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Asymptotic expansions at any time for scalar fractional SDEs of Hurst index $H > 1/2$

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Abstract

We study the asymptotic expansions with respect to h of

$$E[\Delta_h f(X_t)], \quad E[\Delta_h f(X_t)|\mathcal{F}_t^X] \quad \text{and} \quad E[\Delta_h f(X_t)|X_t],$$

where $\Delta_h f(X_t) = f(X_{t+h}) - f(X_t)$, when $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth real function, $t \geq 0$ is a fixed time, X is the solution of a one-dimensional stochastic differential equation driven by a fractional Brownian motion of Hurst index $H > 1/2$ and \mathcal{F}^X is its natural filtration.

Key words: Asymptotic expansion - Fractional Brownian motion - stochastic differential equation - Malliavin calculus.

2000 Mathematics Subject Classification: Primary 60G15, 60G17; Secondary 60F15, 60H07.

1 Introduction

We study the asymptotic expansions with respect to h of

$$\begin{aligned} P_t f(h) &\triangleq E[\Delta_h f(X_t)], \\ \widehat{P}_t f(h) &\triangleq E[\Delta_h f(X_t) | \mathcal{F}_t^X] \\ \widetilde{P}_t f(h) &\triangleq E[\Delta_h f(X_t) | X_t], \end{aligned} \quad (1)$$

with $\Delta_h f(X_t) \triangleq f(X_{t+h}) - f(X_t)$, when $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth real function, $t \geq 0$ is a fixed time, X is the solution to the following fractional stochastic differential equation:

$$X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s, \quad t \in [0, T], \quad (2)$$

and \mathcal{F}^X is its natural filtration. Here, $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are real functions belonging to the space C_b^∞ of all bounded continuous functions having bounded derivatives of all order, while B is a one-dimensional fractional Brownian motion of Hurst index $H \in (1/2, 1)$. When the integral with respect to B is understood in the Young sense, Eq. (2) has a unique pathwise solution X in the set of processes whose paths are Hölder continuous of index $H \in (1 - H, H)$. Moreover, *e.g.* by Theorem 4.3 in [14], we have, for any $g : \mathbb{R} \rightarrow \mathbb{R} \in C_b^\infty$:

$$g(X_t) = g(x) + \int_0^t g'(X_s) \sigma(X_s) dB_s + \int_0^t g'(X_s) b(X_s) ds, \quad t \in [0, T]. \quad (3)$$

The asymptotic expansion of $E[f(X_h)]$ with respect to h has been recently studied in [1, 9]. It corresponds, in our framework, to the case where $t = 0$, since we have obviously

$$E[f(X_h) - f(x)] = P_0 f(h) = \widehat{P}_0 f(h) = \widetilde{P}_0 f(h).$$

In these latter references, the authors work in a multidimensional setting and under the weaker assumption that the Hurst index H of the fractional Brownian motion B is bigger than $1/3$ (the integral with respect to B is then understood in the rough paths sense of Lyons' type for [1] and of Gubinelli's type for [9]). In particular, it is proved in [1, 9] that there exists a family

$$\Gamma = \{\Gamma_{2kH+\ell} : (k, \ell) \in \mathbb{N}^2, (k, \ell) \neq (0, 0)\}$$

of differential operators such that, for any smooth $f : \mathbb{R} \rightarrow \mathbb{R}$, we have the following asymptotic expansion:

$$P_0 f(h) \underset{h \rightarrow 0}{\sim} \sum h^{2kH+\ell} \Gamma_{2kH+\ell}(f, \sigma, b)(x). \quad (4)$$

Moreover, in [9], operators $\Gamma_{2kH+\ell}$ are expressed using trees.

Now, a natural question arises. Can we also obtain an expansion of $P_t f(h)$ when $t \neq 0$? Let us first consider the case where B is the standard Brownian motion (it corresponds to the case where $H = 1/2$). By the Markov property, on one hand, we have $\widehat{P}_t f(h) = \widetilde{P}_t f(h)$ and, on the other hand, we always have $P_t f(h) = E[\widehat{P}_t f(h)]$. Thus, there exist relations between $P_t f(h)$, $\widehat{P}_t f(h)$ and $\widetilde{P}_t f(h)$. Moreover, the asymptotic expansion of $P_t f(h)$ can be obtained as a corollary of that of $P_0 f(h)$, by using the conditional expectation either with respect to the past \mathcal{F}_t^X of X , or with respect to X_t alone, and the strong Markov property.

When $H > 1/2$, B is not Markovian. The situation concerning $P_t f(h)$, $\widehat{P}_t f(h)$ and $\widetilde{P}_t f(h)$ is then completely different and actually more complicated. In particular, we do not have $\widehat{P}_t f(h) = \widetilde{P}_t f(h)$ anymore and we cannot deduce the asymptotic expansion of $P_t f(h)$ from that of $P_0 f(h)$.

The current paper is concerned with the study of possible asymptotic expansions of the various quantities $P_t f(h)$, $\widehat{P}_t f(h)$ and $\widetilde{P}_t f(h)$ when $H > 1/2$. We will see that some non-trivial phenomena appear. More precisely, we will show in Section 3 that $\widehat{P}_t f(h)$ does not admit an asymptotic expansion in the scale of the fractional powers of h when $t \neq 0$. So, concerning $\widehat{P}_t f(h)$, the situations when $t = 0$ and $t > 0$ are really different. On the other hand, unlike $\widehat{P}_t f(h)$, the quantities $P_t f(h)$ and $\widetilde{P}_t f(h)$ admit, when $t \neq 0$, an asymptotic expansion in the scale of the fractional powers of h . However the computation of this expansion is more difficult than in the case where $t = 0$ (as made in [1, 9]). That is why we preferred only consider the one-dimensional case. Indeed, as an illustration, let us consider the trivial equation $dX_t = dB_t$, $t \in [0, T]$, $X_0 = 0$. That is $X_t = B_t$ for every $t \in [0, T]$. We have, by a Taylor expansion:

$$\widetilde{P}_0 f(h) = \sum_{k=1}^n \frac{f^{(k)}(0)}{k!} E[(B_h)^k] + \dots = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{f^{(2k)}(0)}{2^k k!} h^{2Hk} + \dots,$$

while, by a linear Gaussian regression, when $t \neq 0$:

$$\begin{aligned} \widetilde{P}_t f(h) &= E \left[f \left(\left(1 + \frac{H}{t} h - \frac{h^{2H}}{2t^{2H}} + \dots \right) B_t + (h^{2H} + \dots) N \right) - f(B_t) \middle| B_t \right] \\ &= \frac{HB_t f'(B_t)}{t} h - \frac{B_t f'(B_t)}{2t^{2H}} h^{2H} + \dots, \end{aligned}$$

with $N \sim \mathcal{N}(0, 1)$ a random variable independent of B_t .

One of the key points of our strategy relies on the use of a Girsanov transformation and the Malliavin calculus for fractional Brownian motion. We refer to [4, 12] for a deep insight of this topic.

We will restrict the exposition of our asymptotic expansions for the case when $\sigma = 1$. The reason is that, under the following assumption:

- (A) The function σ is elliptic on \mathbb{R} , that is it verifies $\inf_{\mathbb{R}} |\sigma| > 0$,

Eq. (2) can be reduced, using the change of variable formula (3), to a diffusion Y with a constant diffusion coefficient:

$$Y_t = \int_0^{X_t} \frac{dz}{\sigma(z)}.$$

Moreover, since $\int_0^{\cdot} \frac{dz}{\sigma(z)}$ is strictly monotonous from \mathbb{R} to \mathbb{R} under assumption (A), the σ -fields generated by X_t (resp. by X_s , $s \leq t$) and Y_t (resp. by Y_s , $s \leq t$) are the same. Consequently, to assume that $\sigma = 1$ is not at all restrictive since it allows to recover the general case under assumption (A). Consequently, we consider in the sequel that X is the unique solution of

$$X_t = x + \int_0^t b(X_s) ds + B_t, \quad t \in [0, T], \quad (5)$$

with $b \in C_b^\infty$ and $x \in \mathbb{R}$.

The paper is organized as follows. In Section 2 we recall some basic facts about fractional Brownian motion, the Malliavin calculus and fractional stochastic differential equations. In Section 3 we prove that $\widehat{P}_t f(h)$ does not admit an asymptotic expansion with respect to the scale of fractional powers of h , up to order $n \in \mathbb{N}$. Finally, we show in Section 4 that $\widetilde{P}_t f(h)$ admits an asymptotic expansion.

2 Preliminaries

We begin by briefly recalling some basic facts about stochastic calculus with respect to a fractional Brownian motion. One refers to [11, 12] for further details. Let $B = (B_t)_{t \in [0, T]}$ be a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. We mean that B is a centered Gaussian process with the covariance function $E(B_s B_t) = R_H(s, t)$, where

$$R_H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}). \quad (6)$$

We denote by \mathcal{E} the set of step \mathbb{R} -valued functions on $[0, T]$. Let \mathfrak{H} be the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\langle \mathbf{1}_{[0, t]}, \mathbf{1}_{[0, s]} \rangle_{\mathfrak{H}} = R_H(t, s).$$

We denote by $|\cdot|_{\mathfrak{H}}$ the associate norm. The mapping $\mathbf{1}_{[0, t]} \mapsto B_t$ can be extended to an isometry between \mathfrak{H} and the Gaussian space $\mathcal{H}_1(B)$ associated with B . We denote this isometry by $\varphi \mapsto B(\varphi)$.

The covariance kernel $R_H(t, s)$ introduced in (6) can be written as

$$R_H(t, s) = \int_0^{s \wedge t} K_H(s, u) K_H(t, u) du,$$

where $K_H(t, s)$ is the square integrable kernel defined, for $s < t$, by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad (7)$$

with $c_H^2 = \frac{H(2H-1)}{\beta(2-2H, H-1/2)}$ and β the Beta function. By convention, we set $K_H(t, s) = 0$ if $s \geq t$.

We define the operator \mathcal{K}_H on $L^2([0, T])$ by

$$(\mathcal{K}_H h)(t) = \int_0^t K_H(t, s) h(s) ds.$$

Let $\mathcal{K}_H^* : \mathcal{E} \rightarrow L^2([0, T])$ be the linear operator defined by:

$$\mathcal{K}_H^* (\mathbf{1}_{[0, t]}) = K_H(t, \cdot).$$

The following equality holds for any $\phi, \psi \in \mathcal{E}$:

$$\langle \phi, \psi \rangle_{\mathfrak{H}} = \langle \mathcal{K}_H^* \phi, \mathcal{K}_H^* \psi \rangle_{L^2([0, T])} = \mathbb{E} (B(\phi)B(\psi)).$$

Then, \mathcal{K}_H^* provides an isometry between the Hilbert space \mathfrak{H} and a closed subspace of $L^2([0, T])$.

