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LIMIT LAWS FOR TRANSIENT RANDOM WALKS IN RANDOM ENVIRONMENT ON $\mathbb{Z}$

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Abstract. We consider transient random walks in random environment on $\mathbb{Z}$ with zero asymptotic speed. A classical result of Kesten, Kozlov and Spitzer says that the hitting time of the level n converges in law, after a proper normalization, towards a positive stable law, but they do not obtain a description of its parameter. A different proof of this result is presented, that leads to a complete characterization of this stable law. The case of Dirichlet environment turns out to be remarkably explicit.

1. Introduction

One-dimensional random walks in random environment to the nearest neighbors have been introduced in the sixties in order to give a model of DNA replication. Recently, this model has known a strong revival in view of applications to the detection of genetics anomalies (see for instance [6] or [19]). In 1975, Solomon gives, in a seminal work [24], a criterion of transience-recurrence for these walks, and shows that three different regimes can be distinguished: the random walk may be recurrent, or transient with a positive asymptotic speed, but it may also be transient with zero asymptotic speed. This last regime, which does not exist among usual random walks, is probably the one which is the less well understood and its study is the purpose of the present paper.

Let us first remind the main existing results concerning the other regimes. In his paper, Solomon computes the asymptotic speed of transient regimes. In 1982, Sinai states, in [22], a limit theorem in the recurrent case. It turns out that the motion in this case is unusually slow since the position of the walk at time n has to be normalized by $(\log n)^2$ in order to present a non trivial limit. In 1986, the limiting law is characterized independently by Kesten [17] and Golosov [12]. Let us notice here that, beyond the interest of his result, Sinai introduces a very powerful and intuitive tool in the study of one-dimensional random walks in random environment. This tool is the potential, which is a function on $\mathbb{Z}$ canonically associated to the random environment. It turns out to be an usual random walk when the transition probabilities at each site are independent and identically distributed (i.i.d.).

Let us now focus on the works about the transient walk with zero asymptotic speed. The main result was obtained by Kesten, Kozlov and Spitzer in [18] who proved that, when normalized by a suitable power of n, the hitting time of the level n converges towards a positive stable law whose index corresponds to the power of n lying in the

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normalization. Recently, Mayer-Wolf, Roitershtein and Zeitouni [20] generalized this result to the case when the environment is defined by an irreducible Markov chain.

Our purpose is to characterize the positive stable law in the case of i.i.d. transition probabilities. Let us mention here that the stable limiting law has been characterized in the case of diffusions in random potential when the potential is either a Brownian motion with drift [15], [13] or a Lévy process [23], but we remind here that despite the similarities of both models one cannot transport results from the continuous model to the discrete one.

The proof chooses a radically different approach than previous ones dealing with the transient case. While the proofs in [18] and [20] are mainly based on the representation of the trajectory of the walk in terms of branching processes in random environment (with immigration), our approach relies heavily on Sinai's interpretation of a particle living in a random potential. However, in the recurrent case, the potential one has to deal with is a recurrent random walk and Sinai introduces a notion of valleys which does not make sense anymore in our setting where the potential is a (let's say negatively) drifted random walk. Therefore, we introduce a different notion of valley which is closely related to the excursions of this random walk above its past minimum. It turns out that a result of Iglehart [14] gives an equivalent of the tail of the height of these excursions. Now, as soon as one can prove that the hitting time of the level n can be reduced to the time spent by the random walk to cross the high excursions of the potential above its past minimum, between 0 and n, which are well separated in space, an i.i.d. property comes out, and the problem is reduced to the study of the tail of the time spent by the walker to cross a single excursion.

It turns out that this tail involves the expectation of the functional of some meander associated with the random walk defining the potential. Now, this functional is itself related to the constant that appears in Kesten's renewal theorem [16]. These last two facts are contained in [8]. Now, in the case when the transition probabilities follow some Beta distribution a result of Chamayou and Letac [5] gives an explicit formula for this constant which yields finally an explicit formula for the parameter of the positive stable law which is obtained at the limit.

The same technics also allow to derive the convergence of the normalized process to the inverse of a standard stable subordinator. This result can be compared with the scaling limits obtained for the trap model of Bouchaud, see [4] for a review.

Soon after finishing this article, we learnt of an independent work, by Peterson and Zeitouni [21], which, by the study of the fluctuations of the potential, showed that a quenched stable limit law is not possible in the zero asymptotic speed regime.

The paper is organized as follows: the results are stated in Section 2, a detailed sketch of the proof is presented in Section 3, and the rest of the paper is devoted to proofs.

2. Notations and main results

Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of i.i.d. random variables taking values in $(0,1)$ defined on Ω, which stands for the random environment. Denote by P the distribution of ω and by E the corresponding expectation. Conditioning on ω (i.e. choosing an environment), we define the random walk in random environment $(X_n,\, n \geq 0)$ as a nearest-neighbor random walk on $\mathbb{Z}$ with transition probabilities given by ω:

$(X_n,\, n \geq 0)$ is the Markov chain satisfying $X_0 = 0$ and for $n \geq 0$,

$$P_\omega\left(X_{n+1} = x+1 \,|\, X_n = x\right) = \omega_x = 1 - P_\omega\left(X_{n+1} = x-1 \,|\, X_n = x\right).$$

We denote by P_ω the law of $(X_n,\, n \geq 0)$ and E_ω the corresponding expectation. We denote by $\mathbb{P}$ the joint law of $(\omega, (X_n)_{n\geq 0})$. We refer to Zeitouni [25] for an overview of results on random walks in random environment.

In the study of one-dimensional random walks in random environment, an important role is played by a process called the potential, denoted by $V = (V(x),\ x \in \mathbb{Z})$. Let us introduce

$$\rho_i := \frac{1-\omega_i}{\omega_i}, \qquad i \in \mathbb{Z}.$$

Then, the potential is a function of the environment ω, and is defined as follows:

$$V(x) := \begin{cases} \sum_{i=1}^{x} \log \rho_i & \text{if } x \geq 1, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x+1}^{0} \log \rho_i & \text{if } x \leq -1. \end{cases}$$

Furthermore, we consider the weak descending ladder epochs for the potential defined by $e_0 := 0$ and

$$e_i := \inf\{k > e_{i-1} :\ V(k) \leq V(e_{i-1})\}, \qquad i \geq 1,$$

which play a crucial role in our proof. Observe that $(e_i - e_{i-1})_{i\geq 1}$ is a family of i.i.d. random variables. Moreover, classical results of fluctuation theory (see [9], p. 396), tell us that, under assumptions (a)-(b) of Theorem 1,

$$E[e_1] < \infty. \tag{2.1}$$

Now, observe that the $((e_i, e_{i+1}])_{i\geq 0}$ stand for the set of excursions of the potential above its past minimum. Let us introduce H_i, the height of the excursion $(e_i, e_{i+1}]$ defined by $H_i := \max_{e_i \leq k \leq e_{i+1}} \left(V(k) - V(e_i)\right)$, for $i \geq 0$. Note that the $(H_i)_{i\geq 0}$'s are i.i.d. random variables.

We now introduce the hitting time $\tau(x)$ of level x for the random walk $(X_n,\, n \geq 0)$,

$$\tau(x) := \inf\{n \geq 1 :\ X_n = x\}, \quad x \in \mathbb{Z}. \tag{2.2}$$

For $\alpha \in (0,1)$, let $\mathcal{S}_\alpha^{ca}$ be a completely asymmetric stable random variable of index α with Laplace transform, for $\lambda > 0$,

$$E[\mathrm{e}^{-\lambda \mathcal{S}_\alpha^{ca}}] = \mathrm{e}^{-\lambda^\alpha}.$$

Moreover, let us introduce the constant C_K describing the tail of Kesten's renewal series, see [16], defined by $R := \sum_{k\geq 0} \mathrm{e}^{V(k)}$:

$$P\{R > x\} \sim \frac{C_K}{x^\kappa}, \qquad x \to \infty. \tag{2.3}$$

Then the main result of the paper can be stated as follows. The symbols "$\xrightarrow{\text{law}}$" denotes the convergence in distribution.

Theorem 1. *Let $\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of independent and identically distributed random variables such that*

(a) there exists $0 < \kappa < 1$ for which $E\left[\rho_0^\kappa\right] = 1$ and $E\left[\rho_0^\kappa \log^+ \rho_0\right] < \infty$,

(b) the distribution of $\log \rho_0$ *is non-lattice.*

Then, we have, when n *goes to infinity,*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{law}} 2\left(\frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0]\right)^{\frac{1}{\kappa}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{law}} \frac{\sin(\pi\kappa)}{2^\kappa \pi \kappa^2 C_K^2 E[\rho_0^\kappa \log \rho_0]} \left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa.$$

Remark 1. *We think that the method used in this paper could also treat the case* $\kappa = 1$ *(see Section 10 for conjecture and comments).*

The result of Theorem 1 is interesting when C_K is explicitly known. In the case of Dirichlet environment, i.e. when the law of the environment satisfies $\omega_1(\,\mathrm{d}x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{[0,1]}(x)\,\mathrm{d}x$, with $\alpha, \beta > 0$ and $B(\alpha,\beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x$, things can be made much more explicit. The assumption of Theorem 1 correspond to the case where $0 < \alpha - \beta < 1$ and an easy computation leads to $\kappa = \alpha - \beta$.

Corollary 1. *In the case where* ω_1 *has a distribution* $\mathrm{Beta}(\alpha,\beta)$*, with* $0 < \alpha - \beta < 1$*, Theorem 1 applies with* $\kappa = \alpha - \beta$*. Then, we have, when* n *goes to infinity,*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{law}} 2\left(\frac{\pi}{\sin(\pi(\alpha-\beta))} \frac{\psi(\alpha)-\psi(\beta)}{B(\alpha,\beta)^2}\right)^{\frac{1}{\alpha-\beta}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{law}} \frac{\sin(\pi(\alpha-\beta))}{2^{\alpha-\beta}\pi} \frac{B(\alpha,\beta)^2}{\psi(\alpha)-\psi(\beta)} \left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa,$$

where ψ *denotes the classical Digamma function,* $\psi(z) := (\log\Gamma)'(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

In the case where C_K is unknown, it is possible to give a probabilistic representation of the parameter. Actually, we obtain first Theorem 2, from which we deduce Theorem 1. In this aim, let us introduce the classical distribution $\tilde{P}$ associated with the random walk $(V(x), x \in \mathbb{Z})$ under P (denoted by ${}^aP$ in [9], p. 406). If μ denotes the law of $\log \rho_0$, thanks to assumption (a) of Theorem 1 we can define the law $\tilde{\mu} = \rho_0^\kappa \mu$, and the law $\tilde{P} = \tilde{\mu}^{\otimes\mathbb{Z}}$ which is the law of a sequence of i.i.d. random variables with law $\tilde{\mu}$. The definition of κ implies that $\int \log\rho\,\tilde{\mu}(d\rho) > 0$.

Theorem 2. *Let* $\omega := (\omega_i,\, i \in \mathbb{Z})$ *be a family of independent and identically distributed random variables satisfying assumptions* (a)*–*(b) *of Theorem 1. Then, we have, when* n *goes to infinity,*

$$\frac{\tau(n)}{n^{1/\kappa}} \xrightarrow{\text{law}} 2\left(\frac{\pi}{\sin(\pi\kappa)} \frac{E[M^\kappa]^2}{E[e_1]^2} \frac{(1-E[\mathrm{e}^{\kappa V(e_1)}])^2}{E[\rho_0^\kappa \log\rho_0]}\right)^{\frac{1}{\kappa}} \mathcal{S}_\kappa^{ca},$$

$$\frac{X_n}{n^\kappa} \xrightarrow{\text{law}} \frac{\sin(\pi\kappa)}{2^\kappa\pi} \frac{E[e_1]^2}{E[M^\kappa]^2} \frac{E[\rho_0^\kappa \log\rho_0]}{(1-E[\mathrm{e}^{\kappa V(e_1)}])^2} \left(\frac{1}{\mathcal{S}_\kappa^{ca}}\right)^\kappa.$$

where M *has the law of the exponential of a meander, i.e.*

$$M \overset{\text{law}}{=} \sum_{k<0} \mathrm{e}^{-V_k'} + \sum_{k\geq 0} \mathrm{e}^{-V_k''},$$

with $(V'_k)_{k<0}$ *under* $P\{\cdot|V'_k \geq 0,\, \forall k<0\}$ *and independent of* $(V''_k)_{k\geq 0}$ *under* $\tilde{P}\{\cdot|V''_k > 0,\, \forall k>0\}$.

Remark 2. *When* C_K *is not explicit it is better to use the expression of the parameter in terms of* $E[M^\kappa]$ *which is easy to evaluate numerically.*

Our technics also allow to derive the convergence of the normalized process.

Theorem 3. *Under the assumption (a)-(b) of Theorem 1, the law of the process*

$$(S_t^{(n)},\, t\geq 0) := \left(n^{-\kappa} X_{\lfloor nt\rfloor},\, t\geq 0\right),$$

defined on the space of càdlàg functions equipped with the Skorokhod M_1*-topology, converges to the law of*

$$\left(\frac{\sin(\pi\kappa)}{2^\kappa \pi\kappa^2 C_K^2 E\left[\rho_0^\kappa \log\rho_0\right]} Z_t,\, t\geq 0\right),$$

where Z_t *is the inverse of the standard* κ*-stable subordinator.*

In the following, the constant C stands for a positive constant large enough, whose value can change from line to line.

3. Sketch of the proof

Let us start now with the outlines of our proof.

Since assumption (a) of Theorem 1 implies $E[\log\rho_0]<0$, the random walk describing the potential is negatively drifted, so that the random walker will converge almost surely to the region of lowest potential, i.e. to infinity. Along its way, it will have to overcome some obstacles which are represented by the *excursions* of the random potential above its past minimum.

Now, a result of Iglehart [14] says that, under assumptions (a)-(b) of Theorem 1, the tail of the height H of an excursion above its past minimum is given by

$$P\{H>h\} \sim C_I\, \mathrm{e}^{-\kappa h}, \qquad h\to\infty, \tag{3.1}$$

where

$$C_I = \frac{(1-E[\mathrm{e}^{\kappa V(e_1)}])^2}{\kappa E[\rho_0^\kappa \log\rho_0] E[e_1]}, \tag{3.2}$$

with e_1 denoting the endpoint of the first excursion, so that $V(e_1)\leq 0$. Iglehart's result is actually deduced from a former well-known result of Cramer, whose proof was later simplified by Feller [9], concerning the tail of the maximum S of a $\mathbb{N}$-time indexed random walk which claims that

$$P\{S>h\} \sim C_F \mathrm{e}^{-\kappa h}, \qquad h\to\infty. \tag{3.3}$$

Since S is stochastically bigger than H, C_I must be smaller than C_F, and a rather straight argument of Iglehart shows that the ratio between both constants is equal to $1-E[\mathrm{e}^{\kappa V(e_1)}]$.

Recalling (2.1), the law of large numbers implies that the number of excursions between 0 and n is almost surely equivalent to $n/E[e_1]$. We will be therefore interested in the asymptotic of the hitting time of the n-th excursion, we will denote by $\tau(e_n)$.

