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# A PROBABILISTIC REPRESENTATION OF CONSTANTS IN KESTEN'S RENEWAL THEOREM

NATHANAËL ENRIQUEZ, CHRISTOPHE SABOT, AND OLIVIER ZINDY

**Abstract.** The aims of this paper are twofold. Firstly, we derive some probabilistic representation for the constant which appears in the one-dimensional case of Kesten's renewal theorem. Secondly, we estimate the tail of some related random variable which plays an essential role in the description of the stable limit law of one-dimensional transient sub-ballistic random walks in random environment.

## 1. INTRODUCTION

In 1973, Kesten published a famous paper [9] about the tail estimates of renewal series of the form  $\sum_{i \geq 1} A_1 \dots A_{i-1} B_i$ , where  $(A_i)_{i \geq 0}$  is a sequence of non-negative i.i.d.  $d \times d$  random matrices and  $(B_i)_{i \geq 1}$  is a sequence of i.i.d. random vectors of  $\mathbb{R}^d$ . His result states that the tail of the projection of this random vector on every direction is equivalent to  $Ct^{-\kappa}$ , when  $t$  tends to infinity, where  $C$  and  $\kappa$  are positive constants. The constant  $\kappa$  is defined as the solution of the equation  $k(s) = 1$ , with  $k(s) := \lim_{n \rightarrow \infty} \mathbb{E}(\|A_1 \dots A_n\|^s)^{1/n}$ . The proof of his result in the one-dimensional case, even if it is much easier than in dimension  $d \geq 2$ , is already rather complicated.

Even if we are concerned by the one-dimensional case in this paper, let us mention that a significant generalization of Kesten's result, in the multi-dimensional case, was recently achieved by de Saporta, Guivarc'h and Le Page [3], who relaxed the assumption of positivity on  $A_i$ .

In 1991, Goldie [7] relaxed, in dimension  $d = 1$ , the assumption of positivity on the  $A_i$  and simplified Kesten's proof. Furthermore, he obtained a formula for the implicit constant  $C$  in the special case where  $A_i$  is non-negative and  $\kappa$  is an integer.

In 1991, Chamayou and Letac [1] observed that, in dimension  $d = 1$ , if  $A_i$  has the same law as  $(1 - X_i)/X_i$ , with  $X_i$  following a Beta distribution on  $(0, 1)$ , then the law of the series itself is computable so that the constant  $C$  is explicit in this special case also. The following question was then asked. How does one effectively compute the constant  $C$ ?

In our framework, we consider the case  $d = 1$  and we make the following assumptions:  $\rho_i = A_i$  is a sequence of i.i.d. positive random variables,  $B_i = 1$  and there exists  $\kappa$  such that  $\mathbb{E}(\rho_1^\kappa) = 1$ . Moreover, we assume a weak integrability condition and that the law of  $\log \rho_i$ , which has a negative expectation by the previous assumptions, is non-arithmetic. In this context we are interested in the random series

$$R = 1 + \sum_{k \geq 1} \rho_1 \cdots \rho_k.$$

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The previous assumptions ensure that the tail of the renewal series is equivalent to  $C_K t^{-\kappa}$ , when  $t$  tends to infinity. We are now aiming at finding a probabilistic representation of the constant  $C_K$ .

Besides, this work is motivated by the study of one-dimensional random walks in random environment. In [10], Kesten, Kozlov and Spitzer proved, using the tail estimate derived in [9], that when the RWRE is transient with null asymptotic speed, then the behavior depends on an index  $\kappa \leq 1$ : the RWRE  $X_n$  normalized by  $n^{1/\kappa}$  converges in law to  $C_\kappa \left(\frac{1}{\mathcal{S}_\kappa}\right)^\kappa$  where  $\mathcal{S}_\kappa$  is a positive stable random variable with index  $\kappa$ . The computation of the explicit value of  $C_\kappa$  was left open. In [5], the authors derive an explicit expression, either in terms of the Kesten's constant  $C_K$  when it is explicit, or in terms of the expectation of a random series when  $C_K$  is not explicit. To this end, we need to obtain a tail estimate for a random variable closely related to the random series  $R$ , and to relate it to Kesten's constant. This is the other aim of this paper.

The strategy of our proof is based on a coupling argument in the spirit of the coupling argument used to derive the renewal theorem (cf [4], 4.3). We first interpret  $\rho_1 \dots \rho_n$  as the exponential of a random walk  $(V_n, n \geq 0)$ , which is negatively drifted, since  $\mathbb{E}(\log \rho_1) < 0$ . We have now to deal with the series  $R := \sum_{n \geq 0} e^{V_n}$ . One can write

$$R = e^S \sum_{n \geq 0} e^{V_n - S},$$

where  $S$  is the maximum of  $(V_n, n \geq 0)$ . The heuristic is that  $S$  and  $\sum_{n \geq 0} e^{V_n - S}$  are asymptotically independent. The coupling argument is used to derive this asymptotic independence. But, in order to implement this strategy, several difficulties have to be overcome: we first need to condition  $S$  to be large. Moreover, we have to couple conditioned processes: this requires to describe precisely the part of the process  $(V_0, \dots, V_{T_S})$ , where  $T_S$  is the first hitting time of the level  $S$ .

## 2. NOTATIONS AND STATEMENT OF THE RESULTS

Let  $(\rho_i)_{i \in \mathbb{Z}}$  be a sequence of i.i.d. positive random variables with law  $Q = \mu^{\otimes \mathbb{Z}}$ . With the sequence  $(\rho_i)_{i \in \mathbb{Z}}$  we associate the potential  $(V_k)_{k \in \mathbb{Z}}$  defined by

$$V_n := \begin{cases} \sum_{k=1}^n \log \rho_k & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -\sum_{k=n+1}^0 \log \rho_k & \text{if } n \leq -1. \end{cases}$$

Let  $\rho$  having law  $\mu$ . Suppose now that the law  $\mu$  is such that there is  $\kappa > 0$  satisfying

$$(1) \quad \mathbb{E}^\mu(\rho^\kappa) = 1 \quad \text{and} \quad \mathbb{E}^\mu(\rho^\kappa \log^+ \rho) < \infty.$$

Moreover, we assume the distribution of  $\log \rho$  is non-lattice. Then the law  $\mu$  is such that  $\log \rho$  is integrable and that

$$(2) \quad \mathbb{E}^\mu(\log \rho) < 0,$$

which implies

$$\lim_{n \rightarrow \infty} \frac{V_n}{n} = \int \log \rho d\mu < 0,$$

under  $Q$ .

We set

$$S := \max\{V_k, k \geq 0\},$$

and

$$H := \max\{V_k, 0 \leq k \leq T_{\mathbb{R}_-}\},$$

where  $T_{\mathbb{R}_-}$  is the first positive hitting time of  $\mathbb{R}_-$ :

$$T_{\mathbb{R}_-} := \inf\{k > 0, V_k \leq 0\}.$$

The random variable  $S$  is the absolute maximum of the path  $(V_k)_{k \geq 0}$  while  $H$  is the maximum of the first positive excursion. We also set

$$T_S := \inf\{k \geq 0, V_k = S\}, \quad T_H := \inf\{k \geq 0, V_k = H\}.$$

We clearly have,  $Q$ -almost surely,

$$H \leq S < \infty, \quad T_H \leq T_S < \infty.$$

The following tail estimate for  $S$  is a classical consequence of renewal theory, see [6],

$$(3) \quad \mathbb{P}^Q(e^S \geq t) \sim C_F t^{-\kappa},$$

when  $t \rightarrow \infty$ , where

$$C_F = \frac{1 - \mathbb{E}^Q(e^{\kappa V(T_{\mathbb{R}_-)})})}{\kappa \mathbb{E}^\mu(\rho^\kappa \log \rho) \mathbb{E}^Q(T_{\mathbb{R}_-})}.$$

The tail estimate of  $H$  is derived by Iglehart, in [8],

$$(4) \quad \mathbb{P}^Q(e^H \geq t) \sim C_I t^{-\kappa},$$

when  $t \rightarrow \infty$ , where

$$C_I = \frac{(1 - \mathbb{E}^Q(e^{\kappa V(T_{\mathbb{R}_-)})})^2}{\kappa \mathbb{E}^\mu(\rho^\kappa \log \rho) \mathbb{E}^Q(T_{\mathbb{R}_-})} = (1 - \mathbb{E}^Q(e^{\kappa V(T_{\mathbb{R}_-)})}) C_F.$$

Consider now the random variable

$$R := \sum_{n=0}^{\infty} e^{V_n}.$$

This random variable clearly satisfies the following renewal equation

$$R \stackrel{\text{law}}{=} 1 + \rho R,$$

where  $\rho$  is a random variable with law  $\mu$  independent of  $R$ . In [9], Kesten proved (actually his result was more general and concerned by the multidimensional version of this one) that there exists a positive constant  $C_K$  such that

$$(5) \quad \mathbb{P}^Q(R \geq t) \sim C_K t^{-\kappa},$$

when  $t \rightarrow \infty$ . The constant  $C_K$  has been made explicit in some particular cases: for  $\kappa$  integer by Goldie, see [7], and when  $\rho \stackrel{\text{law}}{=} \frac{W}{1-W}$  where  $W$  is a beta variable, by Chamayou and Letac [1]. One aim of this paper is to derive an expression of this constant in terms of the expectation of some simple random variable.

We need now to introduce some Girsanov transform of  $Q$ . Thanks to (1) we can define the law

$$\tilde{\mu} = \rho^\kappa \mu,$$

and the law  $\tilde{Q} = \tilde{\mu}^{\otimes \mathbb{Z}}$  which is the law of a sequence of i.i.d. random variables with law  $\tilde{\mu}$ . The definition of  $\kappa$  implies that

$$\int \log \rho \tilde{\mu}(d\rho) > 0,$$

and thus that

$$\lim_{n \rightarrow \infty} \frac{V_n}{n} = \int \log \rho d\tilde{\mu} > 0,$$

under  $\tilde{Q}$ . Moreover,  $\tilde{Q}$  is a Girsanov transform of  $Q$ , i.e. we have for all  $n$

$$\mathbb{E}^Q(\phi(V_0, \dots, V_n)) = \mathbb{E}^{\tilde{Q}}(e^{-\kappa V_n} \phi(V_0, \dots, V_n)),$$

for any bounded test function  $\phi$ . Let us now introduce the random variable  $M$  defined by

$$(6) \quad M = \sum_{k < 0} e^{-V_k} + \sum_{k \geq 0} e^{-V_k},$$

where  $(V_k)_{k < 0}$  is distributed under  $Q(\cdot | V_k \geq 0, \forall k < 0)$  and independent of  $(V_k)_{k \geq 0}$  which is distributed under  $\tilde{Q}(\cdot | V_k > 0, \forall k > 0)$ .

**Theorem 2.1.** *i) We have the following tail estimate*

$$\mathbb{P}^Q(R \geq t) \sim C_K t^{-\kappa},$$

when  $t \rightarrow \infty$ , where

$$C_K = C_F \mathbb{E}(M^\kappa).$$

*ii) We have*

$$\mathbb{P}^Q(R \geq t; H = S) \sim C_{KI} t^{-\kappa},$$

when  $t \rightarrow \infty$ , where

$$C_{KI} := C_I \mathbb{E}(M^\kappa).$$

**Remark 2.1 :** The conditioning  $H = S$  means that the path  $(V_k)_{k \geq 0}$  never goes above the height of its first excursion. This conditioning appears naturally in [5], and is useful to return the path, cf Section 4.

