On the Periodic Lorentz Gas and the Lorentz Kinetic Equation

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ABSTRACT. We prove that the Boltzmann-Grad limit of the Lorentz gas with periodic distribution of scatterers cannot be described with a linear Boltzmann equation. This is at variance with the case of a Poisson distribution of scatterers, for which the convergence to the linear Boltzmann equation was proved by Gallavotti [*Phys. Rev. (2)* **185**, 308 (1969)]. The arguments presented here complete the analysis in [Golse-Wennberg, *M2AN Modél. Math. et Anal. Numér.* **34**, 1151 (2000)], where the impossibility of a kinetic description was established only in the case of absorbing obstacles. The proof is based on estimates on the distribution of free-path lengths established in [Golse-Wennberg *loc.cit.*] and in [Bourgain-Golse-Wennberg, *Commun. Math. Phys.* **190**, 491 (1998)], and on a classical result on the spectrum of the linear Boltzmann equation which can be found in [Ukai-Point-Ghidouche, *J. Math. Pures Appl. (9)* **57**, 203 (1978)].

1. INTRODUCTION

About 100 years ago, Lorentz [15] proposed the following linear kinetic equation to describe the motion of electrons in a metal:

$$(1.1) \quad (\partial_t + v \cdot \nabla_x + \tfrac{1}{m} F(t, x) \cdot \nabla_v) f(t, x, v) = N_{at} r_{at}^2 |v| \mathcal{C}(f(t, x, \cdot))(v)$$

where $f(t, x, v)$ is the (phase space) density of electrons which, at time t, are located at x and have velocity v. In Eq. (1.1), F is the electric force field, m the mass of the electron, while N_{at} and r_{at} designate respectively the number of metallic atoms per unit volume and the radius of each such atom. Finally $\mathcal{C}(f)$ is the collision integral: it acts on the velocity variable only, and is given, for all continuous $\phi \equiv \phi(v)$ by the formula

$$(1.2) \quad \mathcal{C}(\phi)(v) = \int_{|\omega|=1, v \cdot \omega > 0} \big(\phi(v - 2(v \cdot \omega)\omega) - \phi(v)\big) \cos(v, \omega) d\omega \,.$$

In the case where $F \equiv 0$, Gallavotti [11, 12] proved that Eq. (1.1) describes the Boltzmann-Grad limit of a gas of point particles undergoing elastic collisions on a random (Poisson) configuration of spherical obstacles. His result was successively strengthened by Spohn [18], and by Boldrighini-Bunimovich-Sinai [4].

In the presence of an external, non-zero electric force F and for the same random configuration of absorbing obstacles as in [11, 12], Desvillettes-Ricci [10] proved recently that the Boltzmann-Grad limit of a gas of point particles leads to a non-Markovian equation — see also an earlier, similar observation by Bobylev-Hansen-Piasecki-Hauge [2].

The case of periodic configuration of obstacles, perhaps closer to Lorentz's original ideas, completely differs from the random case. In the case of absorbing obstacles, and without external force F, several results suggest that the Boltzmann-Grad limit is non-Markovian [8, 14, 9]. However, all these results are based on explicit computations that are possible only in the case of absorbing obstacles.

In the present note, we show that neither equation (1.1) nor any variant thereof can describe the Boltzmann-Grad limit of the periodic Lorentz gas with no external force ($F \equiv 0$) and in the case of reflecting obstacles.

The crucial observation (already made in [8, 14]) is that the distribution of first hitting times satisfies, in the periodic case, an inequality stated below as Theorem 2.1 that prevents these first hitting times from begin exponentially distributed. The probabilistic representation of equation (1.1) by a jump process with exponentially distributed jump times plus a drift (see for instance [16]) suggests that the Boltzmann-Grad limit of the periodic Lorentz gas with reflecting obstacles cannot be described by (1.1).

In the present paper, we give a complete proof of this fact (see Theorem 4.1 below), which by the way does not appeal to the probabilistic representation of (1.1). In addition, the method of proof used here provides an explicit estimate of the difference between the single particle phase space density for the periodic Lorentz gas in the Boltzmann-Grad limit and the solution of the Lorentz kinetic equation (1.1) — see the inequality (4.18) and the discussion thereafter. This result has been announced in [13]

Recently, Ricci and Wennberg [17] have considered the following interesting variant of the periodic Lorentz gas studied here. Their microscopic model consists of gas of point particles in a periodic configuration of obstacles that are bigger than in the Boltzmann-Grad

scaling considered in the present paper. However, some of these obstacles are removed with a probability carefully chosen in terms of the obstacle radius so that the mean collision time remains of order one. They proved that the expected single-particle phase-space density so obtained converges to a solution of the Lorentz kinetic equation (1.1). This result suggests that the failure of (1.1) to capture the Boltzmann-Grad limit of the periodic Lorentz gas is a very unstable phenomenon that is specific to the periodic setting and likely to disappear whenever some amount of randomness is injected in the microscopic system.

2. The periodic Lorentz gas

Let $D \in \mathbf{N}$, $D \geq 2$. For each $r \in (0, \frac{1}{2})$, we consider the domain

$$Z_r = \{ x \in \mathbf{R}^D \mid \mathrm{dist}(x, \mathbf{Z}^D) > r \} \,,$$

which is usually referred to as "the billiard table".

