p-Laplace operator and diameter of manifolds.

Jean-Francois Grosjean

P-LAPLACE OPERATOR AND DIAMETER OF MANIFOLDS

Jean-François GROSJEAN

March 22, 2007

GROSJEAN Jean-François, Institut Élie Cartan (Mathématiques), Université Henri Poincaré Nancy I, B.P. 239, F-54506 VANDOEUVRE-LES-NANCY CEDEX, FRANCE.

E-mail: grosjean@iecn.u-nancy.fr

Abstract

Let (M^n, g) be a compact Riemannian manifold without boundary. In this paper, we consider the first nonzero eigenvalue of the p-Laplacian $\lambda_{1,p}(M)$ and we prove that the limit of $\sqrt[p]{\lambda_{1,p}(M)}$ when $p \to \infty$ is $2/d(M)$ where $d(M)$ is the diameter of M. Moreover if (M^n, g) is an oriented compact hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ or $\mathbb{S}^{n+1}$, we prove an upper bound of $\lambda_{1,p}(M)$ in term of the largest principal curvature κ over M. As applications of these results we obtain optimal lower bounds of $d(M)$ in term of the curvature. In particular we prove that if M is a hypersurface of $\mathbb{R}^{n+1}$ then : $d(M) \geq \pi/\kappa$.

Mathematics Subject Classification (2000): 53A07, 53C21.

1 Introduction and main results

Let (M^n, g) be a connected compact Riemannian manifold of dimension n without boundary and let $p > 1$. The first nonzero eigenvalue of the p-Laplace operator is defined by the following variational characterization

$$\lambda_{1,p}(M) = \inf \left\{ \frac{\int_M |du|^p dv}{\int_M |u|^p dv} \ ; \ u \in W^{1,p}(M) \ , \ u \neq 0 \ \text{ and } \ \int_M |u|^{p-2} u dv = 0 \right\}$$

This infimum is achieved by a $C^{1,\alpha}$ eigenfunction f_p (see regularity results in [8] and [9]) which satisfies the Euler-Lagrange equation

$$\Delta_p f_p = \lambda_{1,p}(M) |f_p|^{p-2} f_p$$

where Δ_p is the p-Laplace operator defined by $\Delta_p u = -\text{div}\,(|\text{grad}\, u|^{p-2}\text{grad}\, u)$ and $\text{grad}\, u$ is the gradient of u with respect to the metric g of M.

On the other hand if (Ω^n, g) is a bounded domain of M with boundary $\partial\Omega$, we consider the first Dirichlet eigenvalue defined by

$$\lambda^D_{1,p}(\Omega) = \inf \left\{ \frac{\int_\Omega |du|^p dv}{\int_\Omega |u|^p dv} \ ; \ u \in W^{1,p}_0(\Omega) \ , \ u \neq 0 \right\}$$

As previously, this infimum is achieved by a $C^{1,\alpha}$ eigenfunction f_p which satisfies the same Euler-Lagrange equation with zero boundary values. Moreover, Lindqvist ([5]) proves that f_p is the unique positive eigenfunction associated to $\lambda^D_{1,p}(\Omega)$ up to multiplication by constants.

In [4] the authors proved that $\sqrt[p]{\lambda^D_{1,p}(\Omega)} \longrightarrow \frac{1}{r(\Omega)}$ when $p \longrightarrow \infty$, where $r(\Omega) = \max_{x\in\Omega}(\text{dist}\,(x, \partial\Omega))$ and $\text{dist}\,(x, \partial\Omega)$ denotes the geodesic distance between x and $\partial\Omega$. The main result of this paper is a similar result for the first nonzero eigenvalue of the p-Laplace operator defined on the compact manifolds. Indeed we will prove the following theorem.

Theorem 1.1 *Let (M^n, g) be a compact Riemannian manifold. Then we have*

$$\sqrt[p]{\lambda_{1,p}(M)} \longrightarrow \frac{2}{d(M)} \ \ \textit{when} \ \ p \longrightarrow \infty$$

where $d(M)$ is the diameter of M.

In a second part, considering the case of compact hypersurfaces isometrically immersed in the Euclidean space and the sphere, we will give an upper bound of $\lambda_{1,p}(M)$ (see theorems 3.1 and 3.2) in term of the largest principal curvature. Combining these results with the above theorem 1.1 we get optimal lower bounds of the diameter of hypersurfaces of the Euclidean space and the sphere (see corollaries 3.1 and 3.2).

2 Proof of the main result

In this section we give a proof of the theorem 1.1

PROOF: There exist two points x_1 and x_2 of M so that $d(M) = \text{dist}\,(x_1, x_2)$, where dist denotes the geodesic distance on M. Since the open balls $B(x_1, \frac{d(M)}{2})$ and $B(x_2, \frac{d(M)}{2})$ of center respectively x_1 and x_2 and of radius $\frac{d(M)}{2}$ are disjoint, we have by the min-max principle

$$\lambda_{1,p}(M) \leq \max_i \left(\lambda_{1,p}^D \left(B\left(x_i, \frac{d(M)}{2}\right)\right)\right)$$

Let δ be the function defined on $B(x_i, \frac{d(M)}{2})$ by $\delta(x) = \frac{d(M)}{2} - \text{dist}\,(x_i, x)$. Then δ is in the space $W^{1,p}(M)$ for $p > 1$ and is zero on the boundary. Consequently by the characterization of the first Dirichlet eigenvalue, we get

$$\lambda_{1,p}^D \left(B\left(x_i, \frac{d(M)}{2}\right)\right) \leq \frac{\|d\delta\|_p^p}{\|\delta\|_p^p} \leq \frac{V\left(B\left(x_i, \frac{d(M)}{2}\right)\right)}{\|\delta\|_p^p}$$

Then

$$\limsup \sqrt[p]{\lambda_{1,p}^D \left(B\left(x_i, \frac{d(M)}{2}\right)\right)} \leq \limsup \frac{\sqrt[p]{V\left(B\left(x_i, \frac{d(M)}{2}\right)\right)}}{\|\delta\|_p} = \frac{2}{d(M)}$$

and

$$\limsup \sqrt[p]{\lambda_{1,p}(M)} \leq \frac{2}{d(M)}$$

Now, we will prove that $\liminf \sqrt[p]{\lambda_{1,p}(M)} \geq \frac{2}{d(M)}$. So, let f_p be a first eigenfunction for $\lambda_{1,p}(M)$, $p > 1$ and let ${\Omega_p}^+ = {f_p}^{-1}(\mathbb{R}^+)$ and ${\Omega_p}^- = {f_p}^{-1}(\mathbb{R}^-)$. ${\Omega_p}^+$ and ${\Omega_p}^-$ are nonempty connected open subsets and Matei ([6]) has proved that

$$\lambda_{1,p}(M) = \lambda_{1,p}^D({\Omega_p}^+) = \lambda_{1,p}^D({\Omega_p}^-)$$

