x)$ -power,  $(W, U_2) \subseteq \text{IDir}(x) \pmod{U_1}$ ,  $\tau'(x) = 3$ : contradiction. So  $b = 0$ , then  $V \in \text{IDir}(x) + (U_1) =: W(x)$ . Or  $\text{ord}_x(H(x)^{-1} g^p) > \epsilon(x)$ , then we make a translation over  $X$ , so that  $Y = X - \theta$  and  $\Delta(h; u_1, u_2; w)$  is minimal. Let us denote:

$$\text{IDir}(\text{in}_x(\frac{\partial H(x)^{-1} f}{\partial w})) = (W, cU_1 + dU_2), c, d \in k(x), \text{ not both } 0. \quad (1)$$

The reader sees that, if  $ab \neq 0$ , then the polyhedra  $\Delta(\frac{\partial H(x)^{-1} f}{\partial w}; u_1, u_2; w)$  has two non solvable vertices  $(1, 0)$  and  $(0, 1)$ , so we have **II.1(1)** for  $(Y; u_1, u_2, w)$  and  $cd \neq 0$ . If  $a = 0$ ,  $b \neq 0$ , the polyhedra  $\Delta(\frac{\partial H(x)^{-1} f}{\partial w}; u_1, u_2; w)$  has  $(0, 1)$  for unique vertex of its initial side, this vertex is not solvable, to get (1), we have to modify  $Y$  and  $w$ , but we will not modify  $\text{in}_x(\frac{\partial H(x)^{-1} f}{\partial w})$  nor  $\text{in}_x(w)$ . So we get the assertion.

**II.1.4 Proposition** Let a suitable r.s.p.  $(X, u_1, u_2, u_3)$  of  $x$  with  $\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2)$  and  $\Delta(h; u_1, u_2, u_3; X)$  minimal:  $f := H(x)(u_1 u_2 \phi + \gamma u_3^{\omega(x)+1})$ ,  $\gamma$  invertible and  $\text{ord}_x(u_1 u_2 \phi) = \omega(x)$ . Then  $\kappa(x) = 0$ .

Proof. Indeed,  $\omega(x) = \text{ord}_x(u_1 u_2 \phi) \geq 2$ . Blow up  $x$ , the only possible very near point has for parameters  $X' = \frac{X}{u_3}, u'_1 = \frac{u_1}{u_3}, u'_3 = \frac{u_3}{u_3}, u'_3 = u_3$ . As we are at the origin of a chart, there is no translation to do,  $f' = H(x')(u'_1 u'_2 \phi' + \gamma u'_3)$ :  $\omega(x') \leq 1 < \omega(x)$ .

**II.2 Theorem** Let  $x \in \Sigma_p$  with  $\kappa(x) = 3$  and  $(X, u_1, u_2, v)$  a r.s.p. verifying **II.1(1)**,

(i) if  $C(x) = 0$ ,  $\epsilon(x) = \omega(x)$  then  $\kappa(x) \leq 1$ ,

(ii) if  $C(x) = 0$ ,  $A_1(x) < 1$ ,  $A_2(x) < 1$ , then  $\kappa(x) \leq 1$ .

**II.2.1** Proof of (i): case where  $\epsilon(x) = \omega(x)$ . Either  $A_1 A_2 > 0$ , then by **II.1.4**,  $\kappa(x) = 0$ . Or (with an eventual permutation on  $u_1, u_2$ )  $A_2 = 0$ :  $H(x)^{-1} f = \sum_{0 \leq i \leq 1 + \omega(x)} v^{1 + \omega(x) - i} \phi_i$  with for some  $i = i_0$ ,  $1 \leq i_0 \leq 1 + \omega(x)$ ,  $\phi_{i_0} = \gamma u_1^{i_0 A_1}$ ,  $\gamma$  invertible,  $A_1 < 1$ :  $\text{ord}_x(u_1^{i_0 A_1} u_3^{1 + \omega(x) - i_0}) \leq \omega(x)$ , indeed, there is equality. If  $1 \leq i_0 < 1 + \omega(x)$  or  $\text{div}(u_1) \not\subseteq E$ , then  $\kappa(x) \leq 2$ . If  $i_0 = 1 + \omega(x)$ , then  $A_1 = 1 - \frac{1}{1 + \omega(x)}$ ,  $\text{ord}_{u_1}(\phi_i) \geq i A_1 = \frac{i \omega(x)}{1 + \omega(x)}$ ,  $1 \leq i < 1 + \omega(x)$ , so  $\text{ord}_{u_1}(\phi_i) \geq i$ :

$$f = H(x)(\phi_0 v^{1 + \omega(x)} + \lambda u_1^{\omega(x)} + u_1 \psi), \quad \psi \in (u_1, v)^{\omega(x)}, \quad \lambda \text{ invertible or } H(x)^{-1} g^p = \gamma u_1^{\omega(x)}.$$

As  $\omega'(x) = 2$ ,  $H(x)^{-1} g^p \neq \gamma u_1^{\omega(x)}$ : we have  $\lambda$  invertible. Let  $Y = V(X, v, u_1)$ ,  $(u_1, v)^{\omega(x)} \supset J(f, E) = (v^{\omega(x)}) \text{ mod } (u_1)$ . When  $a(1) + \omega(x) \geq p$ ,  $Y = \text{div}(u_1) \cap \{\omega \geq \omega(x)\}$ :  $Y$  is not a formal curve,  $Y$  is permissible. We blow up  $Y$ , the point  $x'$  of parameters  $X' = \frac{X}{v}, u'_1 = \frac{u_1}{v}, u_2, v$  is the only point above which may be very near to  $x$ , an easy computation show that  $1 \geq \omega(x')$ , we are done except if  $\omega(x) = 1$ . In this last case, we get  $H(x')^{-1} f' = \phi_0 v' + \lambda u_1 + v' \psi'$ ,  $\lambda$  invertible,  $H(x') = u_1^{a(1)} u_2^{a(2)} v^{1 + a(1) - p}$ . Then by chapter 2 **II.5(i)**, the  $\text{cl}_{a_1 + a_2 + 1 + a(1) + 1 - p}(D(u_1^{a(1)} u_2^{a(2)} v^{1 + a(1) - p}(v + \lambda u_1)))$ ,  $D \in \mathcal{D}(E')$  are not all proportional:  $U_1$  and  $V$  are in the directrix of  $x'$ . The curve of ideal  $(X', v', u_1)$  is permissible, it is the preimage of  $x$ , so it is not a formal curve. We blow up this curve: there is no very near point above. If  $a(1) + \omega(x) < p$ , we claim  $\kappa(x) \leq 1$ , let us first state:

**II.2.1.1 Lemma** If  $H(x)^{-1} f = \phi_0 u_3^{a'} u_2^{b'} + u_1 \phi$  with  $\text{ord}_x(u_1 \phi) = \omega(x)$ ,  $\phi_0$  invertible, and  $a', b' \leq \omega(x) - 1$ , then  $\kappa(x) = 0$ .

Proof. Blow up  $x$ , any  $x'$  very near to  $x$  is on the strict transform of  $\text{div}(u_1)$ ,  $x'$  is at the origin of a chart and the pair of exponents  $(a', b')$  becomes  $(a' + b' - \omega(x), b')$  or  $(a', a' + b' - \omega(x))$ , an induction on  $a' + b'$  gives the result.

**II.2.1.2 Lemma** If  $a(1) + \omega(x) < p$ ,  $1 + \omega(x) \not\equiv 0 \text{ mod } (p)$ ,  $\Delta(h; u_1, u_2, v; X)$  minimal,

$$H(x)^{-1} f = \phi_0 v^{1 + \omega(x)} u_2^i + \lambda u_1^{\omega(x)} + u_1 \psi u_2^i, \quad i \leq \omega(x), \quad u_1 \psi \in (u_1, v)^{1 + \omega(x)},$$

$\phi_0$  invertible,  $\text{div}(u_1) \subseteq E$  and  $E = \text{div}(u_1 u_2)$  when  $i > 0$ , then  $\kappa(x) \leq 1$ .

Proof. By induction on  $a(2)$ . When  $i = \omega(x)$ , we blow up  $V(X, u_1, u_2) = V(u_1, u_2) \cap \omega \geq \omega(x)$  which is permissible, the point of parameters  $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u_2, v)$  is the only point which may be very near to  $x$ ,

$$H(x')^{-1} f' = \phi_0 v^{1 + \omega(x)} u_2^{i - \omega(x)} + \lambda u_1^{\omega(x)} + u'_1 \psi u_2^{i - \omega(x)},$$

$H(x') = u_1^{a(1)} u_2^{a(2) + a(1) + \omega(x) - p}$ ,  $a(2) + a(1) + \omega(x) - p < a(2)$ : we get the result by induction on  $a(2)$ .

When  $i < \omega(x)$ , we blow up  $x$ . Every point very near to  $x$  is on the strict transform of  $\text{div}(u_1)$ . In the chart of origin  $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u_2, v' = \frac{v}{u_2})$ , we get  $H(x')^{-1} f' = \phi_0 v'^{1 + \omega(x)} u_2^{i+1} + \lambda u_1^{\omega(x)} + u'_1 \psi u_2^{i - \omega(x)}$ ,  $H(x') = u_1^{a(1)} u_2^{a(2) + a(1) + \omega(x) - p}$ .

When  $i \leq \omega(x) - 2$ , the origin is the only point possibly very near to  $x$ ,  $a(2) + a(1) + \omega(x) - p < a(2)$ : we get the result by induction on  $a(2)$ .

When  $i = \omega(x) - 1$ , at the origin we are in the case above, so  $\kappa(x') \leq 1$ , if  $v'(x') \neq 0$ , let us denote  $Y' = V(X', u'_1, u_2)$ , then  $H(x')^{-1} \frac{\partial f'}{\partial v'} = \gamma' u_2^{\omega(x)} \pmod{(u'_1 u_2)}$  ( $\lambda = \bar{\lambda} \pmod{(u_2)}$  in  $k(x')[[u_1, u_2, u_3]]$ ),  $\gamma'$  invertible, as  $a(1) + \omega(x) \neq 0 \pmod{(p)}$ ,  $H(x')^{-1} \frac{\partial f'}{\partial v'} = \gamma'' u_1^{\omega(x)} \pmod{(u_2)}$ , so  $\text{IDir}(J(f', E', Y')) = (U'_1, U'_2)$ ,  $J(f', E', Y') \subseteq (u'_1, u_2)^{\omega(x)}$ . If we blow up  $Y'$ , there is no very near point.

The last point we have to study is  $(X' = \frac{X}{v}, u'_1 = \frac{u_1}{v}, u'_2 = \frac{u_2}{v}, v)$ . We get  $H(x')^{-1} f' = \phi_0 v^{1+i} u_2^i + \lambda u_1^{\omega(x)} + u'_1 \psi u_2^{i-\omega(x)} v^{1+i}$ . If  $i \leq \omega(x) - 2$ ,  $\kappa(x') = 0$  by **II.2.1.1**, if  $i = \omega(x) - 1$ , we blow up  $V(X', u'_1, v)$  and the reader sees that there is no point very near to  $x'$ :  $\kappa(x') = 1$ .

**II.2.2** Proof of **II.2(ii)**. By **II.2.1**, we have just to look at the case where  $\epsilon(x) = 1 + \omega(x)$ . In that case or there exists  $i_0$ ,  $1 \leq i_0 \leq 1 + \omega(x)$ ,  $\phi_{i_0} = u_1^{i_0 A_1} u_2^{i_0 A_2} \times \text{invertible}$  and,  $u_1^{i A_1} u_2^{i A_2}$  divides  $\phi_i$  for all  $i$ ,  $1 \leq i \leq 1 + \omega(x)$  (as  $A_i < 1$ ,  $i = 1, 2$ , we have  $2 \leq i_0$ ) or  $H(x)^{-1} g^p = u_1^{(1+\omega(x))A_1} u_2^{(1+\omega(x))A_2}$  and  $u_1^{i A_1} u_2^{i A_2}$  divides  $\phi_i$  for all  $i$ ,  $1 \leq i \leq 1 + \omega(x)$ . Condition (1) in **I.1** and  $\epsilon(x) = 1 + \omega(x)$  imply that  $V$  is in the ideal of the directrix of  $x$ . If we blow-up  $x$ , the reader will see that only the origins of the two possible charts, i.e. the points of parameters  $\frac{X}{u_1}, u_1, \frac{u_2}{u_1}, \frac{v}{u_1}$  or  $\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, \frac{v}{u_2}$  may be very near to  $x$  and that the couple  $(A_1, A_2)$  becomes  $(A_1 + A_2 - 1, A_2)$  or  $(A_1, A_1 + A_2 - 1)$ . An induction on  $A_1 + A_2$  and (i) give the result.

**II.3 Theorem** Let  $x \in \Sigma_p$  with  $\kappa(x) = 3$ ,  $(X, u_1, u_2, v)$  a r.s.p. verifying **II.1(1)** and  $u_1 \in S$ ,  $u_2 \in S$ : if we blow-up  $(u_1, u_2)$  and if  $x'$  is a point near to  $x$  with  $\Omega(x') \geq \Omega(x)$ , then  $\Omega(x') = \Omega(x)$ ,  $\kappa(x') = 3$  and there exists a regular system of parameters  $(Z, v_1, v_2, w)$  of  $x'$  verifying (1) such that:

- (i)  $\gamma(v_1, v_2, w) \leq \gamma(u_1, u_2, v)$ ,
- (ii) if  $x'$  is in the chart with origin  $(X, u_1, u'_2 = \frac{u_2}{u_1}, v)$  (so-called first chart), then  $A_1(x') = B$ , and, if  $m(x) = 1$ ,  $\beta(v_1, v_2, w) \leq A_2(u_1, u_2, u_3) + C(u_1, u_2, u_3) \leq \beta(u_1, u_2, u_3)$ , if  $x'$  is not rational over  $x$  and  $\beta(u_1, u_2, u_3) \geq 1$ ,  $\beta(v_1, v_2, w) < \beta(u_1, u_2, u_3)$  or  $\kappa(x') \leq 1$ , if  $u'_2(x') = 0$ ,  $C(x') \leq C(x)$ ,
- (iii) if  $x'$  is the origin of the second chart,  $(x' = (X, u'_1 = \frac{u_1}{u_2}, u_2, v))$ , then  $C(x') \leq C(x)$ , furthermore,  $C(x') \leq \frac{\beta(x)}{2}$  and, if there is equality,  $\Delta(H(x)^{-1} f + H(x)^{-1} g^p; u_1, u_2; v)$  has only two vertices  $(A_1(x), \beta(x))$  and  $(A_1(x) + \frac{\beta(x)}{2}, 0)$ .
- (iv) if  $m(x) = m(x') = 1$  and  $\beta(u_1, u_2, v) > 0$ , then  $\beta(v_1, v_2, w) \leq \beta(u_1, u_2, v)$ ,  $A_1(x) < A_1(x')$ ,
- (v) if  $\gamma(x) \geq 2$  and  $m(x) = 1$ ,  $m(x') = 2$ , then  $\gamma(x') < \gamma(x)$ ,
- (vi) if  $\beta(x) < 1$  and  $m(x) = 1$ ,  $m(x') = 2$ , then  $C(x') < \frac{1}{2}$ ,
- (vii) if  $0 < C(x) < \frac{1}{2}$  and  $m(x) = 2$ ,  $m(x') = 1$ , then  $\beta(x') < 1$ .

**II.3.1 Notations** Let  $f := H(x) \sum_{0 \leq i \leq 1+\omega(x)} u_3^{1+\omega(x)-i} u_1^{b_i} u_2^{c_i} f_i$ ,  $f_i \in k(x)[[u_1, u_2]]$ ,  $1 \leq i \leq 1 + \omega(x)$ ,  $f_0 \in \widehat{S}$ ,  $f_0$  invertible,  $a_0 = b_0 = 0$ ,  $u_j$  does not divide  $f_i \in k(x)[[u_1, u_2]]$ ,  $j = 1, 2$  and  $F_i$  is the initial form of  $f_i$ ,  $d_i = \text{ord}_{(u_1, u_2)}(f_i)$  when  $f_i \neq 0$ .

Proof of (iii).

$$f' = f := H(x')(\phi_0 v^{1+\omega(x)} + \sum_{1 \leq i \leq 1+\omega(x)} v^{1+\omega(x)-i} u_1^{b_i} u_2^{b_i+c_i+d_i} (F_i(u'_1, 1) + u_2 \psi_i)).$$

We are at the origin of a chart, there is no translation to do, etc. As  $\phi_0$  is invertible and  $1 + \omega(x) \neq 0 \pmod{(p)}$ , no translation can spoil the initial form of  $H(x') \phi_0 v^{1+\omega(x)}$ :  $\kappa(x') = 3$ . The transformations on the polyhedras are now well known,  $\beta(x') = \beta(x) + A_1(x)$ ,  $A_1(x') = A_1(x)$ ,  $A_2(x') = B(x)$ , so

$$C(x') \leq \alpha_2(x) - A_1(x) \leq C(x) + A_1(x) - A_1(x) = C(x),$$

$$C(x') \leq \beta(x) + A_1(x) - B(x) = \beta(x) - (B(x) - A_1(x)), \text{ and } C(x') \leq B(x) - A_1(x).$$

So if  $B(x) - A_1(x) \neq \frac{\beta(x)}{2}$ , we get  $C(x') < \frac{\beta(x)}{2}$ . In every case,  $C(x') \leq \frac{\beta(x)}{2}$ . If we have  $C(x') = \frac{\beta(x)}{2}$ , then,  $B(x) - A_1(x) = \frac{\beta(x)}{2}$ , as  $C(x') \leq \alpha_2(x) - A_1(x) \leq B(x) - A_1(x)$ , then  $\alpha_2(x) = B(x)$ , as  $\alpha_2(x) + \beta_2(x) = B(x)$ , we have  $\beta_2(x) = 0$ . So  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; v)$  two vertices  $(A_1(x), \beta(x))$  (smallest abscissa) and  $(A_1(x) + \frac{\beta(x)}{2}, 0)$  (smallest ordinate) which give in  $\Delta(H(x')^{-1}(f, g^p); u'_1, u_2; v)$  the two vertices  $(A_1(x), \beta(x) + A_1(x))$  (smallest abscissa),  $(A_1(x) + \frac{\beta(x)}{2}, A_1(x) + \frac{\beta(x)}{2})$  (smallest ordinate). If  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; v)$  had another vertex, it would be under the side of ends  $(A_1(x), \beta(x))$  and  $(A_1(x) + \frac{\beta(x)}{2}, 0)$ , it would give in  $\Delta(H(x')^{-1}f + H(x')^{-1}g^p; u'_1, u_2; v)$  a vertex  $(x_1, x_2)$  under the side of ends  $(A_1(x), \beta(x) + A_1(x))$ ,  $(A_1(x) + \frac{\beta(x)}{2}, A_1(x) + \frac{\beta(x)}{2})$  which have both  $2A_1(x) + \beta(x)$  for sum of coordinates: we would get  $x_1 + x_2 < 2A_1(x) + \beta(x)$ ,  $C(x') \leq x_1 + x_2 - A_1(x') - A_2(x') < \frac{\beta(x)}{2}$ .

**II.3.2 First chart** The first chart has for origin the point of parameters  $(X, u_1, u'_2 = \frac{u_2}{u_1}, v)$ .

$$f' = f := H(x')(\phi_0 v^{1+\omega(x)} + \sum_{1 \leq i \leq 1+\omega(x)} v^{1+\omega(x)-i} u_1^{b_i+c_i+d_i} u_2^{c_i} (F_i(1, u'_2) + u_1 \psi_i)).$$

Either there is some index  $i_0$  with  $\frac{b_{i_0}+c_{i_0}+d_{i_0}}{i_0} = B = \inf_i \frac{b_i+c_i+d_i}{i}$  (case (a)) or  $B = \frac{\text{ord}_{u_1, u_2}(H(x')^{-1}g^p)}{p}$  (case (b)). When  $x'$  is the origin of the chart, there is no translation to do, (i)(ii)(iii)(iv)(vi) are clear. When  $x'$  is not the origin,  $u'_2(x') \neq 0$ , we may have to do some translations on  $X, v$  to get (1) in  $x'$  for the new parameters. In case (a), we take  $i_0$  minimal. In case (b), nothing happens to  $g$ ,  $\beta(x') \leq \frac{\text{ord}_{u_2}(H(x')^{-1}g^p)}{p} \leq \beta(x)$ . Furthermore, if  $m(x') = 1$ , then  $\beta(x') = 0 = C(x')$ . As the chart we consider contains all the points  $x'$  above  $x$  with  $m(x') = 1$ , (i)(ii)(iii)(iv)(vi) are proven in case (b). In case (a) and not (b), let

$$I_0 := \{i_0 \mid \frac{b_{i_0} + c_{i_0} + d_{i_0}}{i_0} = B = \inf_i \{ \frac{b_i + c_i + d_i}{i} \},$$

for every  $i_0 \in I_0$ ,  $1 + \omega(x) - i_0 = 0 \pmod{p}$ , to get **II.1(1)** in  $x'$ , the eventual translation on  $v$  will change  $v$  in  $v' = v + u_1^a \theta$ ,  $a > B$ ,  $\theta \in k(x')[[u'_1, v_2]]$ ,  $v_2 \in k(x)[u_2]$ , this translation will not touch

$$v^{1+\omega(x)-i_0} u_1^{b_{i_0}+c_{i_0}+d_{i_0}} F_{i_0}(1, u'_2) u_2^{c_{i_0}},$$

for all  $i_0$ , the eventual translation on  $X'$  will add a  $p$ -power. But  $v^{1+\omega(x)-i_0} u_1^{b_{i_0}} u_2^{c_{i_0}} F_{i_0}$  defined the vertices  $(x_1, x_2, x_3)$  of  $\Delta(h; u_1, u_2, v; X)$  with  $x_1 + x_2 + Bx_3 = B(1 + \omega(x)) + a(1) + a(2)$  and  $x_3$  minimal among them,  $u_1^{a(1)} u_2^{a(2)} v^{1+\omega(x)-i_0} u_1^{b_{i_0}} u_2^{c_{i_0}} F_{i_0}$  is not a  $p$ -power,  $A_1(x') = B(x)$ , and, by chapter 1, **II.5.3.2(iii)**,  $\beta(x') \leq \frac{c_{i_0}+d_{i_0}}{i_0} \leq \beta(x)$  and, if  $x'$  is not the origin,

$$i_0 \beta(x') \leq \text{ord}_{x'}(F_{i_0}(1, u'_2) + p\text{-power}) \leq \frac{d_{i_0}}{d} + 1 \leq \frac{C(x)}{d} + \frac{1}{i_0}$$

where  $d$  is the degree of the residual extension. When  $\beta(x) > 1$ ,  $i_0 \beta(x) > i_0$  which implies  $\frac{C(x)}{d} \leq \frac{i_0 \beta(x)}{d} < i_0 \beta(x) - 1$ :  $\beta(x') < \beta(x)$ . When  $\beta(x) = 1$ , then  $d_{i_0} \leq i_0$ , if  $d_{i_0} = i_0$ , then, as  $i_0 \neq 0 \pmod{p}$ ,  $\text{ord}_x(\frac{\partial F_{i_0}(u_1, u_2)}{\partial u_2}) = i_0 - 1$ ,  $\text{ord}_{x'}(F_{i_0}(1, u'_2) + p\text{-power}) \leq \frac{d_{i_0}-1}{d} + 1$ , so  $\beta(x') < 1$  or  $i_0 = 1$  and  $i_0 = 1$  is the only index with  $\frac{b_{i_0}+c_{i_0}+d_{i_0}}{i_0} = B$ . In that last case, I claim that  $\kappa(x') \leq 1$ : indeed  $\omega(x) = 0 \pmod{p}$ ,

$$f' = u_1^{a(1)} (\phi'_0 v^{\omega(x)+1} + \phi'_1 v^{\omega(x)} u_1^{A_1(x')} v_2 + \sum_{2 \leq j \leq \omega(x)+1} v^{\omega(x)+1-j} u_1^{a_j} \phi'_j,$$

$\phi'_0\phi'_1$  invertible,  $\phi'_j \in k(x')[[u'_1, v_2]]$ ,  $2 \leq j \leq \omega(x) + 1$  and  $a_j > jA_1(x')$ . Then, if  $A_1(x') = 0$ ,  $f' = u_1'^{a(1)}v^{\omega(x)}w \bmod u_1'^{a(1)+1}$ , where  $w = \phi'_0v + \phi'_1v_2$ :  $\kappa(x') \leq 2$ , if  $A_1(x') \geq 1$ ,  $V(X, v, u_1) = \text{div}(u_1) \cap \{\omega > 0\}$ , we “blow up  $V(X, v, u_1)$   $A_1(x')$  times”, the point  $y$  of parameters  $(\frac{X}{u_1^{A_1(x')}}; u_1, v_2, \frac{v}{u_1^{A_1(x')}})$  is the only possible very near point and  $u_1'^{a(1)}v^{\omega(x)}w \bmod u_1'$  is the strict transform of  $f'$ :  $\kappa(y) \leq 2$ , in every case,  $\kappa(x') \leq 1$ .