The *Lorentz gas* is the dynamical system corresponding to a cloud of point particles that move freely in Z_r — collisions between particles being neglected — and are specularly reflected at the boundary of each obstacle — the obstacles being the balls of radius r centered at the lattice points, i.e. the connected components of Z_r^c. Since collisions between particles are neglected, one can equivalently consider a single particle whose initial position and velocity are appropriately distributed in $Y_r \times \mathbf{S}^{D-1}$.

In this dynamical system, a particularly important notion is that of "free path length": see the review article by Bunimovich [5], pp. 221-222.

The free path length — or "(forward) exit time" — for a particle starting from $x \in Z_r$ in the direction $v \in \mathbf{S}^{D-1}$ is defined as

$$(2.1) \qquad \tau_r(x, v) = \inf\{ t > 0 \mid x + tv \in \partial Z_r \} \,.$$

The function τ_r is then extended by continuity to the non-characteristic part of the boundary of the phase-space, i.e. to

$$\{ (x, v) \in \partial Z_r \times \mathbf{S}^{D-1} \mid v \cdot n_x \neq 0 \} \,,$$

where n_x designates the inward unit normal field on ∂Z_r (pointing toward Z_r, i.e. outside of the scatterers). Because Z_r is invariant under $\mathbf{Z}^D$-translations, one has

$$\tau_r(x + k, v) = \tau_r(x, v) \text{ for each } (x, v) \in Z_r \times \mathbf{S}^{D-1} \text{ and } k \in \mathbf{Z}^D \,.$$

Hence τ_r can be seen as a $[0, +\infty]$-valued function defined on $Y_r \times \mathbf{S}^{D-1}$ (and a.e. on $\overline{Y}_r \times \mathbf{S}^{D-1}$), where $Y_r = Z_r / \mathbf{Z}^D$.
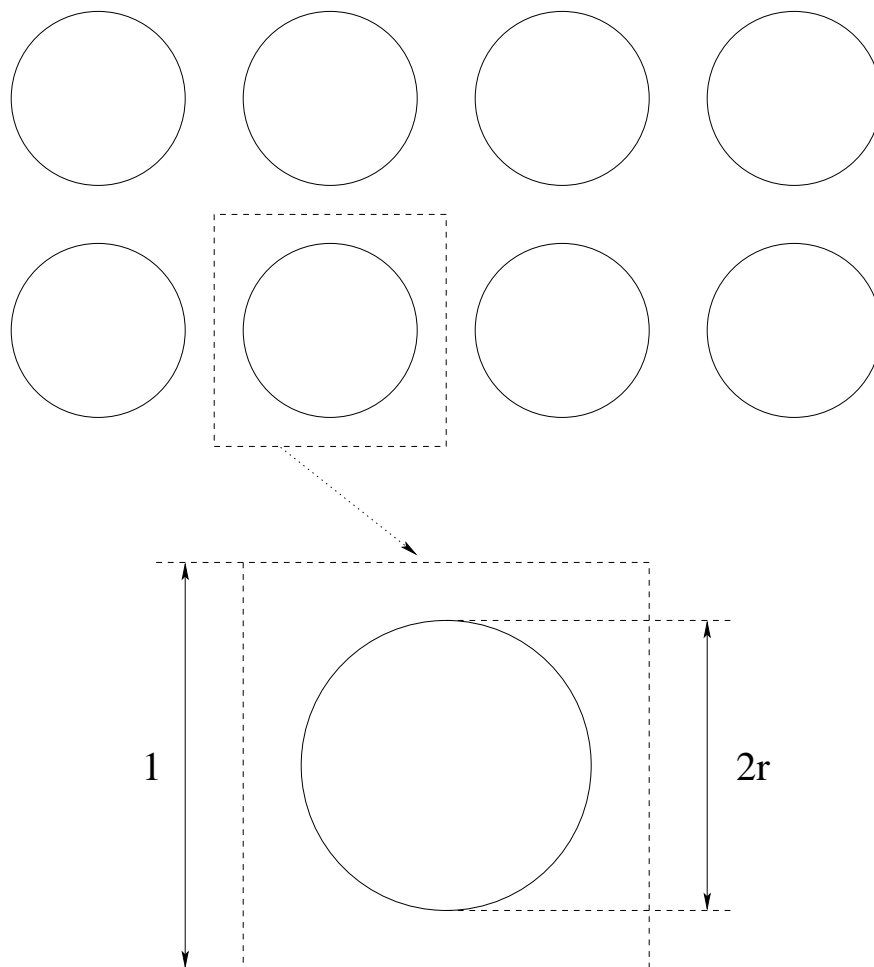
FIGURE 1. The billiard table Z_r and the punctured torus Y_r

Whenever the components of $v \in \mathbf{S}^{D-1}$ are rationally independent — i.e. if $k \cdot v \neq 0$ for each $k \in \mathbf{Z}^D \setminus \{0\}$ — each orbit of the linear flow $x \mapsto x + tv$ is dense on $\mathbf{T}^D = \mathbf{R}^D/\mathbf{Z}^D$, and thus $\tau_r(x, v) < +\infty$ for each $x \in Z_r$.