Let $f_p^+ \in W_0^{1,p}({\Omega_p}^+)$ and $f_p^- \in W_0^{1,p}({\Omega_p}^-)$ be positive eigenfunctions associated respectively to $\lambda_{1,p}^D({\Omega_p}^+)$ and $\lambda_{1,p}^D({\Omega_p}^-)$. We can assume that $\|f_p^+\|_\infty = \|f_p^-\|_\infty = 1$. We extend f_p^+ and f_p^- by 0 respectively outside ${\Omega_p}^+$ and ${\Omega_p}^-$. Then f_p^+ and $f_p^- \in W^{1,p}(M)$. Fix an exponent $q > n$. For $p > q$ by Hölder's inequality we have

$$\left(\int_M |df_p^+|^q dv\right)^{1/q} \leq \left(\int_M |df_p^+|^p dv\right)^{1/p} V(M)^{\frac{p-q}{pq}}$$

$$= \sqrt[p]{\lambda_{1,p}(M)} \left(\int_M |f_p^+|^p dv \right)^{1/p} V(M)^{\frac{p-q}{pq}}$$
$$\leq \sqrt[p]{\lambda_{1,p}(M)} V(M)^{1/q}$$

where in this last inequality, we have used the fact that $\|f_p^+\|_\infty = 1$. Since $\limsup \sqrt[p]{\lambda_{1,p}(M)} \leq \frac{2}{d(M)}$, we conclude that $(f_p^+)_{p \geq q}$ is uniformly bounded in $W^{1,q}(M)$. Then we can select a subsequence $(f_{p_k}^+)$ that converges to a function f_∞^+ weakly in $W^{1,q}(M)$ and by the Sobolev embedding theorems uniformly in $C^0(M)$ (in fact in $C^{0,\alpha}(M)$ for a certain α, $0 < \alpha < 1$).

Now using the weak lower semicontinuity of the norm of $W^{1,q}(M)$ and the fact that $(f_{p_k}^+)$ converges uniformly, we have

$$\|f_\infty^+\|_q + \|df_\infty^+\|_q \leq \liminf_{p_k \to \infty} \left(\|f_{p_k}^+\|_q + \|df_{p_k}^+\|_q \right) = \|f_\infty^+\|_q + \liminf_{p_k \to \infty} \|df_{p_k}^+\|_q$$

Then

$$\|df_\infty^+\|_q \leq \liminf_{p_k \to \infty} \|df_{p_k}^+\|_q \leq V(M)^{1/q} \liminf_{p_k \to \infty} \sqrt[p_k]{\lambda_{1,p_k}(M)}$$

From this we deduce that

$$\|df_\infty^+\|_\infty \leq \liminf_{p_k \to \infty} \sqrt[p_k]{\lambda_{1,p_k}(M)}$$

Now let $\Omega^+ = (f_\infty^+)^{-1}(\mathbb{R}^+)$. Since for any p, $\|f_p^+\|_\infty = 1$ and f_∞^+ is continuous, it follows that Ω^+ is a nonempty subset of M. Let $x \in \Omega^+$ and choose $y \in \partial\Omega^+$ so that $\text{dist}\,(x,y) = \text{dist}\,(x, \partial\Omega^+)$ where $\text{dist}\,(x, \partial\Omega^+)$ denotes the geodesic distance between x and $\partial\Omega^+$. Then, necessarily $f_\infty^+(y) = 0$ and

$$|f_\infty^+(x)| = |f_\infty^+(x) - f_\infty^+(y)| \leq \|df_\infty^+\|_\infty \text{dist}\,(x,y) \leq \|df_\infty^+\|_\infty r(\Omega^+)$$

where for any domain Ω, $r(\Omega) = \max_{x \in \Omega}(\text{dist}\,(x, \partial\Omega))$. Then

$$\frac{1}{r(\Omega^+)} \leq \|df_\infty^+\|_\infty \leq \liminf_{p_k \to \infty} \sqrt[p_k]{\lambda_{1,p_k}(M)}$$

By similar arguments, we can select a subsequence $(f_{p_l}^-)$ of $(f_{p_k}^-)$, so that

$$\max\left(\frac{1}{r(\Omega^+)}, \frac{1}{r(\Omega^-)} \right) \leq \liminf_{p_l \to \infty} \sqrt[p_l]{\lambda_{1,p_l}(M)}$$

Now, for any domain Ω, $r(\Omega)$ is nothing but the radius of the largest ball inscribed in Ω. Moreover it is easy to see that Ω^+ and Ω^- are disjoint. Therefore $\max\left(\frac{1}{r(\Omega^+)}, \frac{1}{r(\Omega^-)} \right) \geq$

$\frac{2}{d(M)}$. This completes the proof, since we can apply this process to any subsequence of f_p^+ and f_p^-.

□

3 Applications

Before giving an application of the asymptotic property of the main theorem 1.1, we will give a kind of generalization of the Reilly's inequality for the $\lambda_{1,p}(M)$ of hypersurfaces of the Euclidean space $\mathbb{R}^{n+1}$ and the sphere $\mathbb{S}^{n+1}$. We recall that the Reilly's inequality gives an optimal upper bound for the first nonzero eigenvalue of the Laplacian in terms of the second fundamental form and the volume of M (see for instance [1], [2], [3] and [7]).

Let (M^n, g) be an oriented compact n-dimensional hypersurface isometrically immersed by ϕ in $\mathbb{R}^{n+1}$ or $\mathbb{S}^{n+1}$ and let B be the second fundamental form of the immersion ϕ. In the following theorems, we will give an upper bound of $\lambda_{1,p}(M)$ in terms of the function $|B|_\infty$ defined for any x by $|B|_\infty(x) = \max\limits_{X \in T_xM, |X|=1} |B_x(X,X)|$ and the Riemannian volume $V(M)$ of M.

Theorem 3.1 *Let (M^n, g) be an oriented compact n-dimensional Riemannian manifold isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. Then for any $p > 1$, there exists a constant $K(p, M)$ depending only on p and M, so that*

1. *$K(p, M) \longrightarrow 1$ when $p \longrightarrow \infty$.*
2. $$\lambda_{1,p}(M)V(M) \leq K(p,M)^p \lambda_{1,p}(\mathbb{S}^n) \| |B|_\infty \|_p^p$$

Similarly, we have for hypersurfaces of $\mathbb{S}^{n+1}$, the following theorem

Theorem 3.2 *Let (M^n, g) be an oriented compact n-dimensional Riemannian manifold isometrically immersed by ϕ in $\mathbb{S}^{n+1}$ and let $R(M)$ be the smallest radius of the ball containing M. Then for any $p > 1$, there exists a constant $K(p, M)$ depending only on p and M, so that*

1. *$K(p, M) \longrightarrow 1$ when $p \longrightarrow \infty$.*
2. $$\lambda_{1,p}(M)V(M) \leq K(p,M)^p \lambda_{1,p}(\mathbb{S}^n) \left\| |B|_\infty + \left(\frac{1 - \cos R(M)}{\sin R(M)} \right) \right\|_p^p$$

We will see in the proofs, that the constant $K(p,M)$ satisfies $K(p,M) \geq 1$, but if M is a geodesic sphere, $K(p,M) = 1$ and the two inequalities are equalities.

