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Cyclic and ruled Lagrangian surfaces in complex Euclidean space

Henri Anciaux and Pascal Romon

Abstract

We study those Lagrangian surfaces in complex Euclidean space which are foliated by circles or by straight lines. The former, which we call *cyclic*, come in three types, each one being described by means of, respectively, a planar curve, a Legendrian curve of the 3-sphere or a Legendrian curve of the anti de Sitter 3-space. We also describe ruled Lagrangian surfaces. Finally we characterize those cyclic and ruled Lagrangian surfaces which are solutions to the self-similar equation of the Mean Curvature Flow.

Keywords: Lagrangian surfaces; circle foliation; Mean Curvature Flow.
2000 MSC: 53D12 (Primary) 53C42 (Secondary)

Introduction

In this paper, we classify the Lagrangian surfaces of $\mathbb{C}^2$ which are foliated either by round circles (henceforth called *cyclic* surfaces) or by straight lines (*ruled* surfaces). This completes a former paper of the authors together with Ildefonso Castro [ACR] in which all Lagrangian submanifolds of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, with $n \geq 3$, which are foliated by round $(n-1)$-spheres were characterized. The reason for the lower bound on the dimension was the following: since the submanifold is Lagrangian, any spherical leaf must be isotropic; when the dimension of this leaf is at least two, it spans a linear space which is itself Lagrangian. This observation simplifies the structure of the problem, roughly speaking by reducing the underlying group structure from $SO(2n)$ to $U(n)$.

However this reduction no longer holds in dimension two, see for instance the Lagrangian cylinder $\{(e^{it}, s) \in \mathbb{C}^2, (s,t) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}\}$: this (Lagrangian) surface is foliated by circles which are contained in non-Lagrangian (actually complex) planes. Other examples are the Hopf tori studied by Pinkall in [P]. As expected the situation is richer in dimension two, and actually cyclic Lagrangian surfaces come in three families which are described in Theorem 1 and 2. In Section 1 we classify the cyclic Lagrangian surfaces when all the centers of the circles coincide. We call those surfaces *centered* cyclic. In Section 2 we treat the general case which amounts to adding a convenient translation term. In Section 3 we apply this characterization to finding self-similar cyclic surfaces, that is those surfaces which are solutions of the following elliptic PDE:

$$H + \lambda X^{\perp} = 0,$$

where H denotes the mean curvature vector of the surface and $X^{\perp}$ the normal component of its position vector. The case of positive (resp. negative) λ corresponds to the case of a self-shrinking (resp. self-expanding) soliton of the Mean Curvature Flow (see [A]). Finally we give in the last Section a description of ruled Lagrangian surfaces (notably self-similar ones) using an analogous method, recovering more simply a known result from Blair [B].

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1 Centered cyclic Lagrangian surfaces

Let Σ a surface of $\mathbb{R}^4$ foliated by circles with common center located at the origin of $\mathbb{R}^4$. Locally, Σ may be parametrized by the following immersion:

$$\begin{array}{rccl} X: & I \times \mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{R}^4 \\ & (s,t) & \mapsto & r(s)(e_1(s)\cos t + e_2(s)\sin t), \end{array}$$

where $r(s)$ is a positive function and $(e_1(s), e_2(s))$ is an orthonormal basis of the plane containing our circle.

From now on we shall assume that Σ is Lagrangian with respect with some complex structure J. We will often identify $\mathbb{R}^4$ with $\mathbb{C}^2$ in such a way that J is the complex multiplication by i. Denote by $K := \langle e_1, Je_2\rangle = -\langle e_1, Je_2\rangle$ the *Kähler angle* of the plane $e_1 \wedge e_2$. Note that the vanishing of K means that $e_1 \wedge e_2$ is also Lagrangian. In this case the analysis done in [ACR] holds

and X takes the following form: $X(s,t) = r(s)e^{i\phi(s)}(\cos t, \sin t)$ making use of the above identification.

Denoting by subscripts the partial derivatives (the prime corresponding also to the derivative in s for functions of just one variable), the Lagrangian assumption is equivalent to $\langle X_s, JX_t\rangle = 0$. Since

$$X_s = r'(e_1 \cos t + e_2 \sin t) + r(e_1' \cos t + e_2' \sin t)$$

$$X_t = r(e_2 \cos t - e_1 \sin t),$$

we see that

$$\begin{aligned}
\langle X_s, JX_t\rangle &= rr'(K\cos^2 t + K\sin^2 t) \\
&\quad + r^2\left(\cos^2 t\langle e_1', Je_2\rangle - \sin^2 t\langle e_2', Je_1\rangle\right) \\
&\quad + r^2 \cos t \sin t\left(\langle e_2', Je_2\rangle - \langle e_1', Je_1\rangle\right) \\
&= rr'K + \frac{r^2}{2}\left(\langle e_1', Je_2\rangle - \langle e_2', Je_1\rangle\right) \\
&\quad + \frac{r^2\cos 2t}{2}\left(\langle e_1', Je_2\rangle + \langle e_2', Je_1\rangle\right) \\
&\quad + \frac{r^2\sin 2t}{2}\left(\langle e_2', Je_2\rangle - \langle e_1', Je_1\rangle\right) \\
&= \frac{2rr'K + r^2K'}{2} + \frac{r^2\cos 2t}{2}\left(\langle e_1', Je_2\rangle + \langle e_2', Je_1\rangle\right) \\
&\quad + \frac{r^2\sin 2t}{2}\left(\langle e_2', Je_2\rangle - \langle e_1', Je_1\rangle\right)
\end{aligned}$$

which holds for all t. The vanishing of the constant term implies that $r^2K = C$ for some real constant C. If this constant vanishes, we recover the case $K = 0$ mentioned above and treated in [ACR] (cf also [A]), so we may assume that $C \neq 0$. Thus r is completely determined by K, and both are non zero.