On the measurable space $Y_r \times \mathbf{S}^{D-1}$ equipped with its Borel σ-algebra, we define μ_r as the probability measure proportional to the Lebesgue measure on $Y_r \times \mathbf{S}^{D-1}$: in other words[1]

$$d\mu_r(y, v) = \frac{dy dv}{|Y_r||\mathbf{S}^{D-1}|} \,.$$

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[1]If A is a measurable d-dimensional set in $\mathbf{R}^D$ ($d \leq D$), we denote by $|A|$ its d-dimensional volume. Here $|Y_r|$ is the Lebesgue measure of any fundamental domain of the quotient space Y_r, i.e. of the unit cube with a ball of radius r removed.

Define the distribution of τ_r under μ_r by the usual formula

$$\Phi_r(t) := \mu_r\left(\{(y,v) \in Y_r \times \mathbf{S}^{D-1} \mid \tau_r(y,v) \geq t\}\right).$$

Theorem 2.1. *Let $D \geq 2$. There exist two positive constants C_1 and C_2 such that, for each $r \in (0, \frac{1}{2})$ and each $t > 1/r^{D-1}$*

$$\frac{C_1}{tr^{D-1}} \leq \Phi_r(t) \leq \frac{C_2}{tr^{D-1}}\,.$$

In the theorem above, the lower bound in the case $D = 2$ and the upper bound for all $D \geq 2$ were proved by Bourgain-Golse-Wennberg [8]; the lower bound was extended to the case of any $D \geq 2$ by Golse-Wennberg [14]. More precise results concerning $\Phi_r(t/r)$ in space dimension $D = 2$ have recently been obtained by Caglioti-Golse [9] and Boca-Zaharescu [3]; however, only the result above (Theorem 2.1) is used in the present paper.

3. The linear Boltzmann equation

The linear Boltzmann equation for a free[2] gas of particles moving at speed 1 is

$$(3.1) \qquad (\partial_t + v \cdot \nabla_x)f(t,x,v) = \mathcal{C}(f(t,x,\cdot))(v)$$

where $f \equiv f(t,x,v)$ is the single particle phase-space density, also known as "distribution function". In other words, $f(t,x,v)$ is the density of particles which, at time t, are located at $x \in \mathbf{R}^D$ and move in the direction $v \in \mathbf{S}^{D-1}$. The term $\mathcal{C}(f)$ is the collision integral; specifically, $\mathcal{C}$ is an integral operator of the form

$$(3.2) \qquad \mathcal{C}(\phi)(v) = \sigma \int_{\mathbf{S}^{D-1}} k(v,w)(\phi(w) - \phi(v))dw$$

where $\sigma > 0$ and $k \in C(\mathbf{S}^{D-1} \times \mathbf{S}^{D-1})$ satisfies the following properties

$$(3.3)$$

$$k(v,w) = k(w,v) > 0\,, \qquad \int_{\mathbf{S}^{D-1}} k(v,w)dv = 1\,, \text{ for all } v, w \in \mathbf{S}^{D-1}\,.$$

Changing variables according to $\omega \mapsto w = v - 2(v \cdot \omega)\omega$ in the Lorentz collision integral for $D = 3$ shows that

$$\frac{1}{\pi}\int_{\omega \in \mathbf{S}^2, v \cdot \omega > 0} \left(\phi(v - 2(v \cdot \omega)\omega) - \phi(v)\right)\cos(v,\omega)d\omega$$

$$= \int_{\mathbf{S}^2} k(v,w)(\phi(w) - \phi(v))dw \text{ with } k(v,w) \equiv \frac{1}{4\pi}\,.$$

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[2]I.e. without external force.

Hence, the Lorentz kinetic model (1.1)-(1.2) is a special case of the linear Boltzmann equation (3.1)-(3.2).

In the sequel, we shall restrict our attention to the case where the linear Boltzmann equation is posed in a periodic box. Without loss of generality, we assume that this periodic box has size 1, and that $\mathbf{S}^{D-1}$ is endowed with its rotationally invariant unit measure, henceforth denoted by dv. Finally we denote by $\langle \cdot \rangle$ the average with respect to both variables x and v:

$$(3.4) \qquad \langle \phi \rangle = \iint_{\mathbf{T}^D \times \mathbf{S}^{D-1}} \phi(x,v) dx dv \,.$$

Consider then the unbounded operator A on $L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})$ defined by

$$(A\phi)(x,v) = -v \cdot \nabla_x \phi(x,v) - \sigma \phi(x,v) + \sigma \int_{\mathbf{S}^{D-1}} k(v,w)\phi(x,w) dw$$

with domain

$$D(A) = \{\phi \in L^2(\mathbf{T}^D \times \mathbf{S}^{D-1}) \,|\, v \cdot \nabla_x \phi \in L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})\} \,.$$

We recall the following result, originally proved in [19] for the more complicated case of the linearization of Boltzmann's equation at a uniform Maxwellian state.