Now, from these above theorems and from the main theorem 1.1 we obtain the following corollaries which give an optimal lower bounds of $d(M)$ in term of the largest principal curvature $\kappa = \max\limits_{x\in M}\left(\max\limits_{X\in T_xM,|X|=1}|B_x(X,X)|\right)$

Corollary 3.1 *Let (M^n,g) be an oriented compact n-dimensional Riemannian manifold isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. Then*

$$d(M) \geq \frac{\pi}{\kappa}$$

Moreover, if (M^n,g) is a geodesic sphere, then equality holds.

We have also an optimal result for hypersurfaces of $\mathbb{S}^{n+1}$

Corollary 3.2 *Let (M^n,g) be an oriented compact n-dimensional Riemannian manifold isometrically immersed by ϕ in $\mathbb{S}^{n+1}$ and let $R(M)$ be the smallest radius of the ball containing M. Then*

$$d(M) \geq \frac{\pi}{\kappa + \left(\dfrac{1-\cos R(M)}{\sin R(M)}\right)}$$

Moreover, if (M^n,g) is a geodesic sphere, then equality holds.

In particular, if (M^n,g) is immersed in the hemisphere $\mathbb{S}^{n+1}_+$, then

$$d(M) \geq \frac{\pi}{\kappa+1}$$

and if (M^n,g) is a greatest sphere, equality holds.

Remark In [6] Matei proved that if the Ricci curvature Ric of an n-dimensional compact Riemannian manifold satisfies $Ric \geq kg$, then $\lambda_{1,p}(M) \geq \lambda_{1,p}\left(\mathbb{S}^n\left(\sqrt{\frac{n-1}{k}}\right)\right)$, where $\mathbb{S}^n\left(\sqrt{\frac{n-1}{k}}\right)$ denotes the sphere of radius $\sqrt{\frac{n-1}{k}}$. Now if we apply the theorem 1.1, we find again the well known Myers inequality

$$d(M) \leq \pi\sqrt{\frac{n-1}{k}}$$

4 Proof of theorems 3.1 and 3.2

Let (M^n, g) be a compact n-dimensional Riemannian manifold isometrically immersed in $\mathbb{R}^{n+1}$ or $\mathbb{S}^{n+1}$ and oriented by the unit normal vector field ν. Then the second fundamental form of the immersion will be defined by $B(X,Y) = \langle \nabla_X \nu, Y \rangle$, where ∇ and $\langle\ ,\ \rangle$ are respectively the Riemannian connection and the inner product of the ambient space $\mathbb{R}^{n+1}$ or $\mathbb{S}^{n+1}$.

On the other hand, we recall that the first Dirichlet eigenfunction f_p of the p-Laplacian on the hemisphere $\mathbb{S}^n_+ = \{x \in \mathbb{S}^n / \text{dist}\,(x, x_0) \leq \pi/2\}$ is necessarily radial ([5]). Actually, it means that there exists a function Φ_p so that $f_p(x) = \Phi_p(\text{dist}\,(x_0, x))$.

Assume now that (M^n, g) is immersed in $\mathbb{R}^{n+1}$. For any $\xi \in \mathbb{S}^n$, let

$$\varphi^+_{p,\xi}(x) = \Phi_p(\arccos \langle \nu_x, \xi \rangle) \chi_{\{\langle \nu_x, \xi \rangle > 0\}}$$

and

$$\varphi^-_{p,\xi}(x) = -\Phi_p(\arccos(-\langle \nu_x, \xi \rangle)) \chi_{\{\langle \nu_x, \xi \rangle < 0\}}$$

where Φ_p is the function defined in the preliminaries and for any subset A of M, χ_A is the characteristic function of A. For more convenience we put $\rho(\nu_x, \xi) = \arccos \langle \nu_x, \xi \rangle$ if $\langle \nu_x, \xi \rangle \geq 0$ and $\rho(\nu_x, \xi) = \arccos(-\langle \nu_x, \xi \rangle)$ if $\langle \nu_x, \xi \rangle < 0$. Moreover we recall that $|B|_\infty$ is the function defined by $|B|_\infty(x) = \max\limits_{X \in T_x M, |X|=1} |B_x(X,X)|$.

First, we will estimate the gradient of $\varphi^+_{p,\xi}$ and $\varphi^-_{p,\xi}$.

Lemma 4.1 *Let (M^n, g) be an oriented compact n-dimensional Riemannian manifold isometrically immersed in $\mathbb{R}^{n+1}$. The functions $\varphi^+_{p,\xi}$ and $\varphi^-_{p,\xi}$ are in the Sobolev space $W^{1,p}(M)$ and satisfy for any $x \in M$*

$$|d\varphi^+_{p,\xi}(x)| \leq |\Phi'_p(\rho(\nu_x, \xi))||B|_\infty(x) \chi_{\{\langle \nu_x, \xi \rangle > 0\}}$$

and

$$|d\varphi^-_{p,\xi}(x)| \leq |\Phi'_p(\rho(\nu_x, \xi))||B|_\infty(x) \chi_{\{\langle \nu_x, \xi \rangle < 0\}}$$

Proof: Let $(e_i)_{1 \leq i \leq n}$ be an orthonormal basis at $x \in M$. At x we have

$$\begin{aligned} e_i(\varphi^+_{p,\xi}) &= -\frac{\Phi'_p(\rho(\nu, \xi))}{\sqrt{1 - \langle \nu, \xi \rangle^2}} e_i \langle \nu, \xi \rangle \chi_{\{\langle \nu_x, \xi \rangle > 0\}} \\ &= -\frac{\Phi'_p(\rho(\nu, \xi))}{|\xi^T|} \langle \nabla_{e_i} \nu, \xi \rangle \chi_{\{\langle \nu_x, \xi \rangle > 0\}} \end{aligned}$$

$$= -\frac{\Phi_p'(\rho(\nu,\xi))}{|\xi^T|} B(e_i, \xi^T)\chi_{\{\langle \nu_x,\xi\rangle>0\}}$$

where ξ^T denotes the orthogonal projection of ξ on T_xM. Then

$$|d\varphi^+_{p,\xi}|^2 = \frac{\Phi_p'(\rho(\nu,\xi))^2}{|\xi^T|^2}|B\xi^T|^2\chi_{\{\langle \nu_x,\xi\rangle>0\}} \leq (\Phi_p'(\rho(\nu,\xi)))^2||B|_\infty|^2\chi_{\{\langle \nu_x,\xi\rangle>0\}}$$

which completes the proof for the first inequality. Obviously, the second is similar. □