In [5], we need some tail estimate on a random variable of the type of  $R$  but with an extra term. Let us introduce the event

$$(7) \quad \mathcal{I} := \{H = S\} \cap \{V_k \geq 0, \forall k \leq 0\},$$

and the random variable

$$Z := e^S M_1 M_2,$$

where

$$M_1 := \sum_{k=-\infty}^{T_S} e^{-V_k},$$

$$M_2 := \sum_{k=0}^{\infty} e^{V_k - S}.$$

**Theorem 2.2.** *We have the following tail estimate*

$$\mathbb{P}^Q(Z \geq t | \mathcal{I}) \sim \frac{1}{\mathbb{P}^Q(H = S)} C_U t^{-\kappa},$$

when  $t \rightarrow \infty$ , where

$$C_U = C_I \mathbb{E}(M^\kappa)^2 = \frac{C_I}{C_F} (C_K)^2.$$

**Remark 2.2 :** The factor  $1/\mathbb{P}^Q(H = S)$  in the last formula just comes from the fact that the event  $\{H = S\}$  is now in the conditioning. But this term will not play any role in [5].

**Remark 2.3 :** In [5] we need to evaluate  $C_U$ : when  $C_K$  is not explicit it is better to use the expression of  $C_U$  in terms of  $\mathbb{E}(M^\kappa)$  which is easy to evaluate numerically. When  $C_K$  is explicit, we use the expression in terms of  $C_K$ .

### Sketch of the proof and organization of the paper

The intuition behind Theorem 2.1 and Theorem 2.2 is the following. Let us first consider  $\mathbb{P}^Q(R \geq t | H = S)$ . The law  $Q(\cdot | \mathcal{I})$  has a symmetry property which implies that the variable  $R = M_2 e^H$  has the same distribution as  $M_1 e^H$  (cf Section 4). Then, the proof of Theorem 2.1 is based on the following arguments.

Firstly, we prove that the variables  $M_1$  and  $e^H$  are asymptotically independent. To this end, we use a delicate coupling argument which works only when  $H$  is conditioned to be large. Therefore, we need to restrict ourselves to large values of  $H$ . To this end, we need to control the value of  $R$  conditioned by  $H$ ; this is done in Section 6. Then, a second difficulty is that we have to couple conditioned processes (namely, the process  $(V_k)$  conditioned to have a first high excursion). We overcome this difficulty by using an explicit description of the law of the path  $(V_0, \dots, V_{T_H})$ . Namely, the path  $(V_0, \dots, V_{T_H})$  behaves like  $V$  under  $\tilde{Q}(\cdot | V_k > 0, \forall k > 0)$  stopped at some random time.

Secondly, we observe that the distribution of  $M_1$  is close to the distribution of  $M$  as a consequence of the above description of the law of  $(V_0, \dots, V_{T_H})$ .

From these two facts, we deduce that  $\mathbb{P}^Q(R \geq t | \mathcal{I}) \simeq \mathbb{P}^Q(M e^H \geq t | \mathcal{I})$ , where  $M$  and  $H$  are roughly independent. Using the tail estimate for  $H$  we get the part ii) of Theorem 2.1. For Theorem 2.2, we proceed similarly: the variable  $Z$  writes  $M_1 R$  and, for large  $H$ , the variables  $M_1$  and  $R$  are asymptotically independent and the law of  $M_1$  is close to the law of  $M$ . Then the estimate on the tail of  $R$  allows to conclude.

Let us now describe the organization of the proofs. In Section 3, we prove that  $M$  has finite moments of all orders and we estimate the rest of the series  $M$ . Section 4 contains some preliminary properties of the law  $Q(\cdot | \mathcal{I})$ , and Section 5 presents a representation of the law of the process  $(V_0, \dots, V_S)$  in terms of the law  $\tilde{Q}$ . Section 6 contains crucial estimates which will allow to restrict ourselves to large values of  $H$ . In Section 7, we detail the coupling arguments which roughly give the asymptotic independence of  $M_1$  and  $e^H M_2$ . Finally, in Section 8 we compile the arguments of the previous sections to prove Theorem 2.2 and Theorem 2.1. In Section 9, we present a Tauberian version of these tail estimates, which is the version we ultimately use in [5].

3. MOMENTS OF  $M$ 

Here is a series of three lemmas about the moment of the exponential functional of the meanders. In this section, we denote by  $\{V \geq -L\}$  the event  $\{V_k \geq -L, k \geq 0\}$ .

**Lemma 3.1.** *There exists  $C > 0$  such that, for all  $L \geq 0$ ,*

$$\mathbb{E}^{\tilde{Q}}\left(\sum_{k \geq 0} e^{-V_k} \mid V \geq -L\right) \leq C e^L.$$

*Proof.* Using Markov inequality, we get

$$\mathbb{E}^{\tilde{Q}}\left(\sum_{k \geq 0} e^{-V_k} \mid V \geq -L\right) \leq \sum_{k \geq 1} \frac{1}{k^2} + \sum_{k \geq 1} \tilde{Q}\left(e^{-V_k} \geq \frac{1}{k^2} \mid V \geq -L\right) e^L.$$

Since  $\mathbb{P}^{\tilde{Q}}(V \geq -L) \geq \mathbb{P}^{\tilde{Q}}(V \geq 0) > 0$ , for all  $L \geq 0$ ,

$$\tilde{Q}\left(e^{-V_k} \geq \frac{1}{k^2} \mid V \geq -L\right) = \tilde{Q}(V_k \leq 2 \log k \mid V \geq -L) \leq C \tilde{Q}(V_k \leq 2 \log k).$$

Now, since large deviations do occur, we get, from Cramer's theory, see [2], that  $\mathbb{E}^{\tilde{Q}}(\log \rho_0) > 0$  implies that the sequence  $\tilde{Q}(V_k \leq 2 \log k)$  is exponentially decreasing.

The sum  $\sum_{k \geq 1} \tilde{Q}(e^{-V_k} \geq \frac{1}{k^2} \mid V \geq -L)$  is therefore bounded uniformly in  $L$ , and the result follows.  $\square$

**Lemma 3.2.** *Under  $\tilde{Q}^{\geq 0} := \tilde{Q}(\cdot \mid V_k \geq 0, \forall k \geq 0)$ , all the moments of  $\sum_{k \geq 0} e^{-V_k}$  are finite.*

*Proof.* Let us treat, for more readability, the case of the second moment. Observe first that

$$\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\left(\sum_{i \geq 0} e^{-V_i}\right)^2\right) \leq 2\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\sum_{i \geq 0} e^{-2V_i} \left(\sum_{j \geq i} e^{-(V_j - V_i)}\right)\right).$$

Applying the Markov property to the process  $V$  under  $\tilde{Q}$  at time  $i$ , we get

$$\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\left(\sum_{i \geq 0} e^{-V_i}\right)^2\right) \leq 2\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\sum_{i \geq 0} e^{-2V_i} \mathbb{E}^{\tilde{Q}}\left[\sum_{l \geq 0} e^{-V'_l} \mid V' \geq -V_i\right]\right),$$

where  $V'$  is a copy of  $V$  independent of  $(V_k)_{0 \leq k \leq i}$ . Now, we use Lemma 3.1 to get the upper bound

$$C\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\sum_{i \geq 0} e^{-2V_i} \times e^{V_i}\right) \leq C\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\sum_{i \geq 0} e^{-V_i}\right),$$

which is finite, again by applying Lemma 3.1. This scheme is then easily extended to higher moments.  $\square$

We will need further a finer result than Lemma 3.1, we state here.

**Lemma 3.3.** *There exists  $C > 0$  such that, for all  $L > 0$  and for all  $\varepsilon' > 0$ , we have*

- if  $\kappa < 1$ ,

$$\mathbb{E}^{\tilde{Q}}\left(\sum_{i \geq 0} e^{-V(i)} \mid V \geq -L\right) \leq C e^{(1-\kappa+\varepsilon')L},$$

- if  $\kappa \geq 1$ ,

$$\mathbb{E}^{\tilde{Q}}\left(\sum_{i \geq 0} e^{-V(i)} \mid V \geq -L\right) \leq C e^{\varepsilon' L}.$$

*Proof.* Let  $\alpha \in [0, 1]$  and define  $T_{(-\infty, -\alpha L]} := \min\{i \geq 0 : V_i \leq -\alpha L\}$ . Let us write

$$\begin{aligned} \sum_{i \geq 0} e^{-V_i} &= \left(\sum_{i \geq 0} e^{-V_i}\right) \mathbf{1}_{\{V > -\alpha L\}} \\ &+ \left(\sum_{i=0}^{T_{(-\infty, -\alpha L]}-1} e^{-V_i} + \sum_{i=T_{(-\infty, -\alpha L]}}^{\infty} e^{-V_i}\right) \mathbf{1}_{\{T_{(-\infty, -\alpha L]} < \infty\}}. \end{aligned}$$

Now, since  $\tilde{Q}(V \geq -A)$  is uniformly bounded from below, for  $A > 0$ , by  $\tilde{Q}(V > 0) > 0$ , we obtain that  $\mathbb{E}^{\tilde{Q}}(\sum_{i \geq 0} e^{-V_i} \mid V \geq -L)$  is less or equal than

$$\begin{aligned} (8) \quad & C \mathbb{E}^{\tilde{Q}}\left(\sum_{i \geq 0} e^{-V_i} \mid V \geq -\alpha L\right) \\ & + C \mathbb{E}^{\tilde{Q}}\left(\sum_{i < T_{(-\infty, -\alpha L]}} e^{-V_i}; T_{(-\infty, -\alpha L]} < \infty; V \geq -L\right) \\ & + C \mathbb{E}^{\tilde{Q}}\left(\sum_{i \geq T_{(-\infty, -\alpha L]}} e^{-V_i}; T_{(-\infty, -\alpha L]} < \infty; V \geq -L\right). \end{aligned}$$

Lemma 3.1 bounds the first term in (8) from above by  $C e^{\alpha L}$ . Furthermore,  $i < T_{(-\infty, -\alpha L]}$  implies  $e^{-V_i} \leq e^{\alpha L}$ . Therefore,  $C e^{\alpha L} \mathbb{E}^{\tilde{Q}}(T_{(-\infty, -\alpha L]} \mathbf{1}_{\{T_{(-\infty, -\alpha L]} < \infty\}})$  is an upper bound for the second term in (8), which is treated as follows,

$$\begin{aligned} \mathbb{E}^{\tilde{Q}}(T_{(-\infty, -\alpha L]} \mathbf{1}_{\{T_{(-\infty, -\alpha L]} < \infty\}}) &\leq \sum_{k \geq 0} k \tilde{Q}(T_{(-\infty, -\alpha L]} = k) \\ &\leq \sum_{k \geq 0} k \tilde{Q}(V_k \leq -\alpha L) \\ &\leq \sum_{k \geq 0} k e^{-k\theta \tilde{I}(-\frac{\alpha L}{k})} e^{-k(1-\theta)\tilde{I}(-\frac{\alpha L}{k})}, \end{aligned}$$

where  $0 < \theta < 1$  and  $\tilde{I}$  denotes the rate function associated with  $\tilde{P}$  which is positive convex and admits a unique minimum on  $\mathbb{R}_+$ . We can therefore bound by below all the terms  $\tilde{I}(-\frac{\alpha L}{k})$  by  $\tilde{I}(0) > 0$ . Moreover, a more sophisticated result yields  $\sup_{x \leq 0} \tilde{I}(x)/x \leq -\kappa$  (see definition of  $\kappa$  and formula (2.2.10) in ([2], p. 28)). Therefore, we obtain

$$\mathbb{E}^{\tilde{Q}}(T_{(-\infty, -\alpha L]} \mathbf{1}_{\{T_{(-\infty, -\alpha L]} < \infty\}}) \leq e^{-\theta \kappa \alpha L} \sum_{k \geq 0} k e^{-k(1-\theta)\tilde{I}(0)} \leq C e^{-\theta \kappa \alpha L}.$$

As a result, the second term in (8) is bounded by  $C e^{(1-\theta \kappa \alpha)L}$ .