3.1. **The general case.** In a first step, we show (see Lemma 10) that $\tau(e_n)$ reduces to the time spent by the walker to climb high excursions, namely, higher than $h_n := \frac{(1-\varepsilon)}{\kappa}\log n$. Let us notice here, that, statistically, by Iglehart's result, no excursion of height larger than $\frac{(1+\varepsilon)}{\kappa}\log n$ can be found among the first n excursions.

It turns out that these excursions are spatially well separated (see Lemma 3), and that there are asymptotically $nP\{H \geq h_n\}$ of these, i.e. $C_I n^\varepsilon$ (see Lemma 2). One can therefore define boxes around, we shall denote by $([a_k, d_k])_{0\leq k\leq C_I n^\varepsilon}$, such that the random walker will have a small probability to go back to a box which was already visited. More precisely, let b_k and c_k denote respectively the starting point of the k-th high excursion and the first time this excursion reaches its maximum, so that the following ranking $a_k \leq b_k \leq c_k \leq d_k$ holds. With an overwhelming probability, for all $k \in [0, C_I n^\varepsilon]$, the walker, once arrived at b_k, will never visit a_k again (see Lemma 9).

In addition, one can prove that the portions of potential between a_k and d_k, we call "deep valleys" are almost i.i.d. The proof of this fact requires the introduction of what we call "$*$-valleys" which are i.i.d., and coincide with the sequence of "deep valleys" with a high probability (see Lemma 5).

Now, gathering these two previous facts, we get that $\tau(e_n)$ can be roughly written:

$$\tau(e_n) = \tau(b_1, d_1) + ... + \tau(b_{C_I n^\varepsilon}, d_{C_I n^\varepsilon}),$$

where the $\tau(b_k, d_k)$'s are i.i.d. random variables representing the time spent by the walker to cross the k-th excursion, i.e. to go from b_k to d_k.

Consequently, considering the Laplace transform of $n^{-1/\kappa}\tau(e_n)$, we are led to the study of the asymptotic when λ goes to 0 of $\mathbb{E}[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}\tau(b_1,d_1)}]^{C_I n^\varepsilon}$ (see Proposition 1).

Now, the passage from b_1 to d_1 can be decomposed into the sum of a random geometrically distributed number of unsuccessful attempts to cross the excursion, followed by a successful attempt. The accurate estimation of the time spent by each (successful and unsuccessful) attempt leads us to consider two $h-$processes where the random walker evolves in two modified potentials, one corresponding to the conditioning on a failure (potential $\widehat{V}$, see Lemma 11), and the other to the conditioning on a success (potential $\overline{V}$, see Lemma 12).

It turns out that the contribution of the last successful attempt to the quantity $\tau(b_1, d_1)$ is negligible so that $\mathbb{E}[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}\tau(b_1,d_1)}]^{C_I n^\varepsilon}$ is approximately equal to

$$E\left[\sum_{k\geq 0}(1-p(\omega))E_\omega[\mathrm{e}^{-\lambda F}]^k p(\omega)^k\right]^{C_I n^\varepsilon} = E\left[\frac{1-p(\omega)}{1-p(\omega)\,E_\omega[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}F}]}\right]^{C_I n^\varepsilon},$$

where F denotes the time of an unsuccessful attempt (failure), and $1-p(\omega)$ denotes the (small) probability of success which is known, by classical arguments, to be equal to $\omega_b \frac{\mathrm{e}^{V(b)}}{\sum_{x=b}^{d-1}\mathrm{e}^{V(x)}}$ (a first step of probability ω_b to go to $b+1$ and then, starting at $b+1$, a probability $\frac{\mathrm{e}^{V(b)}}{\sum_{x=b}^{d-1}\mathrm{e}^{V(x)}}$ to hit d before b).

Now, a key step consists in the fact that the linearization $E_\omega[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}F}] \sim 1 - \frac{\lambda}{n^{1/\kappa}}E_\omega[F]$ can be justified. The error is expressed in terms of $E_\omega[F^2]$ which is explicitly computed (see Lemma 11) and dominated by a function of the maximal fall of the potential during its rise from $V(b)$ to $V(c)$, and the maximal rise of the potential during its fall from $V(c)$ to $V(d)$ which can be uniformly controlled on all the $C_I n^\varepsilon$

boxes (see Lemma 13). We are therefore led to the study of

$$\left(E\left[\frac{1}{1+\frac{\lambda}{n^{1/\kappa}}\frac{p}{1-p}E_\omega[F]}\right]\right)^{C_I n^\varepsilon}.$$

Now, $E_\omega[F]$ is known to be equal to $2\omega_b\sum_{a+1}^{d-1}\mathrm{e}^{-(\widehat{V}(x)-\widehat{V}(b))}$. Therefore we are back to the study of

$$\left(E\left[\frac{1}{1+\frac{2\lambda}{n^{1/\kappa}}\mathrm{e}^H\widehat{M}_1M_2}\right]\right)^{C_I n^\varepsilon},$$

where $H=\mathrm{e}^{V(c)-V(b)}$ denotes the height of an high excursion and where $\widehat{M}_1:=\sum_{a+1}^{d-1}\mathrm{e}^{-(\widehat{V}(x)-\widehat{V}(b))}$ and $M_2:=\sum_b^{d-1}\mathrm{e}^{-(V(x)-V(c))}$ are two functionals of the potential that depends very locally on the potential respectively around the local minimum b and the local maximum c.

Since $V(b)$ and $V(c)$ are locally extremal, these functionals can be assimilated to two functionals of *meanders* associated to the random walk defining the potential. Furthermore, a reversal time argument and the proximity of V and $\widehat{V}$ around b show that these two quantities are asymptotically the same functionals of the same meander. It is defined as follows $M:=\sum_{n\in\mathbb{Z}}\mathrm{e}^{-Y_n}$, where Y_n is the random walk of step $\log\rho$, conditioned to be positive on all $\mathbb{Z}$. This conditioning has to be understood as follows: on $\mathbb{Z}_-$ it is the natural one (we condition on an event having a strictly positive probability), whereas on $\mathbb{Z}_+$ it represents the limit in law of random walks of step $\log\rho$ that are conditioned to overshoot a high level before visiting $\mathbb{R}_-$ (see for instance the paper of Bertoin and Doney [2] and the references therein for detailed discussions on the subject) .

Furthermore, it turns out that the three quantities e^H, $\widehat{M}_1$ and M_2 are asymptotically independent. This delicate step based on coupling arguments, which are adapted from the proof of the renewal theorem for the sum of i.i.d. variables, is treated in the paper [8], see Proposition 7.1. As a consequence, the tail of $\mathrm{e}^H\widehat{M}_1M_2$ can be derived, see Theorem 2.2 in [8], as well as a Tauberian result about $\frac{1}{1+\lambda\mathrm{e}^H\widehat{M}_1M_2}$, see Corollary 9.1 in [8]. This Tauberian result yields to

$$\left(E\left[\frac{1}{1+\frac{2\lambda}{n^{1/\kappa}}\mathrm{e}^H\widehat{M}_1M_2}\right]\right)^{C_I n^\varepsilon}=\exp\Big\{-\Big(2^\kappa\frac{\pi\kappa}{\sin(\pi\kappa)}E[M^\kappa]^2C_I\Big)\lambda^\kappa\Big\}+o(1).$$

where C_I is given in (3.2). Now, one can be tempted to express the functional $E[M^\kappa]$ in terms of the more usual constant C_K, see (2.3). This is the content of Theorem 1.1 in [8], which yields

$$C_K=E[M^\kappa]C_F=E[M^\kappa]\frac{(1-E[\mathrm{e}^{\kappa V(e_1)}])}{\kappa E[\rho_0^\kappa\log\rho_0]E[e_1]}.$$

Therefore, the Laplace transform of $n^{-1/\kappa}\tau(e_n)$ writes

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}\tau(e_n)}] &= \exp\Big\{-\Big(2^\kappa\frac{\pi\kappa}{\sin(\pi\kappa)}\frac{C_K^2C_I}{C_F^2}\Big)\lambda^\kappa\Big\}+o(1)\\ &= \exp\Big\{-\Big(2^\kappa\frac{\pi\kappa^2}{\sin(\pi\kappa)}C_K^2E[\rho_0^\kappa\log\rho_0]E[e_1]\Big)\lambda^\kappa\Big\}+o(1).\end{aligned}$$

Finally, since, by the law of large numbers, e_n/n converges a.s. to $E[e_1]$, we conclude that

$$\mathbb{E}[\mathrm{e}^{-\frac{\lambda}{n^{1/\kappa}}\tau(n)}] = \exp\Big\{ -\Big(2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0]\Big)\lambda^\kappa \Big\} + o(1).$$

Hence, we obtain that the limit is the positive stable law with index κ and parameter $2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0]$.

3.2. **The case of a Dirichlet environment.** In the case of a Dirichlet environment, namely when $\omega_1(dx) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} 1_{[0,1]}(x)dx$, $(\alpha, \beta > 0)$ things can be made much more explicit. The assumptions of Theorem 1 correspond to the case when $0 < \alpha - \beta < 1$ and an easy computation shows that $\kappa = \alpha - \beta$. Now, a classical argument of derivation under the sign integral shows that

$$E[\rho_0^\kappa \log \rho_0] = \psi(\alpha) - \psi(\beta),$$

where ψ denotes the classical Digamma function $\psi(z) := (\log \Gamma)'(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

Furthermore, a work of Chamayou and Letac [5] shows that C_K can be made explicit. Indeed, with the notations of [5], ρ_0 follows the law $\beta^{(2)}_{p,q}(dx) := \frac{1}{B(p,q)} x^{p-1}(1+x)^{-p-q} 1_{\mathbb{R}_+}(x)dx$ with $p = \beta$ and $q = \alpha$. Then, Example 9 of [5] says that $\sum_{k\geq 1} \mathrm{e}^{V(k)}$ follows the law of $\beta^{(2)}_{\beta,\alpha-\beta}$ having density $\frac{1}{B(\alpha,\beta)} x^{\beta-1}(1+x)^{-\alpha} 1_{\mathbb{R}_+}(x)$. But we have $\beta^{(2)}_{\beta,\alpha-\beta}([t,+\infty[) \sim \frac{1}{(\alpha-\beta)B(\alpha,\beta)} \frac{1}{t^{\alpha-\beta}}$, $t \to \infty$. Hence, $C_K = \frac{1}{(\alpha-\beta)B(\alpha,\beta)}$.

The expression of the parameter can be simplified into

$$2^\kappa \frac{\pi\kappa^2}{\sin(\pi\kappa)} C_K^2 E[\rho_0^\kappa \log \rho_0] = \frac{\pi 2^{\alpha-\beta}}{\sin(\pi(\alpha-\beta))} \frac{\psi(\alpha) - \psi(\beta)}{B(\alpha,\beta)^2}.$$

4. Two notions of valleys

Sinai introduced in [22] the notion of valley in a context where the random walk defining the potential was recurrent. We have to do a similar job in our framework where the random walk defining the potential is negatively drifted. The deep valleys we introduce here are closely related to the excursions of the random walk above its past minimum which are higher than a critical height. They consist actually in some portion of potential including these excursions. When the critical height is taken sufficiently large, the excursions are quite seldom and the valleys are likely to be disjoint. In order to deal with almost sure disjoint valleys, we also introduce $*$-valleys which coincide with deep valleys with high probability.

4.1. **The deep valleys.** Let us define the maximal variations of the potential before site x by:

$$V^\uparrow(x) := \max_{0\leq i\leq j\leq x} (V(j) - V(i)), \qquad x \in \mathbb{N},$$
$$V^\downarrow(x) := \min_{0\leq i\leq j\leq x} (V(j) - V(i)), \qquad x \in \mathbb{N}.$$

By extension, we introduce

$$V^\uparrow(x,y) := \max_{x\leq i\leq j\leq y} (V(j) - V(i)), \qquad x < y,$$
$$V^\downarrow(x,y) := \min_{x\leq i\leq j\leq y} (V(j) - V(i)), \qquad x < y.$$

In order to define deep valleys, we extract from the first n excursions of the potential above its minimum, these whose heights are greater than a critical height h_n, defined by

$$h_n := \frac{(1-\varepsilon)}{\kappa}\log n, \tag{4.1}$$

for some $0 < \varepsilon < 1/3$. Let $(\sigma(i))_{i\geq 1}$ be the successive indexes of excursions, whose heights are greater than h_n. More precisely,

$$\begin{aligned}
\sigma(1) &:= \inf\{i \geq 0 : H_i \geq h_n, \}, \\
\sigma(j) &:= \inf\{i > \sigma(j-1) : H_i \geq h_n\}, \qquad j \geq 2, \\
K_n &:= \max\{j \geq 0 : \sigma(j) \leq n\}.
\end{aligned}$$

We consider now some random variables depending only on the environment, which define the deep valleys.

Definition 1. *For $1 \leq j \leq K_n + 1$, let us introduce*

$$\begin{aligned}
b_j &:= e_{\sigma(j)}, \\
a_j &:= \sup\{k \leq b_j : V(k) - V(b_j) \geq D_n\}, \\
T_j^{\uparrow} &:= \inf\{k \geq b_j : V(k) - V(b_j) \geq h_n\}, \\
\overline{d}_j &:= e_{\sigma(j)+1}, \\
c_j &:= \inf\{k \geq b_j : V(k) = \max_{b_j \leq x \leq \overline{d}_j} V(x)\}, \\
d_j &:= \inf\{k \geq \overline{d}_j : V(k) - V(\overline{d}_j) \leq -D_n\}.
\end{aligned}$$

where $D_n := (1 + \frac{1}{\kappa})\log n$. We call (a_j, b_j, c_j, d_j) a deep valley and denote by $H^{(j)}$ the height of the j-th deep valley.

Remark 3. *It may happen that two different deep valleys are not disjoint, even if this event is highly improbable as it will be shown in Lemma 3 and Lemma 4 in Subsection 5.1.*

4.2. **The $*$-valleys.** Let us introduce now a subsequence of the deep valleys defined above. It will turn out that both sequences coincide with probability tending to 1 as n goes to infinity. This will be specified in Lemma 5. Let us first introduce

$$\begin{aligned}
\gamma_1^* &:= \inf\{k \geq 0 : V(k) \leq -D_n\}, \\
T_1^* &:= \inf\{k \geq \gamma_1^* : V^{\uparrow}(\gamma_1^*, k) \geq h_n\}, \\
b_1^* &:= \sup\{k \leq T_1^* : V(k) = \min_{0 \leq x \leq T_1^*} V(x)\}, \\
a_1^* &:= \sup\{k \leq b_1^* : V(k) - V(b_1^*) \geq D_n\}, \\
\overline{d}_1^* &:= \inf\{k \geq T_1^* : V(k) \leq V(b_1^*)\}, \\
c_1^* &:= \inf\{k \geq b_1^* : V(k) = \max_{b_1^* \leq x \leq \overline{d}_1^*} V(x)\}, \\
d_1^* &:= \inf\{k \geq \overline{d}_1^* : V(k) - V(\overline{d}_1^*) \leq -D_n\}.
\end{aligned}$$

Let us define the following sextuplets of points by iteration

$$(\gamma_j^*, a_j^*, b_j^*, T_j^*, c_j^*, \overline{d}_j^*, d_j^*) := (\gamma_1^*, a_1^*, b_1^*, T_1^*, c_1^*, \overline{d}_1^*, d_1^*) \circ \theta_{d_{j-1}^*}, \qquad j \geq 2,$$

where θ_i denotes the i-shift operator.