The process $W = (W_t)_{t \in [0, T]}$ defined by

$$W_t = B((\mathcal{K}_H^*)^{-1}(\mathbf{1}_{[0, t]})) \quad (8)$$

is a Wiener process, and the process B has the following integral representation:

$$B_t = \int_0^t K_H(t, s) dW_s.$$

Hence, for any $\phi \in \mathfrak{H}$,

$$B(\phi) = W(\mathcal{K}_H^* \phi).$$

If $b, \sigma \in C_b^\infty$, then (2) admits a unique solution X in the set of processes whose paths are Hölder continuous of index $\alpha \in (1-H, H)$. Moreover, see *e.g.* [7], X has the following Doss-Sussman's type representation:

$$X_t = \phi(A_t, B_t), \quad t \in [0, T], \quad (9)$$

with ϕ and A given respectively by

$$\frac{\partial \phi}{\partial x_2}(x_1, x_2) = \sigma(\phi(x_1, x_2)), \quad \phi(x_1, 0) = x_1, \quad x_1, x_2 \in \mathbb{R}$$

and

$$A'_t = \exp \left(- \int_0^{B_t} \sigma'(\phi(A_t, s)) ds \right) b(\phi(A_t, B_t)), \quad A_0 = x_0, \quad t \in [0, T].$$

Let $b \in C_b^\infty$ and X be the solution of (5). Following [13], the fractional version of the Girsanov theorem applies and ensures that X is a fractional Brownian motion of Hurst parameter H , under the new probability \mathbf{Q} defined by $d\mathbf{Q} = \eta^{-1}d\mathbf{P}$, where

$$\eta = \exp \left(\int_0^T (\mathcal{K}_H^{-1} \int_0^\cdot b(X_r)dr)(s)dW_s + \frac{1}{2} \int_0^T (\mathcal{K}_H^{-1} \int_0^\cdot b(X_r)dr)^2(s)ds \right). \quad (10)$$

Let \mathcal{S} be the set of all smooth cylindrical random variables, *i.e.* of the form $F = f(B(\phi_1), \dots, B(\phi_n))$ where $n \geq 1$, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function with compact support and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to B is the element of $L^2(\Omega, \mathfrak{H})$ defined by

$$D_s^B F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(\phi_1), \dots, B(\phi_n)) \phi_i(s), \quad s \in [0, T].$$

In particular $D_s^B B_t = \mathbf{1}_{[0,t]}(s)$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 = \mathbb{E}[F^2] + \mathbb{E}[|D.F|_{\mathfrak{H}}^2].$$

The Malliavin derivative D verifies the chain rule: if $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is C_b^1 and if $(F_i)_{i=1, \dots, n}$ is a sequence of elements of $\mathbb{D}^{1,2}$ then $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$ and we have, for any $s \in [0, T]$:

$$D_s \varphi(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) D_s F_i.$$

The divergence operator δ is the adjoint of the derivative operator D . If a random variable $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of the divergence operator, that is if it verifies

$$|\mathbb{E}\langle DF, u \rangle_{\mathfrak{H}}| \leq c_u \|F\|_{L^2} \quad \text{for any } F \in \mathcal{S},$$

then $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}\langle DF, u \rangle_{\mathfrak{H}},$$

for every $F \in \mathbb{D}^{1,2}$.

3 Study of the asymptotic expansion of $\widehat{P}_t f(h)$

Recall that $\widehat{P}_t f(h)$ is defined by (1), with X given by (5).

Definition 1 We say that $\widehat{P}_t f(h)$ admits an asymptotic expansion with respect to the scale of fractional powers of h , up to order $n \in \mathbb{N}$, if there exist some real numbers $0 < \alpha_1 < \dots < \alpha_n$ and some nontrivial random variables $C_1, \dots, C_n \in L^2(\Omega, \mathcal{F}_t^X)$ such that

$$\widehat{P}_t f(h) = C_1 h^{\alpha_1} + \dots + C_n h^{\alpha_n} + o(h^{\alpha_n}), \quad \text{as } h \rightarrow 0,$$

where $o(h^\alpha)$ stands for a random variable of the form $h^\alpha \phi_h$, with $E[\phi_h^2] \rightarrow 0$ as $h \rightarrow 0$.

If $\widehat{P}_t f(h)$ admits an asymptotic expansion in the sense of Definition 1 we must have, in particular, the existence of $\alpha > 0$ verifying

$$\lim_{h \rightarrow 0} h^{-\alpha} \widehat{P}_t f(h) \text{ exists in } L^2(\Omega) \text{ and is not identically zero.}$$

But, we have:

Theorem 1 Let $f : \mathbb{R} \rightarrow \mathbb{R} \in C_b^\infty$ and $t \in (0, T]$. Assume moreover that

$$\text{Leb}(\{x \in \mathbb{R} : f'(x) = 0\}) = 0. \quad (11)$$

Then, as $h \rightarrow 0$, $h^{-\alpha} \widehat{P}_t f(h)$ converges in $L^2(\Omega)$ if and only if $\alpha < H$. In this case, the limit is zero.

Remark 1 Since $\widehat{P}_0 f(h) = \widetilde{P}_0 f(h)$, we refer to Theorem 2 for the case where $t = 0$.

Proof. The proof is divided into two cases:

i) First case: $\alpha \in (0, 1]$. Since $H > 1/2$, let us first remark that $h^{-\alpha} \widehat{P}_t f(h)$ converges in $L^2(\Omega)$ if and only if $h^{-\alpha} f'(X_t) E[X_{t+h} - X_t | \mathcal{F}_t^X]$ converges in $L^2(\Omega)$. Indeed, we use a Taylor expansion:

$$|f(X_{t+h}) - f(X_t) - f'(X_t)(X_{t+h} - X_t)| \leq \frac{1}{2} |f''|_\infty |X_{t+h} - X_t|^2,$$

so that

$$|\widehat{P}_t f(h) - f'(X_t) E[X_{t+h} - X_t | \mathcal{F}_t^X]| \leq \frac{1}{2} |f''|_\infty E[|X_{t+h} - X_t|^2 | \mathcal{F}_t^X].$$

Thus, applying in particular Jensen formula:

$$\begin{aligned} & h^{-2\alpha} E[|\widehat{P}_t f(h) - f'(X_t) E[X_{t+h} - X_t | \mathcal{F}_t^X]|^2] \\ & \leq \frac{1}{4} |f''|_\infty^2 h^{-2\alpha} E[E[|X_{t+h} - X_t|^2 | \mathcal{F}_t^X]]^2 \\ & \leq \frac{1}{4} |f''|_\infty^2 h^{-2\alpha} E[|X_{t+h} - X_t|^4] = O(h^{4H-2\alpha}). \end{aligned}$$

Since $\alpha \leq 1 < 2H$, we can conclude.