The two remaining conditions are

$$\langle e_1', Je_2\rangle = -\langle e_2', Je_1\rangle\ , \quad \langle e_1', Je_1\rangle = \langle e_2', Je_2\rangle. \tag{1}$$

In order to make sense of these, we will identify $\mathbb{R}^4$ with $\mathbb{H}$, in such a way that the complex structure is given by the left multiplication by the quaternion i. Then any element in $SO(4)$ can be written as $x \mapsto pxq^{-1}$ where p, q are two unit quaternions. Notice that right multiplication by q^{-1} corresponds exactly to the elements of $SU(2)$. Since $SO(4)$ acts transitively on pairs of

orthonormal vectors, we may write e_1 and e_2 as the respective images of 1 and i, so that $e_1 = pq^{-1}$ and $e_2 = piq^{-1}$. Note that (p, q) is not uniquely determined; rather we have a gauge freedom by right multiplication by $e^{i\theta}$ on (p, q). Finally, we may assume if needed that $q(0)$ takes any prescribed value, since we consider surfaces up to $U(2)$ congruence.

Then the conditions in (1) read as

$$\begin{cases} \langle p'q^{-1} - pq^{-1}q'q^{-1}, ipiq^{-1}\rangle + \langle p'iq^{-1} - piq^{-1}q'q^{-1}, ipq^{-1}\rangle = 0 \\ \langle p'q^{-1} - pq^{-1}q'q^{-1}, ipq^{-1}\rangle - \langle p'iq^{-1} - piq^{-1}q'q^{-1}, ipiq^{-1}\rangle = 0 \end{cases}$$

so, multiplying on the left by p^{-1} and the right by q, and further by i on the left in the second bracket,

$$\begin{cases} \langle p^{-1}p' - q^{-1}q', p^{-1}ipi\rangle - \langle p^{-1}p' + iq^{-1}q'i, p^{-1}ipi\rangle = 0 \\ \langle p^{-1}p' - q^{-1}q', p^{-1}ip\rangle - \langle p^{-1}p' + iq^{-1}q'i, p^{-1}ip\rangle = 0 \end{cases}$$

that is

$$\begin{cases} \langle q^{-1}q' + iq^{-1}q'i, p^{-1}ipi\rangle = 0 \\ \langle q^{-1}q' + iq^{-1}q'i, p^{-1}ip\rangle = 0. \end{cases}$$

Splitting and multiplying left and right by i in the second bracket yields

$$\begin{cases} \langle q^{-1}q', p^{-1}ipi\rangle - \langle q^{-1}q', ip^{-1}ip\rangle = 0 \\ \langle q^{-1}q', p^{-1}ip\rangle + \langle q^{-1}q', ip^{-1}ipi\rangle = 0 \end{cases}$$

$$\begin{cases} \langle q^{-1}q', p^{-1}ipi - ip^{-1}ip\rangle = 0 \\ \langle q^{-1}q', p^{-1}ip + ip^{-1}ipi\rangle = 0 \end{cases}$$

Writing $p = p_0 + ip_1 + jp_2 + kp_3$ we have

$$u := p^{-1}ipi - ip^{-1}ip = 4(p_0p_2 + p_1p_3)j - 4(p_1p_2 - p_0p_3)k$$

and

$$v := p^{-1}ip + ip^{-1}ipi = -ui = 4(p_1p_2 - p_0p_3)j + 4(p_0p_2 + p_1p_3)k.$$

The two right-hand vectors u, v lie in $\mathrm{Span}(j, k)$ and are either linearly independent (over $\mathbb{R}$) or both zero. So we have two cases:

- $u = v = 0$, i.e.

$$0 = (p_0p_2 + p_1p_3)^2 + (p_1p_2 - p_0p_3)^2 = (p_0^2 + p_1^2)(p_2^2 + p_3^2)$$

hence p lies in $\mathrm{Span}(1,i)$ or $\mathrm{Span}(j,k)$ and conditions in (1) hold. Gauging p we may assume that $p=1$ or $p=j$, and the Kähler angle is then $K=-1$ or $K=+1$ respectively, so that the radius r remains constant (and we may as well assume $r=1$). This case corresponds to Hopf surfaces [P], i.e. inverse images of a curve by the Hopf fibration $\mathbb{S}^3\to\mathbb{S}^2$. After a possible change of variable in t (replacing t by $t+\varphi(s)$ for some function φ), we may assume that the curve $s\mapsto e_1(s)$ is Legendrian.

- u,v are independent vectors and span j,k, thus forcing $q^{-1}q'$ to lie in $\mathrm{Span}(1,i)\cap\mathrm{Im}\mathbb{H}$; using gauge action, we may assume that q is constant, and up to congruence write $q=1$. Reverting to complex coordinates, $e_1=p=\gamma_1+\gamma_2 j\simeq(\gamma_1,\gamma_2)\in\mathbb{S}^3\subset\mathbb{C}^2$ and $e_2=pi\simeq(i\gamma_1,-i\gamma_2)$, while $K=|\gamma_2|^2-|\gamma_1|^2\neq 0$. We may normalize, assuming that $K>0$ (if $K<0$ pick the opposite orientation on the surface) and set $(\alpha_1,\alpha_2):=\frac{1}{\sqrt{K}}(\gamma_1,\gamma_2)$, so that

$$X=\frac{\sqrt{C}}{\sqrt{K}}(\gamma_1(s)e^{it},\gamma_2(s)e^{-it})=\sqrt{C}(\alpha_1e^{it},\alpha_2e^{-it})$$

with $|\alpha_1|^2-|\alpha_2|^2=-1$, i.e. (α_1,α_2) lies in $\mathbb{H}^3_1$, the unit anti-De Sitter space. Again, up to a change in variable, we have a Legendrian curve for the undefinite metric in $\mathbb{C}^{1,1}$, i.e. $\langle\alpha_1',i\alpha_1\rangle-\langle\alpha_2',i\alpha_2\rangle=0$ (see [CLU] or [CC]).