Theorem 3.1. *Let $\sigma > 0$ and $k \in C(\mathbf{S}^{D-1} \times \mathbf{S}^{D-1})$ satisfy the assumptions (3.3). Then the operator A generates a strongly continuous contraction semigroup on $L^2(\mathbf{R}^D \times \mathbf{S}^{D-1})$, and there exists positive constants c and γ such that*

$$\|e^{tA}\phi - \langle \phi \rangle\|_{L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})} \leq ce^{-\gamma t} \|\phi\|_{L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})} \,, \quad t \geq 0 \,,$$

for each $\phi \in L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})$.

4. The non-convergence result

Throughout this section, we denote by ϵ the sequence $(1/n)_{n \in \mathbf{N}^*}$. For some given $r_* \in (0, \frac{1}{2})$, we set $r = r_* \epsilon^{1/(D-1)}$, and we denote by Ω_ϵ the open subset ϵZ_r with this particular choice of r.

4.1. **Reflecting vs. absorbing obstacles.** First we define the billiard flow on the scaled billard table Ω_ϵ. It is a one-parameter group on $\Omega_\epsilon \times \mathbf{S}^{D-1}$ denoted by

$$(4.1) \quad S_t^\epsilon : \Omega_\epsilon \times \mathbf{S}^{D-1} \ni (x,v) \mapsto (X_t^\epsilon(x,v), V_t^\epsilon(x,v)) \in \Omega_\epsilon \times \mathbf{S}^{D-1}$$

and defined in the following manner:

- $S_0^\epsilon(x,v) = (x,v)$,
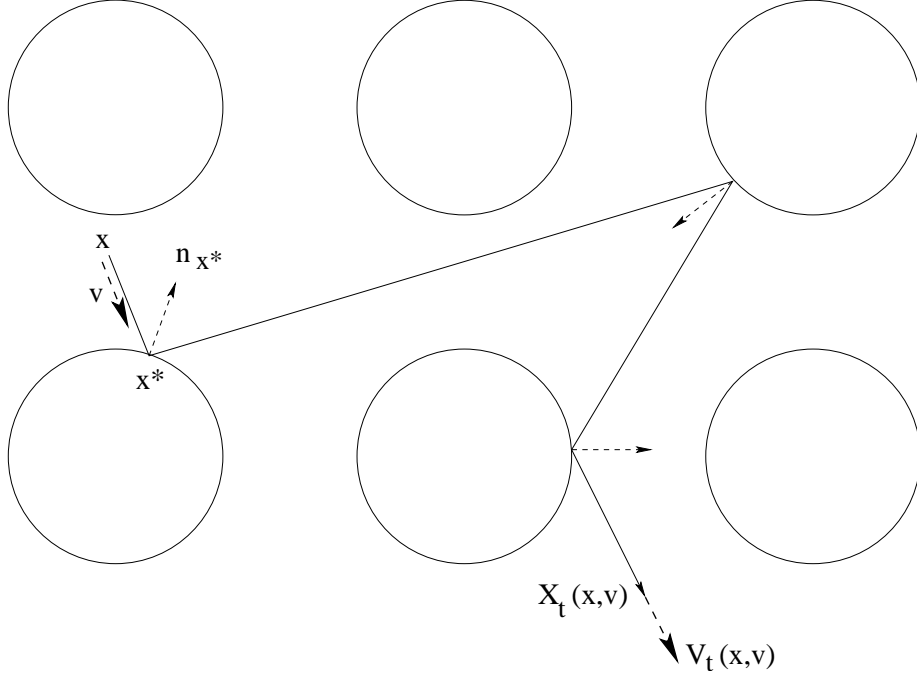
FIGURE 2. The billiard dynamics on Ω_ϵ

- if t is such that $X_t^\epsilon(x,v) \notin \partial\Omega_\epsilon$, then $S_t^\epsilon(x,v)$ is smooth in the time variable near that particular value of t and one has

(4.2) $$\frac{d}{dt}X_t^\epsilon(x,v) = V_t^\epsilon(x,v)\,, \quad \frac{d}{dt}V_t^\epsilon(x,v) = 0\,;$$

- if t is such that $X_t^\epsilon(x,v) = x^* \in \partial\Omega_\epsilon$, then $S_t^\epsilon(x,v)$ has the following jump discontinuity (in the velocity component only)

(4.3) $$X_{t+0}^\epsilon(x,v) = X_{t-0}^\epsilon(x,v)\,, \quad V_{t+0}^\epsilon(x,v) = \mathcal{R}(n_{x^*})V_{t-0}^\epsilon(x,v)\,,$$

where $\mathcal{R}(n_{x^*})$ is the symmetry with respect to the hyperplane orthogonal to the inner unit normal n_{x^*} at $x^* \in \partial\Omega_\epsilon$: in other words

(4.4) $$\mathcal{R}(n_{x^*})\xi = \xi - 2(\xi \cdot n_{x^*})n_{x^*}\,, \quad \xi \in \mathbf{R}^D\,.$$

Notice that this dynamics is mechanically reversible, i.e. for all $t \in \mathbf{R}$, one has

$$S_{-t}^\epsilon(x,v) = S_t^\epsilon(x,-v)\,, \quad (x,v) \in \Omega_\epsilon \times \mathbf{S}^{D-1}\,.$$