By (11) and the fact that X_t has a positive density on \mathbb{R} (see *e.g.* [10], Theorem A), we have that $h^{-\alpha} f'(X_t) E[X_{t+h} - X_t | \mathcal{F}_t^X]$ converges in $L^2(\Omega)$ if and only if $h^{-\alpha} E[X_{t+h} - X_t | \mathcal{F}_t^X]$ converges in $L^2(\Omega)$.

For X given by (5), we have that $\mathcal{F}^X = \mathcal{F}^B$. Indeed, one inclusion is obvious, while the other one can be proved using (9). Moreover, since $b \in C_b^\infty$ and $\alpha < 1$, the term $h^{-\alpha} E[\int_t^{t+h} b(X_s) ds | \mathcal{F}_t^X]$ converges when $h \downarrow 0$. Therefore, we have due to (5) that $h^{-\alpha} E[X_{t+h} - X_t | \mathcal{F}_t^X]$ converges in $L^2(\Omega)$ if and only if $h^{-\alpha} E[B_{t+h} - B_t | \mathcal{F}_t^B]$ converges in $L^2(\Omega)$.

Set

$$Z_h^{(t)} = h^{-\alpha} E[B_{t+h} - B_t | \mathcal{F}_t^B] = h^{-\alpha} \int_0^t (K_H(t+h, s) - K_H(t, s)) dW_s,$$

where the kernel K_H is given by (7) and the Wiener process W is defined by (8). We have

$$\begin{aligned} \text{Var}(Z_h^{(t)}) &= h^{-2\alpha} \int_0^t (K_H(t+h, s) - K_H(t, s))^2 ds \\ &= h^{-2\alpha} c_H^2 \int_0^t s^{1-2H} \left(\int_t^{t+h} (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du \right)^2 ds. \end{aligned}$$

We deduce

$$\begin{aligned} \text{Var}(Z_h^{(t)}) &\geq h^{-2\alpha} \left(\frac{c_H}{H - \frac{1}{2}} \right)^2 t^{2H-1} \int_0^t s^{1-2H} \left((t+h-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}} \right)^2 ds \\ &= h^{-2\alpha} \left(\frac{c_H}{H - \frac{1}{2}} \right)^2 \int_0^t \left(1 - \frac{s}{t} \right)^{1-2H} \left((s+h)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds \\ &= h^{2(H-\alpha)} \left(\frac{c_H}{H - \frac{1}{2}} \right)^2 \int_0^{\frac{t}{h}} \left(1 - \frac{hs}{t} \right)^{1-2H} g^2(s) ds, \end{aligned} \quad (12)$$

with $g(s) = (s+1)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}$. Similarly,

$$\begin{aligned} \text{Var}(Z_h^{(t)}) &\leq h^{-2\alpha} \left(\frac{c_H}{H - \frac{1}{2}} \right)^2 (t+h)^{2H-1} \int_0^t s^{1-2H} \left((t+h-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}} \right)^2 ds \\ &= h^{-2\alpha} \left(1 + \frac{h}{t} \right)^{2H-1} \left(\frac{c_H}{H - \frac{1}{2}} \right)^2 \int_0^t \left(1 - \frac{s}{t} \right)^{1-2H} \left((s+h)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds \\ &= h^{2(H-\alpha)} \left(1 + \frac{h}{t} \right)^{2H-1} \left(\frac{c_H}{H - \frac{1}{2}} \right)^2 \int_0^{\frac{t}{h}} \left(1 - \frac{hs}{t} \right)^{1-2H} g^2(s) ds. \end{aligned} \quad (13)$$

Note that $g^2(s) \sim (H - \frac{1}{2})^2 s^{2H-3}$ as $s \rightarrow +\infty$. So $sg^2(s) \rightarrow 0$, as $s \rightarrow +\infty$, and $\int_0^{+\infty} |g^2(s)| ds < +\infty$ since $2H - 3 < -1$. Since $s \mapsto sg^2(s)$ is bounded

on \mathbb{R}^+ , we have by dominated convergence theorem that

$$\int_0^{\frac{t}{h}} \left(\left(1 - \frac{hs}{t}\right)^{1-2H} - 1 \right) g^2(s) ds = \int_0^1 \frac{(1-u)^{1-2H} - 1}{u} g^2\left(\frac{tu}{h}\right) \frac{tu}{h} du$$

tends to zero as $h \rightarrow 0$. Thus:

$$\lim_{h \rightarrow 0} \int_0^{\frac{t}{h}} \left(1 - \frac{hs}{t}\right)^{1-2H} g^2(s) ds = \int_0^\infty g^2(s) ds < +\infty.$$

Now, combined with (12)-(13), we deduce that

$$\text{Var}(Z_h^{(t)}) \sim h^{2(H-\alpha)} \left(\frac{c_H}{H - \frac{1}{2}} \right)^2 \int_0^\infty g^2(s) ds, \quad \text{as } h \rightarrow 0. \quad (14)$$

If $Z_h^{(t)}$ converges in $L^2(\Omega)$ as $h \rightarrow 0$, then $\lim_{h \rightarrow 0} \text{Var}(Z_h^{(t)})$ exists and is finite. But, thanks to (14), we have that $\lim_{h \rightarrow 0} \text{Var}(Z_h^{(t)}) = +\infty$ when $\alpha > H$. Consequently, $Z_h^{(t)}$ does not converge in $L^2(\Omega)$ as $h \rightarrow 0$ when $\alpha > H$.