Summing up, we have proved the following

Theorem 1 *A centered cyclic Lagrangian surface may be locally parameterized, up to $U(2)$ congruence, by one the following immersions:*

Type I *(complex extensors):*

$$\begin{array}{rccc} X: & I\times\mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{C}^2\\ & (s,t) & \mapsto & r(s)e^{i\phi(s)}(\cos t,\sin t)\end{array}$$

Type II *(Hopf type):*

$$\begin{array}{rccc} X: & I\times\mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{C}^2\\ & (s,t) & \mapsto & ce^{it}(\gamma_1(s),\gamma_2(s))\end{array}$$

where $\gamma=(\gamma_1,\gamma_2)$ is any Legendrian curve of $\mathbb{S}^3$ and c is a real constant,

Type III *(De Sitter type)*

$$\begin{array}{rccl} X: & I\times\mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{C}^2 \\ & (s,t) & \mapsto & c\,(\alpha_1(s)e^{it},\alpha_2(s)e^{-it}) \end{array}$$

where $\alpha=(\alpha_1,\alpha_2)$ is any Legendrian curve in the unit anti-De Sitter space $\mathbb{H}^3_1$ and c is a real constant.

Remark 1 *This analysis applies as well if we do not assume that X is an immersion but only has an isotropic image fibered by circles (we did not use the immersion hypothesis). So the same conclusion holds and will be used in the next Section.*

Remark 2 *Type I surfaces are a particular case of a class of Lagrangian immersions which has been first described in [C1] where they were called* complex extensors.

Remark 3 *In the type III case, it may happen that K is identically ±1. Then we fall back on type II with a Legendrian curve that actually reduces to a single point. The image of X is therefore a circle (lying in a complex plane).*

2 The general case

We now consider a surface Σ of $\mathbb{R}^4$ which is foliated by circles. Locally, Σ may be parametrized by the following immersion (identifying as usual $\mathbb{R}^4$ with $\mathbb{C}^2$):

$$\begin{array}{rccl} Y: & I\times\mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{C}^2 \\ & (s,t) & \mapsto & X(s,t)+V(s), \end{array}$$

where $V(s)$ is a $\mathbb{C}^2$-valued function, and $X(s,t)=r(s)(e_1(s)\cos t+e_2(s)\sin t)$ is a centered surface as in the previous section. Note that we do not assume *a priori* that X is Lagrangian nor that it is always an immersion.

As we have $Y_t=X_t$ and $Y_s=X_s+V'$, the assumption that Y is Lagrangian leads to:

$$\begin{array}{rcl} 0 & = & \langle Y_s,JY_t\rangle=\langle X_s,JX_t\rangle+\langle V',JX_t\rangle \\ & = & \langle X_s,JX_t\rangle+r\cos t\langle V',Je_2\rangle-r\sin t\langle V',Je_1\rangle \end{array}$$

Recall from the previous section that $\langle X_s, JX_t\rangle$ contains only terms in $\cos 2t$ and $\sin 2t$ and a term independent from t. Thus the immersion Y is Lagrangian if and only if: (i) X is cyclic isotropic (cf Remark 1), and (ii) V' belongs to the symplectic orthogonal of $\mathrm{Span}(e_1, e_2)$. Using Theorem 1 we infer:

- for type I surfaces, $\mathrm{Span}(e_1, e_2) = \mathrm{Span}(e^{i\phi}(1,0), e^{i\phi}(0,1))$ is Lagrangian, so its symplectic orthogonal is itself; hence $V'(s) = e^{i\phi(s)}(W_1(s), W_2(s))$ for some real-valued functions W_1, W_2;
- for type II surfaces, $\mathrm{Span}(e_1, e_2)$ is a complex line, so its symplectic orthogonal is the same as its Riemannian orthogonal, which is $\mathrm{Span}((\bar\gamma_2, -\bar\gamma_1), (i\bar\gamma_2, -i\bar\gamma_1))$; V' is determined analogously;
- for type III surfaces, one can check than a basis of the symplectic orthogonal is (f_1, f_2), where $f_1 = (|\alpha_2|^2, \alpha_1\alpha_2)$, $f_2 = (i|\alpha_2|^2, -i\alpha_1\alpha_2)$.