Definition 2. *We call a $*$-valley any quadruplet $(a_j^*, b_j^*, c_j^*, d_j^*)$ for $j \geq 1$. Moreover, we shall denote by K_n^* the number of such $*$-valleys before e_n, i.e. $K_n^* := \sup\{j \geq 0 : T_j^* \leq e_n\}$.*

It will be made of independent and identically distributed portions of potential (up to some translation).

5. Reduction to a single valley

This section is devoted to the proof of Proposition 1 which tells that the study of $\tau(e_n)$ can be reduced to the analysis of the time spent by the random walk to cross the first deep valley. To ease notations, we introduce $\lambda_n := \frac{\lambda}{n^{1/\kappa}}$.

Proposition 1. *For all n large enough, we have*

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_n \tau(e_n)}\right] \in \left[E\left[E_{\omega,|a_1}^{b_1}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right]^{\overline{K}_n} + o(1)\,,\; E\left[E_{\omega,|a_1}^{b_1}\left[\mathrm{e}^{-\lambda_n \tau(d_1)}\right]\right]^{\underline{K}_n} + o(1)\right].$$

where $\underline{K}_n := \lfloor nq_n(1-n^{-\varepsilon/4})\rfloor$, $\overline{K}_n := \lceil nq_n(1+n^{-\varepsilon/4})\rceil$, $q_n := P\{H_0 \geq h_n\}$ and where $E_{\omega,|y}^x$ denotes the quenched law of the random walk in the environment ω, starting at x and reflected at site y.

5.1. **Introducing "good" environments.** Let us define the four following events, that concern exclusively the potential V. The purpose of this subsection is to show that they are realized with an asymptotically overwhelming probability when n goes to infinity. These results will then make it possible to restrict the study of $\tau(e_n)$ to these events.

$$\begin{aligned}
A_1(n) &:= \{e_n < C'n\}\,,\\
A_2(n) &:= \left\{\lfloor nq_n(1-n^{-\varepsilon/4})\rfloor \leq K_n \leq \lceil nq_n(1+n^{-\varepsilon/4})\rceil\right\},\\
A_3(n) &:= \cap_{j=0}^{K_n}\left\{\sigma(j+1)-\sigma(j) \geq n^{1-3\varepsilon}\right\},\\
A_4(n) &:= \cap_{j=1}^{K_n+1}\left\{d_j - a_j \leq C''\log n\right\},
\end{aligned}$$

where $\sigma(0) := 0$ (for convenience of notation) and C', C'' stand for positive constants which will be specified below.

In words, $A_1(n)$ allows us to bound the total length of the first n excursions. The event $A_2(n)$ gives a control on the number of deep valleys. The event $A_3(n)$ ensures that the deep valleys are well separated, while $A_4(n)$ bounds finely the length of each of them.

Let us introduce the following hitting times (for the potential)

$$\begin{aligned}
T_h &:= \min\{x \geq 0 :\, V(x) \geq h\}, \qquad h > 0,\\
T_A &:= \min\{x \geq 0 :\, V(x) \in A\}, \qquad A \subset \mathbb{R}.
\end{aligned}$$

Then, we obtain the following results.

Lemma 1. *The probability $P\{A_1(n)\}$ converges to 1 when n goes to infinity.*

Proof. It is a direct consequence of the law of large numbers as soon as C' is taken bigger than $E[e_1]$. □

Lemma 2. *The probability $P\{A_2(n)\}$ converges to 1 when n goes to infinity.*

In words, Lemma 2 means that K_n "behaves" like $C_I n^\varepsilon$, when n tends to infinity. In particular, (3.1), which yields $q_n \sim \frac{C_I}{n^{1-\varepsilon}}$, and Lemma 2 imply

$$P\{K_n + 1 \geq 2C_I n^\varepsilon\} \to 0, \qquad n \to \infty. \tag{5.1}$$

Proof. At first, observe that

$$P\Big\{\frac{K_n}{nq_n} \geq 1 + n^{-\varepsilon/4}\Big\} = P\{K_n - nq_n \geq n^{1-\varepsilon/4} q_n\} \leq \frac{\mathrm{Var}(K_n)}{n^{2(1-\varepsilon/4)} q_n^2},$$

the inequality being a consequence of Markov inequality and the fact that K_n follows a binomial distribution of parameter (n, q_n). Moreover, $\mathrm{Var}(K_n) = nq_n(1-q_n) \leq nq_n$ implies

$$P\{\frac{K_n}{nq_n} \geq 1 + n^{-\varepsilon/4}\} \leq \frac{1}{n^{1-\varepsilon/2} q_n}.$$

Now, Iglehart's result (see (3.1)) implies $q_n \sim \frac{C_I}{n^{1-\varepsilon}}$, $n \to \infty$. Therefore we get that $P\{\frac{K_n}{nq_n} \leq 1 + n^{-\varepsilon/4}\}$ converges to 1 when n goes to infinity. Using similar arguments, we get the convergence to 1 of $P\{\frac{K_n}{nq_n} \geq 1 - n^{-\varepsilon/4}\}$. □

Lemma 3. *The probability $P\{A_3(n)\}$ converges to 1 when n goes to infinity.*

Proof. We make first the trivial observation that

$$\begin{aligned} P\{A_3(n)\} &\geq P\{\sigma(j+1) - \sigma(j) \geq n^{1-3\varepsilon},\, 0 \leq j \leq \lfloor 2C_I n^\varepsilon \rfloor\,;\, K_n \leq 2C_I n^\varepsilon\} \\ &\geq P\{\sigma(j+1) - \sigma(j) \geq n^{1-3\varepsilon},\, 0 \leq j \leq \lfloor 2C_I n^\varepsilon \rfloor\} - P\{K_n \geq 2C_I n^\varepsilon\}, \end{aligned}$$

the second inequality being a consequence of $P\{A\,;B\} \geq P\{A\} - P\{B^c\}$, for any couple of events A and B. Therefore, recalling (5.1) and using the fact that $(\sigma(j+1) - \sigma(j))_{0 \leq j \leq \lfloor 2C_I n^\varepsilon \rfloor}$ are i.i.d. random variables, it remains to prove that

$$P\{\sigma(1) \geq n^{1-3\varepsilon}\}^{\lfloor 2C_I n^\varepsilon \rfloor} \to 1, \qquad n \to \infty.$$

Since $\sigma(1)$ is a geometrical random variable with parameter q_n, $P\{\sigma(1) \geq n^{1-3\varepsilon}\}$ is equal to $(1-q_n)^{\lceil n^{1-3\varepsilon} \rceil}$, which implies

$$P\{\sigma(1) \geq n^{1-3\varepsilon}\}^{\lfloor 2C_I n^\varepsilon \rfloor} = (1-q_n)^{\lfloor 2C_I n^\varepsilon \rfloor \lceil n^{1-3\varepsilon} \rceil} \geq \exp\left\{-Cn^{1-2\varepsilon} q_n\right\}.$$

Then, the conclusion follows from (3.1), which implies that $q_n \sim C_I / n^{1-\varepsilon}$, $n \to \infty$. □

Lemma 4. *For C'' large enough, The probability $P\{A_4(n)\}$ converges to 1 when n goes to infinity.*

Proof. Looking at the proof of Lemma 3, we have to prove that $P\{d_j - a_j \geq C'' \log n\}$ is equal to a $o(n^{-\varepsilon})$, $n \to \infty$. Moreover, observing that $d_j - a_j = (d_j - \overline{d}_j) + (\overline{d}_j - T_j^{\uparrow}) + (T_j^{\uparrow} - b_j) + (b_j - a_j)$, the proof of Lemma 4 boils down to showing that, for C'' large enough,

$$P\{d_j - \overline{d}_j \geq \frac{C''}{4} \log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{5.2}$$

$$P\{\overline{d}_j - T_j^{\uparrow} \geq \frac{C''}{4} \log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{5.3}$$

$$P\{T_j^{\uparrow} - b_j \geq \frac{C''}{4} \log n\} = o(n^{-\varepsilon}), \qquad n \to \infty, \tag{5.4}$$

$$P\{b_j - a_j \geq \frac{C''}{4} \log n\} = o(n^{-\varepsilon}), \qquad n \to \infty. \tag{5.5}$$

To prove (5.2), we apply the strong Markov property at time $\overline{d}_j$ such that we get $P\{d_j - \overline{d}_j \geq \frac{C''}{4} \log n\} \leq P\{T_{(-\infty, -D_n]} \geq \frac{C''}{4} \log n\}$. Therefore, we have

$$P\{d_j - \overline{d}_j \geq \frac{C''}{4} \log n\} \leq P\{\inf_{0 \leq x \leq \frac{C''}{4} \log n} V(x) > -D_n\} \leq P\{V(\frac{C''}{4} \log n) > -D_n\}.$$

Recalling that $D_n := (1 + \frac{1}{\kappa}) \log n$ and observing that large deviations do occur, we obtain, from Cramer's theory, that $P\{V(\frac{C''}{4} \log n) > -D_n\} \leq \mathrm{e}^{-\frac{C''}{4} \log n \, I(-\frac{4}{C''}(1+\frac{1}{\kappa}))}$, with $I(\cdot)$ the convex rate function associated to V. This inequality implies (5.2) by choosing C'' large enough such that $\frac{C''}{4} I(-\frac{4}{C''}(1 + \frac{1}{\kappa})) > \varepsilon$, which is possible since $I(0) > 0$.

To prove (5.3), observe first that (3.1) implies $P\{H^{(j)} > \frac{(1+\varepsilon')}{\kappa} \log n\} \sim n^{-(\varepsilon' + \varepsilon)} = o(n^{-\varepsilon})$, $n \to \infty$. Therefore, we obtain that $P\{\overline{d}_j - T_j^{\uparrow} \geq \frac{C''}{4} \log n\}$ is less or equal than $P\{T_{(-\infty, -\frac{1+\varepsilon'}{\kappa} \log n]} \geq \frac{C''}{4} \log n\} + o(n^{-\varepsilon})$ and conclude the proof with the same arguments we used to treat (5.2).

To get (5.4), observe first that

$$\begin{aligned} P\{T_j^{\uparrow} - b_j \geq \frac{C''}{4} \log n\} &= P\{T_{h_n} \geq \frac{C''}{4} \log n \,|\, H_0 \geq h_n\} \\ &\leq P\{\frac{C''}{4} \log n \leq T_{h_n} < \infty\} / P\{H_0 \geq h_n\}. \end{aligned}$$

Therefore, Cramer's theory, see [7], yields

$$\begin{aligned} P\{\frac{C''}{4} \log n \leq T_{h_n} < \infty\} &\leq \sum_{k \geq \frac{C''}{4} \log n} P\{V(k) \geq h_n\} \leq \sum_{k \geq \frac{C''}{4} \log n} \mathrm{e}^{-k \, I(\frac{h_n}{k})} \\ &\leq \sum_{k \geq \frac{C''}{4} \log n} \mathrm{e}^{-k \, I(0)} \leq \frac{C}{n^{\frac{C''}{4} I(0)}}, \end{aligned}$$

the second inequality being a consequence of the fact that the convex rate function $I(\cdot)$ is an increasing function on $(m, +\infty)$. Using (3.1), we get, for all large n,

$$P\{T_j^{\uparrow} - b_j \geq \frac{C''}{4} \log n\} \leq \frac{C}{n^{\frac{C''}{4} I(0) - (1-\varepsilon)}},$$

which yields (5.4), by choosing C'' large enough such that $C'' > \frac{4}{I(0)}$.

For (5.5), observe first that $((V(k-b_j)-V(b_j))_{a_j\le k\le b_j}, a_j, b_j)$ has the same distribution as $((V(k))_{a^-\le k\le 0}, a^-, 0)$ under $P\{\cdot|V(k)\ge 0, a^-\le k\le 0\}$, where $a^- := \sup\{k\le 0 : V(k)\ge D_n\}$. Then, since $P\{V(k)\ge 0, k\le 0\} > 0$ and since $(V(-k),\ k\ge 0)$ has the same distribution as $(-V(k),\ k\ge 0)$, we obtain

$$P\{b_j - a_j \ge \frac{C''}{4}\log n\} \le CP\{T_{(-\infty,-D_n]} > \frac{C''}{4}\log n\} \le CP\{V(\frac{C''}{4}\log n) > -D_n\}.$$

Now, the arguments are the same as in the proof of (5.2). □

Defining $A(n) := A_1(n)\cap A_2(n)\cap A_3(n)\cap A_4(n)$, a consequence of Lemma 1, Lemma 2, Lemma 3 and Lemma 4, is that

$$P\{A(n)\} \to 1. \tag{5.6}$$

The following lemma tells us that the $*$-valleys coincide with the sequence of deep valleys with an overwhelming probability when n goes to infinity.

Lemma 5. *If $A^*(n) := \{K_n = K_n^*;\ (a_j,b_j,c_j,d_j) = (a_j^*,b_j^*,c_j^*,d_j^*),\ 1\le j\le K_n\}$, then we have that the probability $P\{A^*(n)\}$ converges to 1, when n goes to infinity.*

Proof. Since, by definition, the $*$-valleys constitute a subsequence of the deep valleys, Lemma 5 is a consequence of Lemma 3 together with Lemma 4. □

Remark 4. *Another meaning of this result is that, with probability tending to 1, two deep valleys are necessarily disjoint.*

5.2. **Preparatory lemmas.** In this subsection, we develop some technical tools allowing us to improve our understanding of the random walk's behavior. In Lemma 8, we prove that, after exiting a deep valley, the random walk will not come back to another deep valley it has already visited, with probability tending to one. Moreover, Lemma 9 specifies that the random walk typically exits from a $*$-valley on the right, while Lemma 10 shows that the time spent between two deep valleys is negligible.

5.2.1. *Preliminary estimates for inter-arrival times.* Let us first give a preliminary result concerning large deviations, more precisely about the convex rate function associated to the potential $V(\cdot)$, denoted by $I(\cdot)$.

Lemma 6. *Under assumptions (a)–(b), we have*

$$\inf_{x\ge 0}\frac{I(x)}{x} = \kappa.$$

Moreover, the minimum is reached at $x_0 := \Lambda'(\kappa)$, with $\Lambda(t) := \log E[\rho_0^t]$.

Proof. Recalling that $I(\cdot)$ is defined by $I(x) := \sup_{t\ge 0}\{tx-\Lambda(t)\}$, for $x\ge 0$, we have $I(x)\ge \kappa x - \Lambda(\kappa) = \kappa x$, since $\Lambda(\kappa) = 0$. Moreover, under assumption (a)–(b), formula (2.2.10) in ([7], p. 28) implies $I(\Lambda'(\kappa)) = \kappa\Lambda'(\kappa)$, which concludes the proof of Lemma 6. □

Let us introduce

$$T^{\uparrow}(h) := \min\{x \geq 0 : V^{\uparrow}(x) \geq h\}, \qquad h > 0,$$
$$T^{\downarrow}(h) := \min\{x \geq 0 : V^{\downarrow}(x) \leq -h\}, \qquad h > 0.$$

Lemma 7. *Under assumptions (a)–(b), we have, for h large enough,*

$$\mathbb{E}_{|0}[\tau_h] \leq C\,\mathrm{e}^h,$$

where $\mathbb{E}_{|0}$ denotes the expectation under the law $\mathbb{P}_{|0}$ of the random walk in the random environment ω (under P) reflected at 0 and $\tau_h := \tau(T^{\uparrow}(h)-1)$.