Now we will show a similar result for hypersurfaces of $\mathbb{S}^{n+1}$. Let (M^n, g) be an oriented compact n-dimensional Riemannian manifold isometrically immersed in $\mathbb{S}^{n+1}$. Let $B(x_0, R(M))$ be the smallest ball of center x_0 and radius $R(M)$ containing M. Let $\xi \in \mathbb{S}^n$ view as the unit sphere of the Euclidean space $T_{x_0}\mathbb{S}^{n+1}$. Let $\tilde{\xi}$ be the unit vector field on $\mathbb{S}^{n+1}$ defined for any $x \in \mathbb{S}^{n+1}$ by

$$\tilde{\xi}_x = \langle \xi, u\rangle \operatorname{grad} r\,|_x + \xi - \langle \xi, u\rangle\, u$$

where $r(x)$ is the geodesic distance on $\mathbb{S}^{n+1}$ between x_0 and x, $\operatorname{grad} r$ is the gradient of r and u is the unit vector of $T_{x_0}\mathbb{S}^{n+1}$ so that $x = \exp_{x_0}(r(x)u)$. Note that $\tilde{\xi}_x$ is nothing but the parallel transport of ξ along the geodesic joining x_0 and x. Therefore $\nabla_{\operatorname{grad} r}\tilde{\xi} = 0$. Moreover if X is a vector of $T_x\mathbb{S}^{n+1}$ so that $\langle X, \operatorname{grad} r\rangle = 0$, then

$$\nabla_X\tilde{\xi} = \left(\frac{1-\cos r}{\sin r}\right)\left(\left\langle \tilde{\xi}, X\right\rangle \operatorname{grad} r - \left\langle \tilde{\xi}, \operatorname{grad} r\right\rangle X\right) \tag{1}$$

As in the Euclidean case, we define two functions $\varphi^+_{p,\xi}$ and $\varphi^-_{p,\xi}$ by

$$\varphi^+_{p,\xi}(x) = \Phi_p\left(\arccos\left\langle \nu_x, \tilde{\xi}_x\right\rangle\right)\chi_{\{\langle \nu_x,\tilde{\xi}_x\rangle>0\}}$$

and

$$\varphi^-_{p,\xi}(x) = -\Phi_p\left(\arccos\left(-\left\langle \nu_x, \tilde{\xi}_x\right\rangle\right)\right)\chi_{\{\langle \nu_x,\tilde{\xi}_x\rangle<0\}}$$

From now we will put $\rho(\nu_x,\xi) = \arccos\left\langle \nu_x, \tilde{\xi}_x\right\rangle$ if $\left\langle \nu_x, \tilde{\xi}_x\right\rangle \geq 0$ and $\rho(\nu_x,\xi) = \arccos(-\left\langle \nu_x, \tilde{\xi}_x\right\rangle)$ if $\left\langle \nu_x, \tilde{\xi}_x\right\rangle < 0$.

Now, we state a lemma similar to the euclidean case.

Lemma 4.2 *Let (M^n, g) be an oriented compact n-dimensional Riemannian manifold isometrically immersed in $\mathbb{S}^{n+1}$ and let $B(x_0, R(M))$ be the smallest ball of center x_0 and radius $R(M)$ containing M. The functions $\varphi^+_{p,\xi}$ and $\varphi^-_{p,\xi}$ are in the Sobolev space $W^{1,p}(M)$ and satisfy for any $x \in M$*

$$|d\varphi^+_{p,\xi}(x)| \leq |\Phi'_p(\rho(\nu_x,\xi))| \left| |B|_\infty(x) + \left(\frac{1-\cos R(M)}{\sin R(M)}\right)\right| \chi_{\{\langle \nu_x, \tilde{\xi}_x\rangle > 0\}}$$

and

$$|d\varphi^-_{p,\xi}(x)| \leq |\Phi'_p(\rho(\nu_x,\xi))| \left| |B|_\infty(x) + \left(\frac{1-\cos R(M)}{\sin R(M)}\right)\right| \chi_{\{\langle \nu_x, \tilde{\xi}_x\rangle < 0\}}$$

PROOF: First we have at x in M

$$|d\varphi^+_{p,\xi}|^2 = \frac{\Phi'_p(\rho(\nu,\xi))^2}{|\tilde{\xi}^T|^2} \left| d\left\langle \nu, \tilde{\xi}\right\rangle\right|^2 \chi_{\{\langle \nu_x, \tilde{\xi}_x\rangle > 0\}}$$

Let $(e_i)_{1\leq i\leq n}$ be an orthonormal frame in a neighborhood of M around x. Assume that e_n is colinear to the gradient of r on M with respect to the metric g. Let $e_n^\star$ be a unit vector of $T_x\mathbb{S}^{n+1}$ orthogonal to $\operatorname{grad} r$ and so that $e_n = \lambda \operatorname{grad} r + \mu e_n^\star$. Then at x we have

$$\begin{aligned}\left| d\left\langle \nu, \tilde{\xi}\right\rangle\right|^2 &= \sum_{i\leq n-1} \left(e_i\left\langle \tilde{\xi}, \nu\right\rangle\right)^2 + \left(e_n\left\langle \tilde{\xi}, \nu\right\rangle\right)^2 \\ &= \sum_{i\leq n-1} \left(\left\langle \nabla_{e_i}\tilde{\xi}, \nu\right\rangle + \left\langle \tilde{\xi}, \nabla_{e_i}\nu\right\rangle\right)^2 + \left(\left\langle \nabla_{\lambda \operatorname{grad} r + \mu e_n^\star}\tilde{\xi}, \nu\right\rangle + \left\langle \tilde{\xi}, \nabla_{e_n}\nu\right\rangle\right)^2\end{aligned}$$

Applying the relation (1) and using the fact that $\nabla_{\operatorname{grad} r}\tilde{\xi} = 0$, we get

$$\begin{aligned}\left| d\left\langle \nu, \tilde{\xi}\right\rangle\right|^2 = \sum_{i\leq n-1} &\left(\left(\frac{1-\cos r}{\sin r}\right)\left\langle \tilde{\xi}, e_i\right\rangle \langle \operatorname{grad} r, \nu\rangle + \left\langle B\tilde{\xi}^T, e_i\right\rangle\right)^2 \\ &+ \left(\mu\left(\frac{1-\cos r}{\sin r}\right)\left\langle \tilde{\xi}, e_n^\star\right\rangle \langle \nabla r, \nu\rangle - \mu\left(\frac{1-\cos r}{\sin r}\right)\left\langle \tilde{\xi}, \nabla r\right\rangle \langle e_n^\star, \nu\rangle + \left\langle B\tilde{\xi}^T, e_n\right\rangle\right)^2\end{aligned}$$

A straightforward computation shows that

$$\mu\left\langle \tilde{\xi}, e_n^\star\right\rangle \langle \operatorname{grad} r, \nu\rangle - \mu\left\langle \tilde{\xi}, \operatorname{grad} r\right\rangle \langle e_n^\star, \nu\rangle = \left\langle \tilde{\xi}^T, e_n\right\rangle \langle \operatorname{grad} r, \nu\rangle$$