If for some  $i_0 \in I_0$ ,  $1 + \omega(x) - i_0 \not\equiv 0 \pmod{p}$ , then an eventual translation on  $v'$  will spoil

$$f'_1 := \sum_{0 \leq i \leq \omega(x), b_i + c_i + d_i = B(x)} (1 + \omega(x) - i)v'^{\omega(x)-i}u_1'^{iA_1(x')}F_i(1, u'_2)$$

which is the initial part of  $H(x')^{-1}\frac{\partial f'}{\partial v'}$  with respect to the valuation  $\nu$  which defines  $A_1(x')$ , i.e.  $\nu(v'^a u_1'^b u_2'^c) = a + A_1(x')b$ . Then by the usual transformation laws on the Newton polyhedron (see **I.8.3.6**),  $\beta(x') \leq \beta(\Delta(f'_1; u_1, u'_2, v')) \leq \frac{1}{p} + \frac{C_{f'_1}}{d}$ , where  $d$  is the degree of the extension of the residual fields of  $x$  and  $x'$ , and, if  $B \notin \mathbb{N}$ ,  $\beta(x') \leq \frac{C_{f'_1}}{d}$ . Furthermore, if  $x'$  is not rational over  $x$ ,  $\beta(\Delta(f'_1; u_1, u'_2, v')) < \beta(\Delta(f_1; u_1, u_2, v)) \leq \beta(x)$ . The reader will use the definition of  $\gamma$  to see that this inequality proves (i)(ii)(iii)(vi) in case (a).

Proof of (vii). When  $0 < C(x) < \frac{1}{2}$ , case (b),  $\beta(x') = 0$ . When  $0 < C(x) < \frac{1}{2}$ , case (a) and  $f'_1 \neq 0$ , by the computations above,  $\beta(x') \leq \frac{1}{p} + C(f'_1) < 1$ . When  $0 < C(x) < \frac{1}{2}$ , case (a) and  $f'_1 = 0$ , then, by the computations above, we have just to consider the case  $1 + \omega(x) - i_0 = 0 \pmod{p}$  and then,  $\beta(x') < 1$  except if  $i_0 = 1$  or  $2$ . In those cases,  $C(x) < \frac{1}{2}$  implies  $\psi_{i_0}$  invertible, so  $i_0\beta(x') \leq 1$ , if  $i_0 = 2$ ,  $\beta(x') \leq \frac{1}{2} < 1$ . If  $i_0 = 1$ , the hypothesis  $C(x) \notin \mathbb{N}$  implies that there exists another index  $i_1$ ,  $2 \leq i_1 \leq 1 + \omega(x)$  with  $b_{i_1} + c_{i_1} + d_{i_1} = i_1B(x)$ ,  $d_{i_1} \leq i_1C(x) < \frac{i_1}{2}$ , as  $f'_1 = 0$ , to get the condition **II.1**(i), the translation on  $v'$  will be  $w = v' + u_1'^a\phi$  with  $\phi \in \widehat{S}'$  and  $a \geq 1 + B(x)$ , so will not touch  $v'^{1+\omega(x)-i_1}u_1'^{b_{i_1}+c_{i_1}+d_{i_1}}F_{i_1}(1, u'_2)u_2'^{c_{i_1}}$ , so  $i_1\beta(x') \leq 1 + d_{i_1} < 1 + \frac{i_1}{2} \leq 2$ , so  $\beta(x') < 1$ .

**II.4 Theorem** *Let  $x \in \Sigma_p$  with  $\kappa(x) = 3$ ,  $(X, u_1, u_2, v)$  a r.s.p. verifying **II.1**(1),  $v \in \widehat{S}$  and possibly  $u_2 \in \widehat{S}$  if  $\text{div}(u_2) \not\subseteq E$ . If  $(\beta < 1, m(x) = 1)$  or  $(\beta < 1, m(x) = 2, A_1(x) < 1)$  and  $C(x) < \frac{1}{2}$ , then  $\kappa(x) \leq 1$ .*

Proof. When  $m(x) = 1$ , then as  $\beta < 1$ , we suppose  $\epsilon(x) = 1 + \omega(x)$ , else  $\kappa(x) = 2$ , furthermore  $a(1) + 1 + \omega(x) \geq p$ : by a sequence of blowing ups centered at  $(X, u_1, v)$ , we reach the case where  $A_1(x) < 1$ .

When  $A_1(X) = 0$ ,  $\beta < 1$  gives  $\kappa(x) \leq 2$ .

When  $0 < A_1(X) < 1$  and  $\omega(x) = \epsilon(x)$ , then  $f = H(x)(v^{1+\omega(x)}\phi_0 + u_1\phi)$ ,  $\text{ord}_x(u_1\phi) = \omega(x)$ . If  $\text{in}_x(u_1\phi) = \lambda u_1^{\omega(x)}$ ,  $\lambda \in k(x)$ , then apply **II.2.1.2** with  $i = 0$ :  $\kappa(x) \leq 1$ . If  $\text{in}_x(u_1\phi) \neq \lambda u_1^{\omega(x)}$ , then  $V$  or  $U_2$  appears in the expansion of  $\text{in}_x(u_1\phi)$ :  $\kappa(x) = 2$  except if  $E = \text{div}(u_1u_2)$  and  $\text{in}_x(u_1\phi) = U_1F(U_1, U_2)$ ,  $F \notin k(x)[U_1]$ : then  $\omega(x) \geq 2$  and  $(U_1, U_2) = \text{IDir}(x)$ . We blow up  $x$ , the only possible very near point is  $x'$  of parameters  $(X' = \frac{X}{v}, u'_1 = \frac{u_1}{v}, u'_2 = \frac{u_2}{v}, v' = v)$ ,  $f' = H(x')(v'\phi_0 + u_1'\phi')$ :  $\omega(x') \leq 1 < \omega(x)$ ,  $\kappa(x) = 0$ .

When  $0 < A_1(X) < 1$  and  $1 + \omega(x) = \epsilon(x)$ , then we blow up  $x$ : by **II.1.2**  $V \in \text{IDir}(x)$ , for the first chart, we get  $A_1(x') = B(x) - 1 \leq \beta(x) + A_1(x) - 1 < A_1(x)$ , for  $\beta(x')$ , when  $m(x) = 1$ , the computations are exactly the same as in **II.3**(iv). When  $m(x) = 2$ , if  $C(x) = 0$  by **II.2**(ii),  $\kappa(x) \leq 1$ , if  $C(x) \neq 0$ , we blow up  $x$ , the computations are exactly the same as in **II.3**(vii):  $\beta(x') < 1 + \lfloor C(x) \rfloor = 1$ ,  $A_1(x') = B(x) - 1 \leq A_1(x) + \beta(x) - 1 < A_1(x)$ . For the second chart,  $A_1(x') = A_1(x)$ ,  $\beta(x') = \beta(x) + A_1(x) - 1 < \beta(x)$ ,  $C(x') \leq \frac{\beta(x)}{2} < \frac{1}{2}$ . An induction on  $(A_1(x), \beta(x))$  gives the result.

**II.5 Corollary** Let  $x \in \Sigma_p$  with  $\kappa(x) = 3$ ,  $(X, u_1, u_2, v)$  a r.s.p. verifying (1) and  $m(x) = 1$  and  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; v)$  has only two vertices  $(A_1, 1)$  and  $(A_1 + \frac{1}{2}, 0)$ , then  $\kappa(x) \leq 1$ .

Proof. We make an induction on  $A_1$ .

**II.5.1** If  $A_1 \geq 1$ , then,  $V(X, v, u_1) = \text{div}(u_1) \cap \{\omega \geq 1\}$ : its an algebraic curve permissible of first kind. We blow up this curve, as  $V \in W(x)$ , there is at most one very near point  $x'$  which has  $(X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = \frac{u_2}{u_1}, v' = \frac{v}{u_1})$  for parameters. Of course, we are at the origin of a chart, there is no translation to do,  $\Delta(H(x')^{-1}(f', g^p); u'_1, u'_2; v')$  is  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; v)$  translated horizontally of  $-1$ . Of course, the monomial  $H(x')\phi_0 v'^{1+\omega(x)}$  occurs in the expansion of  $f'$ , if  $\Omega(x') = \Omega(x)$ , then  $\kappa(x') = 3$ . So we have just to start the induction. From now on,  $A_1(x) < 1$ .

**II.5.2**  $A_1(x) < \frac{1}{2}$ . As  $(A_1 + \frac{1}{2}, 0)$  is a vertex,  $\epsilon(x) = \omega(x)$ , we have  $\kappa(x) = 2$  except maybe in the case where the monomial in  $H(x)^{-1}f$  or  $H(x)^{-1}g^p$  which defines the vertex  $(A_1 + \frac{1}{2}, 0)$  is  $u_1^{(1+\omega(x))(A_1+\frac{1}{2})}$  and  $(1 + \omega(x))(A_1 + \frac{1}{2}) - (1 + \omega(x)) = \omega(x)$ ,  $A_1 + \frac{1}{2} - 1 = \frac{\omega(x)}{1+\omega(x)}$ . In that case,  $\Delta(H(x)^{-1}(f, g^p); u_1, u_2; v)$  has two vertices  $(\frac{\omega(x)-1}{2(1+\omega(x))}, 1)$  and  $(\frac{\omega(x)}{1+\omega(x)}, 0)$ . So in the expansion of  $H(x)^{-1}f$  there is a monomial  $v^{1+\omega(x)-i} u_1^{i \frac{\omega(x)-1}{2(1+\omega(x))}} u_2^i$ .

Either  $\omega(x) = 1$ , then  $V(X, v, u_1)$  is permissible of first kind, it is  $\text{div}(u_1) \cap \{\omega \geq 1\}$ , we blow it up,  $U_1 \in \text{IDir}(x)$ , there is at most one very near point and we get

$$f' = u_1'^{a(1)} v'^{(a(1)+1-p)} (\phi_0 v' + \gamma u_1'),$$

by chapter 2 **II.5**,  $\text{IDir}(x') = (V', u'_1)$ :  $\kappa(x') = 1$ .

Or  $\omega(x) \geq 2$ , then we blow up  $x$ ,  $U_1 \in \text{IDir}(x)$  and, because of  $v^{1+\omega(x)}$ , there is at most one very near point  $x'$  which has for parameters  $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, v' = \frac{v}{u_2})$ . We are at the origin of a chart, etc.... the monomial  $v'^{1+\omega(x)-i} u_1'^i u_2'^{i \frac{\omega(x)-1}{2(1+\omega(x))}} u_2'^{i \frac{\omega(x)-1}{2(1+\omega(x))} + 1}$  occurs in the expansion of  $H(x')^{-1}f'$ . A computation shows that  $i \frac{\omega(x)-1}{2(1+\omega(x))} + i \frac{\omega(x)-1}{2(1+\omega(x))} + 1$  the sum of the exponents of  $u_1'$  and  $u_2'$  is  $< i + 1$  so is  $\leq i$ .

If  $i = 1 + \omega(x)$ , then,  $i \frac{\omega(x)-1}{2(1+\omega(x))} + i \frac{\omega(x)-1}{2(1+\omega(x))} + 1 = \omega(x)$ , we have

$$f' = H(x')(\phi_0 v'^{1+\omega(x)} u_2' + u_1' \phi), \quad E' = \text{div}(u_1' u_2')$$

and in  $x'$   $(u_1' \phi)$  not colinear to  $U_1'^{\omega(x)}$ , if  $\kappa(x') > 2$ , then  $\text{IDir}(x') = (U_1', U_2')$ . We remark that, as the exponent  $i \frac{\omega(x)-1}{2(1+\omega(x))} = \frac{\omega(x)-1}{2}$  is integer,  $\omega(x) \neq 2$ , so  $\omega(x) \geq 3$ . We blow up  $x'$ , the only possible very near point  $y$  has for parameters  $(Y = \frac{X'}{v'}, v_1 = \frac{u_1'}{v'}, v_2 = \frac{u_2'}{v'}, v_3 = v')$ , the monomial  $\phi_0 v'^{1+\omega(x)} u_2'$  becomes  $v_3 v_2$ , so  $\omega(y) \leq 2 < \omega(x)$ .

Let us see the case  $i \leq \omega(x)$ . If  $i \frac{\omega(x)-1}{2(1+\omega(x))} + i \frac{\omega(x)-1}{2(1+\omega(x))} + 1$  the sum of the exponents of  $u_1'$  and  $u_2'$  is  $< i$ , then at worse,  $\Omega(x') = \Omega(x)$  and  $\kappa(x') = 2$ . If  $i \frac{\omega(x)-1}{2(1+\omega(x))} + i \frac{\omega(x)-1}{2(1+\omega(x))} + 1 = i$ , then  $i = \frac{\omega(x)+1}{2}$ , so  $\frac{\omega(x)+1}{2} \in \mathbb{N}$ : the exponent  $i \frac{\omega(x)-1}{2(1+\omega(x))} = \frac{\omega(x)-1}{4}$  is integer,  $\omega(x) \geq 5$ . We blow up  $x'$ , as  $U_1'$  is in its directrix and, because of the monomial  $\phi_0 v'^{1+\omega(x)} u_2'$ , the only possible very near point  $y$  has for parameters  $(Y = \frac{X'}{u_2'}, v_1 = \frac{u_1'}{u_2'}, v_2 = u_2', w = \frac{v'}{u_2'})$ , the monomial  $v'^{1+\omega(x)-i} u_1'^i u_2'^{i \frac{\omega(x)-1}{2(1+\omega(x))}} u_2'^{i \frac{\omega(x)-1}{2(1+\omega(x))} + 1}$  becomes  $w^{\frac{1+\omega(x)}{2}} v_1^{\frac{\omega(x)-1}{4}} v_2$ : a quick computation shows that  $\omega(x) \geq 5$  implies that the sum of the exponents is  $\leq \omega(x)$ , at worse  $\Omega(y) = \Omega(x)$  and  $\kappa(y) = 2$ . End of the case  $A_1 < \frac{1}{2}$ .

**II.5.3**  $\frac{1}{2} \leq A_1 < 1$ . Then  $B = A_1 + C + A_2 \geq 1$ , this implies  $\epsilon(x) = 1 + \omega(x)$ . By **II.1.2(1)**,  $V \in W(x)$ . As  $\Delta(H(x)^{-1}f + H(x)^{-1}g^p; u_1, u_2; v)$  has only two vertices  $(A_1, 1)$  and  $(A_1 + \frac{1}{2}, 0)$ , in the expansion of

$$H(x)^{-1}f = v^{\omega(x)+1}\phi_0 + \sum_{1 \leq i \leq 1+\omega(x)} v^{\omega(x)+1-i} u_1^{b_i} u_2^{c_i} \psi_i,$$

for some  $i = i_0$ ,  $1 \leq i_0 \leq 1 + \omega(x)$ ,

$$u_1^{b_{i_0}} u_2^{c_{i_0}} \psi_{i_0} = (\gamma' + \psi') u_1^{i_0 A_1 + \frac{i_0}{2}}, \quad \gamma' \text{ invertible} \quad (a)$$

with  $\psi' \in (u_1, u_2)$  or

$$H(x)^{-1}g^p = \gamma u_1^{(1+\omega(x))A_1 + \frac{1+\omega(x)}{2}}. \quad (b)$$

We blow up  $x$ . As  $1 + \omega(x) \not\equiv 0 \pmod{p}$ , the monomial  $H(x')\phi_0 v^{1+\omega(x)}$  cannot be destroyed by a translation on  $X'$ :  $\Omega(x') \leq \Omega(x)$ , if equality,  $\kappa(x') = 3$ .

**II.5.4** Let us look at the first chart of origin the point  $(X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = \frac{u_2}{u_1}, v' = \frac{v}{u_1})$ .

$$H(x')^{-1}f' = v'^{\omega(x)+1}\phi_0 + \sum_{1 \leq i \leq 1+\omega(x)} v'^{\omega(x)+1-i} u_1'^{b_i+c_i+d_i-i} (\Psi_i(1, u'_2) + u_1'\psi'_i),$$

in case (b),  $\text{ord}_{x'}(H(x')^{-1}g'^p) = (1 + \omega(x))(A_1 - \frac{1}{2}) < \omega(x)$ , there is no very near point in this chart.

From now on, we are in case (a). For  $i$  with  $b_i + c_i + d_i = iB$  (notations of **II.3.1**), as the initial side of  $\Delta(H(x)^{-1}f; u_1, u_2; v)$  has only  $(A_1 + \frac{1}{2}, 0)$  for vertex:  $c_i = 0$ ,  $u_1^{b_i} \Psi_i = \gamma_i u_1^{i(A_1 + \frac{1}{2})}$ ,  $\gamma_i$  invertible,

$$v'^{\omega(x)+1-i} u_1'^{b_i+c_i+d_i-i} (\Psi_i(1, u'_2) + u_1'\psi'_i) = v'^{\omega(x)+1-i} u_1'^{iA_1 - \frac{i}{2}} (\gamma_i + u_1'\psi'_i). \quad (1)$$

As  $i(A_1 + \frac{1}{2})$  is an exponent in the expansion of  $H(x)^{-1}f$ , it is an integer, say  $N$ . So  $2A_1 + 1 = \frac{2N}{i}$ . As  $A_1 < 1$ ,  $2A_1 + 1 < 3$ , so  $N < \frac{3i}{2}$ . Then, the sum of the exponents

$$\omega(x) + 1 - i + i(A_1 - \frac{1}{2}) = \omega(x) + 1 - i + N - i < \omega(x) + 1 + \frac{3i}{2} - 2i = \omega(x) + 1 - \frac{i}{2}. \quad (2)$$

If there is some  $i > 1$  with  $b_i + c_i + d_i = iB$ ,  $\omega(x) + 1 - i + i(A_1 - \frac{1}{2}) < \omega(x)$ , in the expansion of  $f'$  appears the monomial  $\gamma H(x') v'^{\omega(x)+1-i} u_1'^b$  of (1), with  $b = iA_1 - \frac{i}{2} \leq i - 1$ . Either this monomial is not spoiled by a translation on  $X'$  and  $\omega(x') < \omega(x)$  or it becomes  $H(x') v'^{\omega(x)+1-i} u_1'^b w$  and, because of  $\frac{\partial}{\partial w}$ ,  $\omega(x') < \omega(x)$ . When  $i = 1$  is the only index with  $b_i + c_i + d_i = iB$ , then  $A_1 = \frac{1}{2}$ , in the expansion of  $f'$  appears the monomial  $\gamma H(x') v'^{\omega(x)} = u_1'^{a(1)+\omega(x)+1-p} \gamma v'^{\omega(x)}$  of (1). Either  $\omega(x) \not\equiv 0 \pmod{p}$  or  $a(1) + 1 \not\equiv 0 \pmod{p}$ , then this monomial cannot be spoiled by any translation on  $X'$ , and  $\omega(x') \leq \omega(x)$  if  $\omega(x) \not\equiv 0 \pmod{p}$ , at worse  $\Omega(x') = \Omega(x)$  and  $\kappa(x') = 2$  if  $a(1) + 1 \not\equiv 0 \pmod{p}$ . Or  $\omega(x) \equiv 0 \pmod{p}$  and  $a(1) + 1 \equiv 0 \pmod{p}$ : this implies that  $\bar{\gamma} \in k(x)$  is not a  $p$ -power: if  $x'$  is rational over  $x$ , the monomial cannot be spoiled by any translation on  $X'$ , we conclude as above. Or by a translation, this monomial becomes  $u_1'^{a(1)+\omega(x)+1-p} v'^{\omega(x)} w$  where  $w \in k(x)[u'_2]$  is a parameter at  $x'$ , then, after translation, as there is no other possibility for  $i$ ,  $f'$  becomes

$$f'' = H(x')(v'^{1+\omega(x)}\phi_0 + v'^{\omega(x)}w + u_1'\phi) = H(x')(v'^{\omega(x)}z + u_1'\phi), z = w + \phi_0 v,$$

by chapter 2 **II.1**, we have at worse  $\Omega(x') = \Omega(x)$  and  $\kappa(x') = 2$ .

**II.5.5**  $\frac{1}{2} \leq A_1 < 1$ , origin of the second chart: the point  $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, v' = \frac{v}{u_2})$ . Then  $H(x') = u'_1{}^{a(1)} u'_2{}^{a(1)+1+\omega(x)-p}$ . By the usual transformation laws,  $\Delta(H(x')^{-1} f' + H(x')^{-1} g''^p; u'_1, u'_2; v')$  has only two vertices  $(A_1, A_1)$  and  $(A_1 + \frac{1}{2}, A_1 - \frac{1}{2})$ . As  $2A_1 \geq 1$ , we have  $\epsilon(x') \geq 1 + \omega(x)$ . We blow up  $x'$ . As  $A_1(x') = A_1(x) > 0$ , then  $V \in \text{IDir}(x)$ . Let us look at the first chart of origin  $(Y = \frac{X'}{u'_1}, v_1 = u'_1, v_2 = \frac{u'_2}{u'_1}, w = \frac{v'}{u'_1})$ . We call  $y$  the point we consider in this chart:  $y$  is assumed to be very near to  $x$ . If  $y$  is the origin, then by the usual transformation laws,  $C(y) = 0$ ,  $A_1(y) = B(x') = 2A_1(x) - 1 < 1$ ,  $A_2(y) = A_2(x') = 0$ : by **II.2**,  $\kappa(y) \leq 2$ .

From now on,  $v_2(y) \neq 0$ . As  $1 + \omega(x) \neq 0 \pmod{p}$ , the initial form of  $H(y)\phi_0 v^{1+\omega(x)}$  will not be spoilt by any translation on  $Y$ :  $\kappa(y) \leq 3$ . In the expansion of

$$H(x')^{-1} f' = v'^{\omega(x)+1} \phi_0 + \sum_{1 \leq i \leq 1+\omega(x)} v'^{\omega(x)+1-i} u_1'^{b'_i} u_2'^{c'_i} \psi'_i, \quad d'_i = \text{ord}_x(\psi'_i)$$

there is a monomial defining the vertex  $(A_1, A_1)$ , so for some  $i = i_1$ ,  $1 \leq i_1 \leq 1 + \omega(x)$ ,

$$u_1'^{b'_{i_1}} u_2'^{c'_{i_1}} \psi'_{i_1} = u_1'^{i_1 A_1} u_2'^{c'_{i_1}} (\Psi'_{i_1}(u'_1, u'_2) + \phi), \quad \phi \in (u'_1, u'_2)^{d'_{i_1}+1}, \quad d'_{i_1} + c'_{i_1} = i_1 A_1, \quad (a)$$

with  $\Psi'_{i_1}$  homogeneous of degree  $d'_{i_1}$  or

$$H(x)^{-1} g^p = \gamma u_1'^{(1+\omega(x))A_1} u_2'^{(1+\omega(x))A_1}. \quad (b)$$

In case (b),  $H(y)^{-1} g''^p = \gamma v_1^{(2A_1-1)(1+\omega(x))}$ , as  $2A_1 - 1 < 1$ ,  $\text{ord}_y(H(y)^{-1} g''^p) \leq \omega(x)$ : at worse,  $\omega(y) = \omega(x)$  and  $\omega'(y) = 1 < \omega'(x)$ :  $y$  is not very near to  $x$ .