Proof. Using (Zeitouni [25], formula (2.1.14)), we obtain that $\mathbb{E}_{|0}[\tau_h]$ is bounded from above by $E\big[\sum_{0\leq i\leq j<T^{\uparrow}(h)} \mathrm{e}^{V(j)-V(i)}\big]$. Therefore, since $T^{\uparrow}(h) \leq T^{\uparrow}(h)\circ\theta_i$, for any $i \geq 0$, we obtain

$$\mathbb{E}_{|0}[\tau_h] \leq \sum_{i\geq 0} E\Big[\mathbf{1}_{\{i<T^{\uparrow}(h)\}} \sum_{i\leq j<T^{\uparrow}(h)} \mathrm{e}^{V(j)-V(i)}\Big] \leq \beta_1(h)\,\beta_2(h), \tag{5.7}$$

where

$$\begin{aligned}\beta_1(h) &:= E\big[T^{\uparrow}(h)\big],\\ \beta_2(h) &:= E\Big[\sum_{0\leq j<T^{\uparrow}(h)} \mathrm{e}^{V(j)}\Big].\end{aligned}$$

To bound $\beta_1(h)$, let us introduce the number N of complete excursions before $T^{\uparrow}(h)$, defined by $N = N(h) := \sup\{i \geq 0 : e_i < T^{\uparrow}(h)\}$. Then, we can write $\beta_1(h) = E[\sum_{i=0}^{N-1}(e_i - e_{i-1}) + (T^{\uparrow}(h) - e_N)]$. Observe that the definition of $T^{\uparrow}(h)$ implies that N is a geometrical random variable with parameter $q = q(h) := P\{H \geq h\}$ and recall that, by (3.1), we have $q \sim C_I\,\mathrm{e}^{-\kappa h}$, $h \to \infty$. Therefore, we get, for h large enough,

$$\begin{aligned}\beta_1(h) &\leq \sum_{k\geq 0}(1-q)^k q\big(kE[e_1|H<h] + E[T_h|H\geq h]\big)\\ &\leq C\sum_{k\geq 0}(1-q)^k q\big(kE[e_1] + E[T_h|H\geq h]\big),\end{aligned}$$

the second inequality being a consequence of the fact that $E[e_1] < \infty$ (see (2.1)) together with $P\{H<h\} \to 1$, $h\to\infty$, by (3.1). By obvious calculations, this yields $\beta_1(h) \leq C(1-q)q^{-1}E[e_1] + E[T_h|H\geq h]$, which implies with (3.1) that

$$\beta_1(h) \leq C\mathrm{e}^{\kappa h} + E[T_h|H\geq h]. \tag{5.8}$$

Now, let us bound $E[T_h|H\geq h]$. To this aim, we observe first that $E[T_h|H\geq h] \leq C\mathrm{e}^{\kappa h}\sum_{k\geq 0}(k+1)P\{T_h = k+1\,;\,H\geq h\}$. Then, applying the Markov property at time k, we get

$$\begin{aligned}E[T_h|H\geq h] &\leq C\mathrm{e}^{\kappa h}\sum_{k\geq 0}(k+1)E[\mathbf{1}_{\{0<V(k)<h\}}\mathrm{e}^{-\kappa(h-V(k))}]\\ &\leq C\sum_{k\geq 0}(k+1)\sum_{j=0}^{\lfloor h\rfloor}\mathrm{e}^{\kappa(j+1)}P\{V(k)\geq j\}.\end{aligned}$$

Since large deviations do occur, we obtain by Cramer's theory, see [7], that $P\{V(k) \geq j\} \leq \mathrm{e}^{-kI(\frac{j}{k})}$. Now, recalling that $I(\cdot)$ is an increasing function on $\mathbb{R}^+$ together with Lemma 6, we obtain

$$P\{V(k) \geq j\} \leq \mathrm{e}^{-k\frac{I(0)}{2}}\mathrm{e}^{-\kappa\frac{j}{2}}.$$

Since $I(0) > 0$, this yields that there exists $C > 0$ such that, for all large h,

$$E[T_h|H \geq h] \leq C\mathrm{e}^{\frac{\kappa}{2}h}. \tag{5.9}$$

Assembling (5.8) and (5.9) implies, for h large enough,

$$\beta_1(h) \leq C\mathrm{e}^{\kappa h}. \tag{5.10}$$

In a second step, we bound $\beta_2(h)$. Let us first introduce $\mathcal{E}_k := \{\max_{0\leq j\leq k-1} H_j < h\,;\, H_k \geq h\}$ and write

$$\begin{aligned}\beta_2(h) &= \sum_{k\geq 0} E\Big[\mathbf{1}_{\mathcal{E}_k} \sum_{0\leq j<T^\uparrow(h)} \mathrm{e}^{V(j)}\Big] \\ &= \sum_{k\geq 0}\Big(\sum_{i=0}^{k-1} E\Big[\mathbf{1}_{\mathcal{E}_k}\mathrm{e}^{V(e_j)}J_i\Big] + E\Big[\mathbf{1}_{\mathcal{E}_k}\mathrm{e}^{V(e_k)}\overline{J}_k\Big]\Big),\end{aligned}$$

where $J_i := \sum_{j=e_i}^{e_{i+1}} \mathrm{e}^{V(j)-V(e_i)}$ for $i \geq 0$ and $\overline{J}_k := \sum_{j=e_k}^{T^\uparrow(h)-1} \mathrm{e}^{V(j)-V(e_k)}$ which is well defined on $\mathcal{E}_k$. Observe that $\mathcal{E}_k = \{N(h) = k\}$ and recall that $N(h)$ is a geometrical random variable with parameter $q = q(h) = P\{H \geq h\}$. Then, the Markov property applied at times $(e_j)_{1\leq j\leq k}$ yields that $\beta_2(h)$ is less or equal than

$$\sum_{k\geq 0}(1-q)^k q\Big(E[J_0|H_0 < h]\sum_{j=0}^{k-1} E[\mathrm{e}^{V(e_1)}|H_0 < h]^j + E[\overline{J}_0|H_0 \geq h]E[\mathrm{e}^{V(e_1)}|H_0 < h]^k\Big),$$

which implies that $\beta_2(h)$ is bounded by

$$\frac{1}{1-E[\mathrm{e}^{V(e_1)}|H_0 < h]}E[J_0|H_0 < h] + \frac{q}{1-(1-q)E[\mathrm{e}^{V(e_1)}|H_0 < h]}E[\overline{J}_0|H_0 \geq h].$$

Now, since V is transient to $-\infty$, then H_0 is almost surely finite and $E[\mathrm{e}^{V(e_1)}|H_0 < h] \to E[\mathrm{e}^{V(e_1)}] < 1$, when $h \to \infty$. Recalling that $q = q(h) \to 0$, $h \to \infty$, it follows that

$$\beta_2(h) \leq C\big(E[J_0|H_0 < h] + qE[\overline{J}_0|H_0 \geq h]\big), \tag{5.11}$$

for h large enough.

Let us first bound $E[\overline{J}_0|H_0 \geq h]$. Recall that if μ denotes the law of $\log\rho_0$, thanks to assumption (a) of Theorem 1 we can define the law $\tilde{\mu} = \rho_0^\kappa\mu$, and the law $\tilde{P} = \tilde{\mu}^{\otimes\mathbb{Z}}$ which is the law of a sequence of i.i.d. random variables with law $\tilde{\mu}$. The definition of κ implies that $\int \log\rho\,\tilde{\mu}(d\rho) > 0$. Then, using the Girsanov property between P and

$\tilde{P}$, we can write

$$\begin{aligned} E[\overline{J}_0|H_0 \geq h] &\leq C\mathrm{e}^{\kappa h}\tilde{E}[\mathrm{e}^{-\kappa V(T_h)}\overline{J}_0\mathbf{1}_{\{H_0\geq h\}}] \\ &\leq C\tilde{E}\Big[\mathrm{e}^{-\kappa(V(T_h)-h)}\sum_{k=0}^{T_h-1}\mathrm{e}^{V(k)}\mathbf{1}_{\{H_0\geq h\}}\Big] \\ &\leq C\tilde{E}\Big[\sum_{k=0}^{T_h-1}\mathrm{e}^{V(k)}\mathbf{1}_{\{\min_{0<i<T_h}V(k)>0\}}\Big] \\ &\leq C\tilde{E}\Big[\sum_{k\geq 0}\sum_{p=0}^{\lfloor h\rfloor}\mathrm{e}^{V(k)}\mathbf{1}_{\{p\leq V(k)<p+1\}}\Big] \\ &\leq C\sum_{p=0}^{\lfloor h\rfloor}\mathrm{e}^{p+1}\tilde{E}\Big[\sum_{k\geq 0}\mathbf{1}_{\{p\leq V(k)<p+1\}}\Big]. \end{aligned}$$

Moreover, by Markov property, we have $\tilde{E}[\sum_{k\geq 0}\mathbf{1}_{\{p\leq V(k)<p+1\}}] \leq \tilde{E}[\sum_{k\geq 0}\mathbf{1}_{\{0\leq V(k)<1\}}]$, which is finite since $(V(k))_{k\geq 0}$ has a positive drift under $\tilde{P}$.

Therefore, recalling (5.11) and (3.1), we get

$$\beta_2(h) \leq C(E[J_0|H_0<h] + \mathrm{e}^{(1-\kappa)h}) \tag{5.12}$$

and only have to bound $E[J_0|H_0<h]$. Recall that $R=\sum_{k\geq 0}\mathrm{e}^{V(k)}$ and observe that $J_0 \leq R$. Moreover, let us denote by $E^{\mathcal{I}}[\cdot]$ the expectation under $P^{\mathcal{I}}\{\cdot\} := P\{\cdot|\mathcal{I}\}$, with $\mathcal{I} := \{H=S\}$. Then, we first observe that $E^{\mathcal{I}}[R|H<h] \geq E[R\,\mathbf{1}_{\{H=S<h\}}] \geq E[J_0\mathbf{1}_{\{H=S<h\}}]$. Furthermore, since J_0 depends only on $(V(k)\,;\,0\leq k\leq e_1)$ and since $P\{V(k)\leq 0\,;\,k\geq 0\}>0$, we get, by applying the strong Markov property at time e_1, that $E[J_0\mathbf{1}_{\{H<h\}}] \leq CE^{\mathcal{I}}[R|H<h]$, which implies

$$E[J_0|H<h] \leq CE^{\mathcal{I}}[R|H<h].$$

Therefore, we only have to prove that $E^{\mathcal{I}}[R|H<h] \leq C\mathrm{e}^{(1-\kappa)h}$. To this aim, we recall first that Corollary 6.1 in [8] implies that, $P^{\mathcal{I}}$-almost surely,

$$E^{\mathcal{I}}[R|\lfloor H\rfloor] \leq C\mathrm{e}^{\lfloor H\rfloor}. \tag{5.13}$$

Now, observe that $E^{\mathcal{I}}[R|H<h] \leq CE^{\mathcal{I}}[R\,\mathbf{1}_{\{H<h\}}]$ and let us write

$$\begin{aligned} E^{\mathcal{I}}[R\,\mathbf{1}_{\{H<h\}}] &\leq \sum_{k=0}^{\lfloor h\rfloor}E^{\mathcal{I}}\Big[\mathbf{1}_{\{\lfloor H\rfloor=k\}}E^{\mathcal{I}}[R|\lfloor H\rfloor=k]\Big] \\ &\leq C\sum_{k=0}^{\lfloor h\rfloor}E^{\mathcal{I}}\Big[\mathbf{1}_{\{\lfloor H\rfloor=k\}}\mathrm{e}^{\lfloor H\rfloor}\Big] \\ &\leq C\sum_{k=0}^{\lfloor h\rfloor}\mathrm{e}^{k}P^{\mathcal{I}}\{\lfloor H\rfloor=k\} \\ &\leq C\sum_{k=0}^{\lfloor h\rfloor}\mathrm{e}^{(1-\kappa)k} \leq C\mathrm{e}^{(1-\kappa)h}, \end{aligned} \tag{5.14}$$

the second inequality is a consequence of (5.13) and the fourth inequality due to the fact that $P^{\mathcal{I}}\{\lfloor H\rfloor=k\} \leq c\mathrm{e}^{-\kappa p}$ for some positive constant c. Now assembling (5.7), (5.10), (5.12) and (5.14) concludes the proof of Lemma 7. □

5.2.2. *Important preliminary results.* Before establishing the announced lemmas, we introduce, for any $x, y \in \mathbb{Z}$,

$$\tau(x,y) := \inf\{k \geq 0 :\ X_{\tau(x)+k} = y\}.$$

Then, we have the following results.