For each $t \in \mathbf{R}$, the map S_t^ϵ is invariant under $\mathbf{Z}^D$-translations, in the following sense: for all $k \in \mathbf{Z}^D$ and $(x,v) \in \Omega_\epsilon \times \mathbf{S}^{D-1}$, one has

$$S_t^\epsilon(x+k,v) = (X_t^\epsilon(x,v) + k, V_t^\epsilon(x,v)) = S_t^\epsilon(x,v) + (k,0)\,.$$

Hence S_t^ϵ defines a one-parameter group — still denoted by S_t^ϵ — on the quotient space $\Upsilon_\epsilon \times \mathbf{S}^{D-1}$, where $\Upsilon_\epsilon = \Omega_\epsilon/\mathbf{Z}^D$, and the restriction of the measure $dxdv$ (defined on $\mathbf{T}^D \times \mathbf{S}^{D-1}$) to $\Upsilon_\epsilon \times \mathbf{S}^{D-1}$ is invariant under S_t^ϵ.

Starting from the billiard flow S_t^ϵ on $\Upsilon_\epsilon \times \mathbf{S}^{D-1}$, we define a unitary group $\hat{S}_t^\epsilon$ on $L^2(\Upsilon_\epsilon \times \mathbf{S}^{D-1}, dxdv)$ by the formula

$$(4.5) \qquad \hat{S}_t^\epsilon f(x,v) = f(S_{-t}^\epsilon(x,v)), \quad (x,v) \in \Upsilon_\epsilon \times \mathbf{S}^{D-1}, \; t \in \mathbf{R}.$$

Next we turn to the case of the same billiard table, but with absorbing obstacles. By this, we mean that whenever a particle hits the boundary of Ω_ϵ, it disappears. Equivalently, one may think of Ω_ϵ as a sieve, with particles falling into the holes (the components of Ω_ϵ^c).

The analogue of $\hat{S}_t^\epsilon$ in the case of absorbing obstacles is the contraction semigroup T_t^ϵ defined on $L^2(\Upsilon_\epsilon \times \mathbf{S}^{D-1}, dxdv)$ by the formula

$$(4.6)$$
$$T_t^\epsilon g(x,v) = g(x-tv,v)\mathbf{1}_{\tau_r(x/\epsilon,-v)>t/\epsilon}, \quad (x,v) \in \Upsilon_\epsilon \times \mathbf{S}^{D-1}, \; t \in \mathbf{R},$$

where τ_r is the free path length defined in (2.1) — recall that $r_* \epsilon^{1/(D-1)}$.

Finally, $L^2(\Upsilon_\epsilon \times \mathbf{S}^{D-1}, dxdv)$ can be embedded as a subspace of $L^2(\mathbf{T}^D \times \mathbf{S}^{D-1}, dxdv)$, identifying each function defined a.e. on $\Upsilon_\epsilon \times \mathbf{S}^{D-1}$ with its extension by 0 in the complement of $\Upsilon_\epsilon \times \mathbf{S}^{D-1}$.

The maps $\hat{S}_t^\epsilon$ and T_t^ϵ are related by the following elementary inequality: for each $f \in L^2(\Upsilon_\epsilon \times \mathbf{S}^{D-1}, dxdv)$ and each $t \geq 0$,

$$(4.7) \qquad f \geq 0 \text{ a.e. implies that } \hat{S}_t^\epsilon f(x,v) \geq T_t^\epsilon f(x,v) \text{ a.e.}$$

on $\Upsilon_\epsilon \times \mathbf{S}^{D-1}$. Indeed, the formulas (4.6), (4.5), and (4.2) imply that

$$T_t^\epsilon f(x,v) = f(x-tv,v)\mathbf{1}_{\tau_r(x/\epsilon,-v)>t/\epsilon} = f(S_{-t}^\epsilon(x,v))\mathbf{1}_{\tau_r(x/\epsilon,-v)>t/\epsilon}$$
$$\leq f(S_{-t}^\epsilon(x,v)) = \hat{S}_t^\epsilon f(x,v).$$

Finally, we give the PDE interpretation of the operators $\hat{S}_t^\epsilon$ and T_t^ϵ. The function $F_\epsilon(t,x,v) = \hat{S}_t^\epsilon f(x,v)$ is the solution of

$$\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = 0, \qquad\qquad (x,v) \in \Upsilon_\epsilon \times \mathbf{S}^{D-1},$$
$$F_\epsilon(t,x,v) = F_\epsilon(t,x,\mathcal{R}(n_x)v), \qquad (x,v) \in \partial\Upsilon_\epsilon \times \mathbf{S}^{D-1},$$
$$F_\epsilon\big|_{t=0} = f,$$

(where $\mathcal{R}(n_x)$ designates the reflection with respect to the hyperplane orthogonal to n_x: see fla. (4.4)), while the function $G_\epsilon(t,x,v) =$

$\hat{T}_t^\epsilon g(x, v)$ is the solution of

$$\partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon = 0 \,, \qquad (x, v) \in \Upsilon_\epsilon \times \mathbf{S}^{D-1} \,,$$
$$G_\epsilon(t, x, v) = 0 \,, \qquad x \in \partial\Upsilon_\epsilon \,, \ v \cdot n_x > 0 \,,$$
$$G_\epsilon\big|_{t=0} = g \,.$$