So we conclude this section by the

Theorem 2 *A cyclic Lagrangian surface may be locally parameterized, up to $U(2)$ congruence, by one the following immersions:*

Type I

$$\begin{array}{rccl} Y: & I\times\mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{C}^2 \\ & (s,t) & \mapsto & r(s)e^{i\phi(s)}(\cos t, \sin t) + \int_{s_0}^{s} e^{i\phi(u)}(W_1(u), W_2(u))du \end{array}$$

where W_1, W_2 are real valued; in particular when ϕ is constant, Y stays within the Lagrangian plane $\mathrm{Span}(e^{i\phi}(1,0), e^{i\phi}(0,1))$*;*

Type II

$$\begin{array}{rccl} Y: & I\times\mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{C}^2 \\ & (s,t) & \mapsto & c\,(\gamma_1(s)e^{it}, \gamma_2(s)e^{it}) + \int_{s_0}^{s} W(u)(\bar\gamma_2(u), -\bar\gamma_1(u))du \end{array}$$

where $\gamma = (\gamma_1, \gamma_2)$ is any Legendrian curve of $\mathbb{S}^3$, c is a real constant and W a complex valued function; in particular if γ is constant, then up to congruence, we may assume $\gamma = (1, 0)$ and the immersion becomes $Y(s,t) = (ce^{it}, V_2(s))$. Thus the immersed surface is a Cartesian product of a circle with a planar curve;

Type III

$$\begin{array}{rccl} Y: & I\times\mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{C}^2 \\ & (s,t) & \mapsto & c\,(\alpha_1(s)e^{it},\alpha_2(s)e^{-it})+\int_{s_0}^{s}(W|\alpha_2|^2,\overline{W}\alpha_1\alpha_2)du \end{array}$$

where $\alpha=(\alpha_1,\alpha_2)$ is any Legendrian curve in the unit anti-De Sitter space $\mathbb{H}^3_1$, c is a real constant and W a complex valued function.

Remark 4 *In the type II case, if the curve (γ_1,γ_2) is in addition regular, another basis of the orthogonal space to* $\mathrm{Span}(e_1,e_2)$ *is $((\gamma_1',\gamma_2'),(i\gamma_1',i\gamma_2'))$, so that the immersion may take the alternative form:*

$$Y(s,t)=(\gamma_1(s)e^{it},\gamma_2(s)e^{it})+\int_{s_0}^{s}W(u)(\gamma_1'(u),\gamma_2'(u))du.$$

This is a particular case of Lagrangian immersions which have been recently described in [C2]. This alternative formula will also be useful in the next section.

3 Application to the self-similar equation

In this section we study the self-similar equation in the case of cyclic Lagrangian surfaces and prove the following:

Theorem 3 *A Lagrangian cyclic surface of $\mathbb{C}^2$ which is a soliton of the mean curvature flow, i.e. a solution to the self-similar equation*

$$H+\lambda X^{\perp}=0,$$

for some non-vanishing number λ is locally congruent to an equivariant example described in [A] (in the terminology of the present article, a centered surface of type I) or to the Cartesian product $\mathcal{C}\times\Gamma$ of a circle $\mathcal{C}$ with a planar self-shrinking curve Γ. Such curves have been studied in detail in [AL].

Proof. The proof deals with the three cyclic cases separately: we first prove that a self-similar surface of type I must be centered, thus one of the examples of [A]; then we show that there no self-similar surfaces of type II except the Clifford torus $\mathbb{S}^1\times\mathbb{S}^1$ (which is also a type I surface) and the products of

curves. Finally we see that there are no self-similar surfaces of type III at all.

Case 1: type I surfaces.

A type I surface is parametrized by an immersion of the form:

$$\begin{array}{rccl} X: & I \times \mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{C}^2 \\ & (s,t) & \mapsto & r(s)e^{i\phi(s)}(\cos t, \sin t) + \int_{s_0}^{s} e^{i\phi(u)}(W_1(u), W_2(u))du, \end{array}$$

where $r(s) > 0$. We shall use the following notations: $\gamma(s) = r(s)e^{i\phi(s)}$, and, as we assume that γ is parametrized by arclength, we also denote $\gamma'(s) = e^{i\theta(s)}$.

We start computing the first derivatives of the immersion:

$$X_s = \gamma'(\cos t, \sin t) + e^{i\phi}(W_1, W_2), \qquad X_t = \gamma(-\sin t, \cos t),$$

from which we deduce the expression of the induced metric:

$$E = |X_s|^2 = 1 + |W|^2 + 2\cos(\theta - \phi)(W_1 \cos t + W_2 \sin t),$$

$$F = \langle X_s, X_t \rangle = r\cos(\theta - \phi)(W_2 \cos t - W_1 \sin t), \qquad G = |X_t|^2 = r^2,$$

and a basis of the normal space to the surface:

$$N_t = i\gamma(-\sin t, \cos t), \qquad N_s = i\gamma'(\cos t, \sin t) + ie^{i\phi}(W_1, W_2).$$

We now compute the second derivatives of the immersion, in order to calculate the mean curvature vector:

$$X_{ss} = \gamma''(\cos t, \sin t) + i\phi' e^{i\phi}(W_1, W_2),$$

$$X_{st} = \gamma'(-\sin t, \cos t), \qquad X_{tt} = \gamma(-\cos t, -\sin t).$$

This implies in particular that:

$$\langle X_{ss}, N_t \rangle = (W_2 \cos t - W_1 \sin t)\sin(\theta - \phi),$$

$$\langle X_{tt}, N_t \rangle = 0, \qquad \langle X_{st}, N_t \rangle = r\sin(\theta - \phi).$$

On the other hand, we have:

$$\langle X, N_t \rangle = a\cos t + b\sin t,$$

where $a := \langle i\gamma, \int_{s_0}^{s} W_1(u)e^{i\phi(u)}du\rangle$ and $b := \langle i\gamma, \int_{s_0}^{s} W_2(u)e^{i\phi(u)}du\rangle$.