Lemma 8. *Defining* $DT(n) := A(n) \cap \bigcap_{j=1}^{K_n} \left\{\tau(d_j, b_{j+1}) < \tau(d_j, \overline{d}_j)\right\}$, *we have*

$$P\left\{DT(n)\right\} \to 1, \qquad n \to \infty.$$

Proof. Recalling (5.6), we only have to prove that

$$E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{K_n} P_\omega^{d_j}\{\tau(b_{j+1}) > \tau(\overline{d}_j)\}\right] \to 0. \tag{5.15}$$

By (Zeitouni [25], formula (2.1.4)), we get, for $1 \leq j \leq K_n$ and for all ω in $A(n)$:

$$P_\omega^{d_j}\left\{\tau(b_{j+1}) > \tau(c_j)\right\} = \frac{\sum_{k=d_j}^{b_{j+1}-1} \mathrm{e}^{V(k)}}{\sum_{k=\overline{d}_j}^{b_{j+1}-1} \mathrm{e}^{V(k)}} \leq (b_{j+1} - d_j)\mathrm{e}^{V(d_j)-V(\overline{d}_j)+h_n}.$$

Combining (5.10) and Markov inequality, we easily get that $b_{K_n+1} - d_{K_n} = o(n)$ with probability tending to 1. Moreover, by definition, $V(d_j) - V(\overline{d}_j) \leq -D_n$ for $1 \leq j \leq K_n$, and $b_{j+1} - d_j \leq e_n \leq C'\, n$, for $1 \leq j \leq K_n - 1$ on $A_1(n)$. Therefore, we have

$$E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{K_n} P_\omega^{d_j}\{\tau(b_{j+1}) > \tau(\overline{d}_j)\}\right] \leq C\, nE[K_n]\mathrm{e}^{-D_n+h_n}.$$

Recalling that $D_n = (1+\frac{1}{\kappa})\log n$, $h_n = \frac{1-\varepsilon}{\kappa}\log n$ and since $E[K_n] \leq C\, n^\varepsilon$, we obtain

$$E\left[\mathbf{1}_{A(n)} \sum_{j=1}^{K_n} P_\omega^{d_j}\{\tau(b_{j+1}) > \tau(\overline{d}_j)\}\right] \ \leq \ C\, \mathrm{e}^{\varepsilon(1-1/\kappa)\log n},$$

which implies (5.15). □

Lemma 9. *Defining* $DT^*(n) := \bigcap_{j=1}^{K_n^*} \left\{\tau(b_j^*, d_j^*) < \tau(b_j^*, \gamma_j^*)\right\}$, *we have*

$$P\{DT^*(n)\} \to 1, \qquad n \to \infty.$$

Proof. Since, by definition, the $*$-valleys correspond to the K_n deep valleys on $A^*(n)$, we consider $A^\dagger(n) := A^*(n) \cap A_3(n) \cap A_4^*(n)$ to control the $*$-valleys, where $A_4^*(n)$ is defined by $A_4^*(n) := \cap_{j=1}^{K_n^*} \left\{\gamma_{j+1}^* - a_j^* \leq C'' \log n\right\} \cap \left\{\gamma_1^* \leq C'' \log n\right\}$. Using the same arguments as in the proof of Lemma 4, we can prove that $P\{A_4^*(n)\} \to 1$, $n \to \infty$, for C'' large enough. Then, recalling that Lemma 3 and Lemma 5 imply $P\{A^*(n) \cap A_3(n)\} \to 1$, $n \to \infty$, it remains only to prove that

$$E\left[\mathbf{1}_{A^\dagger(n)} \sum_{j=1}^{K_n} P_\omega^{b_j}\{\tau(d_j) > \tau(\gamma_j^*)\}\right] \to 0. \tag{5.16}$$

Observe that by (Zeitouni [25], formula (2.1.4)) we get, for $1 \leq j \leq K_n$,

$$\begin{aligned} P_\omega^{b_j}\{\tau(d_j) > \tau(\gamma_j^*)\} &\leq (d_j - b_j)\mathrm{e}^{H^{(j)} - (V(\gamma_j^*) - V(b_j))} \\ &\leq C \log n \, \mathrm{e}^{H^{(j)} - (V(\gamma_j^*) - V(b_j))}, \end{aligned}$$

the second inequality being a consequence of $\omega \in A^*(n) \cap A_4^*(n)$. Then, to bound $\mathrm{e}^{H^{(j)} - (V(\gamma_j^*) - V(b_j))}$ from above, observe that (3.1) implies $P\{H^{(j)} > \frac{(1+\varepsilon')}{\kappa} \log n\} \sim n^{-(\varepsilon' + \varepsilon)} = o(n^{-\varepsilon})$, $n \to \infty$, for any $\varepsilon' > 0$, which yields that $P\{\bigcap_{j=1}^{K_n} \{H^{(j)} < \frac{(1+\varepsilon')}{\kappa} \log n\}\}$ tends to 1, when n tends to ∞. Therefore, recalling (5.16), we only have to prove that

$$C \log n \; n^{\frac{(1+\varepsilon')}{\kappa}} E\left[\mathbf{1}_{A^\dagger(n)} \sum_{j=1}^{K_n} \mathrm{e}^{-(V(\gamma_j^*) - V(b_j))} \right] \to 0. \tag{5.17}$$

Since $\gamma_j^* - b_{j-1} \leq C'' \log n$ on $A_4^*(n)$ and $b_j - b_{j-1} \geq n^{1-3\varepsilon}$ on $A_3(n)$, we get $b_j - \gamma_j^* \geq \frac{1}{2} n^{1-3\varepsilon}$ for $2 \leq j \leq K_n$ on $A^\dagger(n)$, for all large n. Similarly, $\gamma_0^* \leq C'' \log n$ on $A_4^*(n)$ and $b_1 \geq n^{1-3\varepsilon}$ on $A_3(n)$ yield $b_1 - \gamma_1^* \geq \frac{1}{2} n^{1-3\varepsilon}$ on $A^\dagger(n)$. Therefore, by definition of b_j and since large deviations do occur, we obtain from Cramer's theory, see [7],

$$\begin{aligned} P\{A^\dagger(n)\,;\, V(b_j) - V(\gamma_{j-1}^*) \geq -n^{\frac{1-3\varepsilon}{2}}\} &\leq P\{V(\frac{1}{2} n^{1-3\varepsilon}) \geq -n^{\frac{1-3\varepsilon}{2}}\} \\ &\leq \mathrm{e}^{-\frac{n^{1-3\varepsilon}}{2} I\left(n^{\frac{-1-3\varepsilon}{2}}\right)} = o(n^{-\varepsilon}), \end{aligned}$$

for any $1 \leq j \leq K_n$. This result implies that the term on the left-hand side in (5.17) is bounded from above by $C \log n \; n^{\frac{(1+\varepsilon')}{\kappa}} E[K_n] \mathrm{e}^{-\frac{n^{1-3\varepsilon}}{2}}$. Then, since $E[K_n] \leq C\, n^\varepsilon$, this concludes the proof of Lemma 9. □

Lemma 10. *For any $0 < \eta < \varepsilon(\frac{1}{\kappa} - 1)$, let us introduce the following event $IA(n) := A(n) \cap \left\{ \sum_{j=1}^{K_n} \tau(d_j, b_{j+1}) < n^{1/\kappa - \eta} \right\}$. Then, we have*

$$P\{IA(n)\} \to 1, \qquad n \to \infty.$$

Proof. Recalling that $P\{K_n \geq 2C_I n^\varepsilon\} \to 0$, $n \to \infty$, and that Lemma 8 implies that $P\{DT(n)\} \to 1$, $n \to \infty$, it only remains to prove

$$\mathbb{P}\left\{ DT(n) \cap \left\{ \sum_{j=1}^{\lfloor 2C_I n^\varepsilon \rfloor} \tau(d_j, b_{j+1}) \geq n^{1/\kappa - \eta} \right\} \right\} \to 0, \qquad n \to \infty.$$

Using Markov inequality, we have to prove that

$$\mathbb{E}\left[\mathbf{1}_{DT(n)} \sum_{j=1}^{\lfloor 2C_I n^\varepsilon \rfloor} \tau(d_j, b_{j+1}) \right] = o\left(\frac{1}{n^{1/\kappa - \eta}} \right), \qquad n \to \infty. \tag{5.18}$$

Furthermore, by definition of the event DT (see Lemma 8), we get

$$\begin{aligned}\mathbb{E}\Big[\mathbf{1}_{DT(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}\tau(d_j,b_{j+1})\Big] &\leq E\Big[\mathbf{1}_{A(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}E^{d_j}_{\omega,|\overline{d}_j}[\tau(b_{j+1})]\Big]\\ &\leq E\Big[\mathbf{1}_{A(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}E^{\overline{d}_j}_{\omega,|\overline{d}_j}[\tau(b_{j+1})]\Big].\end{aligned}$$

Applying successively the strong Markov property at $\overline{d}_{\lfloor 2C_I n^{\varepsilon}\rfloor}, \ldots, \overline{d}_2, \overline{d}_1$, this implies

$$\mathbb{E}\Big[\mathbf{1}_{DT(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}\tau(d_j,b_{j+1})\Big] \leq 2C_I n^{\varepsilon}\mathbb{E}_{|0}[\tau(T^{\uparrow}(h_n)-1)].$$

Therefore, Lemma 7 implies

$$\mathbb{E}\Big[\mathbf{1}_{DT(n)}\sum_{j=1}^{\lfloor 2C_I n^{\varepsilon}\rfloor}\tau(d_j,b_{j+1})\Big] \leq Cn^{\varepsilon}\mathrm{e}^{h_n} \leq Cn^{\frac{1}{\kappa}-\varepsilon\left(\frac{1}{\kappa}-1\right)},$$

which yields (5.18) and concludes the proof, since $0<\eta<\varepsilon(\frac{1}{\kappa}-1)$. □

5.3. **Proof of Proposition 1.** Since the time spent on $\mathbb{Z}_-$ is almost surely finite, we reduce our study to the random walk in random environment reflected at 0 and observe that

$$\mathbb{E}\left[\mathrm{e}^{-\lambda_n\tau(e_n)}\right] = \mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\tau(e_n)}\right] + o(1), \qquad n\to\infty,$$

where $\mathbb{E}_{|0}$ denotes the expectation under the law $\mathbb{P}_{|0}$ of the random walk in the random environment ω (under P) reflected at 0.

Furthermore, by definition, $\tau(e_n)$ satisfies

$$\tau(b_1)+\sum_{j=1}^{K_n-1}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\} \leq \tau(e_n) \leq \tau(b_1)+\sum_{j=1}^{K_n}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\},$$

such that we easily get that $\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\tau(e_n)}\right]$ belongs to

$$\left[\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\left(\tau(b_1)+\sum_{j=1}^{K_n}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\}\right)}\right], \mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\left(\tau(b_1)+\sum_{j=1}^{K_n-1}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\}\right)}\right]\right].$$

Let us first recall that Lemma 8 and Lemma 10 imply that $P\{DT(n)\cap IA(n)\}\to 1$, $n\to\infty$. Then, we get that the lower bound in the previous interval is equal to

$$\begin{aligned}&\mathbb{E}_{|0}\left[\mathbf{1}_{DT(n)\cap IA(n)}\mathrm{e}^{-\lambda_n\left(\tau(b_1)+\sum_{j=1}^{K_n}\{\tau(b_j,d_j)+\tau(d_j,b_{j+1})\}\right)}\right]+o(1)\\ =\ &\mathbb{E}_{|0}\left[\mathbf{1}_{DT(n)\cap IA(n)}\,\mathrm{e}^{-\lambda_n\sum_{j=1}^{K_n}\tau(b_j,d_j)}\right]+o(1)\\ =\ &\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\sum_{j=1}^{K_n}\tau(b_j,d_j)}\right]+o(1).\end{aligned}$$

Then, applying the strong Markov property for the random walk successively at $\tau(b_{K_n})$, $\tau(b_{K_n-1})$, $\dots$, $\tau(b_2)$ and $\tau(b_1)$ we get

$$\begin{aligned}\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\sum_{j=1}^{K_n}\tau(b_j,d_j)}\right] &= E\left[\prod_{j=1}^{K_n}E^{b_j}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n\tau(d_j)}\right]\right]\\ &= E\left[\mathbf{1}_{A^*(n)}\prod_{j=1}^{K_n^*}E^{b_j^*}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right]+o(1)\\ &= E\left[\prod_{j=1}^{K_n^*}E^{b_j^*}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right]+o(1),\end{aligned}$$

the second equality being a consequence of Lemma 5. Then, since Lemma 9 implies $\mathbb{P}\{DT^*(n)\}\to 1$, we have

$$\begin{aligned}\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\sum_{j=1}^{K_n}\tau(b_j,d_j)}\right] &= E\left[\prod_{j=1}^{K_n^*}E^{b_j^*}_{\omega,|0}\left[\mathbf{1}_{DT^*(n)}\,\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right]+o(1)\\ &= E\left[\prod_{j=1}^{K_n^*}E^{b_j^*}_{\omega,|\gamma_j^*}\left[\mathbf{1}_{DT^*(n)}\,\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right]+o(1)\\ &= E\left[\prod_{j=1}^{K_n^*}E^{b_j^*}_{\omega,|\gamma_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right]+o(1),\end{aligned}$$

Since $\mathbb{P}\{K_n=K_n^*\}\to 1$, and $\mathbb{P}\{K_n\le\overline{K}_n\}\to 1$, with $\overline{K}_n=\lceil nq_n(1+n^{-\varepsilon/4})\rceil$, we get

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\tau(e_n)}\right]\ge E\left[\prod_{j=1}^{\overline{K}_n}E^{b_j^*}_{\omega,|\gamma_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right]+o(1).$$

Then, applying the strong Markov property (for the potential V) successively at times $\gamma^*_{\overline{K}_n},\dots,\gamma_2^*$ and observing that the $\left(E^{b_j^*}_{\omega,|\gamma_j^*}\left[\mathrm{e}^{-\lambda_n\tau(d_j^*)}\right]\right)_{1\le j\le\overline{K}_n}$ are i.i.d. random variables, we obtain that

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\tau(e_n)}\right]\ge E\left[E^{b_1^*}_{\omega,|\gamma_1^*}\mathrm{e}^{-\lambda_n\tau(d_1^*)}\right]^{\overline{K}_n}+o(1).$$

Since we can easily prove that $P\{(a_1,b_1,c_1,d_1)\ne(a_1^*,b_1^*,c_1^*,d_1^*)\}=o(n^{-\varepsilon})$, and since $\overline{K}_n=O(n^{\varepsilon})$, $n\to\infty$, the strong Markov property applied at γ_1^* yields

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\tau(e_n)}\right]\ge E\left[E^{b_1}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n\tau(d_1)}\right]\right]^{\overline{K}_n}+o(1).$$

Using similar arguments for the upper bound in the aforementioned interval, we get

$$\mathbb{E}_{|0}\left[\mathrm{e}^{-\lambda_n\tau(e_n)}\right]\in\left[E\left[E^{b_1}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n\tau(d_1)}\right]\right]^{\overline{K}_n}+o(1)\,,\,E\left[E^{b_1}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n\tau(d_1)}\right]\right]^{\underline{K}_n}+o(1)\right].$$

Furthermore, observe that $E\left[E^{b_1}_{\omega,|0}\left[\mathrm{e}^{-\lambda_n\tau(d_1)}\right]\right]=E\left[E^{b_1}_{\omega,|a_1}\left[\mathrm{e}^{-\lambda_n\tau(d_1)}\right]\right]+o(n^{-\varepsilon})$. This is a consequence of Lemma 4, definition of a and the fact that (3.1) implies $P\{H^{(1)}>$

$\frac{(1+\varepsilon')}{\kappa}\log n\} \sim n^{-(\varepsilon'+\varepsilon)} = o(n^{-\varepsilon})$, $n \to \infty$, for any $\varepsilon' > 0$, which gives

$$E\left[P_\omega^{b_1}\left\{\tau(a_1) < \tau(d_1)\right\}\right] \le C\log n\, \mathrm{e}^{\frac{(1+\varepsilon')}{\kappa}\log n - D_n} = o(n^{-\varepsilon}).$$

This concludes the proof of Proposition 1. □

6. Annealed Laplace transform for the exit time from a deep valley

This section is devoted to the proof of the linearization. It involves h-processes theory and "sculpture" of a typical deep valley. To ease notations, we shall use a, b, c, and d instead of a_1, b_1, c_1 and d_1. Moreover, let us introduce, for any random variable $Z \ge 0$,

$$R_n(\lambda, Z) := E\left[\frac{1}{1+\frac{\lambda}{n^{1/\kappa}}Z}\right]. \tag{6.1}$$

Then, the result can be expressed in the following way.

Proposition 2. *For any $\xi > 0$, we have, for all large n,*

$$R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2) + o(n^{-\varepsilon}) \le E\left[E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n\tau(d)}]\right] \le R_n(\mathrm{e}^{-\xi}\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2) + o(n^{-\varepsilon}).$$

where $\widehat{M}_1 := \sum_{x=a+1}^{d-1}\mathrm{e}^{-(\widehat{V}(x)-\widehat{V}(b))}$ and $M_2 := \sum_{x=b}^{d-1}\mathrm{e}^{V(x)-V(c)}$. Note that $\widehat{V}$ is defined in the following subsection.