Finally, concerning the third term in (8), we have that

$$\begin{aligned}
& C \mathbb{E}^{\tilde{Q}} \left( \sum_{i \geq T_{(-\infty, -\alpha L]}} e^{-V_i}; T_{(-\infty, -\alpha L]} < \infty; V \geq -L \right) \\
& \leq C \mathbb{E}^{\tilde{Q}} \left( e^{-V_{T_{(-\infty, -\alpha L]}}} \sum_{i \geq T_{(-\infty, -\alpha L]}} e^{-(V_i - V_{T_{(-\infty, -\alpha L]}})}; T_{(-\infty, -\alpha L]} < \infty; V \geq -L \right) \\
& \leq C \mathbb{E}^{\tilde{Q}} \left( e^{-V_{T_{(-\infty, -\alpha L]}}} \mathbf{1}_{\{T_{(-\infty, -\alpha L]} < \infty\}} \mathbb{E}^{\tilde{Q}} \left( \sum_{i \geq 0} e^{-V'_i} \mid V' \geq -(L + V_{T_{(-\infty, -\alpha L]}}) \right) \right),
\end{aligned}$$

where  $V'_i := V_{T_{(-\infty, -\alpha L]}+i} - V_{T_{(-\infty, -\alpha L]}}$ , for  $i \geq 0$ . The last inequality is a consequence of the strong Markov property applied at  $T_{(-\infty, -\alpha L]}$ , which implies that  $(V'_i, i \geq 0)$  is a copy of  $(V_i, i \geq 0)$  independent of  $(V_i, 0 \leq i \leq T_{(-\infty, -\alpha L]})$ . Then, Lemma 3.1 yields that the third term in (8) is less than

$$\begin{aligned}
& C \mathbb{E}^{\tilde{Q}} \left( e^{-V_{T_{(-\infty, -\alpha L]}}} \mathbf{1}_{\{T_{(-\infty, -\alpha L]} < \infty\}} e^{L + V_{T_{(-\infty, -\alpha L]}}} \right) \\
& \leq C e^L \tilde{Q}(T_{(-\infty, -\alpha L]} < \infty) \leq C e^{(1-\kappa\alpha)L}.
\end{aligned}$$

Since  $\theta < 1$  implies  $1 - \theta\kappa\alpha \geq 1 - \kappa\alpha$ , we optimize the value of  $\alpha$  by taking  $\alpha = -\alpha\kappa\theta + 1$ , i.e.  $\alpha = 1/(1 + \kappa\theta)$ . As a result, we get already a finer result than Lemma 3.1 with a bound  $e^{\frac{L}{1+\kappa\theta}}$  instead of  $e^L$ .

Now, the strategy is to use this ameliorated estimation instead of Lemma 3.1 and repeat the same procedure. Like that, we obtain recursively a sequence of bounds, we denote by  $Ce^{u_n L}$ . The first term in (8) is bounded by  $Ce^{\alpha u_n L}$  whereas the second and the third term are still bounded respectively by  $Ce^{(1-\kappa\alpha\theta)L}$  and  $Ce^{(1-\kappa\alpha)L}$ .

Optimizing in  $\alpha$  again, one chooses  $\alpha u_n = -\alpha\kappa\theta + 1$ , i.e.  $\alpha = \frac{1}{u_n + \kappa\theta}$ . The new exponent is therefore  $u_{n+1} = \alpha u_n = \frac{u_n}{u_n + \kappa\theta}$ . Thus, the sequence  $u_n$  is homographic and converges to a limit satisfying  $l = \frac{l}{l + \kappa\theta}$ . For  $\kappa\theta \leq 1$ , the limit is  $l = 1 - \kappa\theta$  and for  $\kappa\theta \geq 1$ , the limit is 0. Since this result holds for any  $0 < \theta < 1$ , it concludes the proof of Lemma 3.3.  $\square$

**Remark 3.1** : Analogous results as in Lemma 3.1, Lemma 3.2 and Lemma 3.3 apply for  $\sum_{k \geq 0} e^{V_k}$  under  $Q$  and conditionally on the event  $\{V_k \leq L, \forall k \geq 0\}$ .

#### 4. A TIME REVERSAL

Let us denote by  $Q^{\mathcal{I}}$  the conditional law  $Q^{\mathcal{I}}(\cdot) := Q(\cdot | \mathcal{I})$ , where  $\mathcal{I}$  is defined in (7). The law  $Q^{\mathcal{I}}$  has the following symmetry property.

**Lemma 4.1.** *Under  $Q^{\mathcal{I}}$  we have the following equality in law*

$$(V_k)_{k \in \mathbb{Z}} \stackrel{\text{law}}{=} (V_{T_H} - V_{T_H - k})_{k \in \mathbb{Z}}.$$

*Proof.* Let  $\phi$  be a positive test function. We have

$$\begin{aligned} & \mathbb{E}^{Q^{\mathcal{I}}}(\phi((V_{T_H} - V_{T_H-k})_{k \geq 0})) \\ &= \sum_{p=0}^{\infty} \mathbb{E}^{Q^{\mathcal{I}}}(\mathbb{1}_{T_H=p} \phi((V_p - V_{p-k})_{k \geq 0})) \\ &= \frac{1}{\mathbb{P}^Q(\mathcal{I})} \sum_{p=0}^{\infty} \mathbb{E}^Q(\mathbb{1}_{\{V_k \geq 0, \forall k \leq 0\}} \mathbb{1}_{\{V_k \leq V_p, \forall k \geq p\}} \mathbb{1}_{\{0 < V_k < V_p, \forall 0 < k < p\}} \phi((V_p - V_{p-k})_{k \geq 0})). \end{aligned}$$

By construction we have, for all  $p \geq 0$ ,

$$(V_p - V_{p-k})_{k \in \mathbb{Z}} \stackrel{\text{law}}{=} (V_k)_{k \in \mathbb{Z}}.$$

This implies that

$$\begin{aligned} & \mathbb{E}^{Q^{\mathcal{I}}}(\phi((V_{T_H} - V_{T_H-k})_{k \geq 0})) \\ &= \frac{1}{\mathbb{P}^Q(\mathcal{I})} \sum_{p=0}^{\infty} \mathbb{E}^Q(\mathbb{1}_{\{V_k \geq 0, \forall k \leq 0\}} \mathbb{1}_{\{V_k \leq V_p, \forall k \geq p\}} \mathbb{1}_{\{0 < V_k < V_p, \forall 0 < k < p\}} \phi((V_k)_{k \geq 0})) \\ &= \mathbb{E}^{Q^{\mathcal{I}}}(\phi((V_k)_{k \in \mathbb{Z}})). \end{aligned}$$

□

This implies that under  $Q^{\mathcal{I}}$ ,  $R$  has the law of  $e^H M_1$ . This last formula will be useful since the asymptotic independence of  $e^H$  and  $M_1$ , in the limit of large  $H$ , is more visible than the asymptotic independence of  $H$  and  $M_2$  and will be easier to prove.

## 5. THE TWO FACES OF THE MOUNTAIN

It will be convenient to introduce the following notations: we denote by  $Q^{\leq 0}$  the conditional law

$$Q^{\leq 0}(\cdot) = Q(\cdot | V_k \leq 0, \forall k \geq 0),$$

and by  $\tilde{Q}^{>0}$  the conditional law

$$\tilde{Q}^{>0}(\cdot) = \tilde{Q}(\cdot | V_k > 0, \forall k > 0).$$

It will be useful to describe the law of the part of the path  $(V_0, \dots, V_{T_S})$ . Let us introduce some notations. If  $(Y_k)_{k \geq 0}$  is a random process under the law  $\tilde{Q}$ , then  $Y_k \rightarrow +\infty$  a.s. and we can define its strict increasing ladder times  $(e_k)_{k \geq 0}$  by:  $e_0 := 0$ , and

$$e_{k+1} := \inf\{n > e_k, Y_n > Y_{e_k}\}.$$

We define a random variable  $((Y_k)_{k \geq 0}, \Theta)$  with values in  $\mathbb{R}^{\mathbb{N}} \times \mathbb{N}$  as follows: the random process  $(Y_k)_{k \geq 0}$  has a law with density with respect to  $\tilde{Q}$  given by

$$\frac{1}{\mathcal{Z}} \left( \sum_{k=0}^{\infty} e^{-\kappa Y_{e_k}} \right) \tilde{Q},$$

where  $\mathcal{Z}$  is the normalizing constant given by

$$\mathcal{Z} = \frac{1}{1 - \mathbb{E}^{\tilde{Q}}(e^{-\kappa Y_{e_1}})}.$$

Then, conditionally on  $(Y_k)_{k \geq 0}$ ,  $\Theta$  takes one of the value of the strict ladder times with probability

$$\mathbb{P}(\Theta = e_p \mid \sigma((Y_k)_{k \geq 0})) = \frac{e^{-\kappa Y_{e_p}}}{\sum_{k=0}^{\infty} e^{-\kappa Y_{e_k}}}.$$

We denote by  $\hat{Q}$  the law of  $((Y_k)_{k \geq 0}, \Theta)$ . Otherwise stated, it means that, for all test function  $\phi$ ,

$$\mathbb{E}^{\hat{Q}}(\phi(\Theta, (Y_n)_{n \geq 0})) = \frac{1}{\mathcal{Z}} \mathbb{E}^{\hat{Q}} \left( \sum_{k=0}^{\infty} e^{-\kappa Y_{e_k}} \phi(e_k, (Y_n)_{n \geq 0}) \right).$$

**Lemma 5.1.** *The processes  $(V_0, \dots, V_{T_S})$  and  $(V_{T_S+k} - V_{T_S})_{k \geq 0}$  are independent and have the following laws:  $(V_{T_S+k} - V_{T_S})_{k \geq 0}$  has the law  $Q^{\leq 0}$  and*

$$(V_0, \dots, V_{T_S}) \stackrel{law}{=} (Y_0, \dots, Y_{\Theta}),$$

where  $((Y_k)_{k \geq 0}, \Theta)$  has the law  $\hat{Q}$ .

*Proof.* Let  $\psi, \theta$  be positive test functions. Let us compute

$$\begin{aligned} & \mathbb{E}^Q(\psi((V_{T_S+k} - V_{T_S})_{k \geq 0})\theta((V_0, \dots, V_{T_S}))) \\ &= \sum_{p=0}^{\infty} \mathbb{E}^Q(\mathbb{1}_{T_S=p} \psi((V_{p+k} - V_p)_{k \geq 0})\theta((V_0, \dots, V_p))) \\ &= \sum_{p=0}^{\infty} \mathbb{E}^Q(\mathbb{1}_{\{V_k < V_p, \forall 0 \leq k < p\}} \mathbb{1}_{\{V_k \leq V_p, \forall k \geq p\}} \psi((V_{p+k} - V_p)_{k \geq 0})\theta((V_0, \dots, V_p))) \\ &= \sum_{p=0}^{\infty} \mathbb{E}^Q(\mathbb{1}_{\{V_k < V_p, \forall k < p\}}\theta((V_0, \dots, V_p))) \mathbb{E}^Q(\mathbb{1}_{\{V_k \leq 0, \forall k \geq 0\}}\psi((V_k)_{k \geq 0})), \end{aligned}$$

using Markov property at time  $p$ . The second term is equal to

$$\mathbb{P}^Q(V_k \leq 0, \forall k \geq 0) \mathbb{E}^{Q^{\leq 0}}(\psi((V_k)_{k \geq 0})).$$

Let us now consider only the first term. Using the Girsanov property of  $Q$  and  $\tilde{Q}$  we get

$$\begin{aligned} \sum_{p=0}^{\infty} \mathbb{E}^Q(\mathbb{1}_{\{V_k < V_p, \forall k < p\}}\theta((V_0, \dots, V_p))) &= \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}}(\mathbb{1}_{\{V_k < V_p, \forall k < p\}} e^{-\kappa V_p} \theta((V_0, \dots, V_p))) \\ &= \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}}(e^{-\kappa V_{e_p}} \theta((V_0, \dots, V_{e_p}))), \end{aligned}$$

where  $(e_p)_{p \geq 0}$  are the strict increasing ladder times of  $(V_k, k \geq 0)$  as defined above. The last formula is exactly the one we need, and also imply that

$$\frac{1}{\mathcal{Z}} = \mathbb{P}^Q(V_k \leq 0, \forall k \geq 0) = 1 - \mathbb{E}^{\tilde{Q}}(e^{-\kappa V_{e_1}}),$$

(which can also be obtained directly). □

Denote now by  $\hat{Q}^{>0}$  the law

$$\hat{Q}^{>0} = \hat{Q}(\cdot \mid Y_k > 0, \forall k > 0).$$

We will need the following result.