We now assume that the immersion X is self-similar, so there exists a non-vanishing real number λ such that:

$$\langle H, N_t\rangle + \lambda\langle X, N_t\rangle = 0,$$

which is equivalent to

$$\langle X_{ss}, N_t\rangle G + \langle X_{tt}, N_t\rangle E - 2\langle X_{st}, N_t\rangle F = -2\lambda(EG - F^2)\langle X, N_t\rangle.$$

In the latter expression, the left hand side term is linear in $\cos t$ and $\sin t$ and the right hand side term is a polynomial of order 3. Linearizing the latter, we easily see that the coefficient of $\cos 2t$ is $aW_2 + bW_1$ and the one of $\sin 2t$ is $aW_1 - bW_2$. So either W_1 and W_2 vanish, or a and b vanish. We are going to show that actually if a and b vanish, then so do W_1 and W_2.

We first write

$$a = \langle i\gamma, \int_{s_0}^{s} W_1(u)e^{i\phi(u)}\rangle = r\langle ie^{i\phi}, \int_{s_0}^{s} W_1(u)e^{i\phi(u)}du\rangle$$

$$= r\left(-\sin\phi\left(\int_{s_0}^{s} W_1(u)\cos\phi(u)du\right) + \cos\phi\left(\int_{s_0}^{s} W_1(u)\sin\phi(u)du\right)\right) = 0.$$

Thus the derivative of a/r with respect to s must vanish, which yields:

$$\phi'\left(-\cos\phi\left(\int_{s_0}^{s} W_1(u)\cos\phi(u)du\right) - \sin\phi\left(\int_{s_0}^{s} W_1(u)\sin\phi(u)du\right)\right)$$
$$+W_1(s)(-\sin\phi\cos\phi + \cos\phi\sin\phi) = 0.$$

Now either ϕ is (locally) constant, the curve γ is a straight line passing through the origin and the image of the immersion X is a piece of a plane (cf Example 2, page 5 of [ACR]), or one can find find points around which ϕ' does not vanish (locally again). In the latter case we get the following linear system:

$$\begin{cases} -\sin\phi\left(\int_{s_0}^{s} W_1(u)\cos\phi(u)du\right) + \cos\phi\left(\int_{s_0}^{s} W_1(u)\sin\phi(u)du\right) &= 0 \\ -\cos\phi\left(\int_{s_0}^{s} W_1(u)\cos\phi(u)du\right) - \sin\phi\left(\int_{s_0}^{s} W_1(u)\sin\phi(u)du\right) &= 0 \end{cases}$$

It follows that $\int_{s_0}^{s} W_1(u)\cos\phi(u)du$ and $\int_{s_0}^{s} W_1(u)\sin\phi(u)du$ must vanish except on an isolated set of points of I, which in turn implies the vanishing of

W_1. Analogously it can be shown that W_2 vanishes as well, so finally the surface must be centered. Lagrangian, self-similar, centered surfaces have been described in detail in [A]. We only recall here that such surfaces are obtained from planar curves γ which are solutions of the following equation:

$$k = \langle \gamma, N \rangle \left(\frac{1}{|\gamma|^2} - \lambda \right),$$

where k is the curvature of γ and N its unit normal vector. This equation admits a countable family of closed solutions which is parametrised by two relatively prime numbers p and q subject to the condition $p/q \in (1/4, 1/2)$; p is the winding number of the curve and q is the number of maxima of its curvature. Except for the circles, none of these curves is embedded.

Case 2: type II surfaces.

As our discussion is local, we consider the two following cases: either the Legendrian curve γ is regular, or it reduces to a single point, and then the surface is a product of a circle with some plane curve. In the first case, the surface may parametrized by an immersion of the form (cf Remark 4):

$$\begin{array}{rccl} X: & I \times \mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{C}^2 \\ & (s,t) & \mapsto & c(\gamma_1(s)e^{it}, \gamma_2(s)e^{it}) + \int_{s_0}^{s} W(u)(\gamma_1', \gamma_2')du \end{array}$$

where $(\gamma_1(s), \gamma_2(s))$ is some unit speed Legendrian curve of $\mathbb{S}^3$. Without loss of generality we fix $c = 1$. We start computing the first derivatives of the immersion:

$$X_s = (\gamma_1' e^{it}, \gamma_2' e^{it}) + W(\gamma_1', \gamma_2'), \qquad X_t = (i\gamma_1 e^{it}, i\gamma_2 e^{it}),$$

from which we deduce the expression of the induced metric:

$$E = |X_s|^2 = 1 + |W|^2 + 2\langle W, e^{it} \rangle, \qquad F = \langle X_s, X_t \rangle = 0, \qquad G = |X_t|^2 = 1,$$

and a basis of the normal space to the surface:

$$N_s = (i\gamma_1' e^{it}, i\gamma_2' e^{it}) + W(i\gamma_1', i\gamma_2'), \qquad N_t = (-\gamma_1 e^{it}, -\gamma_2 e^{it}).$$

We now compute the second derivatives of the immersion, in order to calculate the mean curvature vector:

$$X_{ss} = (\gamma_1'' e^{it}, \gamma_2'' e^{it}) + W(\gamma_1'', \gamma_2'') + W'(\gamma_1', \gamma_2'),$$
$$X_{st} = (i\gamma_1' e^{it}, i\gamma_2' e^{it}), \qquad X_{tt} = (-\gamma_1 e^{it}, -\gamma_2 e^{it}).$$

We first notice that

$$\langle X, N_t \rangle = -1 + a\cos t + b\sin t,$$

where a and b depend only on the variable s.