6.1. Two h-processes.

In order to estimate $E^b_{\omega,|a}\left[\mathrm{e}^{-\lambda_n\tau(d)}\right]$, we decompose the passage from b to d into the sum of a random geometrically distributed number, denoted by N, of unsuccessful attempts to cross the excursion, followed by a successful attempt. More precisely, since N is a geometrically distributed random variable with parameter $1-p$ satisfying

$$1 - p \;=\; \omega_b \frac{\mathrm{e}^{V(b)}}{\sum_{x=b}^{d-1}\mathrm{e}^{V(x)}}, \tag{6.2}$$

we can write $\tau(d) = \sum_{i=1}^{N} F_i + G$, where the F_i's are the successive i.i.d. failures and G the first success. The accurate estimation of the time spent by each (successful and unsuccessful) attempt leads us to consider two h-processes where the random walker evolves in two modified potentials, one corresponding to the conditioning on a failure (see the potential $\widehat{V}$ and Lemma 11) and the other to the conditioning on a success (see the potential $\bar{V}$ and Lemma 12).

6.1.1. *The failure case: the h-potential $\widehat{V}$.* Let us fix a realization of ω. To introduce the h-potential $\widehat{V}$, we consider the valley $a < b < c < d$ and define $h(x) := P^x_\omega\{\tau(b) < \tau(d)\}$. Therefore, for any $b < x < d$, we define $\widehat{\omega}_x := \omega_x\frac{h(x+1)}{h(x)}$ and similarly $(1-\widehat{\omega}_x) := (1-\omega_x)\frac{h(x-1)}{h(x)}$. We obtain for any $b \le x < y < d$,

$$\widehat{V}(y) - \widehat{V}(x) = (V(y) - V(x)) + \log\left(\frac{h(x)\,h(x+1)}{h(y)\,h(y+1)}\right). \tag{6.3}$$

Using (Zeitouni [25], formula (2.1.4)), we get

$$\frac{h(x)\,h(x+1)}{h(y)\,h(y+1)} = \frac{\sum_{j=x}^{d-1} \mathrm{e}^{V(j)}\ \sum_{j=x+1}^{d-1} \mathrm{e}^{V(j)}}{\sum_{j=y}^{d-1} \mathrm{e}^{V(j)}\ \sum_{j=y+1}^{d-1} \mathrm{e}^{V(j)}} \geq 1. \tag{6.4}$$

Thus we obtain for any $b \leq x < y \leq c$,

$$\widehat{V}(y) - \widehat{V}(x) \geq V(y) - V(x). \tag{6.5}$$

Lemma 11. *For any environment ω, we have*

$$E_\omega\left[F_1\right] = 2\,\omega_b \Bigg(\sum_{i=a+1}^{b-1} \mathrm{e}^{-(V(i)-V(b))} + \sum_{i=b}^{d-1} \mathrm{e}^{-(\widehat{V}(i)-\widehat{V}(b))} \Bigg), \tag{6.6}$$

and

$$E_\omega\left[F_1^2\right] = 4\omega_b\, R^+ + 4(1-\omega_b)\, R^-, \tag{6.7}$$

where

$$\begin{aligned} R^+ &:= \sum_{i=b+1}^{d-1} \left(1 + 2\sum_{j=b}^{i-2} \mathrm{e}^{\widehat{V}(j)-\widehat{V}(i-1)}\right)\left(\mathrm{e}^{-(\widehat{V}(i-1)-\widehat{V}(b))} + 2\sum_{j=i+1}^{d-1} \mathrm{e}^{-(\widehat{V}(j-1)-\widehat{V}(b))}\right), \\ R^- &:= \sum_{i=a+1}^{b-1} \left(1 + 2\sum_{j=i+2}^{b} \mathrm{e}^{V(j)-V(i+1)}\right)\left(\mathrm{e}^{-(V(i+1)-V(b))} + 2\sum_{j=a+1}^{i-1} \mathrm{e}^{-(V(j+1)-V(b))}\right). \end{aligned}$$

Remark 5. *Alili* [1] *and Goldsheid* [11] *prove a similar result for a non-conditioned hitting time. Here we give the proof in order to be self-contained.*

Proof. Let us first introduce

$$\begin{aligned} N_i^+ &:= \sharp\{k < \tau(b) : X_k = i-1, X_{k+1} = i\}, \qquad i > b, \\ N_i^- &:= \sharp\{k < \tau(b) : X_k = i+1, X_{k+1} = i\}, \qquad i < b. \end{aligned}$$

Observe that, under $P_{\widehat{\omega}}$, for $i > b$ and conditionally on $N_i^+ = x$, N_{i+1}^+ is the sum of x independent geometrical random variables with parameter $\widehat{\omega}_i$. It means that $E_{\widehat{\omega}}[N_{i+1}^+ | N_i^+ = x] = \frac{x}{\widehat{\rho}_i}$ and $\mathrm{Var}_{\widehat{\omega}}[N_{i+1}^+ | N_i^+ = x] = \frac{x}{\widehat{\omega}_i \widehat{\rho}_i^2}$. Similarly, under P_ω, for $i < b$ and conditionally on $N_i^- = x$, N_{i-1}^- is the sum of x independent geometrical random variables with parameter $1-\omega_i$. It means that $E_\omega[N_{i-1}^- | N_i^- = x] = x\rho_i$ and $\mathrm{Var}_\omega[N_{i-1}^- | N_i^- = x] = \frac{x\rho_i^2}{(1-\omega_i)}$.

Since

$$E_\omega[F_1] = 2\omega_b\, E_{\widehat{\omega}}[\sum_{b+1}^{d-1} N_i^+] + 2(1-\omega_b)\, E_\omega[\sum_{a+1}^{b-1} N_i^-],$$

an easy calculation yields (6.6).

To calculate $E_\omega[F_1^2]$, observe first that

$$E_\omega[F_1^2] = 4\omega_b\, E_{\widehat{\omega}}\left[(\sum_{i=b+1}^{d-1} N_i^+)^2\right] + 4(1-\omega_b)\, E_\omega\left[(\sum_{i=a+1}^{b-1} N_i^-)^2\right].$$

Then, it remains to prove that $E_{\widehat{\omega}}[(\sum_{b+1}^{d-1} N_i^+)^2] = R^+$ and $E_{\omega}[(\sum_{a+1}^{b-1} N_i^-)^2] = R^-$. We will only treat $E_{\widehat{\omega}}[(\sum_{b+1}^{d-1} N_i^+)^2]$, the case of $E_{\omega}[(\sum_{a+1}^{b-1} N_i^-)^2]$ being similar. We get first

$$E_{\widehat{\omega}}\left[(\sum_{b+1}^{d-1} N_i^+)^2\right] = \sum_{i=b+1}^{d-1} E_{\widehat{\omega}}[(N_i^+)^2] + 2\sum_{i=b+1}^{d-1}\sum_{j=i+1}^{d-1} E_{\widehat{\omega}}[N_i^+ N_j^+]. \tag{6.8}$$

Observe that $E_{\widehat{\omega}}\left[N_i^+ N_j^+\right] = E_{\widehat{\omega}}\left[N_i^+ E_{\widehat{\omega}}\left[N_j^+ \,|\, N_i^+, \dots, N_{j-1}^+\right]\right] = E_{\widehat{\omega}}\left[N_i^+ \frac{N_{j-1}^+}{\widehat{\rho}_{j-1}}\right]$, for $i < j$, so that we get, by iterating,

$$E_{\widehat{\omega}}\left[N_i^+ N_j^+\right] = E_{\widehat{\omega}}\left[(N_i^+)^2\right] \frac{1}{\widehat{\rho}_{j-1}\cdots\widehat{\rho}_i}.$$

Recalling (6.8), this yields

$$\begin{aligned} E_{\widehat{\omega}}\left[(\sum_{b+1}^{d-1} N_i^+)^2\right] &= \sum_{i=b+1}^{d-1} E_{\widehat{\omega}}\left[(N_i^+)^2\right]\left(1 + 2\sum_{j=i+1}^{d-1} \frac{1}{\widehat{\rho}_i\cdots\widehat{\rho}_{j-1}}\right) \\ &= \sum_{i=b+1}^{d-1} E_{\widehat{\omega}}\left[(N_i^+)^2\right]\left(1 + 2\sum_{j=i+1}^{d-1} \mathrm{e}^{-(\widehat{V}(j-1)-\widehat{V}(i-1))}\right). \end{aligned} \tag{6.9}$$

Now, observe that $E_{\widehat{\omega}}\left[(N_i^+)^2\right] = E_{\widehat{\omega}}\left[E_{\widehat{\omega}}\left[(N_i^+)^2 | N_{i-1}^+\right]\right]$, which implies

$$E_{\widehat{\omega}}\left[(N_i^+)^2\right] = E_{\widehat{\omega}}\left[\sum_{k\geq 1} E_{\widehat{\omega}}[G_1^{(i)} + \cdots + G_k^{(i)}]\mathbf{1}_{\{N_{i-1}^+ = k\}}\right].$$

Since the $G_\cdot^{(i)}$'s are i.i.d., we get $E_{\widehat{\omega}}[G_1^{(i)} + \cdots + G_k^{(i)}] = k\mathrm{Var}_{\widehat{\omega}}[G_1^{(i)}] + k^2 E_{\widehat{\omega}}[G_1^{(i)}]^2$. Recalling that $E_{\widehat{\omega}}[G_1^{(i)}] = \frac{1}{\widehat{\rho}_{i-1}}$ and $\mathrm{Var}_{\widehat{\omega}}[G_1^{(i)}] = \frac{1}{\widehat{\omega}_{i-1}\widehat{\rho}_{i-1}^2}$, this yields

$$\begin{aligned} E_{\widehat{\omega}}\left[(N_i^+)^2\right] &= \frac{E_{\widehat{\omega}}\left[N_{i-1}^+\right]}{\widehat{\omega}_{i-1}\widehat{\rho}_{i-1}^2} + \frac{E_{\widehat{\omega}}\left[(N_{i-1}^+)^2\right]}{\widehat{\rho}_{i-1}^2} \\ &= \frac{1}{\widehat{\omega}_{i-1}\widehat{\rho}_{b+1}\cdots\widehat{\rho}_{i-2}\widehat{\rho}_{i-1}^2} + \frac{E_{\widehat{\omega}}\left[(N_{i-1}^+)^2\right]}{\widehat{\rho}_{i-1}^2}. \end{aligned} \tag{6.10}$$

Denoting $W_{b+1} := 1$ and $W_i := (\widehat{\rho}_{b+1}\dots\widehat{\rho}_{i-1})^2 E_{\widehat{\omega}}\left[(N_i^+)^2\right]$ for $b+1 < i < d$, (6.10) becomes

$$W_i - W_{i-1} = \frac{\widehat{\rho}_{b+1}\cdots\widehat{\rho}_{i-1}}{\widehat{\omega}_{i-1}} = \widehat{\rho}_{b+1}\cdots\widehat{\rho}_{i-1} + \widehat{\rho}_{b+1}\cdots\widehat{\rho}_{i-2},$$

the second equality being a consequence of $1/\widehat{\omega}_{i-1} = \widehat{\rho}_{i-1} + 1$. Therefore, we have $W_i = \sum_{b+2}^{i}(W_j - W_{j-1}) + W_{b+1} = \widehat{\rho}_{b+1}\dots\widehat{\rho}_{i-1} + 2(1 + \sum_{b+1}^{i-2}\widehat{\rho}_{b+1}\dots\widehat{\rho}_j)$, which implies

$$\begin{aligned} E_{\widehat{\omega}}\left[(N_i^+)^2\right] &= \frac{1}{\widehat{\rho}_{b+1}\cdots\widehat{\rho}_{i-1}} + 2\sum_{j=b}^{i-2}\frac{\widehat{\rho}_{b+1}\cdots\widehat{\rho}_j}{(\widehat{\rho}_{b+1}\cdots\widehat{\rho}_{i-1})^2} \\ &= \mathrm{e}^{-(\widehat{V}(i-1)-\widehat{V}(b))} + 2\sum_{j=b}^{i-2}\mathrm{e}^{\widehat{V}(j)-2\widehat{V}(i-1)+\widehat{V}(b)}. \end{aligned} \tag{6.11}$$

Assembling (6.9) and (6.11) yields (6.7). □

6.1.2. *The success case: the h-potential $\bar{V}$.* In a similar way, we introduce the h-potential $\bar{V}$ by considering the valley $a < b < c < d$ and defining $g(x) := P^x_\omega\{\tau(d) < \tau(b)\}$. Therefore, for any $b < x < d$, we define $\bar{\omega}_x := \omega_x \frac{g(x+1)}{g(x)}$ and similarly $(1-\bar{\omega}_x) := (1-\omega_x)\frac{g(x-1)}{g(x)}$. We obtain for any $b < x < y \leq d$,

$$\bar{V}(y) - \bar{V}(x) = (V(y) - V(x)) + \log\left(\frac{g(x)\, g(x+1)}{g(y)g(y+1)}\right). \tag{6.12}$$

Recalling (Zeitouni [25], formula (2.1.4)), we have

$$\frac{g(x)\, g(x+1)}{g(y)\, g(y+1)} = \frac{\sum_{j=b}^{x-1} \mathrm{e}^{V(j)} \sum_{j=b}^{x} \mathrm{e}^{V(j)}}{\sum_{j=b}^{y-1} \mathrm{e}^{V(j)} \sum_{j=b}^{y} \mathrm{e}^{V(j)}} \leq 1. \tag{6.13}$$

Therefore, we obtain for any $c \leq x < y \leq d$,

$$\bar{V}(y) - \bar{V}(x) \leq V(y) - V(x). \tag{6.14}$$

Using the same arguments as in the failure case, we get the following result.

Lemma 12. *For any environment ω, we have*

$$E_\omega[G] \leq 1 + \sum_{i=b+1}^{d} \sum_{j=i}^{d} \mathrm{e}^{\bar{V}(j)-\bar{V}(i)}. \tag{6.15}$$

6.2. **Preparatory lemmas.** The study of a typical deep valley involves the following event

$$A_5(n) \;:=\; \left\{\max\{V^\uparrow(a,b)\,;\, -V^\downarrow(b,c)\,;\, V^\uparrow(c,d)\} \leq \delta \log n\right\},$$

where $\delta > \varepsilon/\kappa$. In words, $A_5(n)$ ensures that the potential does not have excessive fluctuations in a typical box. Moreover, we have the following result.