Thus

$$\begin{aligned}\left| d\left\langle \nu, \tilde{\xi}\right\rangle\right|^2 = \sum_{i\leq n-1} &\left(\left(\frac{1-\cos r}{\sin r}\right)\left\langle \tilde{\xi}, e_i\right\rangle \langle \nabla r, \nu\rangle + \left\langle B\tilde{\xi}^T, e_i\right\rangle\right)^2 \\ &+ \left(\left(\frac{1-\cos r}{\sin r}\right)\left\langle \tilde{\xi}, e_n\right\rangle \langle \nabla r, \nu\rangle + \left\langle B\tilde{\xi}^T, e_n\right\rangle\right)^2\end{aligned}$$

$$= |B\tilde{\xi}^T|^2 + 2\left(\frac{1-\cos r}{\sin r}\right)\left\langle B\tilde{\xi}^T, \tilde{\xi}^T\right\rangle \langle \operatorname{grad} r, \nu\rangle + \left(\frac{1-\cos r}{\sin r}\right)^2 |\tilde{\xi}^T|^2 \langle \operatorname{grad} r, \nu\rangle^2$$
$$\leq \left(|B|_\infty + \left(\frac{1-\cos r}{\sin r}\right)\right)^2 |\tilde{\xi}^T|^2$$

Since the function $r \mapsto \frac{1-\cos r}{\sin r}$ is increasing on $(0,\pi)$ it follows that

$$|d\varphi^+_{p,\xi}|^2 \leq \Phi'_p(\rho(\nu,\xi))^2 \left(|B|_\infty + \left(\frac{1-\cos R(M)}{\sin R(M)}\right)\right)^2$$

□

Before proving the theorems 3.1 and 3.2, note that if M is immersed in $\mathbb{R}^{n+1}$, we deduce from the compactness of M that the Gauss application is surjective and for any unit vector $\xi \in \mathbb{R}^{n+1}$ there exists a point x in M so that $\nu_x = \xi$. For hypersurfaces of $\mathbb{S}^{n+1}$ there exists a similar result stated in the following lemma.

Lemma 4.3 *Let (M^n, g) be a compact n-dimensional Riemannian manifold isometrically immersed in $\mathbb{S}^{n+1}$ and oriented by the unit normal vector field ν. Let $B(x_0, R(M))$ be the smallest ball of center x_0 and radius $R(M)$ containing M. Then, for any $\xi \in T_{x_0}\mathbb{S}^{n+1}$ with $|\xi| = 1$, there exists a point $x \in M$ so that $\tilde{\xi}_x = \nu_x$.*

Proof: First note that we have always $R(M) < \pi$. Then M is contained in $\mathbb{S}^{n+1}\backslash\{-x_0\}$. Now consider the stereographic projection

$$\begin{array}{rccc} F & : & \mathbb{S}^{n+1}\backslash\{-x_0\} & \longrightarrow & \mathbb{R}^{n+1} \\ & & (y_0, y_1, ..., y_{n+1}) & \longmapsto & \frac{1}{1-y_0}(y_1, ..., y_{n+1}) \end{array}$$

where the coordinates have been chosen in order to have $x_0 = (-1, 0, ..., 0)$. Let $\xi = (0, \xi_1, ..., \xi_{n+1}) \in T_{x_0}\mathbb{S}^{n+1}$ so that $|\xi| = 1$. For any $x \in \mathbb{S}^{n+1}\backslash\{-x_0\}$, by a simply calculation the coordinates in the chart induced by F of the vector field $\tilde{\xi}_x$ (i.e. the coordinates in $\mathbb{R}^{n+1}$ of $dF_x(\tilde{\xi}_x)$) are $\frac{1}{1+\cos r}(\xi_1, ..., \xi_{n+1})$ where $r = \operatorname{dist}(x_0, x)$. Now, $F(M)$ is an oriented compact hypersurface of $\mathbb{R}^{n+1}$ with a normal vector field N with respect to the canonical metric of $\mathbb{R}^{n+1}$. Then there exists a point $z = F(x)$ so that $N_z = (\xi_1, ..., \xi_{n+1})$. Since F is a conformal embedding, it follows that the coordinates of ν_x are $\frac{1}{1+\cos r}(\xi_1, ..., \xi_{n+1})$ and $\tilde{\xi}_x = \nu_x$.

□

Proof of theorems 3.1 and 3.2: Throughout the proof we will denote by C the function $|B|_\infty$ in the Euclidean case and $|B|_\infty + \left(\frac{1-\cos R(M)}{\sin R(M)}\right)$ in the spherical case. On the other hand, we assume that the functions Φ_p are normalized so that $\Phi_p(0) = 1$. Let us put $\alpha^+_{p,\xi} = \|\varphi^-_{p,\xi}\|_{p-1}$ and $\alpha^-_{p,\xi} = \|\varphi^+_{p,\xi}\|_{p-1}$. Then, observe that

$$\begin{aligned}
&\int_M |\alpha^+_{p,\xi}\varphi^+_{p,\xi} + \alpha^-_{p,\xi}\varphi^-_{p,\xi}|^{p-2}(\alpha^+_{p,\xi}\varphi^+_{p,\xi} + \alpha^-_{p,\xi}\varphi^-_{p,\xi})dv \\
&= \int_M |\alpha^+_{p,\xi}\varphi^+_{p,\xi}|^{p-2}\alpha^+_{p,\xi}\varphi^+_{p,\xi}dv + \int_M |\alpha^-_{p,\xi}\varphi^-_{p,\xi}|^{p-2}\alpha^-_{p,\xi}\varphi^-_{p,\xi}dv \\
&= \|\varphi^-_{p,\xi}\|^{p-1}_{p-1}\int_M |\varphi^+_{p,\xi}|^{p-2}\varphi^+_{p,\xi}dv + \|\varphi^+_{p,\xi}\|^{p-1}_{p-1}\int_M |\varphi^-_{p,\xi}|^{p-2}\varphi^-_{p,\xi}dv
\end{aligned}$$

Since $\varphi^+_{p,\xi} \geq 0$ and $\varphi^-_{p,\xi} \leq 0$, it follows that $\int_M |\varphi^+_{p,\xi}|^{p-2}\varphi^+_{p,\xi}dv = \|\varphi^+_{p,\xi}\|^{p-1}_{p-1}$ and $\int_M |\varphi^-_{p,\xi}|^{p-2}\varphi^-_{p,\xi}dv = -\|\varphi^-_{p,\xi}\|_{p-1}$. This yields that

$$\int_M |\alpha^+_{p,\xi}\varphi^+_{p,\xi} + \alpha^-_{p,\xi}\varphi^-_{p,\xi}|^{p-2}(\alpha^+_{p,\xi}\varphi^+_{p,\xi} + \alpha^-_{p,\xi}\varphi^-_{p,\xi})dv = 0$$