From now on, we are in case (a).

$$H(y)^{-1} f'' = w^{\omega(x)+1} \phi_0 + \sum_{1 \leq i \leq 1+\omega(x)} w^{\omega(x)+1-i} v_1^{b'_i+c'_i+d'_i-i} u_2'^{c'_i} (\Psi'_i(1, u'_2) + v_1 \phi_i).$$

for  $i = i_1$ ,  $w^{\omega(x)+1-i} v_1^{b'_i+c'_i+d'_i-i} u_2'^{c'_i} \Psi'_i(1, u'_2) = w^{\omega(x)+1-i_1} v_1^{i_1(2A_1-1)} u_2'^{c'_{i_1}} \Psi'_{i_1}(1, u'_2)$ .

Case where  $2A_1 = 1$ :  $w^{\omega(x)+1-i_1} v_1^{i_1(2A_1-1)} u_2'^{c'_{i_1}} \Psi'_{i_1}(1, u'_2) = w^{\omega(x)+1-i_1} u_2'^{c'_{i_1}} \Psi'_{i_1}(1, u'_2)$ . If  $1 + \omega(x) - i_1 \neq 0 \pmod{p}$  or  $a(1) + a(1) + \omega(x) + 1 - p + i_1 \neq 0 \pmod{p}$ , a translation on  $Y$  will not spoil  $w^{\omega(x)+1-i_1} u_2'^{c'_{i_1}} \Psi'_{i_1}(1, u'_2)$ ,  $\text{ord}_y(u_2'^{c'_{i_1}} \Psi'_{i_1}(1, u'_2)) \leq d'_{i_1} + c'_{i_1} = \frac{i_1}{2}$ , so  $\epsilon(y) \leq 1 + \omega(x) - i_1 + \frac{i_1}{2} \leq \omega(x)$ : at worse,  $\Omega(y) = \Omega(x)$  and  $\kappa(y) = 2$ . If  $u'_2(x') = 0$ , we are at the origin of a chart: there is no translation to do, we conclude as above. If  $1 + \omega(x) - i_1 = 0 \pmod{p}$  and  $a(1) + a(1) + \omega(x) + 1 - p + i_1 = 0 \pmod{p}$  and  $u'_2(x') \neq 0$ , a translation on  $X'$  may add a  $p$ -power to  $w^{\omega(x)+1-i_1} u_2'^{c'_{i_1}} \Psi'_{i_1}(1, u'_2)$ , by chapter 2 **II.5(ii)**, the  $D(H(x')u_1'^{b'_{i_1}} u_2'^{c'_{i_1}} \Psi'_{i_1}) = D(u_1'^{a(1)} u_2'^{a(1)+1+\omega(x)-p} u_1'^{b'_{i_1}} u_2'^{c'_{i_1}} \Psi'_{i_1})$ ,  $D \in \mathcal{D}$  are not all proportional to  $u_1'^{a(1)} u_2'^{a(1)+1+\omega(x)-p} u_1'^{b'_{i_1}} u_2'^{c'_{i_1}} .d'_{i_1}{}^{\text{th}}$ -power, or  $\Psi'_{i_1}$  is a monomial, so, after an eventual translation,  $w^{\omega(x)+1-i_1} u_2'^{c'_{i_1}} \Psi'_{i_1}(1, u'_2)$  which will become  $w^{\omega(x)+1-i_1} \gamma' v_2^a$ ,  $a \leq d'_{i_1} + c'_{i_1} = \frac{i_1}{2}$ ,  $\gamma'$  invertible, we conclude as above.

From now on,  $\frac{1}{2} < A_1$ . For our index  $i_1$ ,  $\frac{i_1}{2} < i_1 A_1 = b'_{i_1} \in \mathbb{N}$ , so  $i_1 \geq 3$ . Furthermore,  $1 > B - 1 = 2A_1 - 1 > 0$ :  $B - 1 \notin \mathbb{N}$ , so  $w^{\omega(x)+1-i_1} v_1^{i_1(2A_1-1)} u_2'^{c'_{i_1}} \Psi'_{i_1}(1, u'_2) = w^{\omega(x)+1-i_1} u_2'^{c'_{i_1}} \Psi'_{i_1}(1, u'_2)$  will not be spoilt by an eventual translation on  $w$ , after an eventual translation on  $Y$ ,  $w^{\omega(x)+1-i_1} u_2'^{c'_{i_1}} \Psi'_{i_1}(1, u'_2)$  which will become  $w^{\omega(x)+1-i_1} \gamma' v_2^a$ ,  $a \leq d'_{i_1} + c'_{i_1} + 1 = \frac{i_1}{2} + 1$ ,

$\gamma'$  invertible:  $\beta(y) \leq \frac{1}{2} + \frac{1}{i_1} \leq \frac{1}{2} + \frac{1}{3} < 1$ . By **II.4**,  $\kappa(x) \leq 1$ . Finally, we have to look at the origin of the second chart above  $x'$ . Then  $C(y) = 0$ ,  $A_1(y) = A_1(x) < 1$ ,  $A_2(y) = 2A_1(x) - 1 < 1$ : by **II.2**,  $x$  is good.

**II.6 Theorem** *Let  $x \in \Sigma_p$  with  $\kappa(x) = 3$ ,  $(X, u_1, u_2, v)$  a r.s.p. verifying **II.1(1)**,  $v \in \widehat{S}$  and possibly  $u_2 \in \widehat{S}$  if  $\text{div}(u_2) \not\subseteq E$ . If  $C(x) = 0$ , then  $\kappa(x) \leq 1$ .*

Proof. Let us recall that, by **II.2**, if  $\epsilon(x) = \omega(x)$ , or  $(A_1 < 1$  and  $A_2 < 1)$   $\kappa(x) \leq 1$ . So, we suppose:

$$\epsilon(x) = 1 + \omega(x), A_1 \geq 1 \text{ or } A_2 \geq 1.$$

**II.6.1** Case  $a(1) + 1 + \omega(x) < p$  and  $a(2) + 1 + \omega(x) < p$ . Let us denote  $H(x) = u_1^{a(1)} u_2^{a(2)}$ , then  $\text{ord}_x(H(x)f) \geq p$ , so  $a(1)a(2) > 0$ :  $E = \text{div}(u_1 u_2)$ . So  $u_1$  and  $u_2$  play the same role. As  $C(x) = 0$ , we have  $A_i > 0$  for some  $i = 1, 2$ , by **II.1.2**,  $V \in \text{IDir}(x)$ . Let us blow up  $x$ . in the first chart of origin  $(\frac{X}{u_1}, u_1, \frac{u_2}{u_1}, \frac{v}{u_1}) = (X', u'_1, u'_2, v')$ , we get

$$f' = u_1^{a(1)+a(2)+1+\omega(x)-p} u_1^{a(2)} (v'^{1+\omega(x)} \phi_0 + \dots).$$

If  $u'_2(x') \neq 0$ , then  $\text{ord}_{x'}(f') < p$ . If  $u'_2(x') = 0$ , the reader sees that  $\kappa(x') \leq 3$ , if  $x'$  is very near to  $x$ ,  $a(1) + a(2) + 1 + \omega(x) - p < a(2)$ ,  $C(x') = 0$ : an induction on  $a(1) + a(2)$  gives the result. By symetry, it is the same thing in the second chart.

**II.6.2** Case  $1 + \omega(x) \geq p$ . If  $A_1 \geq 1$ ,  $V(X, v, u_1)$  is permissible of first kind, it is  $\text{div}(u_1) \cap \{\omega \geq 1\}$ : it is not formal, we blow it up, the point  $x'$  of parameters  $(\frac{X}{u_1}, u_1, u_2, \frac{v}{u_1})$  is the only point which may be very near to  $x$ , if it is, then  $\kappa(x') = 3$ ,  $A_1(x') = A_1(x) - 1$ ,  $A_2(x') = A_2(x)$ ,  $C(x') = 0$ , we get the result by induction on  $A_1$ . If  $A_2 \geq 1$  and  $\text{div}(u_2) \subseteq E$ , mutatis mutandis, it is the same thing. If  $A_2 \geq 1$  and  $\text{div}(u_2) \not\subseteq E$ , then  $\epsilon(Y) = \omega(x)$  where  $Y = V(X, v, u_2) \not\subseteq E$ : this contradicts the cleaning condition of chapter 1, this case is impossible.

**II.6.3** Case  $1 + \omega(x) < p$  and there exists  $i \in \{1, 2\}$  such that  $a(i) + \omega(x) + 1 \geq p$ . Then  $a(i) > 0$ , so  $\text{div}(u_i) \subseteq E$ . We make a descending induction on

$$(\text{sup}(A_i), \text{sup}(a(j)), n)$$

where  $n$  is 2 if  $(A_1, a(1)) = (A_2, a(2)) = (\text{sup}(A_i), \text{sup}(a(j)), n)$ , else  $n = 1$ .

**II.6.3.1** If there exists  $i \in \{1, 2\}$  such that

$$a(i) + \omega(x) + 1 \geq p \text{ and } A_i \geq 1, \tag{1}$$

we blow up  $V(X, u_i, v)$ . If both  $i = 1$  and  $i = 2$  verifies (1), we choose  $i$  with  $A_i$  maximal. Mutatis mutandis,  $i = 1$ . Then the point  $x'$  of parameters  $(\frac{X}{u_1}, u_1, u_2, \frac{v}{u_1}) = (X', u'_1, u'_2, v')$  is the only point which may be very near to  $x$ . The reader sees that  $\kappa(x') \leq 3$ , if  $x'$  is very near to  $x$ ,  $C(x') = 0$ ,  $A_1(x') = A_1(x) - 1$ ,  $a(1)(x') = a(1)(x) + 1 + \omega(x) - p < a(1)$ :  $(\text{sup}(A_i), \text{sup}(a(j)))$  strictly drops, except if  $A_2(x) = A_1(x)$ ,  $a(1) = a(2)$  where  $n$  becomes 1.

**II.6.3.2** The remaining case. Then for all  $i \in \{1, 2\}$  such that  $a(i) + \omega(x) + 1 \geq p$ ,  $A_i < 1$  and there exists such an  $i$ . So  $\text{div}(u_i) \subseteq E$ , mutatis mutandis,  $i = 1$ . Then,  $A_2 \geq 1$  and  $a(2) + \omega(x) + 1 < p$ . I say that

$$E = \text{div}(u_1 u_2).$$

Suppose  $E = \text{div}(u_1)$ . If  $g = 0$ , then, as  $A_2 \geq 1$ ,  $\epsilon(V(X, u_2, v)) \geq \omega(x) + 1$ ,  $(X, u_2, v) \supset J(f, E)$ , this contradicts the cleaning condition of chapter 1. If  $g \neq 0$ , then  $H(x)^{-1}g^p = \gamma u_1^a$ ,  $a \in \mathbb{N}$ , so  $A_2 = 0$  which contradicts  $A_2 \geq 1$ . We blow up  $x$ , let us look at the first chart of origin the point of parameters  $(\frac{X}{u_1}, u_1, \frac{u_2}{u_1}, \frac{v}{u_1}) = (X', u'_1, u'_2, v')$ . If  $x'$  is the origin, then the reader sees that (cf. **II.3**) that  $\kappa(x') \leq 3$ , if  $x'$  is very near to  $x$ ,  $C(x') = 0$  and  $A_1(x') = A_1(x) + A_2(x) - 1 < A_2(x)$ ,  $A_2(x') = A_2(x)$ ,  $a'(1) = a(1) + a(2) + \omega(x) + 1 - p < a(1)$ ,  $a'(2) = a(2)$ :  $(\sup(A_i), \sup(a(j)))$  strictly drops. If  $u'_2(x') \neq 0$  then  $m(x) = 2$  and  $m(x') = 1$ , by **II.3(vii)**, **II.4**,  $\kappa(x') \leq 1$ . The last point to look at is the point  $x'$  of parameters  $(\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, \frac{v}{u_2}) = (X', u'_1, u'_2, v')$ . The reader sees that (cf. **II.3**) that, if  $x'$  is very near to  $x$ ,  $\kappa(x') \leq 3$ ,  $C(x') = 0$  and  $A_1(x') = A_1(x) < A_2(x)$ ,  $A_2(x') = A_1(x) + A_2(x) - 1 < A_2(x)$ :  $(\sup(A_i), \sup(a(j)), n)$  strictly drops.

## II.7 End of the story

We make an infinite sequence of blowing-ups  $X_i \leftarrow X_{i+1}$  of  $(u_1, u_2)$  for a suitable choice of parameters verifying (1). Let  $x_i \in X_i$ , the centers of the valuation  $\mu$  we are uniformizing. *We are going to prove that there exists some  $i$  such that  $\kappa(x_i) \leq 2$ .* That will end the case  $\kappa(x) = 3$ . By **II.3(i)**, for  $i \gg 0$ ,  $\gamma(x_i) = \gamma(x_{i+n})$ ,  $n > 0$ . Let us call  $\gamma(\mu)$  this value (which depends of  $\mu$  and of choices among the possible  $u_1, u_2$ ).

**II.7.1**  $m(x_i) = 1$  for  $i \gg 0$ . Either  $g \neq 0$ . If, for some  $i$ ,  $\beta(x_i) = 0$ , then  $C(x_i) = 0$ , by **II.2**,  $\kappa(x_i) \leq 1$ . If, for all  $i$ ,  $\beta(x_i) > 0$ , then the sequence  $A_1(x_i) \in \frac{1}{(1+\omega(x))!} \mathbb{N}$  strictly increases, but, as  $g^p = u_1^a$  in  $x_i$ , and that  $u_1$  belongs to the choosen parameters of  $x_{i+1}$ ,  $g^p = u_1^a$  in  $x_{i+1}$ , but  $A_1(x_i) \leq \text{ord}(H(x_i)^{-1}g^p) \leq \frac{a}{1+\omega(x)}$ , this is impossible for  $i \gg 0$ . Or  $g = 0$ . Either for all  $i$  there exists  $j \geq i$  such that  $x_{j+1}$  is not rational over  $x_j$ , then, by **II.3.2**, we reach the case where  $\beta(x_n) < 1$  for some  $n$ , by **II.4**,  $\kappa(x_n) \leq 1$ . Or  $x_{i+1}$  is rational over  $x_i$  for  $i \gg 0$ . As in the corresponding case of  $\kappa(x) = 2$ , there exists  $v = u_2 + \sum_{i \geq 1} \lambda_i u_1^i \in u_2 + k(x)[[u_1]]$  such that  $x_{n_0+i}$  is on the strict transform of  $v$  in  $X_{n_0+i}$ : we have  $(u_1, u_2) \widehat{S}_{n_0} = (u_1, v) \widehat{S}_{n_0}$ . The proof runs along the same lines. So let us choose  $(X, u_1, v, u_3)$  as r.s.p. of  $x_{n_0}$ , let us make a well preparation: we get  $(Y, u_1, v, w)$  as new well prepared r.s.p.. Then, in the sequence of the first line of **II.7**, we stay on the strict transform of  $v$ , the parameters at  $x_{n_0+i}$  are  $(Y, u_1, \frac{v}{u_1}, w)$  and as we are at the origin of the first chart in all the blowing ups,  $(Y, u_1, \frac{v}{u_1}, w)$  is well prepared, so, for  $i \gg 0$ ,  $C(x_i) = 0$ , by **II.6**,  $\kappa(x_i) \leq 1$ .

**II.7.2**  $\gamma(\mu) \geq 2$ . Then, by **II.3(i)(v)**:  $m(x_i) = 1$  for  $i \gg 0$ ,  $\kappa(x_i) \leq 1$ .

**II.7.3**  $\gamma(\mu) = 1$ .

**II.7.3 (i)**  $\gamma(\mu) = 1$ ,  $m(x_i) = 1$  for  $i \gg 0$ . Go to **II.7.1**.

**II.7.3 (ii)**  $\gamma(\mu) = 1$ ,  $m(x_i) = 2$  for  $i \gg 0$ . We are always at the origin of a chart, so, as seen many times, for  $i \gg 0$ ,  $C(x_i) = 0$ , by **II.6**,  $\kappa(x_i) \leq 1$ .

**II.7.3 (iii)**  $\gamma(\mu) = 1$ , there is some  $i$  with  $m(x_i) = 2$  and  $C(x_i) < \frac{1}{2}$ . Let  $j \in \mathbb{N}$  such that  $m(x_i) = m(x_{i+1}) = \dots m(x_{i+j}) = 2$ ,  $m(x_{i+j+1}) = 1$ . Then  $\frac{1}{2} > C(x_i) \geq C(x_{i+1}) \geq \dots \geq C(x_{i+j})$ . By **II.3(vii)** and **II.4**,  $\kappa(x_{i+j+1}) \leq 1$ .

**II.7.3 (iv)**  $\gamma(\mu) = 1$ , there is some  $i$  with  $m(x_i) = 1$  and  $\beta(x_i) < 1$ : by **II.4**,  $\kappa(x_i) \leq 1$ .

**II.7.3 (v)**  $\gamma(\mu) = 1$ , there is some  $i$  with  $m(x_i) = 1$  and  $\beta(x_i) = 1$ . If we are in case **II.7.1**, we are done. Else let  $j \in \mathbb{N}$  such that  $m(x_i) = m(x_{i+1}) = \dots m(x_{i+j}) = 1$ ,  $m(x_{i+j+1}) = 2$ . If there exists  $u$ ,  $0 \leq u \leq j$ ,  $\beta(x_{i+u}) = 0$ ,  $\kappa(x_{i+u}) \leq 1$ . If not, then  $1 = \beta(x_i) \geq \beta(x_{i+1}) \geq \dots \geq \beta(x_{i+j})$ . If  $1 > \beta(x_{i+j})$ , go to **II.4**. If  $1 = \beta(x_{i+j})$  and  $\Delta(H(x_{i+j})^{-1}f_i; u_1, u_2; v)$  has only two vertices  $(A_1, 1)$  and  $(A_1 + \frac{1}{2}, A_1 - \frac{1}{2})$ , then by **II.5**,  $\kappa(x_{i+j}) \leq 1$ . If not, then by **II.3 (iii)**,  $C(x_{i+j}) < \frac{1}{2}$ . Go to **II.7.3 (iii)**.

The reader should be convinced that all the possible cases have been seen.

### III End of transverseness.

We conclude the analysis of those cases of transverseness where  $\text{cl}_{\omega(x)}J(f, E)$  is not contained in the ideal  $(\{U_i \mid \text{div}(u_i) \subseteq E\})$ . This is formalized in definition **III.2** below.

Now that the main cases  $\kappa(x) = 2$  and  $\kappa(x) = 3$  have been dealt with, reduction to  $\kappa(x) \leq 3$  (propositions **III.5** and **III.6**) is based on the following lemma which is an extension of proposition **II.1** of chapter 2.

**III.1 Lemma.** *Assume that  $E = \text{div}(u_1 u_2)$ ,  $\epsilon(x) = \omega(x)$  and  $f = H(x)(\lambda u_1^{\omega(x)} + u_2 \psi)$ , where  $\lambda \in k(x)$ ,  $\lambda \neq 0$  and  $\text{ord}_{u_3}(\psi \bmod(u_1, u_2)) = \omega(x)$ . There exists a sequence of permissible blowing-ups,*

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n$$

*such that either  $x_n$  is not very near  $x$  or  $(\Omega(x) = \Omega(x_n) \text{ and } \kappa(x_n) \leq 3)$ , where  $x_n \in X_n$  is the center of  $\mu$ .*

Proof. Note that  $\kappa(x) = 2$  except possibly if  $\text{cl}_{\omega(x)-1}\psi \in k(x)[U_1, U_2]$ , which we assume from now on. We then have  $\text{VDir}(x) \subseteq \langle U_1, U_2 \rangle$ ,  $\text{VDir}(x) \neq k(x).U_2$ . We have  $\kappa(x) \leq 1$  if  $\omega(x) = 1$  by **II.1(iv)** of chapter 2, so assume that  $\omega(x) \geq 2$ . Let  $e : X' \rightarrow X$  be the blowing up along  $x$  and  $x' \in e^{-1}(x)$  be very near  $x$ .

We first consider the case when  $x' = x'_0 := (X' = \frac{X}{u_3}, u'_1 = \frac{u_1}{u_3}, u'_2 = \frac{u_2}{u_3}, u'_3 = u_3)$ , so  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1 u'_2 u'_3)$ . We are at the origin of a chart, so  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal. In the expansion of  $H(x')^{-1}f'$ , there appears the monomial  $u'_2 u'_3$  with nonzero coefficient. Since  $x'$  is very near  $x$ , we have  $\omega(x) = 2$  and there is an expression

$$\Psi' := \text{cl}_2(H(x')^{-1}f') = \lambda U_1'^2 + \lambda_1 U_1' U_2' + \lambda_2 U_2'^2 + \lambda_3 U_2' U_3',$$

where  $\lambda \lambda_3 \neq 0$ . By lemma **II.1.5** in chapter 2, we have  $\tau(x') = 3$ , so  $\kappa(x) = 0$ .

We now discuss according to  $\text{VDir}(x)$  and consider three cases.

*Case 1.* If  $\text{VDir}(x) = \langle U_1, U_2 \rangle$ , then  $x' = x'_0$  since  $x'$  is very near  $x$ , so the proposition holds.

Now if  $\text{VDir}(x) \neq \langle U_1, U_2 \rangle$  and  $x' \neq x'_0$ , then  $x'$  is in the chart with origin  $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, u'_3 = \frac{u_3}{u_2})$ , so  $E' := (e'^{-1}E)_{\text{red}} = \text{div}(u'_1 u'_2)$ .

*Case 2.* If  $\text{VDir}(x) = k(x).(U_1 + \alpha U_2)$  for some  $\alpha \in k(x)$ ,  $\alpha \neq 0$ , then  $x' = (X', v' := u'_1 + \alpha, u'_2, w' := P(1, u'_3))$ , where  $P \in k(x)[u_2, u_3]$  is homogeneous and unitary in  $u_3$ . We have  $\text{ord}_{\eta(x)}(u_2 \psi) = \omega(x)$  and

$$\text{cl}_{\omega(x)}J(f, E, x) = k(x).(U_1 + \alpha U_2)^{\omega(x)}$$

by assumption, so  $v'^{\omega(x)} \in J(f', E') + (u'_2)$ . Since  $\omega'(x) = 2$  and  $x'$  is very near  $x$ , after possibly performing a translation  $Z' := X' - \theta'$  in order to get  $\Delta(h'; v', u'_2, w'; Z')$  minimal,  $f'$  being changed into  $f'_{Z'} := f' + \theta'^p - \theta' g'^{p-1}$ , we get

$$v'^{\omega(x)} + u'_2 \varphi' = D' f'_{Z'} \in J(f', E') \tag{1}$$

for some  $\varphi' \in \widehat{S}' \simeq k(x')[[v', u'_2, w']]$ .

If  $D' f'_{Z'} \in J(f', E', x')$ , we have  $\kappa(x') = 2$ . If  $\kappa(x') \neq 3$ , it can be assumed that  $D' = \frac{\partial}{\partial w'}$  in (1), we have  $\omega(x) \equiv 0 \pmod{p}$ , in which case  $\kappa(x) = 2$  if  $\epsilon(x') = 1 + \omega(x)$ . Finally, if  $\epsilon(x') = \omega(x)$ , then  $x'$  satisfies the assumptions of chapter 2 **II.1** (w.r.t. the r.s.p.  $(Z', u'_2, v', w')$ ), whose conclusion gives  $\kappa(x') \leq 1$ .