Lemma 13. *For any $\delta > \varepsilon/\kappa$,*

$$P\{A_5(n)\} = 1 - o(n^{-\varepsilon}), \qquad n \to \infty.$$

Proof. We easily observe that the proof of Lemma 13 boils down to showing that

$$P\{V^\uparrow(a,b) \geq \delta \log n\} \;=\; o(n^{-\varepsilon}), \qquad n \to \infty, \tag{6.16}$$

$$P\{-V^\downarrow(b,c) \geq \delta \log n\} \;=\; o(n^{-\varepsilon}), \qquad n \to \infty, \tag{6.17}$$

$$P\{V^\uparrow(c,d) \geq \delta \log n\} \;=\; o(n^{-\varepsilon}), \qquad n \to \infty. \tag{6.18}$$

In order to prove (6.18), let us first observe the following trivial inequality

$$P\{V^\uparrow(c,d) \geq \delta \log n\} \leq P\{V^\uparrow(T_1^\uparrow, d) \geq \delta \log n\}.$$

Looking at the proof of (5.3), we observe that $P\{d - T_1^\uparrow \geq C \log n\} = o(n^{-\varepsilon'})$, for any $\varepsilon' > 0$, by choosing C large enough, depending on ε'. Therefore, we only have to prove that $P\{V^\uparrow(T_1^\uparrow, T_1^\uparrow + C\log n) \geq \delta \log n\} = o(n^{-\varepsilon})$. Then, applying the strong Markov

property at time $T_1^{\uparrow}$, we have to prove that $P\{V^{\uparrow}(0, C\log n) \geq \delta \log n\} = o(n^{-\varepsilon})$. Now, by Cramer's theory, see [7], and Lemma 6, we get

$$\begin{aligned} P\{V^{\uparrow}(0, C\log n) \geq \delta \log n\} &\leq (C\log n)^2 \max_{0\leq k\leq C\log n} P\{V(k) \geq \delta \log n\} \\ &\leq (C\log n)^2 \max_{0\leq k\leq C\log n} \mathrm{e}^{-kI\left(\frac{\delta \log n}{k}\right)} \\ &\leq (C\log n)^2 \exp\{-\kappa\delta \log n\}. \end{aligned}$$

Since $\delta > \varepsilon/\kappa$, this yields (6.18).

To get (6.17), observe first that

$$P\{-V^{\downarrow}(b,c) \geq \delta \log n\} \leq P\{-V^{\downarrow}(b, T_1^{\uparrow}) \geq \delta \log n\} + P\{-V^{\downarrow}(T_1^{\uparrow}, c) \geq \delta \log n\}.$$

The first term on the right-hand side is equal to $P\{V^{\downarrow}(0, T^{\uparrow}(h_n)) \geq \delta \log n | H_0 > h_n\}$. Recalling that (3.1) implies $P\{H_0 > h_n\} \leq Cn^{-(1-\varepsilon)}$ for all large n and observing the trivial inclusion $\left\{V^{\downarrow}(0, T^{\uparrow}(h_n)) \geq \delta \log n\,;\, H_0 > h_n\right\} \subset \left\{T^{\downarrow}(\delta \log n) < T_{h_n} < T_{(-\infty,0]}\right\}$, it follows that $P\{-V^{\downarrow}(b, T_1^{\uparrow}) \geq \delta \log n\}$ is less or equal than

$$\begin{aligned} & Cn^{1-\varepsilon} P\{T^{\downarrow}(\delta \log n) < T_{h_n} < T_{(-\infty,0]}\} \\ \leq\ & Cn^{1-\varepsilon} \sum_{p=\lfloor \delta \log n\rfloor}^{\lfloor h_n\rfloor} P\{M_\delta \in [p, p+1)\,;\, T^{\downarrow}(\delta \log n) < T_{h_n} < T_{(-\infty,0]}\}, \end{aligned}$$

where $M_\delta := \max\{V(k);\, 0 \leq k \leq T^{\downarrow}(\delta \log n)\}$. Applying the strong Markov property at time $T^{\downarrow}(\delta \log n)$ and recalling (3.3) we bound the term of the previous sum, for $\lfloor \delta \log n\rfloor \leq p \leq \lfloor h_n\rfloor$ and all large n, by

$$P\{S \geq p\}\, P\{S \geq h_n - (p - \delta \log n)\} \leq C\mathrm{e}^{-\kappa p}\mathrm{e}^{-\kappa(h_n - p + \delta \log n))},$$

where $S := \sup\{V(k);\, k \geq 0\}$. Thus, we get $P\{-V^{\downarrow}(b, T_1^{\uparrow}) \geq \delta \log n\} \leq C\lfloor h_n\rfloor n^{-\kappa\delta}$, for all large n, which yields $P\{-V^{\downarrow}(b, T_1^{\uparrow}) \geq \delta \log n\} = o(n^{-\varepsilon})$, $n \to \infty$, since $\delta > \varepsilon/\kappa$. Furthermore, applying the strong Markov property at $T_1^{\uparrow}$, we obtain that $P\{-V^{\downarrow}(T_1^{\uparrow}, c) \geq \delta \log n\} \leq P\{-V^{\downarrow}(0, V_{max}) \geq \delta \log n\}$. In a similar way we used before (but easier), we get, by applying the strong Markov property at $T^{\downarrow}(\delta \log n)$, that $P\{-V^{\downarrow}(T_1^{\uparrow}, c) \geq \delta \log n\} \leq n^{-\kappa\delta}$ for all large n. Since $\delta > \varepsilon/\kappa$ this yields (6.17).

For (6.16), observe first that $((V(k-b) - V(b))_{a\leq k\leq b}, a, b)$ has the same distribution as $((V(k))_{a^-\leq k\leq 0}, a^-, 0)$ under $P\{\cdot | V(k) \geq 0\,,\, a^- \leq k \leq 0\}$, where $a^- := \sup\{k \leq 0 : V(k) \geq D_n\}$. Then, since $P\{V(k) \geq 0\,,\, k \leq 0\} > 0$ and since $(V(-k)\,,\, k \geq 0)$ has the same distribution as $(-V(k)\,,\, k \geq 0)$, we obtain

$$P\{V^{\uparrow}(a,b) \geq \delta \log n\} \leq CP\{V^{\uparrow}(0, T_{(-\infty,-D_n]}) \geq \delta \log n\}.$$

Now, the arguments are the same as in the proof of (6.18). □

6.3. **Proof of Proposition 2.** Recall that we can write $\tau(d) = \sum_{i=1}^{N} F_i + G$, where the F_i's are the successive i.i.d. failures and G the first success. Then, denoting F_1

by F, we have

$$\begin{aligned} E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n \tau(d)}] &= E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}] \sum_{k\geq 0} E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]^k (1-p)p^k \\ &= E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}] \frac{1-p}{1-p\, E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]}. \end{aligned} \tag{6.19}$$

In order to replace $E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]$ by $1-\lambda_n E^b_{\omega,|a}[F]$, we observe that $1-\lambda_n E^b_{\omega,|a}[F] \leq E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}] \leq 1-\lambda_n E^b_{\omega,|a}[F] + \frac{\lambda_n^2}{2} E^b_{\omega,|a}[F^2]$, which implies that $E[\frac{1-p}{1-p\, E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]}]$ belongs to

$$\left[E\left[\frac{1-p}{1-p(1-\lambda_n E^b_{\omega,|a}[F])}\right] ;\, E\left[\frac{1-p}{1-p(1-\lambda_n E^b_{\omega,|a}[F] + \frac{\lambda_n^2}{2} E^b_{\omega,|a}[F^2])}\right]\right].$$

Now, we have to bound $\lambda_n E^b_{\omega,|a}[F^2]$ from above. Then, recalling (6.7), which implies $E^b_{\omega,|a}[F^2] \leq 4(R^+ + R^-)$, we only have to bound R^+ and R^-. By definition of R^+, we obtain

$$R^+ \leq (d-b)\left(1 + 2(d-b)\mathrm{e}^{-\widehat{V}^{\downarrow}(b,d)}\right)\left(3(d-b)\max_{b\leq j\leq d} \mathrm{e}^{-(\widehat{V}(j)-\widehat{V}(b))}\right). \tag{6.20}$$

Recalling that the proof of Lemma 4 contains the fact that $P\{d-a \geq C''\log n\} = o(n^{-\varepsilon})$ and that Lemma 13 tells that $P\{A_5(n)\} = 1-o(n^{-\varepsilon})$, we can consider the event $A^{\ddagger}(n) := \{d-a \leq C''\log n\} \cap A_5(n)$, whose probability is greater than $1-o(n^{-\varepsilon})$ for n large enough. It allows us to sculpt the deep valley (a,b,c,d), such that we can bound R^+. We are going to show that the fluctuations of $\widehat{V}$ are, in a sense, related to the fluctuations of V controlled by $A_5(n)$. Indeed, (6.5) yields $\widehat{V}^{\downarrow}(b,c) \geq V^{\downarrow}(b,c) \geq -\delta\log n$ on $A^{\ddagger}(n)$. Moreover, (6.3) together with (6.4) imply that $\widehat{V}(y) - \widehat{V}(x)$ is greater than

$$[V(y) - \max_{y\leq j\leq d-1} V(j)] - [V(x) - \max_{x\leq j\leq d-1} V(j)] - O(\log_2 n),$$

for any $c \leq x \leq y \leq d$, on $A^{\ddagger}(n)$. Since $V(x) - \max_{x\leq j\leq d-1} V(j) \leq 0$ and $V(y) - \max_{y\leq j\leq d-1} V(j) \geq -\delta\log n$ on $A^{\ddagger}(n)$, this yields $\widehat{V}^{\downarrow}(c,d) \geq -\delta\log n - O(\log_2 n)$. Furthermore, since (6.3) and (6.4) imply that $\widehat{V}(c)$ is larger than $\max_{b\leq j\leq c}\widehat{V}(j) - O(\log_2 n)$, assembling $\widehat{V}^{\downarrow}(b,c) \geq -\delta\log n$ with $\widehat{V}^{\downarrow}(c,d) \geq -\delta\log n - O(\log_2 n)$ yield

$$\widehat{V}^{\downarrow}(b,d) \geq -\delta\log n - O(\log_2 n), \tag{6.21}$$

on $A^{\ddagger}(n)$. Therefore, we have, on $A^{\ddagger}(n)$ and for all large n,

$$R^+ \leq C(\log n)^3 n^{\delta} \max_{b\leq j\leq d} \mathrm{e}^{-(\widehat{V}(j)-\widehat{V}(b))}. \tag{6.22}$$

Since $\widehat{V}(b) = V(b)$ and (6.4) implies $\widehat{V}(x) \geq V(x)$, for all $b \leq x \leq c$ (in particular $\widehat{V}(c) \geq V(c)$), it follows from (6.21) that $\widehat{V}(j) - \widehat{V}(b) = (\widehat{V}(j) - \widehat{V}(c)) + (\widehat{V}(c) - \widehat{V}(b)) \geq h_n - \delta\log n - O(\log_2 n)$, which is greater than 0 for n large enough whenever $\delta < (1-\varepsilon)/\kappa$ (it is possible since $\delta > \varepsilon/\kappa$ and $0 < \varepsilon < 1/3$). Therefore, recalling (6.22), we obtain, on $A^{\ddagger}(n)$,

$$R^+ \leq C(\log n)^3 n^{\delta}. \tag{6.23}$$

In a similar way, we prove that $R^- \leq C(\log n)^3 n^\delta$, on $A^\ddagger(n)$, which implies that $\lambda_n E^b_{\omega,|a}[F^2] \leq C(\log n)^3 n^{\delta-\frac{1}{\kappa}}$. Now, observe that, for any $\xi > 0$, $\{\lambda_n E^b_{\omega,|a}[F^2] \leq 2(1-\mathrm{e}^{-\xi})\}$ is included in $A^\ddagger(n)$, such that $\lambda_n E^b_{\omega,|a}[F^2] \leq 2(1-\mathrm{e}^{-\xi}) E^b_{\omega,|a}[F]$ with probability larger than $1-o(n^{-\varepsilon})$. Then, introducing

$$R'_n(\lambda) := E\left[\frac{1}{1+\frac{\lambda}{n^{1/\kappa}}\frac{p}{1-p}E^b_{\omega,|a}[F]}\right],$$

we get, for n large enough,

$$R'_n(\lambda)+o(n^{-\varepsilon}) \leq E\left[\frac{1-p}{1-p\,E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n F}]}\right] \leq R'_n(\mathrm{e}^{-\xi}\lambda)+o(n^{-\varepsilon}). \tag{6.24}$$

In order to bound $E^b_{\omega,|a}\left[\mathrm{e}^{-\lambda_n G}\right]$ by below, we observe that $\mathrm{e}^{-x} \geq 1-x$, for any $x \geq 0$, such that $E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}] \geq 1-\lambda_n E^b_{\omega,|a}[G]$. Therefore, we only have to bound $E^b_{\omega,|a}[G]$ from above. Recalling (6.15), we get $E^b_{\omega,|a}[G] \leq (d-b)^2 \mathrm{e}^{\bar{V}^\uparrow(b,d)}$. Now, let us bound $\bar{V}^\uparrow(b,d)$. We observe first that (6.14) implies $\bar{V}^\uparrow(c,d) \leq V^\uparrow(c,d)$, which yields $\bar{V}^\uparrow(c,d) \leq \delta \log n$ on $A^\ddagger(n)$. Moreover, (6.12) together with (6.13) imply that $\bar{V}(y)-\bar{V}(x)$ is less or equal than

$$[V(y)-\max_{b\leq j\leq y} V(j)] - [V(x)-\max_{b\leq j\leq x} V(j)] + O(\log_2 n),$$

for any $b \leq x \leq y \leq c$, on $A^\ddagger(n)$. Since $V(y)-\max_{b\leq j\leq y} V(j) \leq 0$ and $V(x)-\max_{b\leq j\leq x} V(j) \geq -\delta\log n$ on $A^\ddagger(n)$, this yields $\bar{V}^\uparrow(b,c) \leq \delta\log n + O(\log_2 n)$. Furthermore, (6.14) and the fact that $V(y) \leq V(c)$, for $c \leq y \leq d$, imply that $\bar{V}(y) \leq \bar{V}(c)$ for $c \leq y \leq d$. Therefore, we have

$$\bar{V}^\uparrow(b,d) \leq \delta \log n + O(\log_2 n),$$

on $A^\ddagger(n)$. It means that $E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n G}]$ is greater than $1-o(n^{-\varepsilon})$ on $A^\ddagger(n)$ whenever $\delta < \frac{1}{\kappa}-\varepsilon$, which is possible since $\delta > \varepsilon/\kappa$ and $0 < \varepsilon < 1/3$. Therefore, recalling (6.24), we obtain

$$R'_n(\lambda)+o(n^{-\varepsilon}) \leq E\big[E^b_{\omega,|a}[\mathrm{e}^{-\lambda_n \tau(d)}]\big] \leq R'_n(\mathrm{e}^{-\xi}\lambda)+o(n^{-\varepsilon}). \tag{6.25}$$

Recalling (6.6) and (6.2), we get

$$R_n(\lambda, 2\widehat{M}_1(\mathrm{e}^{H^{(1)}}M_2+\omega_b)) \leq R'_n(\lambda) \leq R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2),$$

where $\widehat{M}_1 := \sum_{x=a+1}^{d-1} \mathrm{e}^{-(\widehat{V}(x)-\widehat{V}(b))}$, $M_2 := \sum_{x=b}^{d-1}\mathrm{e}^{V(x)-V(c)}$ and $R_n(\lambda, Z)$ is defined in (6.1). Furthermore, since $\mathrm{e}^{H^{(1)}} \geq n^{\frac{1-\varepsilon}{\kappa}}$, $M_2 \geq 1$ and $\omega_b \leq 1$ we obtain that, for any $\xi > 0$ and n large enough, $\omega_b \leq (\mathrm{e}^\xi - 1)\mathrm{e}^{H^{(1)}} M_2$. Therefore, we have for all large n,

$$R_n(\mathrm{e}^\xi\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2) \leq R'_n(\lambda) \leq R_n(\lambda, 2\mathrm{e}^{H^{(1)}}\widehat{M}_1 M_2). \tag{6.26}$$

Now, assembling (6.25) and (6.26) concludes the proof of Proposition 2. □

7. Back to canonical meanders

Let us set $S := \max\{V(k)\,;\, k \geq 0\}$, $H := \max\{V(k)\,;\, 0 \leq k \leq T_{\mathbb{R}_-}\} = H_0$, and $T_S := \inf\{k \geq 0 : V(k) = S\}$. Moreover, we define $\mathcal{I}_n := \{H = S \geq h_n\} \cap \{V(k) \geq 0\,, \forall\, k \leq 0\}$, and introduce the random variable $Z := e^S M_1^+ M_2^+$, where

$M_1^+ := \sum_{k=a^-}^{T_{h_n/2}} e^{-V(k)}$ and $M_2^+ := \sum_{k=0}^{d^+} e^{V(k)-S}$, with $a^- = \sup\{k \leq 0 : V(k) \geq D_n\}$ and $d^+ := \inf\{k \geq e_1 : V(k) - V(e_1) \leq -D_n\}$. Then, denoting

$$\mathcal{R}_n(\lambda) := E\left[\frac{1}{1 + n^{-\frac{1}{\kappa}} 2\lambda Z} | \mathcal{I}_n\right],$$

we get the following result.