Conversely, when $\alpha < H$, we have from (14) that $\lim_{h \rightarrow 0} \text{Var}(Z_h^{(t)}) = 0$. Then $Z_h^{(t)} \xrightarrow{L^2} 0$ when $\alpha < H$.

In order to complete the proof of the first case, it remains to consider the case where $\alpha = H$. We first deduce from (14) that $Z_h^{(t)} \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma_H^2)$, as $h \rightarrow 0$, with

$$\sigma_H^2 = \left(\frac{c_H}{H - \frac{1}{2}} \right)^2 \int_0^\infty g^2(s) ds.$$

Let us finally show that the previous limit does not hold in L^2 . Assume for a moment that $Z_h^{(t)}$ converges in $L^2(\Omega)$ as $h \rightarrow 0$. Then, in particular, $\{Z_h^{(t)}\}_{h>0}$ is Cauchy in $L^2(\Omega)$. So, by denoting $Z^{(t)}$ the limit in $L^2(\Omega)$, we have $E[Z_\varepsilon^{(t)} Z_\delta^{(t)}] \rightarrow E[|Z^{(t)}|^2]$ when $\varepsilon, \delta \rightarrow 0$. But, for any fixed $x > 0$, we can show, by using exactly the same transformations as above: as $h \rightarrow 0$,

$$E(Z_{hx}^{(t)} Z_{\frac{h}{x}}^{(t)}) \longrightarrow \left(\frac{c_H}{H - \frac{1}{2}} \right)^2 r(x) = E(|Z^{(t)}|^2),$$

where

$$\begin{aligned} r(x) &= \int_0^\infty \left((s+x)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right) \left((s+\frac{1}{x})^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right) ds \\ &= x \int_0^\infty g(x^2 u) g(u) du. \end{aligned}$$

Consequently, the function r is constant on $]0, +\infty[$. The Cauchy-Schwarz inequality yields:

$$|g|_{L^2}^2 = r(1) = r(\sqrt{2}) = \langle \sqrt{2}g(2\cdot), g \rangle_{L^2} \leq \sqrt{2}|g(2\cdot)|_{L^2}|g|_{L^2} = |g|_{L^2}^2.$$

Thus, we have equality in the previous inequality. We deduce that there exists $\lambda \in \mathbb{R}$ such that $g(2u) = \lambda g(u)$ for all $u \geq 0$. Since $g(0) = 1$ we have $\lambda = 1$. Consequently, for any $u \geq 0$ and any integer n , we obtain

$$g(u) = g\left(\frac{u}{2^n}\right) \xrightarrow{n \rightarrow \infty} g(0) = 1,$$

which is absurd. Therefore, when $\alpha = H$, $Z_h^{(t)}$ does not converge in $L^2(\Omega)$ as $h \rightarrow 0$, which concludes the proof of the first case.

ii) Second case: $\alpha \in (1, +\infty)$. If $h^{-\alpha}\widehat{P}_t f(h)$ converges in $L^2(\Omega)$, then $h^{-1}\widehat{P}_t f(h)$ converges in $L^2(\Omega)$ towards zero. This contradicts the first case, which concludes the proof of Theorem 1. \square

4 Study of the asymptotic expansion of $\widetilde{P}_t f(h)$

Recall that $\widetilde{P}_t f(h)$ is defined by (1), where X is given by (5). The main result of this section is the first point of the following theorem:

Theorem 2 *Let $t \in [0, T]$ and $f : \mathbb{R} \rightarrow \mathbb{R} \in C_b^\infty$. We write \mathcal{N} for $\mathbb{N}^2 \setminus \{(0, 0)\}$. For $(p, q) \in \mathcal{N}$, set*

$$J_{2pH+q} = \{(m, n) \in \mathcal{N} : 2mH + n \leq 2pH + q\}.$$

1. *If $t \neq 0$, there exists a family $\{Z_{2mH+n}^{(t)}\}_{(m,n) \in \mathcal{N}}$ of random variables measurable with respect to X_t such that, for any $(p, q) \in \mathcal{N}$:*

$$\widetilde{P}_t f(h) = \sum_{(m,n) \in J_{2pH+q}} Z_{2mH+n}^{(t)} h^{2mH+n} + o(h^{2pH+q}). \quad (15)$$

2. *If $t = 0$, for any $(p, q) \in \mathcal{N}$, we have:*

$$\begin{aligned} P_0 f(h) &= \widetilde{P}_0 f(h) = \widehat{P}_0 f(h) \\ &= \sum_{(m,n) \in J_{2pH+q}} \left(\sum_{I \in \{0,1\}^{2m+n}, |I|=2m} c_I \Gamma_I(f, b)(x) \right) h^{2mH+n} \\ &\quad + o(h^{2pH+q}), \end{aligned}$$

with c_I and Γ_I respectively defined by (17) and (19) below.

Remark 2 1. In (15), $Z_{0H+1}^{(t)}$ coincides with the stochastic derivative of X with respect to its present t , as defined in [2].

2. The expansion (15) allows to obtain the expansion of $P_t f(h)$ for $t \neq 0$:

$$P_t f(h) = E[\tilde{P}_t f(h)] = \sum_{(m,n) \in J_{2pH+q}} E[Z_{2mH+n}^{(t)}] h^{2mH+n} + o(h^{2pH+q}).$$

The following subsections are devoted to the proof of Theorem 2. Note that a quicker proof of the first assertion could be the following: once $t > 0$ is fixed, we write

$$X_{t+h} = X_t + \int_t^{t+h} b(X_s) ds + \tilde{B}_h^{(t)}, \quad h \geq 0, \quad (16)$$

where $\tilde{B}_h^{(t)} = B_{t+h} - B_t$ is again a fractional Brownian motion. Then, we could think that an expansion for $\tilde{P}_t f(h)$ follows directly from the one for $\tilde{P}_0 f(h)$, simply by a shift. It is unfortunately not the case, due to the fact that the initial value in (16) is not just a *real number* as in the case $t = 0$, but a *random variable*. Consequently, the computation of $E[\tilde{B}_h^{(t)} | X_t]$ is not trivial, since $\tilde{B}_h^{(t)}$ and X_t are not independent.