**Lemma 5.2.** *There exists a positive constant  $c > 0$  such that, for all positive test function  $\psi$ ,*

$$\mathbb{E}^{Q^{\mathcal{I}}}(\psi(V_0, \dots, V_{T_H})) \leq c \mathbb{E}^{\hat{Q}^{>0}}(\psi(Y_0, \dots, Y_{\Theta})).$$

*Proof.* Let  $\Psi$  be a positive test function. Thank to the previous lemma, we have

$$\begin{aligned} & \mathbb{E}^{Q^{\mathcal{I}}}(\Psi(V_0, \dots, V_{T_H})) \\ &= \frac{1}{\mathbb{P}^Q(H = S)} \mathbb{E}^Q(\mathbb{1}_{H=S} \Psi(V_0, \dots, V_{T_H})) \\ &= \frac{1}{\mathcal{Z} \mathbb{P}^Q(H = S)} \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}}(\mathbb{1}_{Y_k > 0, \forall 0 < k \leq e_p} e^{-\kappa Y_{e_p}} \Psi(Y_0, \dots, Y_{e_p})) \\ &= \frac{1}{\mathcal{Z} \mathbb{P}^Q(H = S)} \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}}(\mathbb{1}_{Y_k > 0, \forall k > 0} \frac{1}{\mathbb{P}^{\tilde{Q}}(V_k > -Y_{e_p}, \forall k > 0)} e^{-\kappa Y_{e_p}} \Psi(Y_0, \dots, Y_{e_p})) \\ &\leq \frac{1}{\mathcal{Z} \mathbb{P}^Q(H = S) \mathbb{P}^{\tilde{Q}}(V_k > 0, \forall k > 0)} \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}}(\mathbb{1}_{Y_k > 0, \forall k > 0} e^{-\kappa Y_{e_p}} \Psi(Y_0, \dots, Y_{e_p})) \\ &\leq \frac{1}{\mathbb{P}^Q(H = S) \mathbb{P}^{\tilde{Q}}(V_k > 0, \forall k > 0)} \mathbb{E}^{\hat{Q}^{>0}}(\Psi(Y_0, \dots, Y_{\Theta})), \end{aligned}$$

using the Markov property at time  $e_p$  in the fourth line. This is exactly what we want.  $\square$

## 6. A PRELIMINARY ESTIMATE

To derive the tail estimate of  $R$  or  $Z$  we need to restrict to large values of  $H$ : this will be possible, thank to the following estimate.

**Lemma 6.1.** *For all  $\eta > 0$  there exists a positive constant  $c_\eta$  such that*

$$\mathbb{E}^{Q^{\mathcal{I}}}((M_1)^\eta \mid \lfloor H \rfloor) \leq c_\eta, \quad Q^{\mathcal{I}} \text{ p.s.}$$

where  $\lfloor H \rfloor$  is the integer part of  $H$ .

*Proof.* Since  $(V_k)_{k \leq 0}$  is independent of  $H$  under  $Q^{\mathcal{I}}$ , we have, for all  $p \in \mathbb{N}$ ,

$$\mathbb{E}^{Q^{\mathcal{I}}}((M_1)^\eta \mid \lfloor H \rfloor = p) \leq 2^\eta \left( \mathbb{E}^{Q^{\leq 0}} \left( \left( \sum_{k=0}^{\infty} e^{V_k} \right)^\eta \right) + \mathbb{E}^{Q^{\mathcal{I}}} \left( \left( \sum_{k=0}^{T_H} e^{-V_k} \right)^\eta \mid \lfloor H \rfloor = p \right) \right).$$

The first term on the left-hand side is finite for all  $\eta > 0$  as proved in Section 3. Consider now the last term. Using Lemma 5.2, we get

$$\begin{aligned} & \mathbb{E}^{Q^{\mathcal{I}}}\left(\left(\sum_{k=0}^{T_H} e^{-V_k}\right)^\eta \mid \lfloor H \rfloor = p\right) \\ & \leq \frac{c}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{T_H} e^{-Y_k}\right)^\eta \mathbb{1}_{\lfloor H \rfloor = p}\right) \\ & \leq \frac{c'}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} e^{-\kappa Y_{e_k}} \mathbb{1}_{Y_{e_k} \in [p, p+1[)}\right) \left(\sum_{j=0}^{\infty} e^{-V_j}\right)^\eta\right). \end{aligned}$$

Now, using Cauchy-Schwarz inequality in the last expression, we get

$$\begin{aligned} & \mathbb{E}^{Q^{\mathcal{I}}}\left(\left(\sum_{k=0}^{T_H} e^{-V_k}\right)^\eta \mid \lfloor H \rfloor = p\right) \\ & \leq \frac{c'}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} e^{-\kappa Y_{e_k}} \mathbb{1}_{Y_{e_k} \in [p, p+1[)}\right)^2\right)^{\frac{1}{2}} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} e^{-V_k}\right)^{2\eta}\right)^{\frac{1}{2}} \\ & \leq \frac{c' e^{-\kappa p}}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} \mathbb{1}_{Y_{e_k} \in [p, p+1[)}\right)^2\right)^{\frac{1}{2}} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} e^{-V_k}\right)^{2\eta}\right)^{\frac{1}{2}}. \end{aligned}$$

But the last term is independent of  $p$  and finite by Lemma 3.2. On the other hand, since  $\tilde{Q}(V_k > 0, \forall k > 0) > 0$  and from Markov property, we obtain

$$\mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} \mathbb{1}_{Y_{e_k} \in [p, p+1[)}\right)^2\right) \leq c \mathbb{E}^{\tilde{Q}}\left(\left(\sum_{k=0}^{\infty} \mathbb{1}_{Y_{e_k} \in [p, p+1[)}\right)^2\right) \leq c \mathbb{E}^{\tilde{Q}}\left(\left(\sum_{k=0}^{\infty} \mathbb{1}_{Y_{e_k} \in [0, 1[)}\right)^2\right),$$

which is finite since  $(Y_k)_{k \geq 0}$  has a positive drift under  $\tilde{Q}$ . Finally, using the tail estimate on  $H$ , we know that

$$\begin{aligned} (9) \quad \lim_{p \rightarrow \infty} e^{\kappa p} \mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p) &= \lim_{p \rightarrow \infty} e^{\kappa p} (\mathbb{P}^{Q^{\mathcal{I}}}(H \geq p) - \mathbb{P}(H \geq p + 1)) \\ &= C_I (1 - e^{-\kappa}). \end{aligned}$$

Hence,  $(e^{\kappa p} \mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p))^{-1}$  is a bounded sequence (we do not have to consider the cases where eventually  $\mathbb{P}(\lfloor H \rfloor = p) = 0$  since it is a conditioning by an event of null probability which can be omitted).  $\square$

**Corollary 6.1.** *We have,  $Q^{\mathcal{I}}$ -almost surely,*

$$\mathbb{E}^{Q^{\mathcal{I}}}(Z \mid \lfloor H \rfloor) \leq ec_2 e^{\lfloor H \rfloor}.$$

*Proof.* We have  $Z = M_1 M_2 e^H$ . Using Cauchy-Schwarz inequality and Lemma 6.1 we get

$$\begin{aligned} \mathbb{E}^{Q^{\mathcal{I}}}(Z \mid \lfloor H \rfloor) &\leq e^{\lfloor H \rfloor + 1} \left( \mathbb{E}^{Q^{\mathcal{I}}}((M_1)^2 \mid \lfloor H \rfloor) \mathbb{E}^{Q^{\mathcal{I}}}((M_2)^2 \mid \lfloor H \rfloor) \right)^{\frac{1}{2}} \\ &\leq ec_2 e^{\lfloor H \rfloor}, \end{aligned}$$

since  $M_1$  and  $M_2$  have the same law under  $Q^{\mathcal{I}}$ .  $\square$

**Corollary 6.2.** *Let  $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a function such that*

$$\lim_{t \rightarrow \infty} t^{-1} e^{h(t)} = 0.$$

*Then, we have*

$$\begin{aligned} \mathbb{P}^{Q^{\mathcal{I}}}(R \geq t, H \leq h(t)) &= o(t^{-\kappa}), \\ \mathbb{P}^{Q^{\mathcal{I}}}(Z \geq t, H \leq h(t)) &= o(t^{-\kappa}), \end{aligned}$$

*when  $t$  tends to infinity.*

*Proof.* Let us do the proof for  $Z$ . Let  $\eta$  be a positive real such that

$$\eta > \kappa.$$

We have (all expectations are relative to the measure  $Q^{\mathcal{I}}$ ; so, to simplify the reading, we remove the reference to  $Q^{\mathcal{I}}$  in the following)

$$\begin{aligned} \mathbb{P}^{Q^{\mathcal{I}}}(Z \geq t, H \leq h(t)) &= \mathbb{E}(\mathbb{P}(Z \geq t, H \leq h(t) \mid \lfloor H \rfloor)) \\ &\leq \mathbb{E}(\mathbb{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor} \mathbb{P}(Z \geq t \mid \lfloor H \rfloor)) \\ &\leq \mathbb{E}(\mathbb{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor} \mathbb{P}(M_1 M_2 \geq t e^{-(\lfloor H \rfloor + 1)} \mid \lfloor H \rfloor)) \\ &\leq e^\eta \mathbb{E}(\mathbb{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor} t^{-\eta} e^{\eta \lfloor H \rfloor} \mathbb{E}((M_1 M_2)^\eta \mid \lfloor H \rfloor)) \\ &\leq e^\eta \mathbb{E}(\mathbb{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor} t^{-\eta} e^{\eta \lfloor H \rfloor} \mathbb{E}((M_1)^{2\eta} \mid \lfloor H \rfloor)). \end{aligned}$$

In the last formula, we used Cauchy-Schwarz inequality and the symmetry property of  $Q^{\mathcal{I}}$ , see Lemma 4.1, to obtain

$$\mathbb{E}((M_2)^{2\eta} \mid \lfloor H \rfloor) = \mathbb{E}((M_1)^{2\eta} \mid \lfloor H \rfloor).$$

We can now use the estimate of Lemma 6.1, which gives

$$\begin{aligned} \mathbb{P}^{Q^{\mathcal{I}}}(Z \geq t, H \leq h(t)) &\leq e^\eta c_{2\eta} t^{-\eta} \sum_{p=0}^{\lfloor h(t) \rfloor} e^{\eta p} \mathbb{P}(\lfloor H \rfloor = p) \\ &\leq c' t^{-\eta} \sum_{p=0}^{\lfloor h(t) \rfloor} e^{(\eta - \kappa)p}. \end{aligned}$$

In the last formula, we used the fact that  $\mathbb{P}(\lfloor H \rfloor = p) = O(e^{-\kappa p})$ , see (9). Since we chose  $\eta > \kappa$  we can bound uniformly

$$\mathbb{P}^{Q^{\mathcal{I}}}(Z \geq t, H \leq h(t)) \leq c'' t^{-\eta} e^{(\eta - \kappa)h(t)} = c'' t^{-\kappa} \left( \frac{e^{h(t)}}{t} \right)^{\eta - \kappa}.$$

This gives the result for  $Z$ . Since  $R \leq Z$  we get the result for  $R$ .  $\square$

## 7. THE COUPLING ARGUMENT

We set

$$\begin{aligned} I(t) &:= \mathbb{P}^{Q^{\mathcal{I}}}(e^H M_1 M_2 \geq t), \\ J(t) &:= \mathbb{P}^{Q^{\mathcal{I}}}(e^H M_2 \geq t), \\ K(t) &:= \mathbb{P}^{Q^{\mathcal{I}}}(e^H \geq t). \end{aligned}$$

From the estimate of Iglehart, see [8], we know that

$$K(t) \sim \frac{1}{\mathbb{P}^Q(H = S)} C_I t^{-\kappa},$$

when  $t \rightarrow \infty$ . Indeed, we have

$$\mathbb{P}^{Q^x}(e^H \geq t) = \frac{1}{\mathbb{P}^Q(H = S)}(\mathbb{P}^Q(e^H \geq t) - P^Q(e^H \geq t, S > H)).$$

The second term is clearly of order  $O(t^{-2\kappa})$ , the first term is estimated in [8], cf (4).