On the other hand,

$$2\langle H, N_t \rangle = \frac{\langle X_{ss}, N_t \rangle}{E} + \frac{\langle X_{tt}, N_t \rangle}{G} = \frac{\langle X_{st}, N_s \rangle}{E} + \frac{\langle X_{tt}, N_t \rangle}{G}$$
$$= \frac{1 + \langle W, e^{it} \rangle}{1 + |W|^2 + 2\langle W, e^{it} \rangle} + 1 = \frac{2 + |W|^2 + 3\langle W, e^{it} \rangle}{1 + |W|^2 + 2\langle W, e^{it} \rangle}.$$

Thus the equation $\langle H, N_t \rangle + \lambda \langle X, N_t \rangle = 0$ holds for some non-vanishing constant λ if and only if a and b vanish and $|W| = 1$ or $W = 0$. In particular, $\langle X, N_t \rangle = -1$.

We leave to the reader the easy task to check that if W vanishes, the immersion is self-similar if and only if $\langle \gamma'', J\gamma' \rangle$ vanishes, that is the curve γ has vanishing curvature and thus is a great circle. The corresponding Lagrangian surface is the Clifford torus $\frac{1}{\sqrt{2}}\mathbb{S}^1 \times \frac{1}{\sqrt{2}}\mathbb{S}^1$. So we assume in the remainder that $|W| = 1$. In particular, we may write $W = e^{i\phi}$ and $W' = i\phi' e^{i\phi}$, where ϕ is some real function of the variable s.

We want to look at the other scalar equation $\langle H, N_s \rangle + \lambda \langle X, N_s \rangle = 0$, so we compute

$$\langle X_{ss}, N_s \rangle = \langle \gamma'', J\gamma' \rangle (1 + |W|^2) + \langle iW, W' \rangle + 2\langle W, e^{-it} \rangle \langle \gamma'', J\gamma' \rangle + \langle W', e^{it} \rangle,$$

$$\langle X_{tt}, N_s \rangle = 0.$$

Hence we get

$$\begin{aligned}
2\langle H, N_s\rangle &= \frac{\langle X_{ss}, N_s\rangle}{E} + \frac{\langle X_{tt}, N_s\rangle}{G} = \frac{\langle X_{ss}, N_s\rangle}{1+|W|^2+2\langle W, e^{it}\rangle} + 0 \\
&= \frac{\langle \gamma'', J\gamma'\rangle(1+|W|^2) + \langle iW, W'\rangle + 2\langle W, e^{-it}\rangle\langle \gamma'', J\gamma'\rangle + \langle W', e^{it}\rangle}{1+|W|^2+2\langle W, e^{it}\rangle} \\
&= \frac{2\langle \gamma'', J\gamma'\rangle + \phi' + 2\cos(\phi+t)\langle \gamma'', J\gamma'\rangle - \phi'\sin(\phi-t)}{2+2\cos(\phi-t)} \\
&= \frac{2\langle \gamma'', J\gamma'\rangle + \phi' + (2\langle \gamma'', J\gamma'\rangle\cos\phi - \phi'\sin\phi)\cos t}{2(1+\cos\phi\cos t+\sin\phi\sin t)} \\
&\quad + \frac{(-2\langle \gamma'', J\gamma'\rangle\sin\phi + \phi'\cos\phi)\sin t}{2(1+\cos\phi\cos t+\sin\phi\sin t)}
\end{aligned}$$

On the other hand, it is easy to see that $\langle X, N_s\rangle$ takes the form $a\cos t + b\sin t + c$, where a b and c depend on the variable s. This forces $\langle H, N_s\rangle$ to take a simpler form and implies that

$$\frac{2\langle \gamma'', J\gamma'\rangle + \phi'}{2} = \frac{2\langle \gamma'', J\gamma'\rangle\cos\phi - \phi'\sin\phi}{2\cos\phi} = \frac{-2\langle \gamma'', J\gamma'\rangle\sin\phi + \phi'\cos\phi}{2\sin\phi}$$

In particular we have $\phi' = -\phi'\tan\phi$, which implies that ϕ is constant. It follows that W is constant as well, so $\langle \gamma'', J\gamma'\rangle$ vanishes and the curve γ has vanishing curvature, so it is a great circle of the unit sphere. There is no loss of generality to assume that $\gamma(s) = \frac{1}{\sqrt{2}}(e^{is}, e^{-is})$. Now the immersion takes the following, explicit form:

$$X(s,t) = \frac{1}{\sqrt{2}}(e^{i(s+t)}, e^{i(-s+t)}) + \frac{1}{\sqrt{2}}(-ie^{i(\phi+s)}, ie^{i(\phi-s)}).$$

But in this case it is easy to check $\langle X, N_t\rangle = -1$ does not hold, so we conclude that there is no self-similar type II surface with a regular curve γ and non-vanishing W.

It remains to treat the case of a Cartesian product $\mathcal{C}\times\Gamma$ of a circle $\mathcal{C}$ with an arbitrary curve Γ. It is straightforward that such a product is self-similar if and only if both curves are solutions of the equation

$$k + \lambda\langle X, N\rangle = 0$$

for the same λ. A circle of radius r is trivially self-similar for $\lambda = r^2$. The other self-similar curves with positive λ (self-shrinking curves) have been described in [AL]. Except for the circles, none of them is embedded.

Case 3: type III surfaces.