Now, we can apply the variational characterization of $\lambda_{1,p}(M)$ to the test function $\varphi_{p,\xi} = \alpha^+_{p,\xi}\varphi^+_{p,\xi} + \alpha^-_{p,\xi}\varphi^-_{p,\xi}$ and we get

$$\begin{aligned}
\lambda_{1,p}(M)\int_M |\varphi_{p,\xi}|^p dv &\leq \int_M |d\varphi_{p,\xi}|^p dv \\
&= \int_M |\alpha^+_{p,\xi}|^p |d\varphi^+_{p,\xi}|^p dv + \int_M |\alpha^-_{p,\xi}|^p |d\varphi^-_{p,\xi}|^p dv \\
&\leq |\alpha^+_{p,\xi}|^p \int_M |\Phi'_p(\rho(\nu,\xi))|^p |C|^p \chi^+_\xi dv + |\alpha^-_{p,\xi}|^p \int_M |\Phi'_p(\rho(\nu,\xi))|^p |C|^p \chi^-_\xi dv
\end{aligned}$$

where in this last inequality, we have used the lemma 4.1 and where we have put $\chi^+_\xi(x) = \chi_{\{\langle \nu_x, \xi\rangle > 0\}}$, $\chi^-_\xi(x) = \chi_{\{\langle \nu_x, \xi\rangle < 0\}}$ for hypersurfaces of $\mathbb{R}^{n+1}$ and $\chi^+_\xi(x) = \chi_{\{\langle \nu_x, \tilde{\xi}_x\rangle > 0\}}$, $\chi^-_\xi(x) = \chi_{\{\langle \nu_x, \tilde{\xi}_x\rangle < 0\}}$ for hypersurfaces of $\mathbb{S}^{n+1}$. Now by integrating over $\mathbb{S}^n$ and denoting by $d\sigma$ the volume element of the sphere, we have

$$\begin{aligned}
&\lambda_{1,p}(M)\int_{\xi\in\mathbb{S}^n}\left(\int_M |\varphi_{p,\xi}|^p dv\right) d\sigma \\
&\leq \int_{\xi\in\mathbb{S}^n}\left(|\alpha^+_{p,\xi}|^p \int_M |\Phi'_p(\rho(\nu,\xi))|^p |C|^p \chi^+_\xi dv\right) d\sigma \\
&\qquad\qquad + \int_{\xi\in\mathbb{S}^n}\left(|\alpha^-_{p,\xi}|^p \int_M |\Phi'_p(\rho(\nu,\xi))|^p |C|^p \chi^-_\xi dv\right) d\sigma \\
&= \int_M |C|^p \left(\int_{\xi\in\mathbb{S}^n} |\alpha^+_{p,\xi}|^p |\Phi'_p(\rho(\nu,\xi))|^p \chi^+_\xi d\sigma\right) dv \\
&\qquad\qquad + \int_M |C|^p \left(\int_{\xi\in\mathbb{S}^n} |\alpha^-_{p,\xi}|^p |\Phi'_p(\rho(\nu,\xi))|^p \chi^-_\xi d\sigma\right) dv
\end{aligned}$$

$$\leq \max_{\xi\in\mathbb{S}^n} |\alpha^+_{p,\xi}|^p \int_M |C|^p \left(\int_{\xi\in\mathbb{S}^n} |\Phi'_p(\rho(\nu,\xi))|^p \chi^+_\xi d\sigma \right) dv$$
$$+ \max_{\xi\in\mathbb{S}^n} |\alpha^-_{p,\xi}|^p \int_M |C|^p \left(\int_{\xi\in\mathbb{S}^n} |\Phi'_p(\rho(\nu,\xi))|^p \chi^-_\xi d\sigma \right) dv$$

Now, for a fixed point $x \in M$, we have

$$\int_{\xi\in\mathbb{S}^n} |\Phi'_p(\rho(\nu_x,\xi))|^p \chi^+_\xi(x) d\sigma = \int_{\xi\in\mathbb{S}^n} |\Phi'_p(\rho(\nu_x,\xi))|^p \chi^-_\xi(x) d\sigma = \int_{\mathbb{S}^n_+} |d(\Phi_p \circ r)|^p d\sigma$$

where $r(\xi)$ is nothing but the distance function on the sphere between ν_x and ξ. Now, $\Phi_p \circ r$ is a first eigenfunction associated to the first Dirichlet eigenvalue $\lambda^D_{1,p}(\mathbb{S}^n_+)$ of $\mathbb{S}^n_+$. Therefore

$$\int_{\mathbb{S}^n_+} |d(\Phi_p \circ r)|^p d\sigma = \lambda^D_{1,p}(\mathbb{S}^n_+) \int_{\mathbb{S}^n_+} |\Phi_p \circ r|^p d\sigma$$

And since $\lambda^D_{1,p}(\mathbb{S}^n_+) = \lambda_{1,p}(\mathbb{S}^n)$, we deduce that

$$\lambda_{1,p}(M) \int_{\xi\in\mathbb{S}^n} \left(\int_M |\varphi_{p,\xi}|^p dv \right) d\sigma$$
$$\leq \lambda_{1,p}(\mathbb{S}^n)(\max_{\xi\in\mathbb{S}^n} |\alpha^+_{p,\xi}|^p + \max_{\xi\in\mathbb{S}^n} |\alpha^-_{p,\xi}|^p) \|C\|^p_p \int_{\mathbb{S}^n_+} |\Phi_p \circ r|^p d\sigma \qquad (2)$$

On the other hand

$$\int_{\xi\in\mathbb{S}^n} \left(\int_M |\varphi_{p,\xi}|^p dv \right) d\sigma$$
$$= \int_{\xi\in\mathbb{S}^n} |\alpha^+_{p,\xi}|^p \left(\int_M |\varphi^+_{p,\xi}|^p dv \right) d\sigma + \int_{\xi\in\mathbb{S}^n} |\alpha^-_{p,\xi}|^p \left(\int_M |\varphi^-_{p,\xi}|^p dv \right) d\sigma$$
$$\geq \min_{\xi\in\mathbb{S}^n} |\alpha^+_{p,\xi}|^p \int_M \left(\int_{\xi\in\mathbb{S}^n} |\varphi^+_{p,\xi}|^p d\sigma \right) dv + \min_{\xi\in\mathbb{S}^n} |\alpha^-_{p,\xi}|^p \int_M \left(\int_{\xi\in\mathbb{S}^n} |\varphi^-_{p,\xi}|^p d\sigma \right) dv$$

As previously, we have

$$\int_{\xi\in\mathbb{S}^n} |\varphi^+_{p,\xi}|^p d\sigma = \int_{\xi\in\mathbb{S}^n} |\varphi^-_{p,\xi}|^p d\sigma = \int_{\mathbb{S}^n_+} |\Phi_p \circ r|^p d\sigma$$