Proposition 3. *For any $\xi > 0$, we have, for n large enough,*

$$\mathcal{R}_n(\mathrm{e}^{\xi}\lambda) + o(n^{-\varepsilon}) \leq R_n(\lambda, 2\mathrm{e}^{H^{(1)}} \widehat{M}_1 M_2) \leq \mathcal{R}_n(\mathrm{e}^{-\xi}\lambda) + o(n^{-\varepsilon}).$$

Proof. Step 1: we replace $\widehat{M}_1$ by $\widehat{M}_1^T$.

Recall that $A^{\ddagger}(n) = \{d - a \leq C'' \log n\} \cap A_5(n)$ and that $P\{A^{\ddagger}(n)\} \geq 1 - o(n^{-\varepsilon})$, for all large n. Now, let us introduce $T(\frac{h_n}{2}) := \inf\{k \geq b : V(k) - V(b) \geq h_n/2\}$ and $\widehat{M}_1^T := \sum_{k=a+1}^{T(\frac{h_n}{2})} e^{-(\widehat{V}(k) - \widehat{V}(b))}$. Recalling (6.21), we observe that $\widehat{M}_1 \leq \widehat{M}_1^T + C'' \log n \mathrm{e}^{-\frac{h_n}{2} + \delta \log n}$ on $A^{\ddagger}(n)$. This implies that, for any $\xi > 0$, we have $\widehat{M}_1 - \widehat{M}_1^T \leq (\mathrm{e}^{\xi} - 1)\widehat{M}_1^T$ for all large n, whenever $\delta < \frac{1-\varepsilon}{2\kappa}$, which is possible since $\delta > \varepsilon/\kappa$ and $0 < \varepsilon < 1/3$. Therefore, we obtain, for n large enough,

$$R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}} \widehat{M}_1^T M_2) + o(n^{-\varepsilon}) \leq R_n(\lambda, 2\mathrm{e}^{H^{(1)}} \widehat{M}_1 M_2) \leq R_n(\lambda, 2\mathrm{e}^{H^{(1)}} \widehat{M}_1^T M_2).$$

Step 2: we replace $\widehat{M}_1^T$ by M_1^T.

Let us denote $M_1^T := \sum_{k=a+1}^{T(\frac{h_n}{2})} e^{-(V(k)-V(b))}$. Since $T(\frac{h_n}{2}) \leq c$, (6.5) implies that $\widehat{M}_1^T \leq M_1^T$. Observe that (6.3) with (6.4) imply that $\widehat{V}(y) - \widehat{V}(b) - (V(y) - V(b))$ is less or equal than

$$\log\left(\frac{\sum_{j=b}^{d-1} \mathrm{e}^{V(j)}}{\sum_{j=y}^{d-1} \mathrm{e}^{V(j)}} \frac{\sum_{j=b+1}^{d-1} \mathrm{e}^{V(j)}}{\sum_{j=y+1}^{d-1} \mathrm{e}^{V(j)}}\right) \leq \frac{\sum_{j=b}^{y-1} \mathrm{e}^{V(j)}}{\sum_{j=y}^{d-1} \mathrm{e}^{V(j)}} + \frac{\sum_{j=b+1}^{y} \mathrm{e}^{V(j)}}{\sum_{j=y+1}^{d-1} \mathrm{e}^{V(j)}},$$

for any $b \leq y \leq d$. Therefore, on $A^{\ddagger}(n)$, we obtain $\widehat{V}(y) - \widehat{V}(b) \leq (V(y) - V(b)) + C \log n \mathrm{e}^{-\frac{h_n}{2}}$ for any $b \leq y \leq T(\frac{h_n}{2})$, which yields $\widehat{M}_1^T \geq \exp\{C \log n \, \mathrm{e}^{-\frac{h_n}{2}}\} M_1^T$. Then, for any $\xi > 0$, we obtain that $\widehat{M}_1^T \geq \mathrm{e}^{-\xi} M_1^T$, on $A^{\ddagger}(n)$ and for all large n. This implies

$$R_n(\lambda, 2\mathrm{e}^{H^{(1)}} M_1^T M_2) \leq R_n(\lambda, 2\mathrm{e}^{H^{(1)}} \widehat{M}_1^T M_2) \leq R_n(\mathrm{e}^{-\xi}\lambda, 2\mathrm{e}^{H^{(1)}} M_1^T M_2) + o(n^{-\varepsilon}).$$

Now, assembling Step 1 and Step 2, we get that, for any $\xi > 0$ and n large enough, $R_n(\lambda, 2\mathrm{e}^{H^{(1)}} \widehat{M}_1 M_2)$ belongs to

$$\left[R_n(\mathrm{e}^{\xi}\lambda, 2\mathrm{e}^{H^{(1)}} M_1^T M_2) + o(n^{-\varepsilon}) \, ; \, R_n(\mathrm{e}^{-\xi}\lambda, 2\mathrm{e}^{H^{(1)}} M_1^T M_2) + o(n^{-\varepsilon})\right]. \tag{7.1}$$

Step 3: the "good " conditioning.

Let us first observe that $((V(k-b) - V(b))_{a \leq k \leq d}, a, b, c, d)$ has the same law as $((V(k))_{a^- \leq k \leq d^+}, a^-, 0, T_H, d^+)$ under $P\{\cdot | \mathcal{I}_n'\}$, where $\mathcal{I}_n' := \{H \geq h_n \, ; \, V^{\uparrow}(a^-, 0) \leq h_n \, ; \, V(k) \geq 0 \, , \, a^- \leq k \leq 0\}$. Moreover, we easily obtain that $P\{\{V(k) \geq 0 \, , \, a^- \leq k \leq 0\} \setminus \{V(k) \geq 0 \, , \, k \leq 0\}\} = O(n^{-(1+\kappa)}) = o(n^{-\varepsilon})$, that $P\{\{H \geq h_n\} \setminus \{H = S\}\} = O(n^{-2(1-\varepsilon)}) = o(n^{-\varepsilon})$ and that $P\{V^{\downarrow}(a^-, 0) > h_n\} \leq P\{V^{\downarrow}(a^-, 0) > \delta \log n\} = o(n^{-\varepsilon})$, with the same arguments as in the proof of Lemma 13. Therefore, we have

$P\{\mathcal{I}'_n \triangle \mathcal{I}_n\} = o(n^{-\varepsilon})$. Since $0 \leq R_n(\lambda, Y) \leq 1$, for any $\lambda > 0$ and any positive random variable Y, this yields

$$R_n(\lambda, 2\mathrm{e}^{H^{(1)}} M_1^T M_2) = \mathcal{R}_n(\lambda) + o(n^{-\varepsilon}). \tag{7.2}$$

Now, assembling (7.1) and (7.2) concludes the proof of Proposition 3. □

8. Proof of Theorem 2

Observe first that $\mathcal{R}_n(\lambda)$ can be written

$$\mathcal{R}_n(\lambda) = 1 - E\left[1 - \frac{1}{1 + 2\lambda_n Z} | \mathcal{I}_n\right].$$

Then, we can use Corollary 9.1 and Remark 9.1 in [8], which implies

$$E\left[1 - \frac{1}{1 + 2\lambda_n Z} \,\middle|\, \mathcal{I}_n\right] \sim 2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} \, \frac{E[M^\kappa]^2 C_I}{nP\{H \geq h_n\}} \, \lambda^\kappa, \qquad n \to \infty.$$

Therefore, assembling Proposition 1, Proposition 2, Proposition 3 and recalling that $q_n := P\{H \geq h_n\}$, we get that, for any $\xi > 0$,

$$\begin{aligned}
\liminf_{n\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_n \tau(e_n)}] &\geq \exp\Big\{ - \Big(2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} E[M^\kappa]^2 C_I\Big)(\mathrm{e}^\xi \lambda)^\kappa \Big\}, \\
\limsup_{n\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_n \tau(e_n)}] &\leq \exp\Big\{ - \Big(2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} E[M^\kappa]^2 C_I\Big)(\mathrm{e}^{-\xi} \lambda)^\kappa \Big\}.
\end{aligned}$$

Since this result holds for any $\xi > 0$, we get,

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{-\lambda_n \tau(e_n)}] = \exp\Big\{ - \Big(2^\kappa \frac{\pi\kappa}{\sin(\pi\kappa)} E[M^\kappa]^2 C_I\Big)\lambda^\kappa \Big\}.$$

Now, for the conclusion of the proof of Theorem 2 and for the proofs of Theorem 1 and Corollary 1, we refer to the detailed sketch of the proof, see Section 3. □

9. Proof of Theorem 3

The proof of Theorem 3 relies heavily on the estimates developed previously, so we will not develop in full detail all the arguments. We first extend the convergence result of Theorem 1 to finite dimensional marginals. Consider $(x_1, x_2, \ldots, x_k)$ such that $0 < x_1 < \ldots < x_k$. By the same type of arguments as in the proof of Theorem 1 we can prove that

$$\frac{1}{n^{1/\kappa}} \left(\tau_{\lfloor nx_1 \rfloor}, \tau_{\lfloor nx_2 \rfloor} - \tau_{\lfloor nx_1 \rfloor}, \ldots, \tau_{\lfloor nx_k \rfloor} - \tau_{\lfloor nx_{k-1} \rfloor}\right)$$

converges in distribution to

$$\left(d^{-1/\kappa} x_1^{1/\kappa} S_1, d^{-1/\kappa}(x_2 - x_1)^{1/\kappa} S_2, \ldots, d^{-1/\kappa}(x_k - x_{k-1})^{1/\kappa} S_k\right),$$

where $d = \frac{\sin(\pi\kappa)}{2^\kappa \pi\kappa^2 C_K^2 E[\rho_0^\kappa \log \rho_0]}$ and $S_1, \ldots, S_k$ are independent random variables with the same normalized completely asymmetric stable law with index κ. By a classical inversion argument (as in [18] for the one-dimensional marginal) we can deduce that if $X_n^* := \max\{X_k,\ 0 \leq k \leq n\}$ and $0 < t_1 < \ldots < t_k$, the random vector

$$\frac{1}{n^\kappa} \left(X^*_{\lfloor t_1 n \rfloor}, X^*_{\lfloor t_2 n \rfloor} - X^*_{\lfloor t_1 n \rfloor}, \ldots, X^*_{\lfloor t_k n \rfloor} - X^*_{\lfloor t_{k-1} n \rfloor}\right)$$

converges in distribution to

$$\left(dt_1^\kappa S_1^{-\kappa}, d(t_2-t_1)^\kappa S_2^{-\kappa}, \dots, d(t_k-t_{k-1})^\kappa S_k^{-\kappa}\right).$$

This implies the convergence of the finite dimensional marginals of $X^*_{\lfloor nt\rfloor}/n^\kappa$ to the marginals of the inverse of the stable subordinator of Theorem 3.

Moreover, since $(X^*_{\lfloor nt\rfloor}/n^\kappa,\, t\geq 0)$ is increasing, the same type of arguments as in [3] implies that it converges in distribution in the Skorokhod M_1-topology (we refer to Section 5 of [3] for a detailed discussion).

The last step is to show that with large probability $X_{\lfloor nt\rfloor}/n^\kappa$ and $X^*_{\lfloor nt\rfloor}/n^\kappa$ are uniformly close. Let us now consider

$$n' := n^\kappa \log n.$$

Let T be a positive real number. Since $X^*_{\lfloor nT\rfloor}/n^\kappa$ converges in distribution, the probability $\mathbb{P}\{X^*_{nT}\leq n'\}$ tends to 1, when n tends to infinity.

On the event $DT(n')\cap\{X^*_{nT}\leq n'\}$, we have

$$\max_{k=0,\dots,\lfloor nT\rfloor}|X_k-X_k^*|\leq \max_{j=1,\dots,K_{n'}}|a_j-b_{j+1}|.$$

Indeed, on $DT(n')$, we know that if X_k reaches b_j then it does not backtrack before a_j. Hence $(X_k^*\geq b_j)$ implies that $(X_k\geq a_j)$. This implies the previous estimate. Moreover, using the same arguments as in Subsection 5.1, it is easy to check that

$$\lim_{n\to\infty} P\left\{\max_{j=1,\dots,K_{n'}}|b_{j+1}-a_j|\leq \frac{n^\kappa}{\log n}\right\}=1.$$

This implies that, for all $T>0$ and all $\varepsilon>0$,

$$\lim_{n\to\infty}\mathbb{P}\left\{\max_{t\in[0,T]}\left|\frac{X^*_{\lfloor nt\rfloor}}{n^\kappa}-\frac{X_{\lfloor nt\rfloor}}{n^\kappa}\right|>\varepsilon\right\}=0,$$

which concludes the proof of Theorem 3. □

10. Toward the case $\kappa = 1$

We intend to treat the critical case $\kappa=1$ between the transient ballistic and subballistic cases. This case turns out to be more delicate. Indeed, Lemma 7 is replaced by a weaker statement, which says that $\tau(e_n)$ reduces to the time spent by the walker to climb excursions which are higher than $\alpha\log n$ for α arbitrarily small. Due to this reduced height, the new "high" excursions are much more numerous and are not anymore well separated. The definition of the valleys should then be adapted as well as the "linearization" argument, which is more difficult to carry out. Moreover, a result of Goldie [10] gives an explicit formula for the Kesten's renewal constant, namely $C_K=\frac{1}{E[\rho_0\log\rho_0]}$. As a result, we should obtain, in this case, the following result, which takes a remarkably simple form: $X_n/(\frac{n}{\log n})$ converges in probability to $E[\rho_0\log\rho_0]/2$.

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