We will prove the following key estimates.

**Proposition 7.1.** *For all  $\xi > 0$  there exists a function  $\epsilon_\xi(t) > 0$  such that  $\lim_{t \rightarrow \infty} \epsilon_\xi(t) = 0$  and*

$$e^{-3\xi} \mathbb{E}(J(e^{3\xi} t M^{-1})) (1 - \epsilon_\xi(t)) \leq I(t) \leq e^{3\xi} \mathbb{E}(J(e^{-3\xi} t M^{-1})) (1 + \epsilon_\xi(t)),$$

$$e^{-2\xi} \mathbb{E}(K(e^{2\xi} t M^{-1})) (1 - \epsilon_\xi(t)) \leq J(t) \leq e^{2\xi} \mathbb{E}(K(e^{-2\xi} t M^{-1})) (1 + \epsilon_\xi(t)),$$

where  $M$  is the random variable defined in (6).

We see that Theorem 2.1 ii) is a direct consequence of the second estimate and of the tail estimate for  $K(t)$ . Theorem 2.2 is a consequence of the estimate i) and of the estimate for  $J$ .

*Proof. Step 1:* We first restrict the expectations to large values of  $H$ . Let  $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be any increasing function such that

$$(10) \quad \lim_{t \rightarrow \infty} t^{-1} e^{h(t)} = 0,$$

$$(11) \quad h(t) \geq \frac{9}{10} \log t.$$

From Corollary 6.2, we know that

$$(12) \quad \mathbb{P}^{Q^x}(e^H M_1 M_2 \geq t, H \leq h(t)) = o(t^{-\kappa}) = o(K(t)).$$

Hence, we can restrict ourselves to consider

$$I_h(t) := \mathbb{P}^{Q^x}(e^H M_1 M_2 \geq t \mid H \geq h(t)),$$

$$J_h(t) := \mathbb{P}^{Q^x}(e^H M_2 \geq t \mid H \geq h(t)),$$

*Step 2:* (Truncation of  $M_1, M_2$ ). We need to truncate the sums  $M_1$  and  $M_2$  so that they do not overlap. Under  $Q^x(\cdot \mid H \geq h(t))$  we consider the random variables

$$(13) \quad \tilde{M}_1 := \sum_{-\infty}^{t_1} e^{-V_k},$$

$$(14) \quad \tilde{M}_2 := \sum_{t_2}^{\infty} e^{V_k - S},$$

where

$$t_1 := \inf\{k \geq 0, V_k \geq \frac{1}{3} \log t\} - 1,$$

$$t_2 := \sup\{k \leq T_H, V_k \leq H - \frac{1}{3} \log t\} + 1.$$

Since  $h(t) \geq \frac{9}{10} \log t$ , we have

$$0 \leq t_1 < t_2 \leq T_H.$$

Clearly, by the symmetry property of  $Q^{\mathcal{I}}$ ,  $\tilde{M}_1$  and  $\tilde{M}_2$  have the same law under  $Q^{\mathcal{I}}(\cdot | H \geq h(t))$ . (Observe that the random variables  $\tilde{M}_1$  and  $\tilde{M}_2$  are implicitly defined in terms of the variable  $t$ .)

**Lemma 7.1.** *Let  $\xi$  be a positive real. There exists a constant  $c_\xi > 0$  such that*

$$\mathbb{P}^{Q^{\mathcal{I}}} \left( \tilde{M}_1 \leq e^{-\xi} M_1 \mid H \geq h(t) \right) \leq \begin{cases} c_\xi t^{-\kappa/6} & \text{for } \kappa \leq 1, \\ c_\xi t^{-1/6} & \text{for } \kappa \geq 1. \end{cases}$$

*Proof.* We have, since  $M_1 \geq 1$

$$\begin{aligned} & \mathbb{P}^{Q^{\mathcal{I}}} \left( \tilde{M}_1 \leq e^{-\xi} M_1 \mid H \geq h(t) \right) \\ & \leq \mathbb{P}^{Q^{\mathcal{I}}} \left( M_1 - \tilde{M}_1 \geq 1 - e^{-\xi} \mid H \geq h(t) \right) \\ & \leq \frac{1}{1 - e^{-\xi}} \mathbb{E}^{Q^{\mathcal{I}}} \left( M_1 - \tilde{M}_1 \mid H \geq h(t) \right) \\ & \leq c \frac{e^{-\kappa h(t)}}{\mathbb{P}^{Q^{\mathcal{I}}}(H \geq h(t))} \mathbb{E}^{\tilde{Q}^{>0}} \left( \sum_{k=t_1+1}^{\infty} e^{-Y_k} \left( \sum_{\substack{e_p \geq k, \\ Y_{e_p} \geq h(t)}} e^{-\kappa(Y_{e_p} - h(t))} \right) \right), \end{aligned}$$

where in the last expression we used the result of Lemma 5.2, and the notations of the related section, and where  $c$  is a constant depending on  $\xi$  and on the parameters of the model. Using the fact that  $\mathbb{P}^{Q^{\mathcal{I}}}(H \geq h(t)) \sim C e^{-\kappa h(t)}$ , when  $t \rightarrow \infty$ , the Markov property and the fact that

$$\mathbb{E}^{\tilde{Q}^{>0}} \left( \sum_{\substack{e_p \geq k, \\ Y_{e_p} \geq h(t)}} e^{-\kappa(Y_{e_p} - h(t))} \right) \leq \frac{1}{\mathbb{P}^{\tilde{Q}}(Y_n > 0, \forall n > 0)(1 - \mathbb{E}^{\tilde{Q}}(e^{-\kappa Y_{e_1}}))},$$

independently of  $k$ , we see that

$$\begin{aligned} \mathbb{P}^{Q^{\mathcal{I}}} \left( \tilde{M}_1 \leq e^{-\xi} M_1 \mid H \geq h(t) \right) & \leq c \mathbb{E}^{\tilde{Q}^{>0}} \left( \sum_{k=t_1+1}^{\infty} e^{-Y_k} \right) \\ & \leq c_\xi t^{-\frac{\kappa \wedge 1}{6}}, \end{aligned}$$

using the estimate of Lemma 3.3. □

*Step 3:* (A small modification of the conditioning.) We set

$$\mathcal{I}_h^{(t)} := \mathcal{I} \cap \{S \geq h(t)\} = \{V_k \geq 0, \forall k \leq 0\} \cap \{S = H\} \cap \{S \geq h(t)\},$$

the event by which we condition in  $S_h(t)$ ,  $R_h(t)$ . We set

$$\tilde{\mathcal{I}}_h^{(t)} := \{S \geq h(t)\} \cap \{V_k \geq 0, \forall k \leq 0\} \cap \{V_k > 0, \forall 0 < k < T_{\frac{1}{3} \log t}\},$$

where

$$T_{\frac{1}{3} \log t} := \inf \{k \geq 0, V_k \geq \frac{1}{3} \log t\}.$$

Clearly, we have  $\mathcal{I}_h^{(t)} \subset \tilde{\mathcal{I}}_h^{(t)}$  and

$$\mathbb{P}(\tilde{\mathcal{I}}_h^{(t)} \setminus \mathcal{I}_h^{(t)} \mid \tilde{\mathcal{I}}_h^{(t)}) \leq c t^{-\kappa/3},$$



for a constant  $c > 0$  depending only on the parameters of the model. We set

$$\begin{aligned}\tilde{I}_h(t) &:= \mathbb{P}^Q \left( e^H \tilde{M}_1 \tilde{M}_2 \geq t \mid \tilde{\mathcal{I}}_h^{(t)} \right), \\ \tilde{J}_h(t) &:= \mathbb{P}^Q \left( e^H \tilde{M}_2 \geq t \mid \tilde{\mathcal{I}}_h^{(t)} \right), \\ \tilde{K}_h(t) &:= \mathbb{P}^Q \left( e^H \geq t \mid \tilde{\mathcal{I}}_h^{(t)} \right).\end{aligned}$$

From Step 2 (Lemma 7.1) and Step 3, we see that we have, for all  $\xi > 0$ , the following estimate

$$(15) \quad I_h(e^{2\xi}t) - c_\xi t^{-\frac{\kappa \wedge 1}{6}} \leq \tilde{I}_h(t) \leq I_h(t) + ct^{-\frac{\kappa}{3}},$$

$$(16) \quad J_h(e^\xi t) - c_\xi t^{-\frac{\kappa \wedge 1}{6}} \leq \tilde{J}_h(t) \leq J_h(t) + ct^{-\frac{\kappa}{3}}.$$

*Step 4: (The coupling strategy.)*

Let  $(Y'_k)_{k \geq 0}$  and  $(Y''_k)_{k \geq 0}$  be two independent processes with law

$$\tilde{Q}(\cdot \mid Y_k > 0, 0 < k \leq T_{\frac{1}{3} \log t}).$$

Let us define, for all  $u > 0$ , the hitting times

$$T'_u := \inf\{k \geq 0, Y'_k \geq u\}, \quad T''_u := \inf\{k \geq 0, Y''_k \geq u\}.$$

Set

$$N'_0 := T'_{\frac{1}{3} \log t}, \quad N''_0 := T''_{\frac{1}{3} \log t}.$$

We couple the processes  $(Y'_{N'_0+k})_{k \geq 0}$  and  $(Y''_{N''_0+k})_{k \geq 0}$  as in Durrett (cf [4], (4.3), p. 204): we construct some random times  $K' \geq N'_0$  and  $K'' \geq N''_0$  such that

$$|Y'_{K'} - Y''_{K''}| \leq \xi,$$

and such that  $(Y'_{K'+k} - Y'_{K'})_{k \geq 0}$  and  $(Y''_{K''+k} - Y''_{K''})_{k \geq 0}$  are independent of the  $\sigma$ -field generated by  $Y'_0, \dots, Y'_{K'}$  and  $Y''_0, \dots, Y''_{K''}$ . The method for this  $\xi$ -coupling is the following: we consider some independent Bernoulli random variables  $(\eta'_i)_{i \in \mathbb{N}}$  and  $(\eta''_i)_{i \in \mathbb{N}}$  (with  $\mathbb{P}(\eta'_i = 1) = \mathbb{P}(\eta''_i = 1) = \frac{1}{2}$ ) and we define

$$(Z'_k) = (Y'_{N'_0 + \sum_{i=1}^k \eta'_i}), \quad (Z''_k) = (Y''_{N''_0 + \sum_{i=1}^k \eta''_i}).$$

This extra randomization ensures that the process  $(Z'_k - Z''_k)$  is non arithmetic. Since its expectation is null, there exists a positive random time for which they are at a distance at most  $\xi$  (cf the proof of Chung-Fuchs theorem (2.7), p. 188 and theorem (2.1), p. 183 in [4]). Then we define

$$Y_k = \begin{cases} Y'_k, & \text{when } k \leq K', \\ (Y''_{K''+(k-K')} - Y''_{K''}) + Y'_{K'}, & \text{when } k > K'. \end{cases}$$

Clearly, by construction, since the processes  $Y'$  and  $Y''$  are no longer conditioned when they reach the level  $\frac{1}{3} \log t$ ,  $(Y_k)_{k \geq 0}$  has the law

$$\tilde{Q}(\cdot \mid Y_k > 0, \forall 0 < k < T_{\frac{1}{3} \log t}).$$

We want that  $Y'$  and  $Y''$  couple before they reach the level  $\frac{1}{2} \log t$ , so we set

$$\mathcal{A} = \{K' < T'_{\frac{1}{2} \log t}\} \cap \{K'' < T''_{\frac{1}{2} \log t}\}.$$

Clearly, since the distribution of  $Y'_{N'_0} - \frac{1}{3} \log t$  converges (and the same for  $Y''$ , cf limit theorem (4.10), p. 370 in [6]) and since for all starting points  $Y'_{N'_0}$  and  $Y''_{N''_0}$ ,  $Z'$

and  $Z''$  couple in a finite time almost surely, we have the following result (whose proof is postponed to the end of the section).