*Case 3.* If  $\text{VDir}(x) = k(x).U_1$ , then  $x' = (X', u'_1, u'_2, w' := P(1, u'_3))$ , where  $P \in k(x)[u_2, u_3]$  is homogeneous and unitary in  $u_3$ . Moreover, we have  $\text{ord}_{\eta(x)}(u_2\psi) = 1 + \omega(x)$ . Let  $u'_2{}^{-p}h =: X'^p - X'g'^{p-1} + f'$  and  $\Psi := \text{cl}_{\omega(x)}\psi$ . After possibly performing a translation  $Z' := X' - \theta'$  in order to get  $\Delta(h'; u'_1, u'_2, w'; Z')$  minimal,  $f'$  being changed into  $f'_{Z'} := f' + \theta'^p - \theta'g'^{p-1}$ , we have an expression of the form

$$f'_{Z'} = u'_1{}^{a(1)}u'_2{}^{a(2)+\omega(x)-p}(\lambda'u'_1{}^{\omega(x)} + u'_2\psi'),$$

where either  $\lambda'$  or  $\frac{\partial \lambda'}{\partial w'}$  is a unit and  $\text{ord}_{\eta'(x')}(u'_2\psi') \geq \omega(x)$ , since  $x'$  is very near  $x$ . Note that the form  $U_1^{a(1)}U_2^{a(2)+1}\Psi(0, U_2, U_3)$  is not a  $p^{th}$ -power, since  $\Delta(h; u_1, u_2, u_3; X)$  is minimal. We apply theorem **II.5.3.2**(iii) and (iv) of chapter 1 to the form  $U_2^{a(2)+1}\Psi(0, U_2, U_3)$ , which yields

$$\text{ord}_{\eta'(x')}(\psi' \bmod(u'_1)) \leq \deg \Psi(0, U_2, U_3) = \omega(x), \quad (2)$$

with equality only if  $x'$  is rational over  $x$ . Note that  $\kappa(x') = 2$  if inequality is strict in (2), since then  $\text{cl}_{\omega(x)}f'_{Z'} \notin k(x')[U'_1, U'_2]$ .

There remains to study the equality case in (2). Since  $x'$  is rational over  $x$ ,  $\lambda'$  is a unit in this case. After possibly performing a linear change of coordinates  $w := u_3 + \mu u_2$ , followed by a translation  $Z := X - \theta$  in order to get  $\Delta(h; u_1, u_2, w; Z)$  minimal, it can be assumed that  $x'$  is the origin of the chart, the assumptions in the lemma remaining unchanged. Then  $\Delta(h'; u'_1, u'_2, w'; X')$  is minimal and  $x'$  satisfies the assumption of the lemma, i.e. we can iterate the argument. Let

$$X_1 := \text{Spec}(R'/(h')) \leftarrow X_2 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$$

be the quadratic sequence along  $\mu$ . Since

$$\frac{\mu(u_1)}{\mu(u_2)} < \infty,$$

the center  $x_n$  of  $\mu$  in  $X_n$  is either not very near  $x$ , or has  $\kappa(x_n) = 2$ , or is in case 1 or 2 above for some  $n > 0$  and the conclusion follows.

**III.2 Definition.** We say that  $\kappa(x) = 4$  if  $\text{cl}_{\omega(x)}J(f, E) \not\subseteq (\{U_i \mid \text{div}(u_i) \subseteq E\})$ .

**III.3 Definition.** Assume that  $\kappa(x) = 4$ . We say that  $x$  is good if there exists a sequence of permissible blowing-ups,

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n$$

such that either  $x_n$  is not very near  $x$  or  $(\Omega(x) = \Omega(x_n) \text{ and } \kappa(x_n) \leq 3)$ , where  $x_n \in X_n$  is the center of  $\mu$ .

We do not suppose  $\kappa(x_i) \leq 4$  for  $1 \leq i < n$ .

**III.4 Definition.** Assume that  $\kappa(x) = 4$ . We let

$$\bar{\tau}(x) := \tau(\text{cl}_{\omega(x)}J(f, E) + (\{U_i \mid \text{div}(u_i) \subseteq E\})) \geq 2.$$

**III.5 Proposition.** Assume that  $\kappa(x) = 4$  and  $\bar{\tau}(x) = 2$ . Then  $x$  is good.

Proof. Since  $\bar{\tau}(x) = 2$ , we have  $E = \text{div}(u_1)$ . Let  $F_j := \text{cl}_{\omega(x)}(H(x)^{-1}\frac{\partial f}{\partial u_j})$ , for  $j = 2, 3$ .

**III.5.1** If  $\epsilon(x) = 1 + \omega(x)$ , then after possibly relabelling  $(u_2, u_3)$ , we may assume that  $F_2 \notin (U_1)$ . If  $F_2(0, U_2, U_3) \in k(x)[U_2]$ , we have  $\kappa(x) = 3$ . Otherwise, since  $\bar{\tau}(x) = 2$ , we have

$$F_2(0, U_2, U_3) = \mu_2(U_3 + \alpha U_2)^{\omega(x)}$$

for some  $\alpha, \mu_2 \in k(x)$ ,  $\mu_2 \neq 0$ . Once again,  $\kappa(x) = 3$  if  $\alpha \neq 0$ , so assume that  $\alpha = 0$ . Since  $\bar{\tau}(x) = 2$ , we have  $F_3(0, U_2, U_3) \in k(x)[U_3]$ , whence  $\omega(x) \equiv 0 \pmod{p}$  and this proves that  $\kappa(x) = 2$ .

**III.5.2** If  $\epsilon(x) = \omega(x)$ , we write  $f = H(x)(\Psi + \psi_1)$ , where  $\Psi \in k(x)[u_1, u_2, u_3]_{\omega(x)}$  and  $\text{ord}_{\eta(x)}\psi_1 \geq 1 + \omega(x)$ . Let  $\Psi_1 := \text{cl}_{1+\omega(x)}\psi_1$ . We have  $\kappa(x) \leq 2$  whenever  $\text{cl}_{\omega(x)}(H(x)^{-1}Df) \notin k(x)[U_1]$  for some  $D \in \mathcal{D}(x)$ . Thus it can be assumed that  $\text{VDir}(x) = k(x).U_1$ , so  $\Psi = \lambda u_1^{\omega(x)}$  for some  $\lambda \in k(x)$ ,  $\lambda \neq 0$ . Since  $\bar{\tau}(x) = 2$ , we have  $\kappa(x) = 3$  as in **III.5.1** except if  $F_2(0, U_2, U_3) = \mu_2 U_3^{\omega(x)}$  for some  $\mu_2 \in k(x)$ ,  $\mu_2 \neq 0$ , and  $\omega(x) \equiv 0 \pmod{p}$ , which we assume from now on. Note that the monomial  $U_2 U_3^{\omega(x)}$  necessarily appears with nonzero coefficient in the expansion of  $\Psi_1$ .

Let  $e : X' \rightarrow X$  be the blowing up along  $x$  and  $x' \in e^{-1}(x)$  be very near  $x$ . Since  $\text{VDir}(x) = k(x).U_1$ ,  $x'$  maps to the strict transform of  $\text{div}(u_1)$ .

We first look at the point  $x'$  with coordinates  $(X' = \frac{X}{u_3}, u'_1 = \frac{u_1}{u_3}, u'_2 = \frac{u_2}{u_3}, u'_3 = u_3)$ , so  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1 u'_3)$ . This is the origin of a chart, so  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal, where  $h' := u'_3^{-p}h = X'^p - X'g'^{p-1} + f'$ . In the expansion of  $H(x')^{-1}f'$ , there appears the monomial  $u'_2 u'_3$  with nonzero coefficient, so  $\omega(x') \leq \text{ord}_{\eta'(x')}(H(x')^{-1}\frac{\partial f'}{\partial u'_2}) = 1 < p \leq \omega(x)$ : a contradiction, since  $x'$  is very near  $x$ .

We now consider the case where  $x'$  is in the chart with origin  $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, u'_3 = \frac{u_3}{u_2})$ , so  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1 u'_2)$ . We have

$$u'_2^{-\omega(x)+1} H(x)^{-1} \frac{\partial f}{\partial u_2} \in J(f', E'),$$

where  $h' := u'_2^{-p}h = X'^p - X'g'^{p-1} + f'$ .

We pick  $P \in k(x)[u_2, u_3]$  homogeneous and unitary in  $u_3$  such that  $x' = (X', u'_1, u'_2, v' := P(1, u'_3))$ . After possibly performing a translation  $Z' := X' - \theta'$  in order to get  $\Delta(h'; u'_1, u'_2, v'; Z')$  minimal, there is an expression  $h' = Z'^p - Z'g'^{p-1} + f'_{Z'}$  with

$$u'_2^{-\omega(x)+1} H(x)^{-1} \frac{\partial f}{\partial u_2} \in J(f'_{Z'}, E') + (u'_1),$$

since  $\text{ord}_{\eta(x)}g^p > \text{ord}_{\eta(x)}f$ .

If  $x'$  is not rational over  $x$ , we have

$$\text{ord}_{\eta'(x')}(u'_2^{-\omega(x)+1} H(x)^{-1} \frac{\partial f}{\partial u_2}) \leq 1 + \frac{\omega(x)}{2},$$

so  $\omega(x') < \omega(x)$  except possibly if  $p = 2 = \omega(x)$ . In this last case, since  $\omega(x') = \omega(x) = 2$ , there is a derivation  $D' \in \mathcal{D}'$  such that

$$\text{cl}_2(H(x')^{-1}D'f'_{Z'}) = \mu U'_2 V' + \nu U'_2{}^2 + U'_1 \Psi',$$

where  $\mu, \nu \in k(x')$ ,  $\mu \neq 0$ , and  $\Psi' \in k(x')[U'_2, V']_1$ . As  $p = 2$ ,  $D' \neq \frac{\partial}{\partial v'}$  and we thus have  $H(x')^{-1}D'f'_{Z'} \in J(f'_{Z'}, E', x')$ . So  $\omega(x') \leq \text{ord}_{\eta'(x')}(H(x')^{-1}\frac{\partial f'}{\partial v'}) = 1$ : a contradiction. This proves that  $x'$  is rational over  $x$ .

If  $x'$  is rational over  $x$ , then after changing  $u_3$  into  $v := u_3 + \mu u_2$  for some  $\mu \in k(x)$ , followed by a translation on  $Z := X - \theta$  in order to get  $\Delta(h; u_1, u_2, v; Z)$  minimal, it can be assumed that  $x'$  is the origin of the chart, the assumption in the proposition being unchanged. Then  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal and we have

$$f' = u'_1{}^{a(1)} u'_2{}^{a(1)+\omega(x)-p} (\lambda u'_1{}^{\omega(x)} + u'_2 \psi').$$

Since the monomial  $U_2 U_3^{\omega(x)}$  appears with nonzero coefficient in the expansion of  $\Psi_1$ ,  $x'$  satisfies the assumption of lemma **III.1** and the conclusion follows.

**III.6 Proposition.** *Assume that  $\kappa(x) = 4$  and  $\bar{\tau}(x) = 3$ . Then  $x$  is good.*

Proof. If  $E = \text{div}(u_1 u_2)$ , then  $\kappa(x) \in \{2, 3\}$  (resp.  $\kappa(x) = 3$ ) if  $\epsilon(x) = \omega(x)$  (resp. if  $\epsilon(x) = 1 + \omega(x)$ ). We assume from now on that  $E = \text{div}(u_1)$ .

**III.6.1** If  $\epsilon(x) = \omega(x)$ , we write  $f = H(x)(\Psi + \psi_1)$ , where  $\Psi \in k(x)[u_1, u_2, u_3]_{\omega(x)}$  and  $\text{ord}_{\eta(x)} \psi_1 \geq 1 + \omega(x)$ . Let  $\Psi_1 := \text{cl}_{1+\omega(x)} \psi_1 = Q(U_2, U_3) + U_1 \Psi_2(U_1, U_2, U_3)$ . We have  $\kappa(x) \leq 2$  unless  $\text{VDir}(x) = k(x).U_1$ , i.e.  $\Psi = \lambda U_1^{\omega(x)}$  for some  $\lambda \in k(x)$ ,  $\lambda \neq 0$ . Since  $\bar{\tau}(x) = 3$ , we have

$$\text{VDir}\left(\frac{\partial Q}{\partial U_2}, \frac{\partial Q}{\partial U_3}\right) = \langle U_2, U_3 \rangle. \quad (1)$$

Let  $e : X' \rightarrow X$  be the blowing up along  $x$  and  $x' \in e^{-1}(x)$  be very near  $x$ . Since  $\text{VDir}(x) = k(x).U_1$ ,  $x'$  maps to the strict transform of  $\text{div}(u_1)$ . By symmetry, it can be assumed that  $x'$  is in the chart with origin  $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, u'_3 = \frac{u_3}{u_2})$ , so  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1 u'_2)$ . We have

$$u'_2{}^{-\omega(x)+1} H(x)^{-1} \left( \frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial u_3} \right) \subseteq J(f', E'),$$

where  $h' := u'_2{}^{-p} h = X'^p - X' g'^{p-1} + f'$ . We pick  $P \in k(x)[u_2, u_3]$  homogeneous and unitary in  $u_3$  such that  $x' = (X', u'_1, u'_2, v' := P(1, u'_3))$ . After possibly performing a translation  $Z' := X' - \theta'$  in order to get  $\Delta(h'; u'_1, u'_2, v'; Z')$  minimal, we have an expression  $h' = Z'^p - Z' g'^{p-1} + f'_{Z'}$  with

$$u'_2{}^{-\omega(x)+1} H(x)^{-1} \left( \frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial u_3} \right) \in J(f'_{Z'}, E') + (u'_1 u'_2), \quad (2)$$

since  $\text{ord}_{\eta(x)} g^p > \text{ord}_{\eta(x)} f$ . By (1), we have

$$u'_2{}^{-\omega(x)+1} H(x)^{-1} \left( \frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial u_3} \right) \equiv \gamma' u'_2 v'^{\omega(x)-1} \text{mod } u'_2(u'_1, u'_2), \quad (3)$$

where  $\gamma'$  is a unit, since  $x'$  is very near  $x$ .

If  $\omega(x) = 1$ , (2) and (3) imply that  $(u'_2) \subseteq J(f'_{Z'}, E')$ . On the other hand, we have

$$u'_2{}^{-1} J(f, E, x) \equiv (u'_1) \text{mod } (u'_2).$$

Since  $H(x')^{-1} f'_{Z'} \in (u'_1, u'_2)$ , we get

$$(u'_1, u'_2) = J(f'_{Z'}, E') = J(f'_{Z'}, E', Y')$$

where  $Y' := V(Z', u'_1, u'_2)$ . This property implies in particular that  $Y'$  is permissible of the first kind since  $H(x')^{-1} f'_{Z'} \in (u'_1, u'_2)$  and that no point of the blowing up of  $X'$  along  $Y'$  is very near  $x'$ , so  $x$  is good by **II.5.4.2** (ii) of chapter 1.

If  $x'$  is not rational over  $x$ , then

$$\text{ord}_{\eta'(x')}(u_2'^{-\omega(x)+1}H(x)^{-1}\left(\frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial u_3}\right)) \leq 1 + \frac{\omega(x)}{2},$$

so  $\omega(x') < \omega(x)$  except possibly if  $\omega(x) = 2$  and  $[k(x') : k(x)] = 2$ . In this last case, one contradicts the assumption that  $x'$  is very near  $x$  as in the proof of **III.5** if  $p = 2$ . If  $p \geq 3$ , we have  $\kappa(x') = 2$  if  $\text{cl}_2(H(x')^{-1}f'_{Z'}) \notin k(x')[U'_1, U'_2]$ . Otherwise, we have

$$\text{cl}_2(H(x')^{-1}\frac{\partial f'_{Z'}}{\partial v'}) = \mu U'_2 V' + \nu U_2'^2 + U'_1 \Psi',$$

where  $\mu, \nu \in k(x')$ ,  $\mu \neq 0$ , and  $\Psi' \in k(x')[U'_2, V']_1$ . Since  $p \geq 3$ ,  $k(x')/k(x)$  is separable, so  $\lambda H(x')u_1'^2 \notin (k(x')[[u'_1, u'_2]])^p$  since  $\lambda H(x)u_1^2 \notin (k(x)[[u_1, u_2]])^p$ . Hence  $x'$  satisfies the assumption of lemma **III.1** and the conclusion follows if  $x'$  is not rational over  $x$ .

Assume now that  $x'$  is rational over  $x$ . After performing a linear change of coordinates on  $(u_2, u_3)$ , followed by a translation  $Z := X - \theta$  in order to get  $\Delta(h; u_1, u_2, u_3; Z)$  minimal, it can be assumed that  $x'$  is the origin of the chart, equation (1) remaining valid. Then  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal and we have

$$f' = u_1'^{a(1)} u_2'^{a(1)+\omega(x)-p} (\lambda u_1'^{\omega(x)} + u_2' \psi'). \quad (4)$$

It can be assumed that  $\text{cl}_{\omega(x)}(u_2' \psi') \in k(x')[U'_1, U'_2]$ , since otherwise  $\kappa(x) = 2$ . By (2) and (3) above, we then have

$$H(x')^{-1} \frac{\partial f'}{\partial u_3'} \equiv \gamma' u_2' u_3'^{\omega(x)-1} \pmod{(u_1', u_2'^2)}, \quad (5)$$

so in particular  $\omega(x) \not\equiv 0 \pmod{p}$  and  $x'$  satisfies the assumption of lemma **III.1**, from which the conclusion follows.

**III.6.2** If  $\epsilon(x) = 1 + \omega(x)$ , then, after possibly performing a linear change of coordinates on  $(u_2, u_3)$  and a translation  $Z = X - \theta$  in order to get  $\Delta(h; u_1, u_2, u_3; Z)$  minimal, it can be assumed that  $\langle U_2, U_3 \rangle \supseteq \text{VDir}(x)$ , since  $\bar{\tau}(x) = 3$ . We are done if  $\tau(x) = 3$ , so assume that  $\text{VDir}(x) = \langle U_2, U_3 \rangle$ .

Let  $e : X' \rightarrow X$  be the blowing up along  $x$  and  $x' \in e^{-1}(x)$  be very near  $x$ . Then  $x'$  has coordinates  $(X' = \frac{X}{u_1}, u'_1 = u_1, u'_2 = \frac{u_2}{u_1}, u'_3 = \frac{u_3}{u_1})$  and we have  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1)$ . This is the origin of a chart, so  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal, where  $h' := u_1'^{-p}h = X'^p - X'g'^{p-1} + f'$ . If  $\epsilon(x') = \omega(x')$ , we are done by **III.6.1**. Otherwise, we have  $\epsilon(x') = 1 + \omega(x')$  and  $\bar{\tau}(x') = 3$ , so we may iterate. Let

$$X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow \cdots$$

be the quadratic sequence along  $\mu$ . If  $x$  is not good, we build up a formal curve  $\mathcal{C} := V(\widehat{X}, \widehat{u}'_2, \widehat{u}'_3)$ ,  $\widehat{X} := X - \sum_{n \geq 1} \lambda_n u_1^n$  and  $\widehat{u}'_j := u_j - \sum_{n \geq 1} \mu_{n,j} u_1^n$  for  $j = 2, 3$  such that the center  $x_n$  of  $\mu$  in  $X_n$  lies on the strict transform of  $\mathcal{C}$ . Since  $\Sigma(X) \subseteq \eta^{-1}E$  and  $\mathcal{C} \not\subseteq E$ ,  $x_n$  is a regular point of  $X_n$  for  $n \gg 0$ : a contradiction.

## CHAPTER 4: Resolution when there is tangency

In all this chapter, we assume that  $x \in \Sigma_p$ ,  $\Omega(x) = (\omega(x), 2)$  and that the r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  is such that  $E \subseteq \text{div}(u_1 u_2 u_3)$  and the polyhedron  $\Delta(h; u_1, u_2, u_3; X)$  is minimal.

### I Case $\kappa(x) = 5$

This case is very closed to  $\kappa(x) = 2$ , the invariants are the same, up to a permutation on the indices of  $(u_1, u_2, u_3)$  and the fact that  $u_1$  which, for  $\kappa(x) = 4$ , plays the role of  $u_3$  for  $\kappa(x) = 2$  may divide  $H(x)^{-1}g^p$ . There is a fantastic simplification: *div*( $u_1$ ) has maximal contact for  $\kappa(x) = 5$ . This assertion will be precised in **I.3**.

**I.1 Definition of  $\kappa(x) = 5$ .** We say that  $\kappa(x) = 5$  if  $\Omega(x) = (\omega(x), 2)$  and  $\text{div}(u_1) \subset E$  and there is a derivation  $D \in \mathcal{D}$  with  $H(x)^{-1}Df \equiv u_1^{\omega(x)} \text{mod}(u_2, u_3)$  where, if  $D(\mathfrak{M}) \not\subset \mathfrak{M}$ ,  $\epsilon(x) = 1 + \omega(x)$ .

As usual, assume that  $\kappa(x) = 5$ . We say that  $x$  is good if there exists a sequence of permissible blowing-ups,

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n$$

such that either  $x_n$  is not very near  $x$  or  $(\Omega(x) = \Omega(x_n)$  and  $\kappa(x_n) \leq 4$ ), where  $x_n \in X_n$  is the center of  $\mu$ .

We do not suppose  $\kappa(x_i) \leq 5$  for  $1 \leq i < n$ .

**I.2 Notations** We say that  $(X, u_1, u_2, u_3)$  is prepared if the polyhedra  $\Delta(h; u_1, u_2, u_3; X)$  is minimal and  $u_i \in S$  when  $\text{div}(u_i) \subset E$ ,  $E \subset \text{div}(u_1 u_2 u_3)$ .

There are three different subcases:

case (\*1):  $D(\mathfrak{M}) \subset \mathfrak{M}$ ,  $m(x) = 1$  or  $m(x) = 2$ ,

case (\*2):  $D(\mathfrak{M}) \subset \mathfrak{M}$ ,  $m(x) = 3$ ,

case (\*3):  $D(\mathfrak{M}) \not\subset \mathfrak{M}$ .

When  $m(x) \leq 2$  (cases 1,3), we suppose  $E \subset \text{div}(u_1 u_2)$ . In all cases,  $H(x)^{-1}f = u_1^{\omega(x)}\phi_0 \text{mod}(u_2, u_3)$  with  $\phi_0 \in \widehat{S}$ . In the first and second cases,  $\phi_0$  is invertible. In case 3,  $D(\phi_0)$  invertible, we choose the indices so that  $E \subset \text{div}(u_1 u_2)$ :  $\phi_0 = \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3$ ,  $\gamma_i \in \widehat{S}$ ,  $i = 1, 2, 3$ ,  $\gamma_3$  invertible and  $\text{div}(u_3) \not\subset E$ .

We make the following expansion:

$$H(x)^{-1}f = u_1^{\omega(x)}\phi_0 + \sum_{1 \leq i \leq \omega(x)} u_1^{\omega(x)-i}\phi_i(u_2, u_3),$$

where  $\phi_i(u_2, u_3) \in k(x)[[u_2, u_3]]$ ,  $1 \leq i \leq \omega(x)$ .

We set  $H(x)^{-1}g^p = \gamma u_1^a u_2^b u_3^c$ ,  $\phi_i = u_2^{b(i)} u_3^{c(i)} \psi_i$ ,  $1 \leq i \leq \omega(x)$ ,  $\psi_i(u_2, u_3) \in k(x)[[u_2, u_3]]$ ,  $\psi_i = 0$  or divisible neither by  $u_2$ , nor by  $u_3$ ,  $\gamma = 0$  or  $\gamma$  invertible,  $d(i) = \text{ord}_{(u_2, u_3)}(\psi_i)$ . If  $\psi_i = 0$ , by convention,  $b(i) = c(i) = d(i) = \infty$ .