4.1 Proof of Theorem 2, point (2).

The proof of this part is in fact a direct consequence of Theorem 2.4 in [9]. But, for the sake of completeness on one hand and taking into account that we are dealing with the one-dimensional case on the other hand, we give all the details here. Indeed, contrary to the multidimensional case, it is easy to compute explicitly the appearing coefficients (see Lemma 1 below and compare with Theorem 31 in [1] or Proposition 5.4 in [9]) which has, from our point of view, also its own interest.

The differential operators Γ_I appearing in Theorem 2 are recursively* defined by

$$\Gamma_{(0)}(f, b) = bf', \quad \Gamma_{(1)}(f, b) = f',$$

and, for $I \in \{0, 1\}^k$,

$$\Gamma_{(I,0)}(f, b) = b(\Gamma_I(f, b))', \quad \Gamma_{(I,1)}(f, b) = (\Gamma_I(f, b))', \quad (17)$$

with $(I, 0), (I, 1) \in \{0, 1\}^{k+1}$. The constants c_I are explained as follows. Set

$$dB_t^{(i)} = \begin{cases} dB_t & \text{if } i = 1, \\ dt & \text{if } i = 0. \end{cases} \quad (18)$$

*We can also use a rooted trees approach in order to define the Γ_I 's. See [9] for a thorough study, even in the multidimensional case and $H > 1/3$.

Then, for a sequence

$$I = (i_1, \dots, i_k) \in \{0, 1\}^k,$$

we define

$$c_I = E \left[\int_{\Delta^k[0,1]} dB^I \right] = E \left[\int_0^1 dB_{t_k}^{(i_k)} \int_0^{t_k} dB_{t_{k-1}}^{(i_{k-1})} \dots \int_0^{t_2} dB_{t_1}^{(i_1)} \right]. \quad (19)$$

Set $|I| = \sum_{1 \leq j \leq k} i_j$. Equivalently, $|I|$ denotes the number of integrals with respect to $'dB'$. Note, since B and $-B$ have the same law, that we have

$$c_I = c_I (-1)^{|I|}.$$

Thus c_I vanishes when $|I|$ is odd. In general, the computation of the coefficients c_I can be made as follows:

Lemma 1 *Let $I \in \{0, 1\}^k$. We denote by $J = \{j_1 < \dots < j_m\}$ the set of indices $j \in \{1, \dots, k\}$ such that $dB_t^{(j)} = dt$. We then have that c_I is given by*

$$\int_0^1 dt_{j_m} \dots \int_0^{t_{j_2}} dt_{j_1} E \left[\frac{(B_1 - B_{t_{j_m}})^{k-j_m} (B_{t_{j_1}})^{j_1-1}}{(k-j_m)!(j_1-1)!} \prod_{k=2}^m \frac{(B_{t_{j_k}} - B_{t_{j_{k-1}}})^{j_k-j_{k-1}}}{(j_k-j_{k-1})!} \right].$$

The expectation appearing in the above formula can always be computed using the moment generating function of an m -dimensional Gaussian random variable. For instance, we have

$$\begin{aligned} E \left[\int_0^1 \int_0^{t_3} \int_0^{t_2} dt_1 dB_{t_2} dB_{t_3} \right] &= \frac{1}{2(2H+1)} \\ E \left[\int_0^1 \int_0^{t_3} \int_0^{t_2} dB_{t_1} dt_2 dB_{t_3} \right] &= \frac{2H-1}{2(2H+1)} \\ E \left[\int_0^1 \int_0^{t_3} \int_0^{t_2} dB_{t_1} dB_{t_2} dt_3 \right] &= \frac{1}{2(2H+1)} \\ E \left[\int_0^1 \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} dt_1 dt_2 dB_{t_3} dB_{t_4} \right] &= \frac{1}{2(2H+1)(2H+2)} \\ E \left[\int_0^1 \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} dt_1 dB_{t_2} dt_3 dB_{t_4} \right] &= \frac{H}{(2H+1)(2H+2)} \\ E \left[\int_0^1 \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} dt_1 dB_{t_2} dB_{t_3} dt_4 \right] &= \frac{1}{2(2H+1)(2H+2)} \\ E \left[\int_0^1 \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} dB_{t_1} dt_2 dB_{t_3} dt_4 \right] &= \frac{H}{(2H+1)(2H+2)} \\ E \left[\int_0^1 \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} dB_{t_1} dt_2 dt_3 dB_{t_4} \right] &= \frac{H(2H-1)}{2(2H+1)(2H+2)} \\ E \left[\int_0^1 \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} dB_{t_1} dB_{t_2} dt_3 dt_4 \right] &= \frac{1}{2(2H+1)(2H+2)}. \end{aligned} \quad (20)$$

Lemma 2 When $f : \mathbb{R} \rightarrow \mathbb{R} \in C_b^\infty$, we have

$$\begin{aligned} f(X_h) &= f(x) + \sum_{k=1}^{n-1} \sum_{I_k \in \{0,1\}^k} \Gamma_{I_k}(f, b)(x) \int_{\Delta^k[0,h]} dB^{I_k}(t_1, \dots, t_k) \\ &\quad + \sum_{I_n \in \{0,1\}^n} \int_{\Delta^n[0,h]} \Gamma_{I_n}(f, b)(X_{t_1}) dB^{I_n}(t_1, \dots, t_n), \end{aligned} \quad (21)$$

where, again using the convention (18), for $g : \mathbb{R} \rightarrow \mathbb{R} \in C_b^\infty$,

$$\int_{\Delta^k[0,h]} g(X_{t_1}) dB^{I_k}(t_1, \dots, t_k) \triangleq \int_0^h dB_{t_k}^{(i_k)} \int_0^{t_k} dB_{t_{k-1}}^{(i_{k-1})} \dots \int_0^{t_2} dB_{t_1}^{(i_1)} g(X_{t_1}).$$