**Lemma 7.2.**

$$\lim_{t \rightarrow \infty} \mathbb{P}(\mathcal{A}^c) = 0.$$

We set

$$\eta(t) := \mathbb{P}(\mathcal{A}^c),$$

and we choose  $h(t)$  in terms of  $\eta$  by

$$(17) \quad h(t) = \left( \log t + \frac{1}{2\kappa} \log \eta(t) \right) \vee \left( \frac{9}{10} \log t \right) \vee \left( \left(1 - \frac{1}{7\kappa}\right) \log t \right),$$

where  $\vee$  stands for the maximum of the three values. Clearly,  $h(t)$  satisfies the hypotheses (10), (11).

Consider now two independent processes  $(W_k)_{k \geq 0}$  and  $(W'_k)_{k \geq 0}$  (and independent of  $Y', Y''$ ) with the same law  $Q^{\leq 0}$  (cf Section 5). Let  $e$  be a strict increasing ladder time of  $Y$  and define the process  $V(W, W', Y, e) = (V_k)_{k \in \mathbb{Z}}$  by

$$\begin{cases} (V_k)_{k \leq 0} = (-W_{-k})_{k \leq 0}, \\ (V_k)_{k \geq 0} = (Y_0, \dots, Y_e, Y_e + W'_1, \dots, Y_e + W'_k, \dots). \end{cases}$$

If  $Y_e \geq h(t)$  then clearly  $(V_k)_{k \in \mathbb{Z}}$  belongs to the event  $\tilde{\mathcal{I}}_h^{(t)}$ , and the functional  $\tilde{M}_1$  defined in (13) depends only on  $W$  and  $Y'$ ; we denote it by  $\tilde{M}_1(W, Y')$ . The functional  $\tilde{M}_2$  depends only on  $Y, W', e$ ; we denote it by  $\tilde{M}_2(Y, W', e)$ . Using Lemma 5.1, we see that

$$\tilde{I}_h(t) = \frac{1}{\mathcal{Z}_h(t)} \mathbb{E} \left( \sum_{p=0}^{\infty} e^{-\kappa Y_{e_p}} \mathbb{1}_{Y_{e_p} \geq h(t)} \mathbb{1}_{\tilde{M}_1(W, Y') \tilde{M}_2(Y, W', e_p) e^{Y_{e_p}} \geq t} \right),$$

where  $(e_p)_{p \geq 0}$  is the set of strict increasing ladder times of  $Y$  (cf Section 5) and where  $\mathcal{Z}_h(t)$  is the normalizing constant

$$\mathcal{Z}_h(t) = \mathbb{E} \left( \sum_{p=0}^{\infty} e^{-\kappa Y_{e_p}} \mathbb{1}_{Y_{e_p} \geq h(t)} \right).$$

Clearly,  $\mathcal{Z}_h(t) \sim_{t \rightarrow \infty} ce^{-\kappa h(t)}$ . The variable  $Y_{T_h(t)} - h(t)$  is indeed the residual waiting time of the renewal process defined by the values of the process  $Y$  at the successive increasing ladder epochs. Hence, it converges in distribution by the limit theorem (4.10) in ([6], p. 370).

On the coupling event  $\mathcal{A}$ , we have

$$\begin{aligned} Y''_{e_p - K' + K''} - \xi &\leq Y_{e_p} \leq Y''_{e_p - K' + K''} + \xi, \\ \tilde{M}_2(Y, W', e_p) &= \tilde{M}_2(Y'', W', e_p - K' + K''), \end{aligned}$$

for all ladder time  $e_p$  such that  $Y_{e_p} \geq h(t)$  (indeed  $h(t) \geq \frac{9}{10} \log t$ ) and where  $\tilde{M}_2(Y'', W', e_p - K' + K'')$  is the functional obtained from the concatenation of the processes  $Y''$  and  $W'$  at time  $e_p - K' + K''$ , as it is done for  $\tilde{M}_2(Y, W', e_p)$ . The first set of inequalities implies that, on the coupling event  $\mathcal{A}$ , the set  $\{e_p - K' + K'', Y_{e_p} \geq h(t)\}$

is included in the set of strict increasing ladder times of  $Y''$  larger than  $h(t) - \xi$ . So we have

$$\begin{aligned} \tilde{I}_h(t) &\leq \frac{e^{\kappa\xi}}{\mathcal{Z}_h(t)} \mathbb{E} \left( \mathbb{1}_{\mathcal{A}} \left( \sum_{p=0}^{\infty} e^{-\kappa Y''_{e_p}} \mathbb{1}_{Y''_{e_p} \geq h(t) - \xi} \mathbb{1}_{\tilde{M}_1(W, Y') \tilde{M}_2(Y'', W', e_p'') \exp(Y''_{e_p}'') \geq te^{-\xi}} \right) \right) \\ &\quad + \frac{e^{-\kappa h(t)}}{\mathcal{Z}_h(t)} \mathbb{E} \left( \mathbb{1}_{\mathcal{A}^c} \left( \sum_{p=0}^{\infty} e^{-\kappa(Y_{e_p} - h(t))} \mathbb{1}_{Y_{e_p} \geq h(t)} \right) \right), \end{aligned}$$

where  $(e_p'')_{p \geq 0}$  denote the strict increasing ladder times for the process  $Y''$ . Since the process  $\{Y_{e_p}, Y_{e_p} \geq h(t)\}$  depends on the event  $\mathcal{A}$  only through the value of  $Y_{T_{h(t)}}$ , we see that the second term is less or equal than

$$(18) \quad \frac{1}{1 - \mathbb{E} \tilde{Q}(e^{-\kappa Y_{e_1}})} \frac{e^{-\kappa h(t)}}{\mathcal{Z}_h(t)} \mathbb{P}(\mathcal{A}^c) \leq c \mathbb{P}(\mathcal{A}^c).$$

Now, the first term is lower than

$$(19) \quad e^{\kappa\xi} \frac{\mathcal{Z}_{h-\xi}(t)}{\mathcal{Z}_h(t)} \mathbb{P}(e^{S''} \tilde{M}_1(W, Y') \tilde{M}_2'' \geq te^{-\xi}) \leq e^{3\kappa\xi} \mathbb{P}(e^{S''} \tilde{M}_1(W, Y') \tilde{M}_2'' \geq te^{-\xi}),$$

for  $t$  large enough (using the equivalent of  $\mathcal{Z}_h(t)$ ), where  $S''$  and  $\tilde{M}_2''$  are relative to a process  $V''$  independent of  $W, Y'$  and with law  $Q(\cdot | \tilde{\mathcal{I}}_{h-\xi}^{(t)})$ . Moreover, let us introduce  $M_2'' := \sum_{k=0}^{\infty} e^{V_k'' - S''}$ . We need now to replace the truncated sum  $\tilde{M}_1$  by the meander  $M$ . Using the fact that  $\mathbb{P}(\exists k > 0 : Y_k' \leq 0) \leq ct^{-\kappa/3}$ , we see that

$$(20) \quad \begin{aligned} \mathbb{P}(e^{S''} \tilde{M}_1(W, Y') \tilde{M}_2'' \geq te^{-\xi}) &\leq \mathbb{P}(e^{S''} M_2'' M \geq te^{-\xi}) + ct^{-\kappa/6} \\ &\leq \mathbb{E}(J_{h-\xi}(e^{-\xi} t / M)) + c' t^{-\kappa/6}, \end{aligned}$$

the second inequality being a consequence of  $\mathbb{P}(\tilde{\mathcal{I}}_h^{(t)} \setminus \mathcal{I}_h^{(t)} | \tilde{\mathcal{I}}_h^{(t)}) \leq ct^{-\kappa/3}$  and  $M$  the random variable defined in (6) and independent of  $V''$ . Finally, considering the choice made for  $h(t)$  (cf (17)), we have

$$t^{-\frac{\kappa\Lambda 1}{6}} \mathbb{P}^{Q^{\mathcal{I}}}(H \geq h(t)) = o(t^{-\kappa}),$$

$$\mathbb{P}(\mathcal{A}^c) \mathbb{P}^{Q^{\mathcal{I}}}(H \geq h(t)) \leq ct^{-\kappa} \sqrt{\mathbb{P}(\mathcal{A}^c)} = o(t^{-\kappa}).$$

Putting everything together (i.e., the estimates (12), (15), (18), (19), (20))

$$\begin{aligned} I(t) &\leq \mathbb{P}(H \geq h(t)) I_h(t) + o(t^{-\kappa}) \\ &\leq \mathbb{P}(H \geq h(t)) (\tilde{I}_h(e^{-2\xi} t) + ct^{-\frac{\kappa\Lambda 1}{6}}) + o(t^{-\kappa}) \\ &\leq \mathbb{P}(H \geq h(t)) (e^{3\kappa\xi} \mathbb{E}(J_{h-\xi}(e^{-3\xi} t / M)) + c \mathbb{P}(\mathcal{A}^c)) + o(t^{-\kappa}) \\ &\leq e^{3\kappa\xi} \mathbb{P}(RM \geq te^{-3\xi}, H \geq h(t) - \xi) + o(t^{-\kappa}), \end{aligned}$$

where  $R$  and  $M$  are independent processes with laws defined in Section 2 (indeed, in the last inequality,  $\mathbb{P}(H \geq h(t)) \mathbb{P}(\mathcal{A}^c) \leq \sqrt{\mathbb{P}(\mathcal{A}^c)} t^{-\kappa} = o(t^{-\kappa})$ ). Now, proceeding exactly as in Corollary 6.2, we see that

$$\mathbb{P}(RM \geq t, H < h(t) - \xi) = o(t^{-\kappa}),$$

(indeed, the only difference is that  $M_1$  is replaced by  $M$  and that  $M$  and  $R$  are independent). Finally, we proved that

$$I(t) \leq e^{3\kappa\xi} \mathbb{E}(J(e^{-3\xi} t / M)) + o(t^{-\kappa}).$$

The lower estimate is similar. We first have, since the set  $\{e_p - K' + K'', Y_{e_p} \geq h(t)\}$  includes the set of strict increasing ladder times of  $Y''$  larger than  $h(t) + \xi$ :

$$\tilde{I}_h(t) \geq \frac{e^{-\kappa\xi}}{\mathcal{Z}_h(t)} \mathbb{E} \left( \mathbb{1}_{\mathcal{A}} \left( \sum_{p=0}^{\infty} e^{-\kappa Y_{e_p}''} \mathbb{1}_{Y_{e_p}'' \geq h(t) + \xi} \mathbb{1}_{\tilde{M}_1(W, Y') \tilde{M}_2(Y'', W', e_p'') \exp(Y_{e_p}'' \geq t e^\xi)} \right) \right).$$

Hence, by the same argument as above

$$\tilde{I}_h^{(t)} \geq e^{-3\kappa\xi} \mathbb{P}(e^{S''} \tilde{M}_1(W, Y') \tilde{M}_2'' \geq e^\xi t) + c\mathbb{P}(\mathcal{A}^c),$$

where  $S''$  and  $\tilde{M}_2''$  are relative to a process  $V''$  independent of  $W$  and  $Y'$  and with law  $Q(\cdot \mid \tilde{I}_{h+\xi}^{(t)})$ . Using, now the fact that  $Y'_k > 0$  for all  $k > 0$  with probability at least  $1 - ct^{-\kappa/3}$  and the fact that  $\tilde{M}_2 \geq e^{-\xi} M_2$  with probability at least  $1 - ct^{-\kappa/6}$ , and the estimate on the tail of the sum  $\sum e^{-Y'_k}$  (of Section 3) we see that

$$\tilde{I}_h^{(t)} \geq e^{-3\kappa\xi} \mathbb{P}(M e^{S''} M_2'' \geq e^{3\xi} t) + o(t^{-\kappa/6}) + c\mathbb{P}(\mathcal{A}^c),$$

where  $M$  is the random variable defined in (6) and independent of  $V''$ . Then, we conclude as previously.