4.2. **Main result.** The paper by Lorentz [15] described in the introduction suggests the following question, in the case of space dimension $D = 3$:

"Let $f^{in} \in L^2(\mathbf{T}^3 \times \mathbf{S}^2)$. Does $\hat{S}_t^\epsilon(f^{in}\mathbf{1}_{\Upsilon_\epsilon \times \mathbf{S}^2})$ (or any subsequence thereof) converge in $L^\infty((0, +\infty) \times \mathbf{T}^3 \times \mathbf{S}^2)$ weak-* as $\epsilon \to 0$ to the solution f of (1.1) on $\mathbf{T}^3 \times \mathbf{S}^2$ with $N_{at} = 1$, $r_{at} = r_*$, and with initial data f^{in}?"

The answer to that question is negative, as shown by the following theorem.

Theorem 4.1. *Assume that the space dimension satisfies $D \geq 2$. There exist initial data $f^{in} \in L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})$ such that, for any $\sigma > 0$ and any function $k \in C(\mathbf{S}^{D-1} \times \mathbf{S}^{D-1})$ satisfying (3.3), no subsequence of $\hat{S}_t^\epsilon(f^{in}\mathbf{1}_{\Upsilon_\epsilon \times \mathbf{S}^{D-1}})$ converges in $L^\infty((0, +\infty) \times \mathbf{T}^D \times \mathbf{S}^{D-1})$ weak-* to the solution of*

$$(4.8) \quad (\partial_t + v \cdot \nabla_x) f(t, x, v) = \sigma \int_{\mathbf{S}^{D-1}} k(v, w) \left(f(t, x, w) - f(t, x, v) \right) dw$$

on $\mathbf{T}^D \times \mathbf{S}^{D-1}$, with initial data

$$(4.9) \qquad\qquad f\big|_{t=0} = f^{in} \,.$$

4.3. **Proof of Theorem 4.1.** We define

$$\Phi_\epsilon(t, x, v) = \mathbf{1}_{\epsilon\tau_r(x/\epsilon, -v) > t} \,.$$

Since $0 \leq \Phi_\epsilon \leq 1$, the sequence Φ_ϵ is relatively weakly-* compact in $L^\infty(\mathbf{R}_+ \times \mathbf{T}^D \times \mathbf{S}^{D-1})$. From now on, we consider a subsequence of $\epsilon = 1/n$, denoted ϵ', so that $\Phi_{\epsilon'} \to \Phi$ in $L^\infty(\mathbf{R}_+ \times \mathbf{T}^D \times \mathbf{S}^{D-1})$ weak-*, and denote $r' = r_*\epsilon'^{1/D-1}$. Then, the function Φ is independent of x, i.e.

$$(4.10) \qquad\qquad \Phi \equiv \Phi(t, v) \,,$$

as shown by the following classical lemma.

Lemma 4.2. *Let Z be a separable locally compact metric space, endowed with a Borel measure m. Let $u_n(x, z) \equiv U_n(nx, z)$, where $(U_n)_{n \geq 1}$ is a bounded sequence of elements of $L^\infty(\mathbf{T}^D \times Z)$. Any weak-* limit point of the sequence $(u_n)_{n \geq 1}$ as $n \to +\infty$ is a function of z alone (i.e. independent of x).*

We postpone the proof of Lemma 4.2 until the end of this section.

In the sequel, we consider an arbitrary initial data $\rho \equiv \rho(x) \in L^\infty(\mathbf{T}^D)$ that is independent of v and a.e. nonnegative. We define

$$(4.11) \qquad f_\epsilon(t, x, v) = \hat{S}_t^\epsilon(\rho \mathbf{1}_{\Upsilon_\epsilon \times \mathbf{S}^{D-1}}) = \rho(X_{-t}^\epsilon(x, v))\mathbf{1}_{\Upsilon_\epsilon}(x)$$

and

$$(4.12) \qquad g_\epsilon(t, x, v) = T_t^\epsilon(\rho \mathbf{1}_{\Upsilon_\epsilon \times \mathbf{S}^{D-1}}) = \rho(x - tv)\Phi_\epsilon(t, x, v)\mathbf{1}_{\Upsilon_\epsilon}(x) \,.$$

Since $\mathbf{1}_{\Upsilon_\epsilon} \to 1$ a.e. on $\mathbf{T}^D \times \mathbf{S}^{D-1}$ and $|\mathbf{1}_{\Upsilon_\epsilon}| \leq 1$, one has, by dominated convergence,

$$(4.13) \qquad g_{\epsilon'} \to \rho(x - tv)\Phi(t, v) \text{ in } L^\infty(\mathbf{R}_+ \times \mathbf{T}^D \times \mathbf{S}^{D-1}) \text{ weak-*} \,.$$