**I.2.1 Cases (\*1) or (\*2).**

As for  $\kappa(x) = 2$ , we set

If  $a < \omega(x)$ ,

$$A_2 = \inf \left\{ \frac{b}{\omega(x)-a}, \frac{b(i)}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$A_3 = \inf \left\{ \frac{c}{\omega(x)-a}, \frac{c(i)}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$B = \inf \left\{ \frac{\text{ord}_{(u_2, u_3)}(H(x)^{-1}g^p)}{\omega(x)-a}, \frac{\text{ord}_{(u_2, u_3)}(\phi_i)}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$\text{if } A_2 = \frac{b}{\omega(x)-a}, \beta := \inf \left\{ \frac{c}{\omega(x)-a}, \frac{\text{ord}_{u_3}(\phi_i \bmod u_2^{iA_2})}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$\text{if } A_2 \neq \frac{b}{\omega(x)-a}, \beta := \inf \left\{ \frac{\text{ord}_{u_3}(\phi_i \bmod u_2^{iA_2})}{i}, 1 \leq i \leq \omega(x) \right\}.$$

If  $a \geq \omega(x)$  (for example, if  $g = 0$ ),

$$A_2 = \inf \left\{ \frac{b(i)}{i}, 1 \leq i \leq \omega(x) \right\}, A_3 = \inf \left\{ \frac{c(i)}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$B = \inf \left\{ \frac{\text{ord}_{(u_2, u_3)}(\phi_i)}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$\beta := \inf \left\{ \frac{\text{ord}_{u_3}(\phi_i \bmod u_2^{iA_2})}{i}, 1 \leq i \leq \omega(x) \right\}.$$

With the convention  $\text{ord}_{u_3}(\phi_i) \bmod u_2^{iA_2} = +\infty$  if  $\frac{\text{ord}_{u_2}(\phi_i)}{i} > A_2$ .

In every case  $C := B - A_2 - A_3$ .

### I.2.2 Case (\*3).

If  $a < \omega(x)$ ,

$$A_2 = \inf \left\{ \frac{b}{\omega(x)-a}, \frac{b(i)}{i}, 1 \leq i \leq \omega(x) \right\}, A_{33} = \inf \left\{ \frac{-1}{\omega(x)-a}, \frac{c(i)-1}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$B_3 = \inf \left\{ \frac{\text{ord}_{(u_2, u_3)}(H(x)^{-1}g^p)-1}{\omega(x)-a}, \frac{\text{ord}_{(u_2, u_3)}(\phi_i)-1}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$\text{if } A_2 = \frac{b}{\omega(x)-a}, \beta_3 := \inf \left\{ \frac{c-1}{\omega(x)-a}, \frac{\text{ord}_{u_3}(\phi_i \bmod u_2^{iA_2})-1}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$\text{if } A_2 \neq \frac{b}{\omega(x)-a}, \beta_3 := \inf \left\{ \frac{\text{ord}_{u_3}(\phi_i \bmod u_2^{iA_2})-1}{i}, 1 \leq i \leq \omega(x) \right\}.$$

If  $a \geq \omega(x)$  (for example, if  $g = 0$ ),

$$A_2 = \inf \left\{ \frac{b(i)}{i}, 1 \leq i \leq \omega(x) \right\}, A_{33} = \inf \left\{ \frac{c(i)-1}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$B_3 = \inf \left\{ \frac{\text{ord}_{(u_2, u_3)}(\phi_i)-1}{i}, 1 \leq i \leq \omega(x) \right\},$$

$$\beta_3 := \inf \left\{ \frac{\text{ord}_{u_3}(\phi_i \bmod u_2^{iA_2})-1}{i}, 1 \leq i \leq \omega(x) \right\}.$$

In every case, we call  $i_1$  or  $i_1(x)$  the smallest  $i$ ,  $1 \leq i \leq \omega(x)$ , such that the monomial  $\lambda_{\omega(x)-i, iA_1(x), i\beta_3(x)} u_1^{\omega(x)-i} u_2^{iA_1(x)} u_3^{i\beta_3(x)}$  appears in the expansion of  $H(x)^{-1}g^p$  or of  $H(x)^{-1}f$  with  $H(x)\lambda_{\omega(x)-i, iA_1(x), i\beta_3(x)} u_1^{\omega(x)-i} u_2^{iA_1(x)} u_3^{i\beta_3(x)}$  not a  $p$ -power,  $\lambda_{\omega(x)-i, iA_1(x), i\beta_3(x)} \in k(x) - \{0\}$ .

**I.2.3** In cases (\*1)(\*2) (resp. (\*3)), the vertices of the side of points  $x = (x_2, x_3)$  of  $\Delta(H(x)^{-1}(f, g^p); u_2, u_3; u_1)$  (resp.  $\Delta(H(x)^{-1}u_2^{-1}(f, g^p); u_2, u_3; u_1)$ ) of equation  $x_2 + x_3 = B$  (resp.  $x_2 + x_3 = B_3$ ) are denoted  $(\alpha_2, \beta_2)$  and  $(\alpha_3, \beta_3)$ ,  $\alpha_2 \leq \alpha_3$  (resp.  $(\alpha_{32}, \beta_{32})$  and  $(\alpha_{33}, \beta_{33})$ ,  $\alpha_{32} \leq \alpha_{33}$ ).

Obviously,  $C, A_j, i_0, B$  and  $\beta$  depend on the choice of prepared r.s.p.  $(X, u_1, u_2, u_3)$ , if there is no possible confusion, we write  $C$  or  $C(x)$  instead  $C(u_1, u_2, v)$ ,  $A_j$  or  $A_j(x)$  instead of  $A_j(u_1, u_2, v)$ ,  $j = 2, 3$ , etc.

**I.2.4** Let  $\gamma(u_1, u_2, u_3) = \sup([\beta], 1)$  in case (\*1),  $\gamma(u_1, u_2, u_3) = 1 + [C]$  in case (\*2),  $\gamma(u_1, u_2, u_3) = 1 + [\beta_3]$  in case (\*3). We note  $\gamma(x) = \gamma(u_1, u_2, u_3)$  for short.

**I.3 Theorem** Assume  $\kappa(x) = 5$  and  $\kappa(x) > 4$ . We blow up the origin, then all  $x'$  above  $x$  with  $\kappa(x') \geq 5$  are on the strict transform of  $\text{div}(u_1)$ .

Furthermore,  $x$  is good or,

for all these  $x'$ ,  $\kappa(x') = 5$ , and we have **I.2** for some r.s.p. at  $x'$  of the form

$$(X' = (X/u_i) + u_i\theta, u_1/u_i, u_i, P(1, u_j/u_i)), \theta \in \widehat{S}', \{i, j\} = \{2, 3\}, P \in k(x)[U_2, U_3]$$

$P$  homogeneous and irreducible.

Moreover, if  $x$  is in case (\*1) or (\*2) and  $(a(1) + \omega(x) \neq 0 \bmod p)$  or  $(a(2) + a(3) \neq 0 \bmod p)$  or  $(x$  in case (\*1) and  $x'$  separable over  $x)$ , then  $x'$  is in case (\*1) or (\*2).

Proof. Assume  $x'$  is very near  $x$  is not on the strict transform of  $\text{div}(u_1)$ , then  $u_j^{-\omega(x)}H(x)^{-1}Df \in J(f', E')\text{mod}(u_j)$ ,  $j = 1, 2, 3$ , as  $\text{in}_x(H(x)^{-1}Df) = U_1^{\omega(x)} + \sum_{1 \leq i \leq \omega(x)} U_1^{\omega(x)-i} F_i(U_2, U_3)$ ,

$u_j^{-\omega(x)}H(x)^{-1}Df = v^{\omega(x)} + \sum_{1 \leq i \leq \omega(x)} \lambda_i v^{\omega(x)-i} w^i$  where  $X/u_i, v, w, u_i$  is a r.s.p. at  $x'$  and  $v$  transverse to  $E'$ : then  $\kappa(x') \leq 4$ . From now on,  $x'$  is on the strict transform of  $\text{div}(u_1)$ .

**I.3.1** *Case (\*1) or (\*2).*

There is no problem at the origin of each chart. So we look at a point  $x'$  in the chart of origin  $(X' = X/u_2, u'_1 = u_1/u_2, u'_2 = u_2, u'_3 = u_3/u_2), u'_3(x') \neq 0$ .

After a possible translation on  $X/u_2$ , we get, with usual notations:  $H(x')^{-1}f' = u_2\phi + F(u'_1, u'_3)$  and  $F(u'_1, u'_3) = \sum_{0 \leq i \leq \omega(x)} \lambda_i u_1^{\omega(x)-i} v^{c(i)}$ , where  $v = P(1, u'_3) \in k(x)[u'_3]$ ,  $\lambda_i \in S'$ ,  $\lambda_i$  invertible or 0,  $\lambda_0$  invertible,  $c(0) = 0$  or 1 and  $c(i) \geq i$ .

When

$$\text{ord}_{x'}(u_2\phi) \geq \text{ord}_{x'}(F(u'_1, u'_3)), \quad (1)$$

we get the result. Else, we have

$$a(1) + \omega(x) = 0 \pmod{p}, \quad a(2) + a(3) = 0 \pmod{p}, \quad (2)$$

$$\text{ord}_{x'}(u_2\phi) = \omega(x), \quad \text{ord}_{x'}(F(u'_1, u'_3)) = 1 + \omega(x). \quad (3)$$

When  $\lambda_i = 0$ , for  $i > 0$ , we have  $\kappa(x') \leq 1$  by chapter 2 **II.1**. From now on, we suppose that one  $\lambda_i \neq 0$ , for  $i > 0$ . Then, for some  $i$ ,  $c(i) = i + 1$ ,  $a(1) + \omega(x) = 0 \pmod{p}$ , so  $i = 0 \pmod{p}$  and  $\text{ord}_{x'}(\frac{\partial F(u'_1, u'_3)}{\partial v}) = \omega(x)$ , so  $\kappa(x') \leq 4$ .

**I.3.2** *Case (\*3).* After an eventual translation on  $X/u_i$  which becomes  $X' = X/u_i + \theta$ , we get, with usual notations:  $H(x')^{-1}f' = u_i\phi + F(u'_1, u'_j)$  and in the expansion of  $F(u'_1, u'_j)$  there is the monomial  $u_1^{\omega(x)}$  or  $u_1^{\omega(x)}v$  where  $v = P(1, u'_j)$ . We look only at the case  $i = 2, j = 3$ , the origin of the second chart is left to the reader. As above, we reach the case

$$H(x')^{-1}f' = u_2\phi + F(u'_1, u'_3), \quad \text{ord}_{x'}(u_2\phi) = \omega(x), \quad \text{ord}_{x'}(F) = 1 + \omega(x),$$

$x'$  rational over  $x$ . As  $E \subset \text{div}(u_1 u_2)$ ,

$$F(u'_1, u'_3) = \sum_{0 \leq i \leq \omega(x)} \lambda_i u_1^{\omega(x)-i} v^{i+1} + \lambda' u_1^{\omega(x)+1},$$

$\lambda_0$  invertible,  $\lambda_i \in S'$ ,  $\lambda', \lambda_i$  invertible or  $= 0$ , if  $\lambda_i = 0$  for  $i > 0$ , we get  $\kappa(x') \leq 1$  by chapter 2 **II.1**. Else, if some  $\lambda_i \neq 0$  for  $i > 0$ , we want to prove that we have  $\kappa(x') \leq 4$ . We suppose  $\kappa(x') > 4$ , then  $\text{in}_{x'}(u_2\phi) \in k(x')[U'_1, U'_2]$ .

We look first at the case where  $x''$  is in the chart of origin  $(X'' = X'/u'_1, u''_1 = u'_1, u''_2 = u'_2/u'_1, v' = v/u'_1)$ .

As  $\kappa(x') > 2$ , as  $x''$  is very near  $x$ ,  $\text{in}_{x''}(u_2\phi) = \gamma U_2^\omega(x)$ ,  $\gamma$  invertible. We get, after an possible translation on  $X''$  to minimalize the characteristic polyhedra of  $u_1^{-p}h'$ :

$$f'' = u_1^{a'} u_2^{b'} (u_2 \phi'' + \gamma' u_1^e w^e), \quad \gamma' \text{ invertible},$$

$\text{ord}_{x''}(u_2\phi'') = \omega(x)$  when  $x''$  is separable over  $x'$  or  $a \neq 0 \pmod{p}$  or  $b \neq 0 \pmod{p}$ .

As  $x''$  is very near to  $x$ ,  $\text{ord}_{x''}(H(x'')^{-1}g'') \geq 1 + \omega(x)$ . When  $\text{ord}_{x''}(H(x'')^{-1}g'') = 1 + \omega(x)$ , the reader sees that  $\kappa(x'') \leq 1$ . From now on,  $\text{ord}_{x''}(H(x'')^{-1}g'') > 1 + \omega(x)$ , so the possible translation on  $X''$  to minimalize the characteristic polyhedra of  $u_1^{-p}h'$  just adds  $p$ -powers modulo  $\mathfrak{M}''^{2+\omega(x)}$  to  $f' := u_1^{-p}f$ . As  $\frac{\partial F(u'_1, u'_3)}{\partial v} \neq 0$ , its order is  $\omega(x)$ , the extension  $D$  of  $\frac{\partial}{\partial v}$  is in

$\mathcal{D}(E'')$ , as  $u_1''^{-\omega(x)+1} \frac{\partial F(u_1', u_3')}{\partial v} = u_1'' w^c \times \text{invertible}$  with  $c \leq \frac{\omega(x)}{d}$  where  $d$  is the degree of the residual extension, we get  $e \leq 1 + c \leq 1 + \frac{\omega(x)}{d}$ . If all are equal, then  $\text{in}_{x'}(\frac{\partial F(u_1', u_3')}{\partial v})$  is equal, up to a multiplication by an invertible, to a power of an irreducible homogeneous polynomial of  $k(x')[U_1', V]$ , as  $\text{ord}_v(\text{in}_{x'} F(u_1', u_3')) < \omega(x)$ , this implies  $\text{deg}_v(\text{in}_{x'} F(u_1', u_3')) = 1 + \omega(x)$ :  $\kappa(x') \leq 3$ . A contradiction. So  $e < 1 + \frac{\omega(x)}{d}$ ,  $e \leq \omega(x)$ . So when  $\text{ord}_{x''}(u_2 \phi'') = \omega(x)$ , we get  $\kappa(x'') \leq 2$  by chapter 2, **II.1**.

Let us look at the case where  $x''$  is not rational over  $x$ ,  $\text{ord}_{x''}(u_2 \phi'') = \omega(x) + 1$ . Then  $e < 1 + \frac{\omega(x)}{d}$  and  $x''$  very near to  $x$  lead to  $e \neq 0 \pmod{p}$ ,  $\omega(x'') = 1$ ,  $e = 0$  or  $1$ . In both cases  $F(u_1', u_3') = \lambda u_1' v + \lambda' u_1'^{\omega(x)+1} \pmod{(u_2')}$ ,  $\lambda$  invertible, by chapter 2, **II.1**,  $\kappa(x') \leq 1$ , a contradiction.

At the origin of the other chart, we denote  $X'' = X'/v$ ,  $v_1 = u_1'/v$ ,  $v_2 = u_2'/v$ ,  $v_3 = v$ , then when  $x''$  is very near to  $x$ ,

$$v^{-\omega(x)+1} H(x)^{-1} \frac{\partial f'}{\partial v} = v_3 \psi \in J(f'', E') = J(f'', E'', x'')$$

with  $\psi \in S''$ ,  $\text{ord}_{x''}(\psi) = \omega(x) - 1$ , so  $V_3 \in \text{VDir}(x'')$ , the reader will see that  $\tau(x'') = 3$  and will end the proof.

Proof of the last assertion. Assume  $x$  is in *Case (\*1) or (\*2) and  $(a(1) + \omega(x) \neq 0 \pmod{p})$  or  $a(2) + a(3) \neq 0 \pmod{p}$  or  $(x$  in case (\*1) and  $x'$  separable over  $x)$* .

The last assertion is clear when  $x'$  is the origin of a chart. Else,  $x'$  is in the chart of origin ( $X' = X/u_2$ ,  $u_1' = u_1/u_2$ ,  $u_2' = u_2$ ,  $u_3' = u_3/u_2$ ) and  $X'(x') = u_1'(x') = 0$ ,  $u_3'(x') \neq 0$ . Then, in  $f'$  appears the monomial

$$u_1'^{a(1)} u_2'^{a(1)+a(2)+a(3)+\omega(x)-p} u_3'^{a(3)} \times \phi_0 u_1'^{\omega(x)} = u_2^{-p} \times u_1^{a(1)} u_2^{a(2)} u_3^{a(3)} \times \phi_0 u_1^{\omega(x)},$$

$\phi_0$  invertible. If  $a(1) + \omega(x) \neq 0 \pmod{p}$  or  $a(2) + a(3) \neq 0 \pmod{p}$ , this monomial will not be spoilt by any translation on  $X'$ , else,  $\phi_0$  is not a  $p^{\text{th}}$ -power in  $k(x)$ , if  $x$  is in case (\*1) and  $x'$  is separable over  $x$ ,  $a(3) = 0$ , again, no translation will touch this monomial.

**I.3.3 Theorem** *Let us suppose  $\kappa(x) = 5$ ,  $\text{div}(u_1 u_2) \subset E$ ,  $x$  not good and  $A_2(x) \geq 1$ .*

*In addition, we suppose  $A_2(x) > 1$  or  $\beta(x) \leq 1$  if  $x$  is in case (\*3).*

*Then  $V(X, u_1, u_2)$  is permissible. We blow it up.*

*Then the point  $x' = (\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, u_3) = (X', u_1', u_2', u_3')$  is the only point above  $x$  which may be very near to  $x$  with  $\kappa(x') \geq 5$ , if it is, then  $\kappa(x') \leq 5$ , if  $\kappa(x') = 5$ , then if  $x$  is in case (\*i),  $i = 1, 2, 3$ ,  $x'$  is in case (\*i),  $(A_2(x'), \beta(x')) = (A_2(x) - 1, \beta(x))$  and, in case (\*3),  $(A_2(x'), \beta_3(x')) = (A_2(x) - 1, \beta_3(x))$ .*

*Furthermore,  $\text{IDir}(x) = (U_1)$ .*

Proof.

It is clear that  $V(X, u_1, u_2)$  is  $\text{div}(u_1) \cap \text{div}(u_2) \cap \{h = 0\}$ : it is not formal. So  $V(X, u_1, u_2)$  is permissible.

We have

$$H(x)^{-1} f \in (u_1, u_2).$$

So  $U_1 \in \text{IDir}(x) \pmod{(U_2)}$ .

If  $U_1 \notin \text{IDir}(x)$ , then  $\text{IDir}(H(x)^{-1} Df) = (U_1 + \lambda U_2)$ ,  $\lambda \in k(x) - \{0\}$ , for some  $D \in \mathcal{D}$ ,  $D(\mathfrak{M}) \subset \mathfrak{M}$  in cases (\*1)(\*2). We have just to look at the chart where  $u_1$  generates the exceptional divisor:  $u_1^{-\omega(x)} H(x)^{-1} Df = (1 + \lambda u_2/u_1)^{\omega(x)} \pmod{(u_1)}$ ,  $(1 + \lambda u_2/u_1)(x') = 0$ ,  $\kappa(x') \leq 4$ .

The remaining case is when  $U_1 \in \text{IDir}(x)$ . The only point  $x'$  we have to look at has

$$\left(\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, u_3\right) = (X', u'_1, u'_2, u'_3)$$

for parameters. We are at the origin of a chart, etc. The only difficult case is when  $x$  is in case (\*3) and  $\epsilon(x') = \omega(x)$  and  $x'$  very near to  $x$ .

Then

$$\operatorname{in}_x(H(x)^{-1}f) = U_1^{\omega(x)}(aU_1 + bU_2 + cU_3) + \sum_{1 \leq i \leq \omega(x)} \lambda_i U_1^{\omega(x)-i} U_2^{i+1},$$

$a, b, c, \lambda_i \in k(x)$ ,  $c \neq 0$ ,

$$H(x')^{-1}f' = u'_1{}^{\omega(x)}(cu'_3 + u'_2(au'_1 + b)) + u'_2 \sum_{1 \leq i \leq \omega(x)} \lambda_i u'_1{}^{\omega(x)-i} + \phi,$$

$\phi \in (u'_2, u'_3)$ ,  $\operatorname{ord}_{x'}(\phi) = \omega(x)$ . If  $\phi \in (u'_2)$ , by chapter 2 **II.1**, we are done: this ends the case  $A_2 > 1$ . If  $A_2 = 1$  and  $\beta(x) = 1$ , then in the expansion of  $\phi$ , there is the monomial  $u'_1{}^{\omega(x)-i} u'_3{}^i$ ,  $1 \leq i \leq \omega(x)$ : so  $\kappa(x') \leq 2$ . If  $A_2 = 1$  and  $\beta(x) < 1$ , then  $\epsilon(x') < \omega(x)$ :  $\kappa(x) = 1$ , a contradiction.

The end of the proof is clear.

**I.4 Theorem** *With hypotheses and notations of I.2, assume  $x$  is in case (\*1) or (\*2). We blow-up  $x$  and  $x'$  is a closed point of the first chart.*

*If  $u'_3(x') = 0$ , then  $x'$  is in case (\*1) or (\*2) and  $C(x') \leq C(x)$ ,  $\beta(x') \leq \beta(x)$ ,  $A_2(x') = B(x) - 1$ .*

*From now on,  $u'_3(x') \neq 0$ , we have*

$$\gamma(x') \leq \gamma(x), \quad \beta(x') < \lfloor \frac{C(x)}{d} \rfloor + 1, \quad (1)$$

and, if  $x$  is in case (\*1),

$$\beta(x') \leq \beta(x) \text{ or } C(x) = 0. \quad (2)$$

*If  $x'$  is not rational over  $x$  and  $\gamma(x) \geq 2$ , then  $\gamma(x') < \gamma(x)$ , except in the following case:*

*$m(x) \leq 2$ ,  $\beta(x) = 2$  where we get  $\beta(x') < 2$  and, if  $x'$  is in case (\*3),  $\beta_3(x') = 1$ ,  $p = 2$ ,  $a(1) + \omega(x) = 0 \pmod{p}$  and  $i_1(x') = 0 \pmod{p}$  (notations of **I.2.3**).*

Proof.

If  $u'_3(x') = 0$ , we are at the origin of the chart, there is no translation to do all the assertions are easy consequences of the transformation laws on the polyhedra. From now on:

$$u'_3(x') \neq 0.$$

Let us prove (1) and (2). Let  $\mu_0$  the monomial valuation given by  $\mu_0(u_1^a u_2^b u_3^c) = a + \frac{b+c}{B(x)}$ .

$$\operatorname{in}_{\mu_0}(H(x)^{-1}f) = U_1^{\omega(x)} \bar{\phi}_0 + \sum_{1 \leq i \leq \omega(x)} U_1^{\omega(x)-i} U_2^{b_i} U_3^{c_i} F_i(U_2, U_3), F_i \in k(x)[U_2, U_3], \quad (3)$$

$F_i = 0$  or  $F_i$  homogeneous of degree  $iB(x) - b_i - c_i$ .

Then, by **I.3**,  $(X', u'_1, u'_2, v) = (\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, P(\frac{u_3}{u_2}))$  is a r.s.p. of  $x'$ , with  $P \in k(x)[\frac{u_3}{u_2}]$ .

$$\operatorname{in}_{\mu_1}(H(x')^{-1}f') = U_1^{\omega(x)} \bar{\phi}'_0 + \sum_{1 \leq i \leq \omega(x)} U_1^{\omega(x)-i} U_2^{i(B(x)-1)} U_3^{c_i} F_i(1, U'_3), F_i \in k(x)[U_2, U_3],$$

is the initial form of  $H(x')^{-1}f'$  with respect to monomial valuation given by  $\mu_1(u_1^a u_2^b v^c) = a + \frac{b}{B(x)-1}$ .