A type III surface is parametrized by an immersion of the form:

$$\begin{array}{rccl} X: & I\times\mathbb{R}/2\pi\mathbb{Z} & \to & \mathbb{R}^4 \\ & (s,t) & \mapsto & c(\alpha_1(s)e^{it},\alpha_2(s)e^{-it})+\int_{s_0}^{s}(W|\alpha_2|^2,\overline{W}\alpha_1\alpha_2)du, \end{array}$$

where (α_1,α_2) is a Legendrian curve of $\mathbb{H}^3_1$, in particular we may assume that

$$|\alpha_1'|^2-|\alpha_2'|^2=1 \quad \langle\alpha_1',i\alpha\rangle-\langle\alpha_2',i\alpha_2\rangle=0.$$

Observe that this implies the following identity (cf [CC]): $|\alpha_1|=|\alpha_2'|$. Again, without loss of generality we fix $c=1$.

We start computing the first derivatives of the immersion:

$$X_s=(\alpha_1'e^{it},\alpha_2'e^{-it})+(W|\alpha_2|^2,\overline{W}\alpha_1\alpha_2), \quad X_t=(i\alpha_1e^{it},-i\alpha_2e^{-it}),$$

from which we deduce the expression of the induced metric:

$$E=1+2|\alpha_1|^2+|W|^2(1+|\alpha_1|^2)(1+2|\alpha_1|^2)+|\alpha_2|^2\langle\alpha_1'\bar{\alpha_1},We^{-it}\rangle+|\alpha_1|^2\langle\alpha_2'\bar{\alpha_2},\overline{W}e^{it}\rangle,$$

$$F=-2(1+|\alpha_1|^2)(\mathrm{Im}\,(\alpha_1\overline{W})\cos t+\mathrm{Re}\,(\alpha_1\overline{W})\sin t),$$

$$G=1+2|\alpha_1|^2,$$

and a basis of the normal space to the surface:

$$N_s=(i\alpha_1'e^{it},i\alpha_2'e^{-it})+(iW|\alpha_2|^2,i\overline{W}\alpha_1\alpha_2),$$

$$N_t=(-\alpha_1e^{it},\alpha_2e^{-it}).$$

We now compute the second derivatives of the immersion:

$$X_{ss}=(\alpha_1''e^{it},\alpha_2''e^{-it})+\frac{d}{ds}(W|\alpha_2|^2,\overline{W}\alpha_1\alpha_2),$$

$$X_{st}=(i\alpha_1'e^{it},-i\alpha_2'e^{-it}), \qquad X_{tt}=(-\alpha_1e^{it},-\alpha_2e^{-it});$$

and some coefficients of the second fundamental form:

$$\langle X_{tt},N_t\rangle=|\alpha_1|^2-|\alpha_2|^2=-1, \qquad \langle X_{st},N_t\rangle=2\langle i\alpha_1',\alpha_1\rangle;$$

Moreover it is easy to see that the coefficient $\langle X_{ss},N_t\rangle$ is an affine function of $\cos t$ and $\sin t$.

On the other hand, we have

$$\langle X, N_t\rangle = 1 + a\cos t + b\sin t,$$

where a and b depend only on the variable s.

We now assume that $\langle H, N_t\rangle + \lambda\langle X, N_t\rangle = 0$ holds for some non-vanishing constant λ; this implies

$$G\langle X_{ss}, N_t\rangle + E\langle X_{tt}, N_t\rangle - 2F\langle X_{st}, N_t\rangle = -\lambda(1 + a\cos t + b\sin t)(EG - F^2).$$

In the latter expression, the left hand side term is linear in $\cos t$ and $\sin t$ and the right hand side is a polynomial of degree 3. Linearizing the latter and introducing the notation $F = \tilde{a}\cos t + \tilde{b}\sin t$, we easily see that the coefficient of $\cos 3t$ is (up to a multiplicative constant) $a\tilde{a}^2 - 2b\tilde{a}\tilde{b}$ and the one of $\sin 3t$ is $2a\tilde{a}b - b\tilde{b}^2$. Thus we get the following system:

$$\left\{\begin{array}{rcl} a\tilde{a}^2 - 2b\tilde{a}\tilde{b} & = & 0 \\ 2a\tilde{a}b - b\tilde{b}^2 & = & 0 \end{array}\right.$$

This implies that either $\tilde{a}$ and $\tilde{b}$ vanish, or a and b vanish. However, in the latter case, the coefficient of $\cos 2t$ is (up to a multiplicative constant) $\tilde{a}^2 + \tilde{b}^2$ and the one of $\sin 2t$ is $2\tilde{a}\tilde{b}$ thus again $\tilde{a}$ and $\tilde{b}$ must vanish. We conclude by observing that

$$\tilde{b} + i\tilde{a} = -2(1 + |\alpha_1|^2)\alpha_1\overline{W},$$

so either α_1 or W vanishes. In the first case, as the discussion is local we may assume that α_1 vanishes identically, and the immersion takes the form

$$Y(s,t) = (V_1(s), \alpha_2 e^{-it}),$$

so the immersed surface is a product of curves; this case was already treated in the former section (type II surfaces).

To complete the proof it remains to show that a centered type III surface cannot be self-similar. We shall use the following result which can be found in [CC] (Proposition 2.1, page 3 and Corollary 3.5, page 9) and that we state here in accordance to our own notations:

Proposition 1 *([CC]) Let $\gamma = (\gamma_1, \gamma_2)$ be unit speed Legendrian curve in $\mathbb{S}^3$ and $\alpha = (\alpha_1, \alpha_2)$ a unit speed Legendrian curve in $\mathbb{H}^3_1$.*

Then the following immersion

$$\begin{array}{rccl} X: & I_1 \times I_2 & \to & \mathbb{C}^2 \\ & (s,t) & \mapsto & (\alpha_1(s)\gamma_1(t), \alpha_2(s)\gamma_2(t)), \end{array}$$

is conformal and Lagrangian; moreover, its mean curvature vector is given by

$$H = e^{-2u}(k_\alpha JX_s + k_\gamma JX_t),$$

where e^{-2u} is the conformal factor and k_α and k_γ are the curvature functions of α and γ respectively.