To prove the estimate on  $J(t)$  and  $K(t)$  we proceed exactly in the same way: we first remark that by the property of time reversal (see Lemma 4.1), we have

$$J(t) = \mathbb{P}^{Q^X}(e^H M_1 \geq t).$$

The situation is then even simpler, we just have to decouple  $M_1$  and  $e^H$ .  $\square$

*Proof.* (of Lemma 7.2). Denote by  $F_{y', y''}(u)$  the probability that  $Z'$  and  $Z''$  couple before the level  $\frac{1}{3} \log t + u$  knowing that  $Y'_{N'_0} = \frac{1}{3} \log t + y'$  and  $Y''_{N''_0} = \frac{1}{3} \log t + y''$ . By the arguments above,  $F_{y', y''}(u)$  tends to 1 when  $u$  tends to infinity. Let  $A > 0$ , we first prove that this convergence is uniform in  $y', y''$  on the compact  $y' \leq A, y'' \leq A$ . For this we consider the set  $\mathcal{S} = (\mathbb{N} \cdot \frac{\xi}{4}) \cap [0, A]$ , and for  $y', y''$  in  $\mathcal{S} \times \mathcal{S}$  the function  $\hat{F}_{y', y''}(u)$  the probability that  $Z'$  and  $Z''$  starting from the points  $Y'_{N'_0} = \frac{1}{3} \log t + y'$  and  $Y''_{N''_0} = \frac{1}{3} \log t + y''$  couple at a distance  $\xi/2$ , before the level  $\frac{1}{3} \log t + u - \xi$ . Let

$$\phi(u) = \inf_{y' \in \mathcal{S}, y'' \in \mathcal{S}} \hat{F}_{y', y''}(u).$$

Clearly  $\phi(u) \rightarrow 1$  when  $u \rightarrow \infty$  and  $F_{y', y''}(u) \geq \phi(u)$ , whenever  $y'$  and  $y''$  are in  $[0, A]$ . This implies that

$$\liminf_{t \rightarrow \infty} \mathbb{P}(\mathcal{A}) \geq \liminf_{A \rightarrow \infty} \liminf_{t \rightarrow \infty} \left( \mathbb{P}(Y'_{N'_0} - \frac{1}{3} \log t \leq A) \right)^2.$$

Moreover,  $\mathbb{P}^{\tilde{Q}}(Y'_k > 0, 0 < k \leq T_{\frac{1}{3} \log t}) \geq \mathbb{P}^{\tilde{Q}}(Y'_k > 0, k \geq 0) > 0$  implies

$$\mathbb{P}(Y'_{N'_0} - \frac{1}{3} \log t \geq A) = \mathbb{P}^{\tilde{Q}}(V_{T_{\frac{1}{3} \log t}} - \frac{1}{3} \log t \geq A \mid V > 0) \leq c\mathbb{P}^{\tilde{Q}}(V_{T_{\frac{1}{3} \log t}} - \frac{1}{3} \log t \geq A),$$

where here  $V$  is the canonical process under  $\tilde{Q}$ . Therefore, since  $V_{T_{\frac{1}{3} \log t}} - \frac{1}{3} \log t$  converges in law (under  $\tilde{Q}$ ) to a finite random variable when  $t$  tends to infinity (see limit theorem (4.10), p. 370 in [6] or Example 4.4 part II, page 214 in [4]), this yields  $\liminf_{t \rightarrow \infty} \mathbb{P}(\mathcal{A}) = 1$ .  $\square$

## 8. PROOF OF THEOREM 2.1 AND THEOREM 2.2

*Proof.* (of Theorem 2.1, *ii*) and Theorem 2.2). Let  $\xi > 0$ . By Proposition 7.1, we have, for all  $A > 0$  and for  $t$  large enough,

$$J(t) \leq e^{3\xi} \left( \mathbb{E}(K(e^{-2\xi}tM^{-1})\mathbb{1}_{M \leq A}) + \mathbb{E}(K(e^{-2\xi}tM^{-1})\mathbb{1}_{M > A}) \right).$$

On the first term, for  $t$  large enough, we can bound from above  $K(e^{-2\xi}tM^{-1})$  by  $(\frac{C_I}{\mathbb{P}Q(H=S)} + \xi)(te^{-2\xi}M^{-1})^{-\kappa}$ . For the second term we can use a uniform bound  $K(t) \leq ct^{-\kappa}$ . Thus we get

$$J(t) \leq e^{3(1+\kappa)\xi} \left( \frac{C_I}{\mathbb{P}Q(H=S)} + \xi \right) t^{-\kappa} (\mathbb{E}(M^\kappa)\mathbb{1}_{M \leq A}) + ct^{-\kappa} \mathbb{E}(M^\kappa\mathbb{1}_{M > A}).$$

Since  $M^\kappa$  is integrable, letting  $A$  tend to  $\infty$ , then  $\xi$  tends to 0, we get the upper bound

$$\limsup_{t \rightarrow \infty} t^\kappa J(t) \leq \frac{C_{KI}}{\mathbb{P}Q(H=S)}.$$

For the lower bound it is the same. The proof of Theorem 2.2 is the same: we use the estimate *i*) of Proposition 7.1 and the tail estimate for  $J$ .  $\square$

*Proof.* (of Theorem 2.1, *i*)).

Let us first recall (5) and Theorem 2.1, *i*), which tells that

$$(21) \quad Q(R > t; H = S) = \frac{C_{KI}}{t^\kappa} + o(t^{-\kappa}), \quad t \rightarrow \infty,$$

where  $C_{KI} = C_I \mathbb{E}(M^\kappa)$ . Then, introducing

$$KI := \sum_{0 \leq k \leq T_{R_-}} e^{V_k}, \quad O_1 := -V_{T_{R_-}},$$

Theorem 2.1, *i*) is a consequence of Theorem 2.1, *ii*) together with the two following lemmas.

**Lemma 8.1.** *We have*

$$(22) \quad Q(KI > t) = \frac{C_{KI}}{t^\kappa} + o(t^{-\kappa}), \quad t \rightarrow \infty.$$

*Proof.* Firstly, observe that  $KI \leq R$  implies  $Q(KI > t; H = S) \leq Q(R > t; H = S)$ . Moreover, Corollary 6.2 implies  $Q(KI > t; e^H = e^S \leq t^{2/3}) = o(t^{-\kappa})$ ,  $t \rightarrow \infty$ , since  $KI \leq R$ . Furthermore, we have  $0 \leq Q(KI > t; e^H > t^{2/3}) - Q(KI > t; e^H = e^S > t^{2/3}) \leq Q(H \neq S; e^H > t^{2/3}) = o(t^{-\kappa})$ ,  $t \rightarrow \infty$ . Therefore, we obtain, when  $t \rightarrow \infty$ ,

$$(23) \quad Q(R > t; H = S) \geq Q(KI > t; e^H > t^{2/3}) + o(t^{-\kappa}).$$

Since, by Corollary 6.2,  $Q(KI > t; e^H \leq t^{2/3}) = o(t^{-\kappa})$ ,  $t \rightarrow \infty$ , we get

$$(24) \quad Q(KI > t; e^H > t^{2/3}) = Q(KI > t) + o(t^{-\kappa}),$$

when  $t \rightarrow \infty$ . Then, assembling (23) and (24) yields

$$(25) \quad Q(R > t; H = S) \geq Q(KI > t) + o(t^{-\kappa}), \quad t \rightarrow \infty.$$

On the other hand, observe that Corollary 6.2 implies that  $Q(R > t; H = S) = Q(R > t; e^H = e^S > t^{2/3}) + o(t^{-\kappa})$ ,  $t \rightarrow \infty$ . Moreover, since we have  $R = KI + e^{O_1} R'$ ,

with  $R'$  a random variable independent of  $KI$  and  $O_1$ , having the same law as  $R$ , we obtain that  $Q(R > t; e^H = e^S > t^{2/3}) \leq Q_1 + Q_2$ , where

$$\begin{aligned} Q_1 &:= Q(KI \leq t - t^{2/3}; R' > t^{2/3}; e^H > t^{2/3}), \\ Q_2 &:= Q(KI > t - t^{2/3}; R > t; e^H = e^S > t^{2/3}). \end{aligned}$$

Now, since  $R'$  and  $H$  are independent, we get  $Q_1 \leq Q(e^H > t^{2/3})Q(R' > t^{2/3}) = o(t^{-\kappa})$ ,  $t \rightarrow \infty$ . Moreover, we easily have  $Q_2 \leq Q(KI > t - t^{2/3})$ . Therefore

$$(26) \quad Q(R > t; H = S) \leq Q(KI > t - t^{2/3}) + o(t^{-\kappa}), \quad t \rightarrow \infty.$$

Recalling (21) and assembling (25) and (26) concludes the proof of Lemma 8.1.  $\square$

**Lemma 8.2.**  $C_{KI}$  satisfies

$$C_{KI} = (1 - \mathbb{E}^Q(e^{-\kappa O_1}))C_K.$$

*Proof.* First, observe that  $Q(R > t) = Q(KI > t) + P_1 + P_2$ , where

$$\begin{aligned} P_1 &:= Q(KI + e^{-O_1}R' > t; t^{1/2} < KI \leq t), \\ P_2 &:= Q(KI + e^{-O_1}R' > t; KI \leq t^{1/2}), \end{aligned}$$

with  $R'$  a random variable independent of  $KI$  and  $O_1$ , with the same law as  $R$ .

Now, let us prove that  $P_1$  is negligible. Observe first that, since  $O_1 \geq 0$  by definition, we have  $P_1 \leq Q(R' > t - KI; t^{1/2} < KI \leq t)$ . Therefore  $0 \leq P_1 \leq P'_1 + P''_1$ , where

$$\begin{aligned} P'_1 &:= Q(R' > t - KI; t - t^{2/3} < KI \leq t), \\ P''_1 &:= Q(R' > t - KI; t^{1/2} < KI \leq t - t^{2/3}). \end{aligned}$$

Since  $R'$  and  $KI$  are independent, (5) and (22) yield  $P''_1 \leq Q(R' > t^{2/3})Q(KI > t^{1/2}) = o(t^\kappa)$ ,  $t \rightarrow \infty$ . Furthermore, we have

$$\begin{aligned} P'_1 &\leq Q(t - t^{2/3} < KI \leq t) \\ &\leq Q(KI > t - t^{2/3}) - Q(KI > t) \\ &= Q(KI > t) \left( \frac{Q(KI > t - t^{2/3})}{Q(KI > t)} - 1 \right). \end{aligned}$$

Therefore (22) implies  $P'_1 = o(t^{-\kappa})$ ,  $t \rightarrow \infty$ . Then, we obtain  $P_1 = o(t^{-\kappa})$ ,  $t \rightarrow \infty$ .