If  $\mu_0(H(x)^{-1}g^p) = \omega(x)$  and  $\text{ord}_{u_1}(H(x)^{-1}g^p) < \omega(x)$ , let  $H(x)^{-1}g^p = \gamma u_1^{\omega(x)-i_0} u_2^{b i_0} u_3^{c i_0}$ ,

$$H(x')^{-1}g'^p = \gamma' u_1'^{\omega(x)-i_0} u_2'^{i_0(B(x)-1)},$$

$\gamma'$  invertible. Then

$$(A_2(x'), \beta(x')) = (B(x) - 1, 0).$$

If  $\mu_0(H(x)^{-1}g^p) > \omega(x)$  or  $\text{ord}_{u_1}(H(x)^{-1}g^p) \leq \omega(x)$ , then we call

$$i_0 = \sup\{i | u_1^{a(1)} u_2^{a(2)} u_3^{a(3)} u_1^{\omega(x)-i} u_2^{b_i} u_3^{c_i} F_i \notin k(x)[u_1, u_2, u_3]^p\}. \quad (4)$$

When we may start the minimization of  $\Delta(h'; u_1', u_2', v; X')$  with the cleaning of the vertex with  $\mu_1$  minimal and first coordinate minimal, we add a  $p^{\text{th}}$ -power to

$$u_1'^{a(1)} u_2'^{a(1)+a(2)+a(3)+\omega(x)-p} u_3'^{a(3)} u_1'^{\omega(x)-i_0} u_2'^{i_0(B(x)-1)} F_{i_0}(1, u_3')$$

which becomes

$$u_1'^{a(1)} u_2'^{a(1)+a(2)+a(3)+\omega(x)-p} u_1'^{\omega(x)-i_0} u_2'^{i_0(B(x)-1)} \gamma_{i_0} v^{e(i_0)}, \quad \gamma_{i_0} \text{ invertible,}$$

with  $e(i_0) \leq 1 + \frac{\deg(F_{i_0})}{d}$ ,  $d$  is the degree of the residual extension,  $e(i_0) \leq \deg(F_{i_0})$  if  $c_{i_0} = 0$ ,

$$e(i_0) \leq \frac{\deg(F_{i_0}) + c(i_0)}{d} + 1 \leq \frac{i_0 \beta(x)}{d} + 1 \quad (5)$$

in general. Either  $\beta(x') < \frac{e(i_0)}{i_0}$  and we get all our assertions or  $\beta(x') = \frac{e(i_0)}{i_0}$  and, if  $e(i_0) = \frac{\deg(F_{i_0}) + c(i_0)}{d} + 1$ , by the following remark,  $a(1) + \omega(x) - i_0(x') = 0 \pmod{p}$ .

Let us remark that, if there exists  $i$  such that  $F_i \neq 0$  and  $a(1) + \omega(x) - i \neq 0 \pmod{p}$ , then

$$u_1'^{a(1)} u_2'^{a(1)+a(2)+a(3)+\omega(x)-p} u_3'^{a(3)} u_1'^{\omega(x)-i} u_2'^{i(B(x)-1)} F_i(1, u_3')$$

will not be spoilt by a translation on  $X'$  and we get  $\beta(x') \leq \frac{\deg(F_i)}{id} \leq \frac{C(x)}{d}$ .

This gives  $A_2(x') = B(x) - 1$  and all the assertions in the case where  $(a(1) + \omega(x) - i_0 \neq 0 \pmod{p})$  or  $a(2) + a(3) + i_0 B(x) \neq 0 \pmod{p}$ . The other assertions are clear except may be the case  $\gamma(x) = 2 = \beta(x)$ ,  $x$  in case (\*1). By (5), we get  $\gamma(x') = 1$  when  $d \geq 3$ .

From now on  $d = 2 = \beta(x)$ ,  $a(1) + \omega(x) - i_0 = 0 \pmod{p}$ ,  $a(2) + i_0 B(x) = 0 \pmod{p}$ .

Then either  $\frac{\partial u_3^{c(i_0)} F_{i_0}}{\partial u_3} \neq 0$  it has degree  $\leq 2i_0 - 1$ , so  $e(i_0) \leq \frac{2i_0-1}{2} + 1 = i_0 + \frac{1}{2}$ , as  $e(i_0) \in \mathbb{N}$ , we get  $e(i_0) \leq i_0$ , we are done. Or  $\frac{\partial u_3^{c(i_0)} F_{i_0}}{\partial u_3} = 0$ , there exists  $D \in \mathcal{D}(E, x)$  with  $D(u_3^{c(i_0)} F_{i_0})$  of degree  $\leq 2i_0$ , if  $D(\mathfrak{M}') \subset \mathfrak{M}'$ , we get  $e(i_0) \leq i_0$ , else  $x'$  is inseparable over  $x$ ,  $p = d = 2$ ,  $e(i_0) \leq i_0 + 1$  and, in case equality,  $i_0 + 1 \neq 0 \pmod{2}$ , etc. The reader ends the proof.

Let us remark that, if we blow up  $x$  and that  $C(x) = 0$ ,  $A_2(x) < 1$  and  $A_3(x) < 1$ , then  $\omega(x') < \omega(x)$ . Indeed, in that case,  $B(x) = A_2(x) + A_3(x) < 2$ , either  $H(x')^{-1}g'^p = \gamma' u_1'^{\omega(x)-i_0} u_2'^{i_0(B(x)-1)}$  which has order  $< \omega(x)$  or  $u_1'^{\omega(x)-i_0} u_2'^{i_0(B(x)-1)} \gamma_{i_0} v^{e(i_0)}$  has order  $< \omega(x)$ .

The next corollary is already proven.

**I.4.1 Corollary** *With hypotheses and notations of I.2, we blow up  $x$ . If  $x$  is in case (\*1) or (\*2) and if  $x'$  is a point in the first chart very near to  $x$  with  $u'_3(x) \neq 0$ , either  $\mu_0(H(x)^{-1}g^p) = \omega(x)$  and  $\text{ord}_{u_1}(H(x)^{-1}g^p) < \omega(x)$ , then  $\beta(x') = 0$ , or  $\mu_0(H(x)^{-1}g^p) > \omega(x)$  or  $\text{ord}_{u_1}(H(x)^{-1}g^p) \geq \omega(x)$ , then  $\beta(x') \leq (1 + \frac{\deg(F_{i_0})}{d})$ , where  $i_0, F_i$  are defined just above in (3)(4).*

**I.4.2 Corollary** *With hypotheses and notations of I.2, if  $x$  is in case (\*1) or (\*2), if  $C(x) = 0$ ,  $A_2(x) < 1$  and  $A_3(x) < 1$ , then  $x$  is good.*

*Proof.* We remark that  $B(x) = C(x) + A_2(x) + A_3(x) \geq 1$ , so  $C(x) = 0$ ,  $A_2(x) < 1$  and  $A_3(x) < 1$  imply  $0 < A_2(x)$  and  $0 < A_3(x)$ , so we blow up  $x$ , we note that either  $\mu_0(H(x)^{-1}g^p) = \omega(x)$  and  $\text{ord}_{u_1}(H(x)^{-1}g^p) < \omega(x)$  or  $\deg(F_{i_0}) = 0$ , we apply the last lines of the proof of I.4: if  $x'$  is not at the origin of a chart,  $\omega(x') < \omega(x)$ , if  $x'$  is at the origin of a chart,  $A_2(x') + A_3(x') < A_2(x) + A_3(x)$ , an induction on  $A_2(x) + A_3(x)$  gives the result.

**I.5 Theorem** *With hypotheses and notations of I.2, we suppose  $x$  is in case (\*3) and  $x$  is not good. We blow-up  $x$ .*

*Let  $x'$  be a closed point very near to  $x$  in the chart of origin  $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, u'_3 = \frac{u_3}{u_2})$  (first chart).*

*(i) When  $u'_3(x') \neq 0$  and  $(a_1 + \omega(x) \neq 0 \pmod{p})$  or  $a(2) + 1 \neq 0 \pmod{p}$ ,  $x'$  is in case (\*1).*

*(ii) If  $x'$  is in case (\*3), then  $\beta_3(x') \leq \beta_3(x)$ , the inequality is strict if  $1 \leq \beta_3(x)$  and  $x'$  is not rational over  $x$ .*

*(iii) When  $\beta_3(x) = 1$  and  $i_1(x) = 0 \pmod{p}$ , if  $x'$  is not rational over  $x$ , then  $\gamma(x') = 1$ , if  $x'$  is rational over  $x$ , then  $\beta(x') < 2$  and, if  $x'$  is in case (\*3) and  $\gamma(x') = 2$ , then  $\beta_3(x') = 1$  and  $i_1(x') = 0 \pmod{p}$ . (For the definition of  $i_1(x)$ , see the end of I.2.2.)*

*(iv) In every case we have*

$$\gamma(x') \leq \gamma(x), \quad A_2(x') = B_3(x) - 1.$$

*Proof.*

We make the blowing up. By I.3,  $(X', u'_1, u'_2, v) = (\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, P(\frac{u_3}{u_2}))$  is a r.s.p. of  $x'$  on the strict transform of  $\text{div}(u_1)$ , with  $P \in k(x)[\frac{u_3}{u_2}]$ .

The term

$$u_1^{a(1)} u_2^{a(2)} \times u_1^{\omega(x)} \phi_0 = u_1^{a(1)} u_2^{a(2)} \times u_1^{\omega(x)} (\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3)$$

in the expansion of  $f$  gives in  $f'$ :

$$u_1'^{a(1)} u_2'^{a(1)+a(2)+\omega(x)+1-p} \times u_1'^{\omega(x)} (\gamma_1 u'_1 + \gamma_2 + \gamma_3 u'_3)$$

If  $u'_3(x') \neq 0$ , as  $\gamma_3$  is invertible, the monomial

$$u_1'^{a(1)} u_2'^{a(1)+a(2)+\omega(x)+1-p} u_1'^{\omega(x)} \gamma_3 u'_3$$

defines the vertex of  $\Delta(h'; u'_1, u'_2, v; X')$  with minimal second and third coordinates, if  $a(1) + \omega(x) \neq 0 \pmod{p}$  or  $a(2) + 1 \neq 0 \pmod{p}$ , this vertex is not solvable,  $x'$  is in case (\*1).

Let  $\mu_0$  the monomial valuation given by  $\mu_0(u_1^a u_2^b u_3^c) = a + \frac{b+c}{B_3(x)}$ .

$$\text{in}_{\mu_0}(H(x)^{-1}u_3^{-1}f) = U_1^{\omega(x)} U_3^{-1} \bar{\phi}_0 + \sum_{1 \leq i \leq \omega(x)} U_1^{\omega(x)-i} U_2^{b_i} U_3^{-1} F_i(U_2, U_3), \quad F_i \in k(x)[U_2, U_3], \quad (1)$$

$F_i = 0$  or  $F_i$  homogeneous of degree  $iB_3(x) - b_i + 1$ .

$$\text{in}_{\mu_1}(H(x')^{-1}f') = U_1^{\omega(x)}\bar{\phi}_0 + \sum_{1 \leq i \leq \omega(x)} U_1^{\omega(x)-i} U_2^{i(B_3(x)-1)} F_i(1, u'_3), F_i \in k(x)[U_2, U_3],$$

is the initial form of  $H(x')^{-1}f'$  with respect to monomial valuation given by  $\mu_1(u_1^a u_2^b v^c) = a + \frac{b}{B_3(x)-1}$ . If  $\mu_0(H(x)^{-1}u_3^{-1}g^p) = \omega(x)$  and  $\text{ord}_{u_1}(H(x)^{-1}g^p) < \omega(x)$ , then

$$H(x')^{-1}u_3^{-1}g'^p = \gamma' u_1^{\omega(x)-i_0} u_3^{-1} u_2^{i_0(B_3(x)-1)}$$

$$(A_2(x'), \beta(x')) = (B_3(x) - 1, 0). \quad (2)$$

Furthermore, in the case of the blowing up of  $x$ , if  $B_3(x) = 1$ , we get  $\epsilon(x') \leq \text{ord}_{x'}(H(x')^{-1}g'^p) \leq 0$ :  $x'$  is quasi ordinary.

If  $\mu_0(H(x)^{-1}g^p) > \omega(x)$  or  $\text{ord}_{u_1}(H(x)^{-1}g^p) \geq \omega(x)$ , we call  $i_0 = \sup\{i | H(x)u_1^{\omega(x)-i} u_2^{b_i} F_i \neq p^{\text{th}} - \text{power}\}$ . By computations as above in the proof of **I.4**, we get  $A_2(x') = B_3(x) - 1$  and

$$\beta(x') \leq \frac{\deg(F_{i_0})}{i_0}. \quad (3)$$

This gives (iv).

If  $x'$  is not rational over  $x$ , then  $\beta(x') \leq \frac{\deg(F_{i_0})}{i_0} + \frac{1}{i_0}$ , the inequality is strict when  $a(1) + \omega(x) - i_0 \not\equiv 0 \pmod{p}$  or when  $\frac{\deg(F_{i_0})}{d} + 1 \in i_0\mathbb{N}$ . Either  $x'$  is in case (\*2), (2)(3) give the result, or  $x'$  is in case (\*3), by (3),  $\beta_3(x') \leq \frac{\deg(F_{i_0})-1}{i_0}$ . This proves (ii).

To end the proof of (iii), we have to look at the case  $\beta_3 = 1$ ,  $i_1(x) = 0 \pmod{p}$ . If  $i_0 \neq i_1(x)$ , then  $\deg(F_{i_0}) + b_{i_0} - i_0 A_1 < i_0 \beta_3 = i_0$ , so  $\deg(F_{i_0}) < i_0$ , we get the result. If  $i_0 = i_1(x)$ , then,  $\beta(x') \leq 1 + \frac{1}{i_0} < 2$  and, if  $x'$  is in case (\*3),  $i_0 \beta_3(x') \leq \deg(F_{i_0}) - 1 \leq i_0$ , the inequality is strict if  $x'$  is not rational over  $x$ . Furthermore, the index  $i_0$  is the smallest  $i$  such that the factor of  $\lambda u_1^{\omega(x)-i} u_2^{iA_1(x')} v^a$ ,  $a \in \mathbb{N}$ ,  $\lambda \in k(x')$ , appears in the expansion  $H(x')^{-1}f'$  and  $H(x')\lambda u_1^{\omega(x)-i} u_2^{iA_1(x')} v^a$  is not a  $p$ -power, so if  $i_0 \beta_3(x') = \text{ord}(F_{i_0}(1, u'_3)) - 1$ ,  $i_0 = i_1(x')$ .

**I.5.1 Proposition** *With hypotheses and notations of I.2, if  $x$  is in case (\*3) with  $\beta_3(x) \leq 0$ , then,  $x$  is good.*

Proof. We make an induction on  $A_2(x)$ . If  $A_2(x) < 1$ , then the monomial  $u_1^{\omega(x)-i} u_2^{iA_2(x)} u_3^{i\beta_3(x)+1}$  occurs in the expansion of  $H(x)^{-1}f$ , its order is  $\leq \omega(x) - 1$ : a contradiction.

If  $A_2(x) \geq 1$ , then  $V(X, u_1, u_2)$  is permissible of second kind, we apply **I.3.3**: either the point  $x'$  is not very near to  $x$ , either  $\kappa(x') \leq 4$  or it is very near to  $x$  with case (\*3) and  $(A_2(x'), \beta_3(x')) = (A_2(x) - 1, \beta_3(x))$ .

**I.5.2 Lemma** *With hypotheses and notations of I.2, assume  $x$  is in case (\*1) or (\*2), we blow up  $x$ .*

If  $x' = (\frac{X}{u_3}, \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3)$  is very near  $x$ , then  $\kappa(x') = 5$ ,  $x'$  is in case (\*2),  $\Delta(h'; \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3; \frac{X}{u_3})$  is minimal. We have  $(A_2(x'), \beta(x')) = (A_2(x), A_2(x) + \beta(x) - 1)$ ,  $(\alpha_2(x), B(x) - 1)$ , is the vertex of smallest ordinate of  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2; u'_3)$ . Furthermore

(i) if  $x$  is in case (\*1),  $C(x') \leq \frac{\beta(x)}{2}$ . If there is equality, then  $\Delta(H(x')^{-1}f' + H(x')^{-1}g'^p; u'_1, u'_2; u'_3)$  has only two vertices  $(A_1(x), \beta(x) + A_1(x) - 1)$ ,  $(\alpha_2(x), B(x) - 1)$ ,  $(\alpha_2(x), B(x))$  which are the ends of its initial side,

(ii) in every case,  $C(x') \leq C(x)$ ,  $\gamma(x') \leq \gamma(x)$ .

Proof. Recopy **I.8.8** in chapter 3.

**I.5.3 Lemma** *With hypotheses and notations of **I.2**, if  $x$  is in case (\*3) and  $\kappa(x) > 4$ ,  $\text{div}(u_2) \subset E$  and we blow up  $x$ . Assume  $x' = (X', u'_1, u'_2, u'_3) = (\frac{X}{u_3}, \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3)$  if  $x'$  is very near  $x$ , then  $\kappa(x') = 5$   $x'$  is in case (\*2),  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal. We have  $(A_2(x'), \beta(x')) = (A_2(x), A_2(x) + \beta_3(x) - 1)$ ,  $(\alpha_3(x'), B_3(x') - 1)$  is the vertex of smallest ordinate of  $\Delta(H(x')^{-1}(f', g'^p); u'_1, u'_2, u'_3)$ ,*

$$A_3(x') = B_3(x) - 1, C(x') \leq 1 + \beta_3(x'), \gamma(x') \leq \gamma(x),$$

with strict inequality when  $3 \leq \gamma(x)$ . Furthermore

(i) if  $B_3(x) - A_2(x) \geq 1$ , then  $\gamma(x') < \gamma(x)$ ,

(ii) if  $\beta_3(x) = 1$ ,  $a(1) + \omega(x) = 0 \pmod{p}$ , then  $x'$  is good or  $\gamma(x') < \gamma(x) = 2$  or  $\gamma(x'') = 1$ , where  $x''$  is the center of the valuation  $\mu$  in the blowing up of  $x'$ ,

(iii) if  $B_3(x) - A_2(x) < 1$  and  $\beta_3(x) \neq 1$  and  $\gamma(x) = 2$ , then the following holds: either  $(a(1) + \omega(x) \neq 0 \pmod{p})$ , either  $(x''$  is in case (\*1) and  $\beta(x'') < 2$ ) or  $(x''$  is in case (\*3) and  $\beta_3(x'') \leq 1)$ , where  $x''$  is the center of  $\mu$  in the blowing up  $X''$  of  $X'$  along  $x'$ .

Proof. We are at the origin of the second chart, there is no translation to do on  $X'$ , etc. For (i)(ii), we cannot recopy directly chapter 3 **I.8.9.1** which uses chapter 3 **I.8.5**: we have no corresponding proposition. What remains valid (mutatis mutandis) is that the vertex of smallest ordinate of  $\Delta(\sum_{1 \leq i \leq s} H(x')^{-1}(f', g'^p); u'_2, u'_3; u'_1)$  is  $(\alpha_3(x'), \alpha_3(x') + \beta_3(x') - 1)$ , so  $C(x') \leq \alpha_3(x') - A_2(x)$ . And, the vertex of smallest abscissa is  $(A_2(x), \beta_3(x) + A_1(x) - 1)$ , so

$$C(x') \leq \beta_3(x) + A_2(x) - B_3(x) = \beta_3(x) - (B_3(x) - A_2(x)) \quad (1).$$

This gives the first assertions of the lemma, (i) and also (ii) when  $B_3(x) - A_2(x) > 0$ .

Furthermore  $C(x') < 1 + \lfloor \beta_3(x') \rfloor$  ( $\Rightarrow \gamma(x') \leq \gamma(x)$ ) if  $B_3(x) - A_2(x) \geq 0$ .

As  $\alpha_3(x) \leq 1 + B_3(x)$ ,

$$C(x') \leq \alpha_3(x) - A_2(x) \leq 1 + B_3(x) - A_2(x),$$

if  $B_3(x) - A_2(x) < 0$  or  $(B_3(x) - A_2(x) = 0$  and  $\alpha_3(x) < 1 + B_3(x))$ , we get  $C(x') < 1$ . So we get  $C(x') < 1 + \lfloor \beta_3(x') \rfloor$  and we get also (ii) when  $B_3(x) - A_2(x) < 0$  or  $(B_3(x) - A_2(x) = 0$  and  $\alpha_3(x) < 1 + B_3(x))$ .

To end the proof of (ii), we have to consider the case  $C(x') = 1$ ,  $B_3(x) - A_2(x) = 0$  and  $\alpha_3(x) = 1 + B_3(x)$ , this means that  $\beta_3(x) = -1$ : the monomial of  $H(x)^{-1}u_3^{-1}f$  or of  $H(x)^{-1}u_3^{-1}g^p$  which defines  $(\alpha_3(x), \beta_3(x))$  is  $u_1^{\omega(x)-1}u_2^{A_2+1}u_3^{-1}$ . So

$$A_2 \in \mathbb{N}, B_3 = \alpha_3 + \beta_3 \in \mathbb{N}$$

$\Delta(H(x')^{-1}(f', g'^p); u'_2, u'_3; u'_1)$  has only two vertices

$$(A_2, 1 + A_2), (1 + A_2, A_2) \in \mathbb{N}^2$$

and in the expansion of  $H(x')^{-1}f'$  or  $H(x')^{-1}g'^p$  appears the monomial  $u_1^{\omega(x)-1}u_2^{1+A_2}u_3^{A_2}$ . If  $m(x'') = 2$ , if we go back to the proof of **I.4**, we have, in the first case line above,  $i_0 = 1$ , so  $a(1) + \omega(x) - i_0 \neq 0 \pmod{p}$ , so  $\beta(x'') \leq C(x') = 1$ : this leads to  $\gamma(x'') = 1$ , in the second case, we have  $\beta(x'') = 0$ . If  $m(x'') = 3$ , we get  $C(x'') = 0$ .

For (iii), by the same arguments as in chapter 3 **I.8.9.3**, we get that if  $\mu_0$  the monomial valuation given by  $\mu_0(u_1^a u_2^b u_3^c) = a + \frac{b+c}{B(x')}$ .

$$\text{in}_{\mu_0}(H(x')^{-1}f') = U_1^{\omega(x)} \bar{\gamma}_3 + \sum_{1 \leq i \leq \omega(x)} U_1^{\omega(x)-i} U_2^{b_i} U_3^{c_i} F_i(U_2', U_3'), F_i \in k(x)[U_2', U_3'], \quad (1)$$

$F_i = 0$  or  $F_i$  homogeneous of degree  $d_i \leq i$ .

If  $x''$  is the origin of a chart above  $x'$ , then, recopy chapter 3 **I.8.9.3**: the proof is made just by looking at polyhedrons. If  $x''$  is not the origin of the first chart, then,  $i\beta(x'') \leq d_i + 1$  or  $\beta(x'') = 0$  if all the  $F_i = 0$ .

So we get  $\gamma(x'') = 1$  if  $\inf\{\frac{d_i}{i} | H(x')u_2^{b_i}u_3^{c_i}F_i \notin k(x')[u_1', u_2', u_3']^p\} < 1$ .

If for all  $i$  with  $H(x')u_2^{b_i}u_3^{c_i}F_i \notin k(x')[u_1', u_2', u_3']^p$ ,  $d_i = i$ , then we get  $\beta(x'') \leq 1 + \frac{1}{i}$ , this gives the result except if there is only  $F_1 \neq 0$ ,  $x''$  in case (\*1) and that  $\beta(x'') = i + 1 = 2 > d_1 = 1$ . This means that  $a(1) + \omega(x) - 1 = 0 \pmod{p}$ . In particular  $a(1) + \omega(x) \neq 0 \pmod{p}$ .