Thus

$$\int_{\xi\in\mathbb{S}^n}\left(\int_M|\varphi_{p,\xi}|^p dv\right)d\sigma\geq(\min_{\xi\in\mathbb{S}^n}|\alpha^+_{p,\xi}|^p+\min_{\xi\in\mathbb{S}^n}|\alpha^-_{p,\xi}|^p)V(M)\int_{\mathbb{S}^n_+}|\Phi_p\circ r|^p d\sigma \qquad (3)$$

Now note that the functions $\xi\mapsto\alpha^+_{p,\xi}$ and $\xi\mapsto\alpha^-_{p,\xi}$ are lower semicontinuous on $\mathbb{S}^n$. Indeed if $\xi_k\to\xi$, we have by using the Fatou's lemma,

$$\begin{aligned}\liminf(\alpha^+_{p,\xi_k})^{p-1}&=\liminf\int_M\Phi_p(\rho(\nu,\xi_k))^{p-1}\chi^+_{\xi_k}dv\\&\geq\int_M\liminf\left(\Phi_p(\rho(\nu,\xi_k))^{p-1}\chi^+_{\xi_k}\right)dv\\&\geq\int_M\Phi_p(\rho(\nu,\xi))^{p-1}\chi^+_\xi dv=(\alpha^+_{p,\xi})^{p-1}\end{aligned}$$

Since M is compact there exists $x\in M$ so that $\nu_x=\xi$ in the Euclidean case and $\nu_x=\tilde{\xi}_x$ in the spherical case (see the lemma 4.3). Therefore we have necessarily $\alpha^+_{p,\xi}$, $\alpha^-_{p,\xi}>0$, for any $\xi\in\mathbb{S}^n$, and by the lower semicontinuity of $\xi\mapsto\alpha^+_{p,\xi}$ and $\xi\mapsto\alpha^-_{p,\xi}$, we deduce that $\min_{\xi\in\mathbb{S}^n}|\alpha^+_{p,\xi}|^p+\min_{\xi\in\mathbb{S}^n}|\alpha^-_{p,\xi}|^p>0$. Consequently, combining (2) with (3), we obtain that

$$\lambda_{1,p}(M)V(M)\leq\left(\frac{\max_{\xi\in\mathbb{S}^n}|\alpha^+_{p,\xi}|^p+\max_{\xi\in\mathbb{S}^n}|\alpha^-_{p,\xi}|^p}{\min_{\xi\in\mathbb{S}^n}|\alpha^+_{p,\xi}|^p+\min_{\xi\in\mathbb{S}^n}|\alpha^-_{p,\xi}|^p}\right)\lambda_{1,p}(\mathbb{S}^n)\|C\|_p^p$$

Let us put $K(p,M)=\left(\frac{\max_{\xi\in\mathbb{S}^n}|\alpha^+_{p,\xi}|^p+\max_{\xi\in\mathbb{S}^n}|\alpha^-_{p,\xi}|^p}{\min_{\xi\in\mathbb{S}^n}|\alpha^+_{p,\xi}|^p+\min_{\xi\in\mathbb{S}^n}|\alpha^-_{p,\xi}|^p}\right)^{1/p}$. To complete the proof, we need to show that $\lim_{p\to\infty}K(p,M)=1$. For this we state the

Lemma 4.4 *For any $\xi\in\mathbb{S}^n$, we have*

$$\lim_{p\to\infty}\alpha^+_{p,\xi}=1\ \ and\ \ \lim_{p\to\infty}\alpha^-_{p,\xi}=1$$

Moreover the convergence is uniform with respect to ξ.

PROOF: Since Φ_p is decreasing on $(0,\frac{\pi}{2})$ (see [6]), we deduce from the fact that $\Phi_p(0)=1$, that

$$\int_{\mathbb{S}^n_+}|\Phi_p\circ r|^p d\sigma\leq V(\mathbb{S}^n_+)$$

Then fix an exponent $q>n$. For $p>q$, by Hölder's inequality we have

$$\left(\int_{\mathbb{S}^n_+}|d(\Phi_p\circ r)|^q d\sigma\right)^{1/q}\leq\left(\int_{\mathbb{S}^n_+}|d(\Phi_p\circ r)|^p d\sigma\right)^{1/p}V(\mathbb{S}^n_+)^{\frac{p-q}{pq}}$$

$$= \lambda_{1,p}^D(\mathbb{S}_+^n)^{1/p} \left(\int_{\mathbb{S}_+^n} |\Phi_p \circ r|^p d\sigma \right)^{1/p} V(\mathbb{S}_+^n)^{\frac{p-q}{pq}}$$
$$\leq \lambda_{1,p}(\mathbb{S}^n)^{1/p} V(\mathbb{S}_+^n)^{1/q}$$

Then $\Phi_p \circ r$ is uniformly bounded in $W_0^{1,q}(\mathbb{S}_+^n)$. Then we can select a subsequence $(p_k)_{k\geq 1})$ so that by the Sobolev embedding theorems $(\Phi_{p_k} \circ r)_k$ converges uniformly to a radial continuous function $\Phi_\infty \circ r$. Note that $\Phi_\infty(0) = 1$. Now, let us consider

$$T_{\xi,\varepsilon} = \left\{ x \in M / 1 - \frac{\varepsilon}{3} < \Phi_\infty(\rho(\nu_x, \xi)) \leq 1 \right\} \cap \{ x \in M / \langle \nu_x, \xi \rangle > 0 \}$$

for hypersurfaces of $\mathbb{R}^{n+1}$ and

$$T_{\xi,\varepsilon} = \left\{ x \in M / 1 - \frac{\varepsilon}{3} < \Phi_\infty(\rho(\nu_x, \xi)) \leq 1 \right\} \cap \left\{ x \in M / \left\langle \nu_x, \tilde{\xi}_x \right\rangle > 0 \right\}$$

for hypersurfaces of $\mathbb{S}^{n+1}$. Since M is compact, for any $\xi \in \mathbb{S}^n$, there exists $x \in M$ so that $\xi = \nu_x$ in the Euclidean case and $\tilde{\xi}_x = \nu_x$ for the spherical case. This implies that $V(T_{\xi,\varepsilon}) > 0$. Clearly, the function $\xi \mapsto V(T_{\xi,\varepsilon})$ is lower semicontinuous. From the compactness of $\mathbb{S}^n$, it follows that $\xi \mapsto V(T_{\xi,\varepsilon})$ achieves its minimum at a point ξ_0 of $\mathbb{S}^n$ and $\min_{\xi \in \mathbb{S}^n} V(T_{\xi,\varepsilon}) = V(T_{\xi_0,\varepsilon}) = V_\varepsilon > 0$. On the other hand, since $\Phi_{p_k} \to \Phi_\infty$ uniformly, there exists k_1 so that $\forall k \geq k_1$ and $\forall t \in [0, \frac{\pi}{2}]$, $\Phi_\infty(t) - \frac{\varepsilon}{3} \leq \Phi_{p_k}(t)$. Then

$$\alpha_{p_k,\xi}^+ = \|\varphi_{p_k,\xi}^+\|_{p_k - 1} = \left(\int_M |\Phi_{p_k}(\rho(\nu, \xi))|^{p_k - 1} dv \right)^{\frac{1}{p_k - 1}}$$
$$\geq \left(\int_{T_{\xi,\varepsilon}} |\Phi_{p_k}(\rho(\nu, \xi))|^{p_k - 1} dv \right)^{\frac{1}{p_k - 1}} \geq \left(1 - \frac{2\varepsilon}{3} \right) V_\varepsilon^{\frac{1}{p_k - 1}}$$