Proof. Applying (3) already twice, we can write

$$\begin{aligned} f(X_h) &= f(x) + \int_0^h f'(X_s) dB_s + \int_0^h (bf')(X_s) ds \\ &= f(x) + \Gamma_{(1)}(f, b)(x) B_h + \Gamma_{(0)}(f, b)(x) h \\ &\quad + \sum_{I_2 \in \{0,1\}^2} \int_{\Delta^2[0,h]} \Gamma_{I_2}(f, b)(X_{t_1}) dB^{I_2}(t_1, t_2). \end{aligned}$$

Applying (3) repeatedly we finally obtain (21). \square

The remainder can be bounded by the following lemma:

Lemma 3 If $n \geq 2$, $\varepsilon > 0$ (small enough) and $g : \mathbb{R} \rightarrow \mathbb{R} \in C_b^\infty$ are fixed, we have,

$$\sum_{I_n \in \{0,1\}^n} E \left| \int_{\Delta^n[0,h]} g(X_{t_1}) dB^{I_n}(t_1, \dots, t_n) \right| = O(h^{nH-\varepsilon}).$$

Proof. It is a direct application of Theorem 2.2 in [8], combined with the Garsia, Rodemich and Rumsey Lemma [6]. \square

Thus, in order to obtain the asymptotic expansion of $\tilde{P}_t f(h)$, Lemmas 2 and 3 say that it is sufficient to compute

$$E \left[\int_{\Delta^k[0,h]} dB^{I_k}(t_1, \dots, t_k) \right],$$

for any $I_k \in \{0,1\}^k$, with $1 \leq k \leq n-1$. By the self-similarity and the stationarity of fractional Brownian motion, we have that

$$\int_{\Delta^k[0,h]} dB^{I_k}(t_1, \dots, t_k) \stackrel{\mathcal{L}}{=} h^{H|I_k|+k-|I_k|} \int_{\Delta^k[0,1]} dB^{I_k}(t_1, \dots, t_k).$$

Hence it follows

$$\mathbb{E} \left[\int_{\Delta^k[0,h]} dB^{I_k}(t_1, \dots, t_k) \right] = h^{H|I_k|+k-|I_k|} c_{I_k}$$

and the proof of point (2) of Theorem 2 is a consequence of Lemmas 2 and 3 above.

4.2 Proof of Theorem 2, point (1).

Let $f : \mathbb{R} \rightarrow \mathbb{R} \in C_b^\infty$ and X be the solution of (5). We then know (see Section 2) that X is a fractional Brownian motion of Hurst index H , under the new probability \mathbf{Q} defined by $d\mathbf{Q} = \eta^{-1}d\mathbf{P}$, with η given by (10). Since $b : \mathbb{R} \rightarrow \mathbb{R} \in C_b^\infty$, remark that $\eta \in \mathbb{D}^{1,2}$. Moreover, the following well-known formula holds for any $\xi \in L^2(\mathbf{P}) \cap L^2(\mathbf{Q})$:

$$E[\xi|X_t] = \frac{E^{\mathbf{Q}}[\eta\xi|X_t]}{E^{\mathbf{Q}}[\eta|X_t]}.$$

In particular,

$$E[f(X_{t+h}) - f(X_t)|X_t] = \frac{E^{\mathbf{Q}}[\eta(f(X_{t+h}) - f(X_t))|X_t]}{E^{\mathbf{Q}}[\eta|X_t]}.$$

Now, we need the following technical lemma:

Lemma 4 *Let $\zeta \in \mathbb{D}^{1,2}(\mathfrak{H})$ be a random variable. Then, for any $h > 0$, the conditional expectation $E[\zeta(f(B_{t+h}) - f(B_t))|B_t]$ is equal to*

$$\begin{aligned} & H(2H-1)f'(B_t) \int_0^T du E[D_u\zeta|B_t] \int_t^{t+h} |v-u|^{2H-2} dv \\ & + \frac{1}{2}t^{-2H}f'(B_t)(B_t E[\zeta|B_t] - E[\langle D\zeta, \mathbf{1}_{[0,t]} \rangle_{\mathfrak{H}}|B_t])(h^{2H} - (t+h)^{2H} + t^{2H}) \\ & - \frac{H}{2}t^{-2H}f''(B_t)E[\zeta|B_t] \int_t^{t+h} ((v-t)^{2H-1} - v^{2H-1})(t^{2H} + v^{2H} - (v-t)^{2H}) dv \\ & + \frac{1}{2}f''(B_t)E[\zeta|B_t]((t+h)^{2H} - t^{2H}) \\ & + H(2H-1) \int_0^T du \int_t^{t+h} |v-u|^{2H-2} E[D_u\zeta(f'(B_v) - f'(B_t))|B_t] dv \\ & + Ht^{-2H}B_t \int_t^{t+h} ((v-t)^{2H-1} - v^{2H-1}) E[\zeta(f'(B_v) - f'(B_t))|B_t] dv \\ & - Ht^{-2H} \int_t^{t+h} ((v-t)^{2H-1} - v^{2H-1}) E[\langle D\zeta, \mathbf{1}_{[0,t]} \rangle_{\mathfrak{H}}(f'(B_v) - f'(B_t))|B_t] dv \end{aligned}$$

$$\begin{aligned}
& -\frac{H}{2}t^{-2H} \int_t^{t+h} ((v-t)^{2H-1} - v^{2H-1}) (t^{2H} + v^{2H} - (v-t)^{2H}) E[\zeta(f''(B_v) - f''(B_t))|B_t] dv \\
& + H \int_t^{t+h} E[\zeta(f''(B_v) - f''(B_t))|B_t] v^{2H-1} dv.
\end{aligned} \tag{22}$$