Now, let us estimate  $P_2$ . Observe that  $\underline{P}_2 \leq P_2 \leq \overline{P}_2$ , where

$$\begin{aligned} \underline{P}_2 &:= Q(e^{-O_1}R' > t; KI \leq t^{1/2}), \\ \overline{P}_2 &:= Q(e^{-O_1}R' > t - t^{1/2}). \end{aligned}$$

Since  $R'$  and  $O_1$  are independent, (5) yields

$$(27) \quad \overline{P}_2 = \frac{\mathbb{E}^Q(e^{-\kappa O_1})C_K}{t^\kappa} + o(t^{-\kappa}), \quad t \rightarrow \infty.$$

Therefore, it only remains to estimate  $\underline{P}_2$ . Since  $R'$  is independent of  $KI$  and  $O_1$ , we obtain for any  $\varepsilon > 0$  and  $t$  large enough,

$$(1 - \varepsilon) C_K \mathbb{E}^Q \left( \mathbf{1}_{\{KI \leq t^{1/2}\}} \frac{e^{-\kappa O_1}}{t^\kappa} \right) \leq \underline{P}_2 \leq (1 + \varepsilon) C_K \mathbb{E}^Q \left( \mathbf{1}_{\{KI \leq t^{1/2}\}} \frac{e^{-\kappa O_1}}{t^\kappa} \right).$$

Moreover,

$$\mathbb{E}^Q \left( \mathbf{1}_{\{KI \leq t^{1/2}\}} \frac{e^{-\kappa O_1}}{t^\kappa} \right) = \frac{\mathbb{E}^Q(e^{-\kappa O_1})}{t^\kappa} - \mathbb{E}^Q \left( \mathbf{1}_{\{KI > t^{1/2}\}} \frac{e^{-\kappa O_1}}{t^\kappa} \right),$$

and the second term on the right-hand side is less or equal than  $t^{-\kappa}Q(KI > t^{1/2}) = o(t^{-\kappa})$ ,  $t \rightarrow \infty$ . Thus

$$(28) \quad P_2 = \frac{\mathbb{E}^Q(e^{-\kappa O_1})C_K}{t^\kappa} + o(t^{-\kappa}), \quad t \rightarrow \infty.$$

Assembling (27) and (28) yields  $P_2 = \frac{\mathbb{E}^Q(e^{-\kappa O_1})C_K}{t^\kappa} + o(t^{-\kappa})$ ,  $t \rightarrow \infty$ . Therefore, recalling (5), (22) and  $Q(R > t) = Q(KI > t) + P_1 + P_2$ , we obtain  $C_{KI} = (1 - \mathbb{E}^Q(e^{-\kappa O_1}))C_K$ , which concludes the proof of Lemma 8.2.  $\square$

Since Theorem 2.1, *ii*) together with Lemma 8.1 and Lemma 8.2 yield  $C_{KI} = C_I \mathbb{E}^Q[M^\kappa] = (1 - \mathbb{E}^Q[e^{-\kappa O_1}])C_K$ , we get  $C_K = C_I \mathbb{E}^Q[M^\kappa](1 - \mathbb{E}^Q[e^{-\kappa O_1}])^{-1}$ . Now, recalling that  $C_I = (1 - \mathbb{E}^Q[e^{-\kappa O_1}])C_F$ , this concludes the proof of Theorem 2.1, *i*).  $\square$

## 9. A TAUBERIAN RESULT

**Corollary 9.1.** *Let  $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be such that*

$$\lim_{\lambda \rightarrow 0} \lambda e^{h(\lambda)} = 0, \quad \lim_{\lambda \rightarrow 0} h(\lambda) = \infty.$$

*Then, for  $\kappa < 1$ ,*

$$\mathbb{E}^Q \left( 1 - \frac{1}{1 + \lambda Z} \mid \mathcal{I}_h^{(\lambda)} \right) \sim \frac{1}{\mathbb{P}^Q(H \geq h(\lambda))} \frac{\pi \kappa}{\sin(\pi \kappa)} C_U \lambda^\kappa,$$

*when  $\lambda \rightarrow 0$ , where  $\mathcal{I}_h^{(\lambda)}$  is the event*

$$\mathcal{I}_h^{(\lambda)} = \mathcal{I} \cap \{H \geq h(\lambda)\} = \{V_k \geq 0, \forall k \leq 0\} \cap \{H = S \geq h(\lambda)\}.$$

*Proof.* Clearly, we have

$$\mathbb{E}^Q \left( 1 - \frac{1}{1 + \lambda Z} \mid \mathcal{I}_h^{(\lambda)} \right) = \frac{\mathbb{P}^Q(H = S)}{\mathbb{P}^Q(H = S \geq h(\lambda))} \mathbb{E}^{Q^\mathcal{I}} \left( \mathbb{1}_{H \geq h(\lambda)} \left( 1 - \frac{1}{1 + \lambda Z} \right) \right)$$

Since  $\mathbb{P}^Q(H = S \geq h(\lambda)) \sim \mathbb{P}^Q(H \geq h(\lambda))$  we consider now

$$\mathbb{E}^{Q^\mathcal{I}} \left( \mathbb{1}_{H \geq h(\lambda)} \left( 1 - \frac{1}{1 + \lambda Z} \right) \right)$$

We will forget in the following the reference to the law  $Q^\mathcal{I}$ , and simply write  $\mathbb{E}$  for the expectation with respect to  $Q^\mathcal{I}$ . We have

$$(29) \quad \begin{aligned} & \mathbb{E} \left( \mathbb{1}_{H \geq h(\lambda)} \left( 1 - \frac{1}{1 + \lambda Z} \right) \right) \\ &= \mathbb{E} \left( \mathbb{1}_{Z \geq e^{h(\lambda)}} \left( 1 - \frac{1}{1 + \lambda Z} \right) \right) - \mathbb{E} \left( \mathbb{1}_{e^H < e^{h(\lambda)} \leq Z} \left( 1 - \frac{1}{1 + \lambda Z} \right) \right). \end{aligned}$$

For  $\kappa < 1$ , the second term can be bounded by

$$\begin{aligned} \mathbb{E}\left(\mathbb{1}_{e^H < e^{h(\lambda)} \leq Z} \left(1 - \frac{1}{1 + \lambda Z}\right)\right) &\leq \sum_{p=0}^{\lfloor h(\lambda) \rfloor} \mathbb{E}\left(\mathbb{1}_{[H]=p} \frac{\lambda Z}{1 + \lambda Z}\right) \\ &= \sum_{p=0}^{\lfloor h(\lambda) \rfloor} \mathbb{E}\left(\mathbb{1}_{[H]=p} \mathbb{E}\left(\frac{\lambda Z}{1 + \lambda Z} \mid [H] = p\right)\right) \\ &\leq \sum_{p=0}^{\lfloor h(\lambda) \rfloor} \mathbb{E}\left(\mathbb{1}_{[H]=p} \frac{c\lambda e^p}{1 + c\lambda e^p}\right), \end{aligned}$$

where, in the last inequality, we used Jensen inequality and Corollary 6.1, and where  $c$  denotes a constant independent of  $\lambda$  (which may change from line to line). Now, since  $\mathbb{P}([H] = p) \leq ce^{-\kappa p}$  for a positive constant  $c$ , we get that

$$\begin{aligned} \mathbb{E}\left(\mathbb{1}_{e^H < e^{h(\lambda)} \leq Z} \left(1 - \frac{1}{1 + \lambda Z}\right)\right) &\leq c\lambda \sum_{p=0}^{\lfloor h(\lambda) \rfloor} e^{(1-\kappa)p} \leq c'\lambda e^{(1-\kappa)h(\lambda)} \\ &\leq c'\lambda^\kappa (\lambda e^{h(\lambda)})^{1-\kappa} = o(\lambda^\kappa), \end{aligned}$$

for  $\kappa < 1$ , since  $\lambda e^{h(\lambda)} \rightarrow 0$ ,  $\lambda \rightarrow 0$ .

By integration by part, we see that the first term of (29) is equal to

$$\begin{aligned} &\mathbb{E}\left(\mathbb{1}_{Z \geq h(\lambda)} \left(1 - \frac{1}{1 + \lambda Z}\right)\right) \\ &= \left[\frac{\lambda z}{1 + \lambda z} \mathbb{P}(Z \geq z)\right]_{e^{h(\lambda)}}^{\infty} + \int_{e^{h(\lambda)}}^{\infty} \frac{\lambda}{(1 + \lambda z)^2} \mathbb{P}(Z \geq z) dz. \end{aligned}$$

The first term is lower than

$$c\lambda e^{(1-\kappa)h(\lambda)} = c\lambda^\kappa (\lambda e^{h(\lambda)})^{1-\kappa} = o(\lambda^\kappa),$$

for  $\kappa < 1$ . For the second term, let us suppose first that

$$h(\lambda) \rightarrow \infty.$$

we can estimate  $\mathbb{P}(Z \geq z)$  by

$$\left(\frac{C_U}{\mathbb{P}Q(H=S)} - \eta\right)z^{-\kappa} \leq \mathbb{P}(Z \geq z) \leq \left(\frac{C_U}{\mathbb{P}Q(H=S)} + \eta\right)z^{-\kappa},$$

for any  $\eta$ , when  $\lambda$  is sufficiently small. Hence we are lead to compute the integral

$$\int_{e^{h(\lambda)}}^{\infty} \frac{\lambda}{1 + \lambda z} z^{-\kappa} dz = \lambda^\kappa \int_{\frac{\lambda e^{h(\lambda)}}{1 + \lambda e^{h(\lambda)}}}^1 x^{-\kappa} (1-x)^\kappa dx,$$

(making the change of variables  $x = \lambda z / (1 + \lambda z)$ ). For  $\kappa < 1$  this integral converges, when  $\lambda \rightarrow 0$ , to

$$\Gamma(\kappa + 1)\Gamma(-\kappa + 1) = \frac{\pi\kappa}{\sin(\pi\kappa)}.$$

If  $h(\lambda)$  does not converge to  $\infty$  when  $\lambda$  tends to 0, then we can take  $\bar{h}(\lambda)$  such that  $h(\lambda) \leq \bar{h}(\lambda)$ , and such that  $\bar{h}(\lambda) \rightarrow \infty$ ,  $\lambda e^{\bar{h}(\lambda)} \rightarrow 0$ . The part of the integral between  $h(\lambda)$  and  $\bar{h}(\lambda)$  is of order  $o(\lambda^\kappa)$ . For the part between  $\bar{h}(\lambda)$  and  $\infty$  it is the previous estimate (this is essentially the same as proving Tauber's theorem).  $\square$



**Remark 9.1** : Let us make a final remark useful for [5]. If we truncate the series  $M_1$  on the right and on the left when  $V_k$  reaches the level  $A > 0$ , and if we truncate  $M_2$  when  $H - V_k$  reaches the level  $A$  then the results of Theorem 2.2 and Corollary 9.1 remain valid just by replacing in the tail estimate  $M$  by the meander truncated at level  $A$ . More precisely, let  $A > 0$  and consider

$$\overline{M}_1 = \sum_{k=t_1^-}^{t_1^+} e^{-V_k}, \quad \overline{M}_2 = \sum_{k=t_2^-}^{t_2^+} e^{V_k-H}$$

where

$$t_1^- = \sup\{k \leq 0, V_k \geq A\}, \quad t_1^+ = \inf\{k \geq 0, V_k \geq A\} \wedge T_H$$

$$t_2^- = \sup\{k \leq T_H, H - V_k \geq A\} \vee 0, \quad t_2^+ = \inf\{k \geq T_H, V_k \geq A\}$$

then the results of Theorem 2.2 and Corollary 9.1 remain valid when we consider  $\overline{Z} = e^H \overline{M}_1 \overline{M}_2$  instead of  $Z$ , if we replace in the tail estimate  $M$  by  $\overline{M} = \sum_{t_-}^{t_+} e^{-V_k}$  where  $t_-$  and  $t_+$  are the hitting times of the level  $A$  on the left and on the right. Indeed, in the proof of Theorem 2.2 we see that considering the truncated  $\overline{M}_1$  and  $\overline{M}_2$  only simplifies the proof: we don't need to truncate  $M_1$  and  $M_2$  as we did. In particular, it implies that in Corollary 9.1 we can truncate  $M_1$  and  $M_2$  at a level  $\underline{h}(\lambda) \leq h(\lambda)$ : if  $\underline{h}(\lambda)$  tends to  $\infty$ , we have exactly the same result.

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LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES, CNRS UMR 7599, UNIVERSITÉ  
PARIS 6, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05

*E-mail address:* `enriquez@ccr.jussieu.fr`

INSTITUT CAMILLE JORDAN, CNRS UMR 5208, UNIVERSITÉ DE LYON 1, UNIVERSITÉ DE  
LYON, 43, BOULEVARD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX

*E-mail address:* `sabot@math.univ-lyon1.fr`

LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES, CNRS UMR 7599, UNIVERSITÉ  
PARIS 6, 4 PLACE JUSSIEU, 75252 PARIS CEDEX 05

*E-mail address:* `zindy@ccr.jussieu.fr`