On the other hand, (4.11) shows that $\|f_\epsilon\|_{L^\infty} \leq \|\rho\|_{L^\infty}$; therefore, possibly after extraction of a subsequence (still denoted ϵ'), one has $f_{\epsilon'} \to f$ in $L^\infty(\mathbf{R}_+ \times \mathbf{T}^D \times \mathbf{S}^{D-1})$ weak-*. We recall that $\rho \geq 0$ a.e. on $\mathbf{T}^D$. Therefore, because of the inequality (4.7) and of the weak-* limit (4.13), one has

$$(4.14) \qquad f(t, x, v) \geq \rho(x - tv)\Phi(t, v) \,.$$

In particular,
(4.15)
$$\iint_{\mathbf{T}^D \times \mathbf{S}^{D-1}} f(t, x, v)^2 dx dv \geq \iint_{\mathbf{T}^D \times \mathbf{S}^{D-1}} \rho(x - tv)^2 \Phi(t, v)^2 dx dv$$

$$= \int_{\mathbf{T}^D} \rho(y)^2 dy \int_{\mathbf{S}^{D-1}} \Phi(t, v)^2 dv$$

$$\geq \|\rho\|_{L^2}^2 \left(\int_{\mathbf{S}^{D-1}} \Phi(t, v) dv \right)^2 \,,$$

where the last inequality follows from Jensen's inequality. By assumption, $\Phi_{\epsilon'} \to \Phi$ in $L^\infty(\mathbf{R}_+ \times \mathbf{T}^D \times \mathbf{S}^{D-1})$ weak-*; because of (4.10), one has

$$\iint_{\mathbf{T}^D \times \mathbf{S}^{D-1}} \Phi_{\epsilon'}(t, x, v) dx dv \to \int_{\mathbf{S}^{D-1}} \Phi(t, v) dv \text{ in } L^\infty(\mathbf{R}_+) \text{ weak-*}.$$

Furthermore, by Theorem 2.1,

$$\iint_{\mathbf{T}^D \times \mathbf{S}^{D-1}} \Phi_{\epsilon'}(t, x, v) dx dv$$

$$= \operatorname{meas}(\{(x, v) \in \Upsilon_{\epsilon'} \times \mathbf{S}^{D-1} \,|\, \tau_{r'}(x/\epsilon', -v) > t/\epsilon'\})$$

$$= \operatorname{meas}(\{(y, v) \in Y_{r'} \times \mathbf{S}^{D-1} \,|\, \tau_{r'}(y, -v) > t/\epsilon'\})$$

$$\geq \frac{C_1}{\frac{t}{\epsilon'} r'^{D-1}} = \frac{C_1}{t r_*^{D-1}}, \qquad \text{for all } t > 1/r_*^{D-1} \,.$$

Hence, the inequality (4.15) becomes

(4.16)
$$\left(\iint_{\mathbf{T}^D \times \mathbf{S}^{D-1}} f(t,x,v)^2 dx dv\right)^{1/2} \geq \frac{C_1}{tr_*^{D-1}}\|\rho\|_{L^2}, \quad \text{for all } t > 1/r_*^{D-1}.$$

Assume that f is the solution to (4.8)-(4.9) with $f^{in} = \rho$. By Theorem 3.1, one has

$$\left\| f(t,\cdot,\cdot) - \int_{\mathbf{T}^D} \rho(x)dx \right\|_{L^2(\mathbf{T}^D \times \mathbf{S}^{D-1})} \leq ce^{-\gamma t}\|\rho\|_{L^2}$$

for all $t \geq 0$. In particular, for all $t \geq 0$, one has

(4.17) $$\left(\iint_{\mathbf{T}^D \times \mathbf{S}^{D-1}} f(t,x,v)^2 dx dv\right)^{1/2} \leq \int_{\mathbf{T}^D} \rho(x)dx + ce^{-\gamma t}\|\rho\|_{L^2}.$$

In conclusion, if f is the solution to (4.8)-(4.9), then the initial data ρ (assuming it is not a.e. 0) must satisfy the inequality

(4.18) $$\frac{C_1}{tr_*^{D-1}} \leq \frac{\|\rho\|_{L^1}}{\|\rho\|_{L^2}} + ce^{-\gamma t}, \quad \text{for all } t > 1/r_*^{D-1}.$$

At this point, we recall that this inequality holds for any arbitrary $\rho \in L^\infty(\mathbf{T}^D)$ such that $\rho \geq 0$ a.e.: hence the ratio $\|\rho\|_{L^1}/\|\rho\|_{L^2}$ can be made arbitrarily small. For instance, one can choose ρ as follows: pick b, a bump function on $\mathbf{R}^D$ satisfying

$$0 \leq b \leq 1, \quad \text{supp}(b) \subset [-\tfrac{1}{4}, \tfrac{1}{4}]^D.$$

For $m \in \mathbf{N}^*$, define ρ to be the unique $\mathbf{Z}^D$-periodic function such that

$$\rho\big|_{[-\frac{1}{2}, \frac{1}{2})^D} : x \mapsto b(mx).$$

Then

$$\|\rho\|_{L^1(\mathbf{T}^D)} = m^{-D}\|b\|_{L^1(\mathbf{R}^D)}, \quad \|\rho\|_{L^2(\mathbf{T}^D)} = m^{-D/2}\|b\|_{L^2(\mathbf{R}^D)},$$

so that

$$\frac{\|\rho\|_{L^1(\mathbf{T}^D)}}{\|\rho\|_{L^2(\mathbf{T}^D)}} = m^{-D/2} \to 0 \text{ as } m \to +\infty.$$

Since the ratio $\|\rho\|_{L^1}/\|\rho\|_{L^2}$ can be made arbitrarily small, the inequality (4.18) would entail

$$\frac{C_1}{tr_*^{D-1}} \leq ce^{-\gamma t}, \quad \text{for all } t > 1/r_*^{D-1},$$

which is manifestly wrong for t large enough — specifically, larger than the unique zero of the function $t \mapsto C_1 e^{\gamma t} - cr_*^{D-1}t$, denoted by $t_*(C_1, c, \gamma, r_*)$.