**I.6 Theorem** Let  $x \in \Sigma_p$  with  $\kappa(x) = 5$ ,  $m(x) \geq 2$ . Then  $x$  is good if one of the following is true:

- (i) case (\*1) and  $\beta(x) \leq 1$ ,
- (ii) case (\*2) and  $\beta(x) < 1$ ,
- (iii) case (\*3),  $\beta 3(x) < 1$ .

Proof. We have  $A_2(x) > 0$ . Indeed  $A_2(x) = 0$  implies that in the expansion of  $f$  appears the monomial  $u_1^{\omega(x)-i} u_3^{i\beta(x)}$ ,  $1 \leq i \leq \omega(x)$ . This is impossible in case (\*2), in case (\*1),  $\kappa(x) \leq 2$ , there is nothing to prove. In case (\*3),  $\beta 3(x) < 1$  implies  $\beta(x) \leq 1$ , this contradicts  $\epsilon(x) = 1 + \omega(x)$ .

**I.6.1** Case  $0 < A_2(x) < 1$ . We blow up  $x$ .

First chart, cases (\*1)(\*2). We get  $A_2(x') \leq A_2(x) + \beta(x) - 1 \leq A_2(x)$ ,  $x'$  verifies the hypotheses of **I.6** when  $x'$  is the origin, as  $\gamma(x') \leq \gamma(x)$ ,  $x'$  verifies the hypotheses of **I.6** when  $x'$  is not the origin.

In case (\*3), if  $\kappa(x') = 5$ , we can apply **I.4, I.5**: we get  $A_2(x') \leq A_2(x) + \beta(x) - 1 < A_2(x)$  or  $A_2(x') = A_2(x) + \beta 3(x) - 1 < A_2(x)$ , as  $\gamma(x') \leq \gamma(x)$ ,  $x'$  verifies the hypotheses of **I.6**.

Case  $0 < A_2(x) < 1$ , second chart.

We get  $A_2(x') = A_2(x)$ . Furthermore,  $\beta(x') \leq A_2(x) + \beta(x) - 1 < \beta(x)$  in cases (\*1)(\*2),  $\beta(x') \leq A_2(x) + \beta 3(x) - 1 < \beta 3(x)$  in cases (\*3):  $x'$  verifies the hypotheses of **I.6**.

End of the case  $0 < A_2(x) < 1$ . Then, we blow up  $x'$  and we go on if  $\Omega$  does not strictly drop, etc. We associate to  $x$  the couple  $(A_2, \beta)$  if  $x$  is in case (\*1) or (\*2),  $(A_2, \beta 3)$  in case (\*3). This couple strictly drops for the lexicographical ordering except maybe if  $x$  is in case (\*1) and  $\beta(x') = 1$ , which implies  $x'$  rational over  $x$ .

If the sequence of blowing ups is infinite, all the  $x(i)$  centers of  $\mu$  in  $X(n)$  are in case (\*1) with  $\beta(x(n)) = 1$ . All the  $x(n+1)$  are rational over  $x(n)$ , we can choose  $v$ , eventually  $v \in \widehat{S}$ ,  $v = \sum \lambda_i u_3^i \in k(x)[[u_3]]$ , such that  $x(n+1)$  are on the strict transform of a curve  $\mathcal{C} = V(X, u_1, v)$  which is contained in  $\Sigma_p$  and which gets permissible for  $n \gg 0$ . We conclude by the usual argument.

**I.6.2** Case  $1 \leq A_2(x)$ . We blow up  $V(X, u_1, u_2)$ , by **I.3.3**, we get the result by induction on  $A_2$ .

**I.7 Theorem** Let  $x \in \Sigma_p$   $\kappa(x) = 5$ ,  $x$  in case (\*1) or (\*2). If for a r.s.p.  $(X, u_1, u_2, u_3)$  verifying the conditions of **I.2**,  $C(x) = 0$  and, possibly  $u_3 \in \widehat{S}$  if  $\text{div}(u_3) \not\subset E$ , then  $x$  is good.

Proof. The case  $A_2(x) < 1$  and  $A_3(x) < 1$ , has been made in **I.4.2**. From now on,  $A_i \geq 1$  for some  $i$ ,  $i = 2$  or  $i = 3$ .

**I.7.1** Case  $a(1) + a(2) + \omega(x) < p$  and  $a(1) + a(3) + \omega(x) < p$ . Let us denote  $H(x) = u_1^{a(1)} u_2^{a(2)} u_3^{a(3)}$ , then  $\text{ord}_x(H(x)f) \geq p$ , so  $a(2)a(3) > 0$ :  $E = \text{div}(u_1 u_2 u_3)$ . So  $u_2$  and  $u_3$  play the same role. Let

us blow up  $x$ . We just look at the first chart. By **I.3**, a point  $x'$  very near to  $x$  is on the strict transform of  $\text{div}(u_1)$ . By **I.4**,  $H(x') = u_1^{a(1)} u_2^{a(1)+a(2)+a(3)+\omega(x)-p} u_3^{a(3)}$ ,  $\epsilon(x') \geq \omega(x)$  implies  $u_3'(x') = 0$ ,  $x'$  is the origin of the first chart and an induction on  $a(2) + a(3)$  gives the result.

**I.7.2** Other cases, let us test the blowing up of  $x$ .

By **I.3**, every point  $x'$  very near to  $x$  is on the strict transform of  $\text{div}(u_1)$ . As  $C(x) = 0$ , if  $x'$  is not the origin of a chart, we get  $\beta(x') \leq 1$ , so  $x'$  is good. If  $x'$  is the origin of a chart we get  $C(x') = 0$ . The components of  $\omega > 0$  at  $x'$  in the strict transform of  $\text{div}(u_1)$  are the strict transforms of those going through  $x$  plus, may be a projective line which projects on  $\text{Spec}(S)$  on the intersection of the strict transform of  $\text{div}(u_1)$  and the new exceptional component. After a finite sequence of blowing ups, *we may suppose that,  $\text{div}(u_2) \subset E$  and if  $\text{div}(u_3) \not\subset E$ , there is at most one component of  $\{\omega > 0\} \cap \text{div}(u_1)$  not contained in  $\text{div}(u_2)$ .*

**I.7.3** Case  $a(1) + \omega(x) \geq p$ . Once the condition above obtained, we make an induction on  $A_2 + A_3$ . If  $A_2 \geq 1$ ,  $V(X, u_1, u_2)$  is permissible: it is  $V(u_1, u_2) \cap \{\omega \geq 1\}$ , it is not formal and we blow it up. The only possible very near point is  $x'$ , the point on the strict transform of  $\text{div}(u_1)$  and  $(A_2(x'), A_3(x')) = (A_2(x), A_3(x) - 1)$ ,  $C(x') = 0$ .

If  $A_2 < 1$  and  $A_3 \geq 1$ ,  $\epsilon(V(X, u_1, u_3)) = \omega(x)$ ,  $V(X, u_1, u_3)$  is not formal, it is the component of  $\{\omega \geq 1\} \cap \text{div}(u_1)$  not contained in  $\text{div}(u_2)$ , we blow it up and we conclude as above.

**I.7.4** Case  $a(1) + \omega(x) < p$  and there exists  $i \in \{2, 3\}$  such that  $a(i) + \omega(x) \geq p$ . Then  $a(i) > 0$ :  $\text{div}(u_i) \subset E$ . We make a descending induction on  $(\sup\{A_i, i = 2, 3\}, \sup\{a(j), j = 2, 3\})$  for  $\leq_{lex}$ .

**I.7.4.1** If there exists  $i \in \{2, 3\}$  such that

$$a(i) + \omega(x) + a(1) \geq p \text{ and } A_i \geq 1, \quad (1)$$

we blow up  $V(X, u_i, u_1)$ . If both  $i = 2$  and  $i = 3$  verifies (1), we choose  $i$  with  $(A_i, a(i))$  maximal. Mutatis mutandis,  $i = 2$ . Then the point  $x'$  of parameters  $(\frac{X}{u_2}, \frac{u_1}{u_2}, u_2, u_3)$  is the only point which may be very near to  $x$ . The reader sees that  $\kappa(x') \leq 5$ , if  $x'$  is very near to  $x$ ,  $C(x') = 0$  and  $A_2(x') = A_2(x) - 1$ ,  $a_2(x') = a(2) + \omega(x) + a(1) - p < a(2)$ :  $(\sup(A_i), \sup(a(j)))$  strictly drops except if  $(A_2, a(2)) = (A_3, a(3))$ , in that case we blow up  $(\frac{X}{u_2}, \frac{u_1}{u_2}, u_3)$  and  $(\sup(A_i), \sup(a(j)))$  strictly drops.

**I.7.4.2** The remaining case. There is one  $i \in \{2, 3\}$  such that  $a(i) + \omega(x) + a(1) \geq p$ ,  $A_i < 1$ : so  $\text{div}(u_i) \subset E$ , mutatis mutandis,  $i = 2$ . Then,

$$A_3 \geq 1, \quad a(3) + a(1) + \omega(x) < p, \quad (\sup(A_i), \sup(a(j))) = (A_3, a(2)), \quad a(2) + a(1) + \omega(x) \geq p.$$

We blow up  $x$ , as seen in **I.7.2**, we have to look only at the origins of the first and second chart. In the first chart, if  $x'$  is very near to  $x$ , we get  $A_3(x') = A_3(x)$ ,  $A_2(x') = A_3(x) + A_2(x) - 1 < A_3(x)$ ,  $a'(2) = a(2) + a(3) + \omega(x) + a(1) - p < a(2)$ ,  $a'(3) = a(3) < a(2)$ :  $(\sup(A_i), \sup(a(j)))$  strictly drops.

In the second chart, if  $x'$  is very near to  $x$ , we get  $A_3(x') = A_3(x) + A_2(x) - 1 < A_3(x)$ ,  $A_2(x') = A_2(x) < A_3(x)$ ,  $(\sup(A_i), \sup(a(j)))$  strictly drops.

**I.8 Theorem** *Let  $x \in \Sigma_p$  with  $\kappa(x) = 5$ ,  $\kappa(x) > 4$ ,  $x$  in case (\*3) with  $\text{div}(u_2) \subset E$  and  $\phi_0 \in (u_1, u_3)\widehat{S}$ . If for a r.s.p.  $(X, u_1, u_2, u_3)$  verifying the conditions of **I.2**,  $A_{33}(x) = \beta_3(x)$ , possibly,  $u_3 \in \widehat{S}$ , then  $x$  is good.*

Proof. We see that  $A_{33}(x) = \beta_3(x)$  implies  $B_3(x) = A_{33}(x) + A_2(x)$ , so  $A_2 > 0$  or  $A_{33} > 0$ .

**I.8.1** We blow up  $x$ , by **I.3**, every point  $x'$  very near to  $x$  is on the strict transform of  $\text{div}(u_1)$ .

As  $A_{33}(x) = \beta_3(x)$ , in the proof of **I.5**,  $\deg(F_{i_0}) = 0$ : if  $x'$  is not the origin of a chart, we get  $\beta(x') \leq 1$ , so  $x'$  is good. If  $x'$  is the origin of the first chart and is very near to  $x$ , as  $\phi_0 \in (u_1, u_3)$ , if  $\epsilon(x') = 1 + \omega(x)$ ,  $x'$  is in case (\*3),  $A_{33}(x') = \beta_3(x')$ .

If  $\epsilon(x') = \omega(x)$ ,  $H(x')^{-1}f' = u_1^{\omega(x)}\phi_0 + \psi$ ,  $\text{ord}_{x'}(\psi) = \omega(x)$ , as  $A_3(x) = \beta_3(x)$ , either  $u_2$  divides  $\psi$ , by chapter 2 **II.1**,  $\kappa(x') \leq 1$  or  $\frac{u_3}{u_2}$  divides  $\psi$ , then as  $\text{div}(\frac{u_3}{u_2}) \not\subset E'$ ,  $\kappa(x') = 2$ .

If  $x'$  is the origin of a the second chart,  $x'$  is in case (\*1) or (\*2) and, by **I.5.3**,  $(A_2(x'), \beta(x')) = (A_2(x), A_2(x) + \beta_3(x) - 1)$ ,  $A_3(x') = B_3(x) - 1 = A_3(x) + A_2(x) - 1 = A_2(x) + \beta_3(x) - 1 = \beta(x')$ :  $C(x') = 0$ , by **I.7**,  $x'$  is good.

**I.8.2** So we are only interested in  $x'$  the origin of the first chart. If  $x'$  is not in a case of goodness seen above, then  $x'$  verifies the hypotheses of **I.8**. We blow it up and we go on... Then we create a sequence of points  $x = x_0 \leftarrow x_1 = x' \leftarrow x_2 \leftarrow \dots$  all on the strict transform of  $V(X, u_1, u_3)$ : by the usual argument, this sequence is finite.

## II End of the case $\kappa(x) = 5$

**II.1 Theorem** *Let  $x \in \Sigma_p$  with  $\kappa(x) = 5$ , then  $x$  is good.*

Proof of **II.1** .

We make an infinite sequence of blowing-ups  $X_i \leftarrow X_{i+1}$  along the centers  $x_i \in X_i$ , the centers of the valuation  $\mu$ , we suppose that  $x_{i+1}$  is very near to  $x_i$  for all  $i \leq 0$ . For  $i \geq 1$ ,  $E(i)$  has at least two components.

We are going to prove that there exists some  $i$  such that  $x_i$  is good. That will end the case  $\kappa(x) = 5$ .

As  $\gamma(x_i) \geq \gamma(x_{i+1})$ , for  $i \gg 0$ ,  $\gamma(x_i) = \gamma(x_{i+n})$ ,  $n > 0$ . Let us call  $\gamma(\mu)$  this value (which depends of  $\mu$  and of choices among the prepared parameters at each step).

**II.2**  $m(x_i) = 2$  for all  $i \gg 0$ .

Case where for  $n_0 \gg 0$  all the  $x_{n_0+i}$  are rational over  $x_{n_0}$ . By **I.4**, with  $n_0$  bigger if necessary, we can suppose that they are all in the same case (\*1) or (\*3). By a translation on the indices, we make  $n_0 = 0$ ,  $x = x_0$ .

So there exists  $v = u_3 + \sum_{a \geq 1} \lambda_a u_2^a \in k(x)[[u_2]]$  such that the projection of the  $x_i$  over  $\text{Spec} S$  are all on the strict transform of  $v$ , as  $(u_2, u_3)\widehat{S} = (u_2, v)\widehat{S}$ , we replace the couple  $(u_2, u_3)$  by  $(u_2, v)$ : all the  $x_i$  are origins of the first chart, the reader sees that for  $i \gg 0$  we reach the hypotheses of **I.7** if they are all in case (\*1):  $x_i$  is good. If they are all in case (\*3), it means that  $\phi_0 \in (u_1, v)\widehat{S}$  after a while, we reach the hypotheses of **I.8**:  $x_i$  is good.

Case where for every  $i$  there exists  $j \geq i$  such that  $x_{j+1}$  is not rational over  $x_i$ .

By **I.4**, **I.5**, for some  $i$ , we reach one of the four cases:

- (i)  $\kappa(x_i) \leq 4$ ,
- (ii)  $\gamma(x_i) = 1$ ,  $m(x_i) = 2$ : by **I.6**,  $x_i$  is good,
- (iii)  $x_i$  in case (\*1) and  $\beta(x_i) < 2$ , then by **I.4**, **I.5.2**, for the smallest  $j' \geq i$  such that  $(x_{j'}$  is not rational over  $x_i$  or  $m(x_{j'}) = 3$ ),  $\gamma(x_{j'}) = 1$ : by **I.6**, if  $m(x_{j'}) = 2$ ,  $x_j$  is good, if  $m(x_{j'}) = 3$ , then for the smallest  $j \geq j'$  such that  $m(x_j) = 2$ ,  $\gamma(x_j) = 1$ : by **I.6**  $x_j$  is good,
- (iv)  $x_i$  in case (\*3) and  $\beta_3(x_i) = 1$  and  $i_1(x_i) = 0 \pmod{p}$ , then by **I.5**, **I.5.3**, for the smallest  $j' \geq i$  such that  $(x_{j'}$  is not rational over  $x_i$  or  $m(x_{j'}) = 3$ ),  $\gamma(x_{j'}) = 1$  or  $\gamma(x_{j'+1}) = 1$  or  $a(1) + \omega(x) \neq 0 \pmod{p}$ : in the last case, this means for any  $n > 0$ ,  $x_{j'+n}$  will be in case (\*1)(\*2), by **I.4** we will reach (ii), in the other cases, we conclude as above by **I.6**.

**II.3**  $m(x_i) = 3$  for  $i \gg 0$ .

We are always at the origin of a chart, so by all the  $x_i$  are in case (\*2), by the usual transformation laws on polyhedrons, for  $i \gg 0$ ,  $C(x_i) = 0$ , by **I.7**,  $x_i$  is good.

**Till the end of II, we assume that neither the assumption of II.2, nor II.3 is satisfied.**

**II.4** For all  $n \in \mathbb{N}$ , there is some  $i \geq n$  with  $m(x_i) = 2$  and  $m(x_{i+1}) = 3$ , the  $x_i$  are rational over  $x_0$ ,  $0 < i$ .

**II.4.1** With the hypothesis of **II.4** and with  $a(1) + \omega(x) \neq 0 \pmod{p}$ .

Then, by **I.4**, **I.5.2**, there exists  $i > 0$  such that  $x_i$  is in case (\*2) and  $x_{i+j}$  in case (\*1) or (\*2), for all  $j$ ,  $1 \leq j$ . Furthermore,  $\gamma(\mu) = 1$ . Then, for  $j \gg 0$  such that  $m(x_{i+j}) = 2$ , we have  $\beta(x_{i+j}) \leq 1$ , by **I.6**,  $x_{i+j}$  is good.

**II.4.2** With the hypothesis of **II.4** and  $a(1) + \omega(x) = 0 \pmod{p}$ .

Then  $\gamma(\mu) \leq 2$ . All the  $x_i$  are rational over  $x = x_0$  and  $\gamma(x_i) = \gamma(\mu)$ ,  $i \geq 0$ .

If for  $i \gg 0$  all the  $x_i$  are in case (\*1) or (\*2), as above we can apply **I.4** and **I.5.2**,  $\gamma(\mu) = 1$ . Then, for  $j$  such that  $m(x_{i+j}) = 2$ , we have  $\beta(x_{i+j}) \leq 1$ , by **I.6**,  $x_{i+j}$  is good.

Last case: for all  $n \in \mathbb{N}$ , there is some  $i \geq n$  with  $x_i$  in case (\*3) and some  $j \geq n$  with  $x_j$  in case (\*2). If  $\gamma(\mu) = 1$ , then by **I.6**, for  $i \gg 0$ ,  $x_i$  is good.

**II.4.3** From now on,  $\gamma(\mu) = 2$ .

Let  $i$  such that  $m(x_i) = 2$  and  $m(x_{i+1}) = 3$ .

Either  $x_i$  is in case (\*1), as  $\gamma(x_i) = \gamma(x_{i+1}) = \gamma(\mu) = 2$ , by **I.5.2**,  $\beta(x_i) = 2$ ,  $\Delta(H(x')^{-1}f' + H(x')^{-1}g'^p; u'_1, u'_2; u'_3)$  has only two vertices  $(A_1(x), \beta(x) + A_1(x) - 1)$ ,  $(\alpha_2(x), B(x) - 1)$ ,  $(\alpha_2(x), B(x))$  which are the ends of its initial side, where  $x = x_i$ ,  $x' = x_{i+1}$ .

Then  $C(x_{i+1}) = 1$ , if  $m(x_{i+2}) = 3$ ,  $C(x_{i+2}) = 0$ : contradicts  $\gamma(\mu) = 2$ .

So  $m(x_{i+2}) = 2$ , with the notations of **I.4(4)**, either  $a(1) + \omega(x) - i_0 \neq 0 \pmod{p}$ : we get  $i_0\beta(x_{i+2}) \leq \deg(F_{i_0}) \leq i_0$ : contradicts  $\gamma(\mu) = 2$ . So  $a(1) + \omega(x) - i_0 = 0 \pmod{p}$ ,  $i_0 = 0 \pmod{p}$  and  $\beta(x_{i+2}) \leq 1 + \frac{1}{i_0} \leq 1 + \frac{1}{p}$ . If  $x_{i+2}$  is in case (\*1), by **I.4**, all the  $x_{i+j}$  with  $m(x_{i+j}) = m(x_{i+j-1}) = 2$ ,  $j \geq 3$ , are in case (\*1) with

$$\beta(x_{i+j}) \leq \beta(x_{i+2}) \leq 1 + \frac{1}{p} < 2.$$

So for the smallest  $j_0$  such that  $m(x_{i+j_0}) = 3$ , we get  $C(x_{i+j_0}) < \frac{\beta(x_{i+2})}{2} < 1$ , this contradicts  $\gamma(\mu) = 2$ .

So  $x_{i+2}$  is in case (\*3), we get  $i_0\beta_3(x_{i+2}) \leq \deg(F_{i_0}) \leq i_0$ , as  $\gamma(\mu) = 2$ ,  $\beta_3(x_{i+2}) = 1$ , which implies:  $a(1) + \omega(x) - i_0 = 0 \pmod{p}$  (end of the proof of **I.4**, same notations) so  $i_0 = 0 \pmod{p}$ :  $x_{i+2}$  is in case **II.2(iv)** above. There exists some  $j > i + 2$  such that  $x_j$  is in case (\*1) or  $m(x_j) = 3$ , let  $j_0$  be the smallest. When  $x_{j_0}$  is in case (\*1),  $\beta(x_{j_0}) < 2$ , then for the smallest  $j' \geq j_0$  such that  $m(x_{j'}) = 3$ , we get  $\gamma(x_{j'}) = 1$ : contradiction. When  $m(x_{j_0}) = 3$ , by **I.5.3**,  $\gamma(x_{j_0}) = 1$  or  $x_{j_0}$  is good, etc.: **I.5.3(ii)(iii)** ends the proof.

### III End of the proof of the main theorem.

In this last section, we reduce the local uniformization problem when  $\omega'(x) = 2$  to one of the previously studied cases, i.e.  $\kappa(x) \leq 5$ .

So  $\kappa(x) = 6$  means “no expansion of  $h$  gives  $\kappa(x) \in \{2, 3, 4, 5\}$ ”.

$x$  is said to be good if the quadratic sequence along  $\mu$  makes  $(\iota(x), \kappa(x))_{\text{lex}}$  strictly drop.

We always assume that the r.s.p.  $(X, u_1, u_2, u_3)$  of  $\widehat{R}$  is such that  $E \subseteq \text{div}(u_1 u_2 u_3)$  and  $\Delta(h; u_1, u_2, u_3; X)$  is minimal.

**III.1 Definition.** Assume that  $\kappa(x) > 5$ . Let

$$X =: X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n$$

be the quadratic sequence along  $\mu$ , i.e.  $X_i$  is the blowing up along the center  $x_{i-1}$  of  $\mu$  in  $X_{i-1}$  for  $i \geq 1$ .

We say that  $x$  is good, if  $x_n \notin \Sigma_p(X_n)$ , or  $\Omega(x_n) < \Omega(x)$  or  $(\Omega(x_n) = \Omega(x) \text{ and } \kappa(x_n) \leq 5)$  for some  $n \geq 1$ .

The final theorem of this paper is then:

**III.2 Theorem.** *Assume that  $\kappa(x) > 5$ . Then  $x$  is good.*

We first study the possible occurrences for  $\text{VDir}(x)$  when  $\kappa(x) > 5$ . Recall the definition of  $\text{VDir}(x)$  and  $\tau'(x)$  in chapter 2, **II.1.3** and corollary **II.1.4**.