Proof. Let $g : \mathbb{R} \rightarrow \mathbb{R} \in C_b^1$. We can write using the Itô formula (5.44) p.294 in [11] and basics identities of Malliavin calculus:

$$\begin{aligned}
& E[\zeta g(B_t)(f(B_{t+h}) - f(B_t))] \\
& = E[\zeta g(B_t)\delta(f'(B.)\mathbf{1}_{[t,t+h]})] + H \int_t^{t+h} E[\zeta g(B_t)f''(B_v)] v^{2H-1} dv \\
& = E[g(B_t)\langle D\zeta, \mathbf{1}_{[t,t+h]}f'(B.) \rangle_{\mathfrak{H}}] + E[\zeta g'(B_t)\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[t,t+h]}f'(B.) \rangle_{\mathfrak{H}}] \\
& \quad + H \int_t^{t+h} E[\zeta g(B_t)f''(B_v)] v^{2H-1} dv \\
& = H(2H-1) \int_0^T du \int_t^{t+h} |v-u|^{2H-2} E[f'(B_v)g(B_t)D_u\zeta] dv \\
& \quad + H(2H-1) \int_0^t du \int_t^{t+h} (v-u)^{2H-2} E[\zeta f'(B_v)g'(B_t)] dv \\
& \quad + H \int_t^{t+h} E[\zeta g(B_t)f''(B_v)] v^{2H-1} dv.
\end{aligned}$$

But

$$\begin{aligned}
E[\zeta g(B_t)f'(B_v)B_t] & = E[\langle D(\zeta g(B_t)f'(B_v)), \mathbf{1}_{[0,t]} \rangle_{\mathfrak{H}}] \\
& = E[g(B_t)f'(B_v)\langle D\zeta, \mathbf{1}_{[0,t]} \rangle_{\mathfrak{H}}] + E[\zeta g'(B_t)f'(B_v)]t^{2H} \\
& \quad + \frac{1}{2}E[\zeta g(B_t)f''(B_v)](t^{2H} + v^{2H} - (v-t)^{2H}).
\end{aligned}$$

Consequently

$$\begin{aligned}
& E[\zeta g(B_t)(f(B_{t+h}) - f(B_t))] \\
& = H(2H-1) \int_0^T du \int_t^{t+h} |v-u|^{2H-2} E[f'(B_v)g(B_t)D_u\zeta] dv \\
& \quad + H(2H-1)t^{-2H} \int_0^t du \int_t^{t+h} (v-u)^{2H-2} E[\zeta g(B_t)f'(B_v)B_t] dv \\
& \quad - H(2H-1)t^{-2H} \int_0^t du \int_t^{t+h} (v-u)^{2H-2} E[g(B_t)f'(B_v)\langle D\zeta, \mathbf{1}_{[0,t]} \rangle_{\mathfrak{H}}] dv \\
& \quad - \frac{1}{2}H(2H-1)t^{-2H} \int_0^t du \int_t^{t+h} (v-u)^{2H-2} E[\zeta g(B_t)f''(B_v)](t^{2H} + v^{2H} - (v-t)^{2H}) dv \\
& \quad + H \int_t^{t+h} E[\zeta g(B_t)f''(B_v)] v^{2H-1} dv.
\end{aligned}$$

We deduce:

$$\begin{aligned}
& E [\zeta(f(B_{t+h}) - f(B_t)) | B_t] \\
= & H(2H - 1) \int_0^T du \int_t^{t+h} |v - u|^{2H-2} E[f'(B_v) D_u \zeta | B_t] dv \\
& + Ht^{-2H} B_t \int_t^{t+h} ((v - t)^{2H-1} - v^{2H-1}) E [\zeta f'(B_v) | B_t] dv \\
& - Ht^{-2H} \int_t^{t+h} ((v - t)^{2H-1} - v^{2H-1}) E[f'(B_v) \langle D\zeta, \mathbf{1}_{[0,t]} \rangle_{\mathfrak{H}} | B_t] dv \\
& - \frac{H}{2} t^{-2H} \int_t^{t+h} ((v - t)^{2H-1} - v^{2H-1}) (t^{2H} + v^{2H} - (v - t)^{2H}) E[\zeta f''(B_v) | B_t] dv \\
& + H \int_t^{t+h} E [\zeta f''(B_v) | B_t] v^{2H-1} dv.
\end{aligned}$$

Finally, (22) follows. \square

First, apply the previous lemma with $\zeta = \eta$, $E = E^{\mathbf{Q}}$ and $B = X$, with η given by (10), $d\mathbf{Q} = \eta^{-1} d\mathbf{P}$ and X given by (5). Remark that $\eta \in \mathbb{D}^{\infty,2}$ (see e.g. Lemma 6.3.1 in [11], and [2] for the expression of Malliavin derivatives via the transfer principle). In particular, we can deduce that each random variable V_k , recursively defined by $V_0 = \eta$ and $V_{k+1} = \langle DV_k, \mathbf{1}_{[0,t]} \rangle_{\mathfrak{H}}$ for $k \geq 0$, belongs to $\mathbb{D}^{1,2}$.

In (22), the deterministic terms $\int_t^{t+h} |v - u|^{2H-2} dv$, $(t + h)^{2H} - t^{2H}$ and

$$\int_t^{t+h} ((v - t)^{2H-1} - v^{2H-1}) (t^{2H} + v^{2H} - (v - t)^{2H}) dv$$

have a Taylor expansion in h of the type (15). Lemma 4 allows to obtain the first term of the asymptotic expansion using that $\int_t^{t+h} \phi(s) ds = h\phi(t) + o(h)$ for any continuous function ϕ . By a recursive argument using again Lemma 4, we finally deduce that (15) holds, which concludes the proof of Theorem 2.

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