Hence, the assumption that, for each nonnegative $\rho \equiv \rho(x) \in L^\infty(\mathbf{T}^D)$, there exists a subsequence of $\hat{S}_t^\epsilon(\rho \mathbf{1}_{\Upsilon_\epsilon \times \mathbf{S}^{D-1}})$ converging in L^∞ weak-* to the solution of (4.8)-(4.9) with $f^{in} = \rho$ is wrong. This concludes the proof of Theorem 4.1, once Lemma 4.2 is proved.

4.4. **Proof of Lemma 4.2.** Assume that

$$u_{n_q} \to u \text{ in } L^\infty(\mathbf{T}^D \times Z) \text{ weak-* as } n_q \to +\infty\,.$$

For each $k \in \mathbf{Z}^D$, one has

$$\hat{u}_{n_q}(k, z) = \int_{\mathbf{T}^D} e^{-i2\pi k \cdot x} u_{n_k}(x, z) dx = \hat{U}_{n_q}(k, z) \quad \text{if } n_q | k\,,$$
$$= 0 \qquad \text{otherwise.}$$

Since the assumed convergence entails

$$\hat{u}_{n_q}(k, \cdot) \to \hat{u}(k, \cdot) \text{ in } L^\infty(Z) \text{ weak-* for each } k \in \mathbf{Z}^D\,,$$

this shows that

$$\hat{u}(k, \cdot) = 0 \text{ unless } k = 0\,.$$

Hence u is independent of x, as announced.

5. FINAL REMARKS

We have demonstrated the impossibility of representing the Boltzmann-Grad limit of the periodic Lorentz gas by a linear Boltzmann equation.

Notice that this impossibility results solely from the lower bound on the distribution of free path lengths in Theorem 2.1. This is in fact not too surprising since the probabilistic representation of equation (1.1) involves in particular exponentially distributed jump times for the velocity process.

Notice also the choice of initial data in Theorem 4.1. Since one knows that particles moving in appropriately chosen rational directions may not encounter any obstacles on the billiard table Z_r, it may seem somewhat surprising that the contradiction in Theorem 4.1 is obtained by considering isotropic initial data (i.e. initial data that are independent of the angle variable) instead of pencils of particles concentrated in phase-space on those rational directions that avoid all obstacles. In fact, the contribution of such rational directions is already taken into account in the lower bound in Theorem 2.1. Besides, the case of isotropic initial densities is somewhat more natural in the context of Theorem 4.1, as it corresponds to local equilibria for the Lorentz kinetic model (1.1).

As for the hydrodynamic (diffusion) limit of the periodic Lorentz gas with finite horizon, it was proved in [1] that the case of an isotropic reflection law at the surface of each obstacle can be treated by PDE techniques, avoiding the heavy machinery from ergodic theory required to handle the case of specular reflection and developed by Bunimovich-Sinai — see [7], and also [6].

In the case of the Boltzmann-Grad limit however, changing the collision process does not affect the result in Theorem 4.1, since the obstruction to using the linear Boltzmann equation comes from particles that travel too far before encountering an obstacle for the first time. For such particles, the nature of the collision process is obviously of no importance. Hence Theorem 4.1 holds verbatim if one replaces $\hat{S}_t^\epsilon(f^{in}\mathbf{1}_{\Upsilon_\epsilon \times \mathbf{S}^{D-1}})$ with the solution of

$$\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = 0 \,, \quad x \in \Upsilon_\epsilon \,,\ v \in \mathbf{S}^{D-1} \,,$$

$$f_\epsilon(t)\big|_{\Sigma_+^\epsilon} = \mathcal{K}(f_\epsilon(t)\big|_{\Sigma_-^\epsilon}) \,,$$

$$f_\epsilon\big|_{t=0} = f^{in}\mathbf{1}_{\Upsilon_\epsilon \times \mathbf{S}^{D-1}} \,,$$

where

$$\Sigma_\pm^\epsilon = \{(x,v) \in \partial\Upsilon_\epsilon \times \mathbf{S}^{D-1} \,|\, \pm v \cdot n_x > 0\}$$

and $\mathcal{K}$ is any linear operator from $L^\infty(\Sigma_-^\epsilon)$ to $L^\infty(\Sigma_+^\epsilon)$ that preserves the cone of positive functions.

By the same token, the same result as in Theorem 4.1 holds without change if the obstacles are not assumed to be spherical, or even identical, but instead of arbitrary shapes, provided that they can be included in balls of radius $r = r_*\epsilon^{D/(D-1)}$ centered at the points of $\epsilon\mathbf{Z}^D$.

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