**III.3 Lemma.** *Assume that  $\kappa(x) > 5$ . The following holds:*

- (i)  $\tau(x) = \tau'(x) = 2$ . If  $e : X' \rightarrow X$  is the blowing up along  $x$ , there is thus at most one  $x' \in e^{-1}(x)$  very near  $x$  and  $x'$  is rational over  $x$ ;
- (ii) if  $E = \text{div}(u_1)$ , then  $\epsilon(x) = 1 + \omega(x)$  and  $U_1 \in \text{VDir}(x)$ .

Proof. If  $\tau'(x) = 3$ , then  $\kappa(x) = 0$  by chapter 2 **II.1.4**.

If  $\tau(x) = 1$ , then  $\text{VDir}(x) = k(x) \cdot (\alpha_1 U_1 + \alpha_2 U_2 + \alpha_3 U_3)$ ,  $\alpha_i \in k(x)$  for  $i = 1, 2, 3$ . If there exists  $i$  with  $\alpha_i \neq 0$  and  $\text{div}(u_i) \not\subseteq E$ , then there is transverseness:  $\kappa(x) \in \{2, 4\}$ ; otherwise, there is tangency:  $\kappa(x) = 5$ . This proves (i).

Assume that  $E = \text{div}(u_1)$ . If  $\epsilon(x) = \omega(x)$ , let  $\Psi := \text{in}_x(H(x)^{-1}f)$ . We have  $\kappa(x) = 2$  unless  $\Psi \in k(x) \cdot U_1^{\omega(x)}$ : then  $\kappa(x) = 5$ . If  $\epsilon(x) = 1 + \omega(x)$ , We have  $\kappa(x) = 4$  unless  $\text{cl}_{\omega(x)} J(f, E) \subseteq (U_1)$ . Then  $U_1 \in \text{VDir}(x)$ .

**III.4 Lemma.** *Assume that  $\kappa(x) > 5$  and  $E = \text{div}(u_1 u_2)$ . If  $\text{VDir}(x) = \langle U_1 + \lambda U_2, U_3 \rangle$ ,  $\lambda \neq 0$ , then  $\epsilon(x) = 1 + \omega(x)$  and  $x$  is good.*

Proof. If  $\epsilon(x) = \omega(x)$ , we have  $\kappa(x) = 2$  by definition, since  $U_3 \in \text{VDir}(x)$ . So  $\epsilon(x) = 1 + \omega(x)$ . Let  $e : X' \rightarrow X$  be the blowing up along  $x$ . If  $x' \in e^{-1}(x)$  is very near  $x$ , then  $x'$  has coordinates  $(X' := \frac{X}{u_2}, v'_1 := \frac{u_1}{u_2} + \lambda, u'_2 := u_2, u'_3 := \frac{u_3}{u_2})$ . Therefore  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_2)$ . By **III.3(ii)**,  $x$  is good if  $\epsilon(x') = \omega(x')$ . If  $\epsilon(x') = 1 + \omega(x')$ , then

$$u_2^{-\omega(x)} J(f, E) \subseteq J(f', E'),$$

where  $h' := u_2^{-p}h = X'^p - X'g'^{p-1} + f'$ . We may have to perform a translation  $Z' := X' - \theta'$  in order to get  $\Delta(h'; v'_1, u'_2, u'_3; Z')$  minimal,  $f'$  being changed into  $f'_{Z'} := f' + \theta'^p - \theta'g'^{p-1}$ . Since  $\tau'(x) = 2$ ,  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \epsilon(x)$  and therefore  $u'_2$  divides  $H(x')^{-1}g'^p$ , so we have

$$J(f'_{Z'}, E') \equiv J(f', E') \pmod{(u'_2)}.$$

This implies  $\text{cl}_{\omega(x)} J(f'_{Z'}, E') \not\subseteq (U'_2)$ , so  $\kappa(x') = 4$  and  $x$  is good.

**III.5 Lemma.** *Assume that  $\kappa(x) > 5$  and  $\text{div}(u_1) \subseteq E \subseteq \text{div}(u_1 u_2)$ . If  $\text{VDir}(x) = \langle U_1, U_3 \rangle$ , then  $\epsilon(x) = 1 + \omega(x)$  and  $x$  is good.*

Proof. As in **III.4**,  $U_3 \in \text{VDir}(x)$  and  $\kappa(x) > 2$  implies  $\epsilon(x) = 1 + \omega(x)$ . Let  $e : X' \rightarrow X$  be the blowing up along  $x$ . Since  $x' \in e^{-1}(x)$  is very near  $x$ ,  $x'$  has coordinates  $(X' := \frac{X}{u_2}, u'_1 := \frac{u_1}{u_2}, u'_2 := u_2, u'_3 := \frac{u_3}{u_2})$ . We are at the origin of a chart, the polyhedron  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is thus minimal. Let us denote

$$H(x)^{-1}f =: \Psi(u_1, u_2, u_3) + \phi,$$

with  $H(x) = u_1^{a(1)} u_2^{a(2)}$  ( $a(2) = 0$  if  $E = \text{div}(u_1)$ ),  $\Psi \in k(x)[u_1, u_2, u_3]$  homogeneous of degree  $\epsilon(x)$  and  $\text{ord}_{\eta(x)} \phi > \epsilon(x)$ . Then  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1 u'_2)$ ,  $H(x') = u_1'^{a(1)} u_2'^{a(1)+a(2)+\omega(x)+1-p}$ , and

$$H(x')^{-1}f' = \Psi(u'_1, 1, u'_3) + u'_2 \phi',$$

with  $\phi' \in \widehat{S}'$ . We consider three cases:

*Case 1:*  $\text{ord}_{\eta'(x')} \Psi(u'_1, 1, u'_3) \leq \omega(x)$ . Then equality holds and  $\omega(x') = \epsilon(x')$ . We have  $\kappa(x') = 5$  if  $\text{cl}_{\omega(x)} \Psi(u'_1, 1, u'_3) \in k(x)[U'_1]$ . Otherwise,  $\text{cl}_{\omega(x)} \Psi(u'_1, 1, u'_3) \in k(x)[U'_1, U'_3]$ , so  $\text{VDir}(x') \not\subseteq \langle U'_1, U'_2 \rangle$ :  $\kappa(x') = 2$  and  $x$  is good.

From now on, we assume that  $\text{ord}_{\eta'(x')} \Psi(u'_1, 1, u'_3) = 1 + \omega(x) = \epsilon(x)$ , i.e.  $\Psi \in k(x)[U_1, U_3]$ . We have  $\text{ord}_{\eta'(x')}(u'_2 \phi') \geq \omega(x') = \omega(x)$ . Moreover, since  $\kappa(x) \neq 3$ , we actually have

$$\text{VDir}(x) = \text{VDir}\left(\frac{\partial \Psi}{\partial U_3}\right) = \langle U_1, U_3 \rangle. \quad (1)$$

*Case 2:*  $\text{ord}_{\eta'(x')}(u'_2 \phi') = \omega(x)$ . In particular, we have  $\epsilon(x') = \omega(x')$ . We denote  $\text{in}_{x'}(u'_2 \phi') = U'_2 \Phi'(U'_1, U'_2, U'_3)$ . If  $\Phi' \in k(x)[U'_2]$ , then  $\kappa(x') = 5$ ; if  $\Phi' \notin k(x)[U'_1, U'_2]$ , then  $\kappa(x') = 2$ . It can thus be assumed that

$$\Phi' \in k(x)[U'_1, U'_2] \setminus k(x)[U'_2]. \quad (2)$$

Then  $\text{VDir}(x') = \langle U'_1, U'_2 \rangle$ . Let  $e' : X'' \rightarrow X'$  be the blowing up along  $x'$ . Since  $x'' \in E'' := (e'^{-1}E')$ <sub>red</sub> is very near  $x'$ ,  $x''$  has coordinates  $(X'' := \frac{X'}{u'_3}, u''_1 := \frac{u'_1}{u'_3}, u''_2 := \frac{u'_2}{u'_3}, u''_3 := u'_3)$ ,  $E'' = \text{div}(u''_1 u''_2 u''_3)$  and the polyhedron  $\Delta(h''; u''_1, u''_2, u''_3; X'')$  is minimal. Then

$$H(x'')^{-1} f'' = u''_3 \Psi(u''_1, 1, 1) + u''_2 \Phi'(u''_1, u''_2, 1) + u''_2 u''_3 \phi'',$$

with  $\phi'' \in \widehat{S}''$ . By (1),  $\deg_{U_3} \Psi(U_1, U_3) \geq 2$ , so  $\text{ord}_{\eta''(x'')} \Psi(u''_1, 1, 1) = 1 + \omega(x) - \deg_{U_3} \Psi(U_1, U_3) \leq \omega(x) - 1$ . Since  $x''$  is very near  $x$ , equality holds and we get

$$u''_3 \Psi(u''_1, 1, 1) = \gamma'' u''_3 u''_1^{\omega(x)-1},$$

where  $\gamma'' \in \widehat{S}''$  is a unit. Then  $\epsilon(x'') = \omega(x)$ ,

$$\Psi'' := \text{cl}_{\omega(x)}(H(x'')^{-1} f'') = \lambda U''_3 U''_1^{\omega(x)-1} + U''_2 \Phi'(U''_1, U''_2, 1) + U''_2 U''_3 \Phi''(U''_1, U''_2, U''_3), \quad (3)$$

where  $\lambda \neq 0$ ,  $H(x'') = u''_1^{a(1)} u''_2^{b(2)} u''_3^{a(1)+b(2)+\omega(x)-p}$  and  $b(2) := a(1) + a(2) + \omega(x) + 1 - p$ .

We claim that  $\tau(x'') = 3$ , which implies that  $\kappa(x) = \kappa(x'') = 0$  by chapter 2 **II.1.2**, contradicting the assumption  $\kappa(x) > 5$ . To prove the claim, let us denote

$$\Psi'' =: \sum_{1 \leq j \leq \omega(x)} U''_1^{\omega(x)-j} \Psi_j(U''_2, U''_3).$$

By (3),  $\Psi'' \notin k(x)[U''_2, U''_3]$  and  $\deg_{U''_1} \Psi'' < \omega(x) = \deg \Psi''$ , so we have  $\tau(x'') \geq 2$ . If  $\tau(x'') = 2$ , then  $\text{VDir}(x'') = \langle U''_1 + \alpha_2 U''_2 + \alpha_3 U''_3, \beta_2 U''_2 + \beta_3 U''_3 \rangle$  for some  $\alpha_i, \beta_i \in k(x)$ ,  $i = 2, 3$  with  $(\beta_2, \beta_3) \neq (0, 0)$ . Note that, since  $\Psi_1 \notin k(x)[U_2]$ , we must have  $\beta_3 \neq 0$ . Without loss of generality, it can be assumed that  $\alpha_3 = 0$  and  $\beta_3 = 1$ , i.e.

$$\text{VDir}(x'') = \langle U''_1 + \alpha_2 U''_2, \beta_2 U''_2 + U''_3 \rangle. \quad (4)$$

If  $\beta_2 = 0$ , then with notations about derivations as in chapter 1 **II.3**, we have

$$F_i := H(x'')^{-1} \lambda''_i \frac{\partial H(x'') u''_2 \Phi'(u''_1, u''_2, 1)}{\partial \lambda''_i} \in k(x) \cdot (u''_1 + \alpha_2 u''_2)^{\omega(x)}$$

for each  $i$ ,  $1 \leq i \leq s$ . By (2) and (3),  $F_{i_0} \neq 0$  for some  $i_0$ ,  $1 \leq i_0 \leq s$  and  $U_2''$  divides  $F_{i_0}$ : this contradicts (4). Hence  $\beta_2 \neq 0$ .

By (4), we have

$$H(x'')^{-1} \lambda_i'' \frac{\partial H(x'') u_1''^{\omega(x)-1} \Psi_1(u_2'', u_3'')}{\partial \lambda_i''} \in k(x) \cdot u_1''^{\omega(x)-1} (u_3'' + \beta_2 u_2'') \quad (5)$$

for each  $i$ ,  $1 \leq i \leq s$ . Since  $\lambda \neq 0$ , we have  $\Psi_1 \neq 0$ . By (5),  $\Psi_1 = \mu_2 U_2'' + \mu_3 U_3''$  with  $\mu_2 \mu_3 \neq 0$  and we may apply chapter 2 **II.5(i)** (with  $a = 1$  and  $F = \Psi_1$ ). This yields  $a(1) + \omega(x) - 1 \equiv 0 \pmod{p}$ ,  $b(2)(b(2) + 1) \not\equiv 0 \pmod{p}$ , and  $2(\widehat{b(2)} + 1) = p$ , where  $\widehat{b(2)} \in \{0, \dots, p-1\}$  denotes the remainder of the division of the integer  $b(2)$  by  $p$ . The latter condition implies  $p = 2$ , incompatible with  $b(2)(b(2) + 1) \not\equiv 0 \pmod{p}$ : a contradiction and the claim is proved.

*Case 3:*  $\text{ord}_{\eta'(x')}(u_2' \phi') > \omega(x)$ . Then  $\epsilon(x') = 1 + \omega(x)$  and

$$\frac{\partial \Psi(u_1', 1, u_3')}{\partial u_3'} \in J(f', E') \pmod{(u_2')}.$$

By (1), we thus have

$$\text{VDir}(x') + k(x') \cdot U_2' = \langle U_1', U_2', U_3' \rangle.$$

Since  $\kappa(x) > 5$ , we must have  $\tau'(x') = 2$ , so  $\text{VDir}(x') = \langle U_1' + \lambda U_2', U_3' \rangle$  for some  $\lambda \in k(x)$ , after possibly changing coordinates to  $(u_1', u_2', v_3' := u_3' + \alpha_1 u_2')$  and letting  $Z' := X' - \theta'$  in order to get  $\Delta(h'; u_1', u_2', v_3'; Z')$  minimal. If  $\lambda \neq 0$ , then  $x$  is good by **III.4**. If  $x$  is not good, then  $\kappa(x') > 5$ ,  $\text{VDir}(x') = \langle U_1', U_3' \rangle$  and  $E' = \text{div}(u_1' u_2')$ , so  $x'$  verifies the hypotheses of **III.5** and we iterate the process.

Let  $X = X_0 \leftarrow X_1 \leftarrow \dots \leftarrow X_{n-1} \leftarrow X_n \leftarrow \dots$  be the quadratic sequence along  $\mu$ . There exists a series  $v_3 = u_3 + \sum_{j \geq 2} \alpha_j u_2^j$ ,  $\alpha_j \in k(x)$ , and  $Z \in \widehat{R}$  with the following properties:

- (a) the polyhedron  $\Delta(h; u_1, u_2, v_3; Z)$  is minimal;
- (b) if  $x_n$  is very near  $x$  and  $\kappa(x_n) > 5$ , then  $x_n$  is on the strict transform  $Y_n$  of  $Y := V(Z, u_1, v_3) \subseteq \text{Spec}(\widehat{R}/(h))$  in  $X_n$ .

As pointed out several times in this paper, (b) implies that  $n < \frac{\mu(u_1)}{\mu(u_2)}$ : a contradiction, since the value group of  $\mu$  is Archimedean. Hence  $x$  is good.

**III.5.1 Corollary.** *If  $\kappa(x) > 5$  and either  $E = \text{div}(u_1)$  or  $(E = \text{div}(u_1 u_2)$  and  $\text{VDir}(x) \neq \langle U_1, U_2 \rangle$ ), then  $x$  is good.*

*Proof.* This follows from **III.3** and **III.5** (resp. **III.4** and **III.5**) if  $E = \text{div}(u_1)$  (resp.  $E = \text{div}(u_1 u_2)$ ).

**III.5.2 Lemma.** *Assume that  $\kappa(x) > 5$ ,  $E = \text{div}(u_1 u_2)$  and  $\text{VDir}(x) = \langle U_1, U_2 \rangle$ . Let  $e : X' \rightarrow X$  be the blowing up along  $x$  and  $x' \in e^{-1}(x)$  be very near  $x$ . Then  $\epsilon(x') = \omega(x')$ .*

*Proof.* Since  $\text{VDir}(x) = \langle U_1, U_2 \rangle$ ,  $x'$  has coordinates  $(X' = \frac{X}{u_3}, u_1' = \frac{u_1}{u_3}, u_2' = \frac{u_2}{u_3}, u_3' = u_3)$ , so  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u_1' u_2' u_3')$ . In particular,  $\epsilon(x') = \omega(x')$ .

**III.5.3 Remark.** Corollary **III.5.1** and lemma **III.5.2** reduce theorem **I.2** to the case where  $\text{div}(u_1 u_2) \subseteq E \subseteq \text{div}(u_1 u_2 u_3)$ ,  $\text{VDir}(x) \subseteq \langle \{U_i \mid \text{div}(u_i) \subseteq E\} \rangle$  and  $\epsilon(x) = \omega(x)$ . Then theorem **III.2** is a consequence of propositions **III.6.2** and **III.6.3** below.

**III.6 Definition.** Assume that  $\kappa(x) = 6$ ,  $\text{div}(u_1u_2) \subseteq E \subseteq \text{div}(u_1u_2u_3)$ ,  $\text{VDir}(x) \subseteq\subseteq \{U_i \mid \text{div}(u_i) \subseteq E\} >$  and  $\epsilon(x) = \omega(x)$ . We let  $c(x) = 2$  if  $\text{VDir}(x) = \langle U_{i_1}, U_{i_2} \rangle$  for some  $i_1, i_2$  such that  $\text{div}(u_{i_1}u_{i_2}) \subseteq E$ . Otherwise, let  $c(x) = 3$ .

**III.6.1 Lemma.** Assume that  $\kappa(x) = 6$ ,  $E = \text{div}(u_1u_2)$ ,  $\text{VDir}(x) = \langle U_1, U_2 \rangle$  and  $\epsilon(x) = \omega(x)$ . Then  $F_3 := \text{cl}_{\omega(x)}(H(x)^{-1} \frac{\partial f}{\partial u_3}) \in k(x)[U_1, U_2]$ .

Proof. Let  $e : X' \rightarrow X$  be the blowing up along  $x$ . The only point  $x' \in e^{-1}(x)$  very near  $x$  has parameters  $(X' = \frac{X}{u_3}, u'_1 = \frac{u_1}{u_3}, u'_2 = \frac{u_2}{u_3}, u'_3 = u_3)$ , so the polyhedron  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal. We have  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1u'_2u'_3)$ , so  $J(f', E') = J(f', E', x')$  and

$$f'_3 := u'_3(u'_3)^{-\omega(x)} H(x)^{-1} \frac{\partial f}{\partial u_3} \in J(f', E', x').$$

Since  $x'$  is very near  $x$ ,  $\text{ord}_{\eta'(x')} f'_3 \geq \omega(x)$ . We have

$$f'_3 \equiv u'_3 F'_3(u'_1, u'_2, 1) \pmod{(u'_3)^2},$$

with  $\text{ord}_{\eta'(x')} F'_3(u'_1, u'_2, 1) \leq \omega(x) - 1$  if  $F_3 \notin k(x)[U_1, U_2]$ . Therefore equality holds and  $U'_3 \in \text{VDir}(x')$ .

If  $\varphi \in J(f, E, x)$ , then  $u_3^{-\omega(x)} \Phi(u'_1, u'_2) \in J(f', E', x') + (u'_3)$ , where  $\Phi := \text{cl}_{\omega(x)} \varphi$ . Therefore

$$\langle U'_1, U'_2, U'_3 \rangle \supseteq k(x').U'_3 + \text{VDir}(\{\Phi(U'_1, U'_2) \mid \varphi \in J(f, E, x)\}) \subseteq \text{VDir}(x'),$$

since  $\text{VDir}(x) = \langle U_1, U_2 \rangle$ . This implies  $\tau(x') = 3$ , so  $\kappa(x) = \kappa(x') = 0$ : a contradiction.

**III.6.2 Proposition.** Let  $x$  be as in definition III.6. If  $c(x) = 3$ , then  $x$  is good.

Proof. Necessarily  $E = \text{div}(u_1u_2u_3)$ . Let  $e : X' \rightarrow X$  be the blowing up along  $x$ . As  $c(x) = 3$ , the center  $x'$  of  $\mu$  in  $X'$  is not on the strict transform of any two components of  $E$ , so  $m(x') \leq 2$ . By III.5.1, it can be assumed that  $m(x') = 2$ . After possibly renumbering coordinates, it can be assumed that

$$\text{VDir}(x) = \langle U_1, U_3 + \lambda U_2 \rangle$$

with  $\lambda \neq 0$ . Then  $x' = (X' := \frac{X}{u_2}, u'_1 := \frac{u_1}{u_2}, u'_2 := u_2, v'_3 := \frac{u_3}{u_2} + \lambda)$  and  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1u'_2)$ .

Let  $\varphi \in J(f, E, x)$  be such that  $\Phi := \text{cl}_{\omega(x)} \varphi \notin k(x)[U_1]$ . Then  $u_2^{-\omega(x)} \varphi \in J(f', E')$  and therefore

$$\Phi(u'_1, 1, v'_3 - \lambda) \in J(f'_{Z'}, E') + (u'_2), \quad (1)$$

where  $Z' := X' - \theta'$ ,  $h' := u_2^{-p}h = Z'^p - Z'g'^{p-1} + f'_{Z'}$  and  $\Delta(h'; u'_1, u'_2, v'_3; Z')$  is minimal (since  $\epsilon(x) = \omega(x)$ ,  $\text{ord}_{\eta(x)}(H(x)^{-1}g^p) > \omega(x)$  in here).

If  $\epsilon(x') = 1 + \omega(x)$ , then  $x'$  (hence  $x$ ) is good by (1) and III.5.1, since  $\Phi \notin k(x)[U_1]$ .

If  $\epsilon(x') = \omega(x)$  and  $x$  is not good, then  $\text{VDir}(x') = \langle U'_1, U'_2 \rangle$  by III.5.1, so

$$\Phi(v_1, 1, v_3 - \lambda) \notin J(f'_{Z'}, E', x') + (u'_2).$$

Since  $E' = \text{div}(u'_1u'_2)$ , (1) implies that

$$(H(x')^{-1} \frac{\partial f'_{Z'}}{\partial v'_3}) \equiv (\Phi(u'_1, 1, v'_3 - \lambda)) \pmod{(u'_1, u'_2)},$$

a contradiction by **III.6.1**.

**III.6.3 Proposition.** *Let  $x$  be as in definition **III.6**. If  $c(x) = 2$ , then  $x$  is good.*

Proof. After possibly renumbering coordinates, it can be assumed that  $\text{VDir}(x) = \langle U_1, U_2 \rangle$ . Let  $e : X' \rightarrow X$  be the blowing up along  $x$ . Since the center  $x'$  of  $\mu$  in  $X'$  is very near  $x$ ,  $x' = (X' := \frac{X}{u_3}, u'_1 := \frac{u_1}{u_3}, u'_2 := \frac{u_2}{u_3}, u'_3 := u_3)$ . We have  $E' := (e^{-1}E)_{\text{red}} = \text{div}(u'_1 u'_2 u'_3)$  and  $\Delta(h'; u'_1, u'_2, u'_3; X')$  is minimal. Moreover  $J(f', E', x') = u_3^{-\omega(x)} J(f, E, x)$ , since  $m(x') = 3$ , so we have  $\text{VDir}(x') \equiv \langle U'_1, U'_2 \rangle \pmod{U'_3}$ . We are done by **III.6.2** unless

$$\text{VDir}(x') = \langle U'_1, U'_2 \rangle,$$

i.e.  $x'$  satisfies again the assumptions of **III.6.3** with the same numbering of variables if  $\kappa(x') > 5$ .

Let

$$X = X_0 \leftarrow X_1 \leftarrow \cdots X_{n-1} \leftarrow X_n \leftarrow \cdots$$

be the quadratic sequence along  $\mu$ . We cannot have  $\kappa(x_i) > 5$  for  $i \geq \frac{\mu(u_1)}{\mu(u_3)}$ , so  $x$  is good and the conclusion follows.

“On n’est jamais, jamais assez fort pour ce calcul” (Comtesse Maxime de la Falaise).

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