Now, there exists k_2, so that $\forall k \geq k_2$, $\left(1 - \frac{2\varepsilon}{3}\right) V_\varepsilon^{\frac{1}{p_k - 1}} \geq 1 - \varepsilon$. Moreover, from the normalization of Φ_p, we have $\|\varphi_{p_k,\xi}^+\|_{p_k - 1} \leq V(M)^{\frac{1}{p_k - 1}}$ and there exists k_3 so that $\forall k \geq k_3$, $V(M)^{\frac{1}{p_k - 1}} \leq 1 + \varepsilon$. Therefore, for any $\xi \in \mathbb{S}^n$, we have for any $k \geq \max(k_1, k_2, k_3)$

$$|\|\varphi_{p_k,\xi}^+\|_{p_k - 1} - 1| \leq \varepsilon$$

Finally, we have shown

$$\sup_{\xi \in \mathbb{S}^n} |\alpha_{p_k,\xi}^+ - 1| \to 0$$

and this is true for any other subsequence $(p_k)_k$. This completes the proof.

□

Now, we can compute the limit of $K(p,M)$. We have

$$1 \leq K(p,M) \leq \frac{2^{1/p} V(M)^{\frac{1}{p-1}}}{\min\limits_{\xi \in \mathbb{S}^n} \alpha^+_{p,\xi}}$$

And from the above lemma, it follows that $\lim\limits_{p\to\infty} \min\limits_{\xi\in\mathbb{S}^n} \alpha^+_{p,\xi} = 1$.

□

Remark We can establish for hypersurfaces of the hyperbolic space $\mathbb{H}^{n+1}$ a lower bound of $d(M)$ similar to the spherical case. Indeed, we can prove that if (M^n, g) is an oriented compact n-dimensional Riemannian manifold isometrically immersed in $\mathbb{H}^{n+1}$ and if $R(M)$ is the smallest radius of the ball containing M. Then

$$d(M) \geq \frac{\pi}{\kappa + \left(\dfrac{\cosh R(M) - 1}{\sinh R(M)}\right)}$$

But this inequality is clearly non optimal for geodesic spheres. To prove the above inequality, let $B(x_0, R(M))$ be the smallest ball of center x_0 and radius $R(M)$ containing M and consider the Minkowsky model for $\mathbb{H}^{n+1}$. Namely $\mathbb{H}^{n+1} = \{-y_0^2 + \sum\limits_{1\leq i\leq n+1} y_i^2 = -1, y_0 > 0\}$ and its metric is induced by the quadratic form $q(y,z) = -y_0 z_0 + \sum\limits_{1\leq i\leq n+1} y_i z_i$. Assuming that $x_0 = (1, 0, ..., 0)$, let $\xi \in T_{x_0}\mathbb{H}^{n+1}$, $|\xi| = 1$ and $\tilde{\xi}$ be the unit vector field defined for any $x \in \mathbb{H}^{n+1}$ by

$$\tilde{\xi}_x = \langle \xi, u\rangle \operatorname{grad} r\,|_x + \xi - \langle \xi, u\rangle\, u$$

where $r(x)$ is the geodesic distance on $\mathbb{H}^{n+1}$ between x_0 and x, $\operatorname{grad} r$ is the gradient of r and u is the unit vector of $T_{x_0}\mathbb{H}^{n+1}$ so that $x = \exp_{x_0}(r(x)u)$. As in the spherical case $\tilde{\xi}_x$ is the parallel transport of ξ along the geodesic joining x_0 and x. If X is a vector of $T_x\mathbb{H}^{n+1}$ so that $\langle X, \operatorname{grad} r\rangle = 0$, then

$$\nabla_X \tilde{\xi} = \left(\frac{\cosh r - 1}{\sinh r}\right)\left(-\left\langle \tilde{\xi}, X\right\rangle \operatorname{grad} r - \left\langle \tilde{\xi}, \operatorname{grad} r\right\rangle X\right)$$

where ∇ denotes the Riemannian connection of $\mathbb{H}^{n+1}$. Now by similar computations, we obtain

$$\left| d\left\langle \nu, \tilde{\xi}\right\rangle\right|^2 = |B\tilde{\xi}^T|^2 + 2\left(\frac{\cosh r - 1}{\sinh r}\right)\left\langle B\tilde{\xi}^T, \tilde{\xi}^T\right\rangle \langle \operatorname{grad} r, \nu\rangle + \left(\frac{\cosh r - 1}{\sinh r}\right)^2 |\tilde{\xi}^T|^2 \langle \operatorname{grad} r, \nu\rangle^2$$

But contrary to the spherical case we can only show a non optimal estimate

$$\left|d\left\langle \nu, \tilde{\xi}\right\rangle\right|^2 \leq \left(|B|_\infty + \left(\frac{\cosh r - 1}{\sinh r}\right)\right)^2 |\tilde{\xi}^T|^2.$$

References

[1] El Soufi A.,Ilias S. Une inegalité du type "Reilly" pour les sous-variétés de l'espace hyperbolique, *Comm. Math. Helv.*, **67**, (1992), 167-181.

[2] Grosjean J. F. Upper bounds for the first eigenvalue of the Laplacian on compact submanifolds, *Pac. J. Math.*, **206**, n0 1, (2002), 93-111.

[3] Heintze E. Extrinsic upper bound for λ_1, *Math. Ann.*, **280**, (1988), 389-402.

[4] Juutinen P., Lindqvist P., Manfredi J. J. The ∞-Eigenvalue Problem, *Arch. Rational Mech. Anal.*, **148**, (1999), 89-105.

[5] Lindqvist P. On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, *Proc. Am. Math. Soc.*, **109**, No 1, (1990), 157-164.

[6] Matei A.-M. First eigenvalue for the p-Laplace operator, *Nonlinear Anal., Theory Methods Appl.*, **39A**, No.8, (2000), 1051-1068.

[7] Reilly R. On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space, *Comment. Math. Helv.*,**52**,(1977),525-533.

[8] Serrin J. Local behavior of solutions of quasi-linear equations, *Acta Math.*, **111**, (1964), 247-302 (1964).

[9] Tolksdorf P. Regularity for a more general class of quasilinear elliptic equations, *J. Diff. Equ.*, **51**, (1984), 126-150.