By taking $\gamma(t) = (\frac{1}{\sqrt{2}}e^{it}, \frac{1}{\sqrt{2}}e^{-it})$ in the immersion above, we recover exactly a centered type III immersion. In particular, γ is a great circle of $\mathbb{S}^3$ and has vanishing curvature. This implies that $\langle H, JX_t\rangle$ vanishes. On the other hand, we calculate

$$\langle X, JX_t\rangle = \frac{1}{2}\langle(\alpha_1 e^{it}, \alpha_2 e^{-it}), (-\alpha_1 e^{it}, \alpha_2 e^{-it})\rangle = \frac{1}{2}(-|\alpha_1|^2 + |\alpha_2|^2) = \frac{1}{2}.$$

So we deduce that this immersion X cannot be solution of the self-similar equation $H + \lambda X^\perp = 0$.

4 Ruled Lagrangian surfaces

Let Σ be a smooth, ruled, Lagrangian surface. If the rulings are parallel, it is straightforward that Σ is a Cartesian product $\mathcal{L} \times \Gamma$ of some straight line $\mathcal{L}$, and some planar curve Γ, such that $\mathcal{L} \subset P_1$ and $\Gamma \subset P_2$, where P_1 and P_2 are two orthogonal, complex planes. So from now on, our discussion being local, we shall assume that the rulings are not parallel.

Locally, Σ may be parametrized by the following immersion:

$$\begin{array}{rccl} X: & I \times \mathbb{R} & \to & \mathbb{C}^2 \\ & (s,t) & \mapsto & \gamma(s)t + V(s), \end{array}$$

where $\gamma(s)$ is a unit speed curve of $\mathbb{S}^3$ and $V(s) \in \mathbb{C}^2$, and we have

$$X_s = \gamma' t + V', \qquad X_t = \gamma.$$

We now claim that, without loss of generality, we may assume that X_s and X_t are orthogonal: to see this, we reparametrize the surface by $\hat{X}(s,t) = X(s,t) + \gamma(s)T(s)$, where $T(s)$ is some real function. We observe that the images of X and $\hat{X}$ are the same. Then we compute

$$\hat{X}_s = X_s + \gamma' T + \gamma T', \qquad \hat{X}_t = X_t = \gamma.$$

Thus by choosing T such that $T' = -\langle X_s, X_t \rangle$, we may reparametrize our surface such that $\hat{X}_s$ and $\hat{X}_t$ are orthogonal.

The Lagrangian assumption amounts to

$$\omega(X_s, X_t) = \langle X_s, JX_t \rangle = t\langle \gamma', J\gamma \rangle + \langle V', J\gamma \rangle = 0.$$

The vanishing of $\langle \gamma', J\gamma \rangle$ means that the curve γ is Legendrian. It follows that $(\gamma, J\gamma, \gamma', J\gamma')$ is an orthonormal basis of $\mathbb{R}^4 \simeq \mathbb{C}^2$. Thus the conditions $\langle V', J\gamma \rangle = 0$ and $\langle X_s, X_t \rangle = \langle V', \gamma \rangle = 0$ imply that there exists a planar curve $\alpha(s) = x(s) + iy(s)$ such that $V' = x\gamma' + yJ\gamma' = \alpha\gamma'$. So we have shown the first part of the following:

Theorem 4 *A smooth, ruled, Lagrangian surface of $\mathbb{C}^2$ is either the Cartesian product $\mathcal{L} \times \Gamma$ of a straight line $\mathcal{L}$ with a planar curve Γ or may be locally parametrized by an immersion of the form*

$$\begin{array}{rccl} X: & I \times \mathbb{R} & \to & \mathbb{C}^2 \\ & (s,t) & \mapsto & \gamma(s)t + \int_{s_0}^{s} \alpha(u)(\gamma_1'(u), \gamma_2'(u))du, \end{array}$$

where $\gamma = (\gamma_1, \gamma_2)$ is some Legendrian curve of $\mathbb{S}^3$ and $\alpha = (x, y)$ is some planar curve. Moreover, the only self-similar ruled Lagrangian surfaces of $\mathbb{C}^2$ are Cartesian products $\mathcal{L} \times \Gamma$, where $\mathcal{L}$ is a straight line and Γ is a planar self-similar curve (cf [AL]).

Proof. We have already characterized ruled Lagrangian surfaces, so it remains to study the self-similar equation.

We start computing the first derivatives of the immersion:

$$X_t(s,t) = \gamma(s), \qquad X_s(s,t) = (1 + \alpha(s))(\gamma_1'(s), \gamma_2'(s))$$

from which we deduce the expression of the induced metric:

$$E = (t + x)^2 + y^2, \qquad G = 1, \qquad F = 0,$$

and a basis of the normal bundle:

$$N_t = i\gamma(s), \qquad N_s = i(1+\alpha(s))(\gamma_1'(s), \gamma_2'(s)).$$

We now compute some second derivatives:

$$X_{st} = \gamma'(s), \qquad X_{tt} = 0.$$

We deduce that

$$2\langle H, N_t\rangle = \frac{\langle X_{ss}, N_t\rangle}{E} + \frac{\langle X_{tt}, N_t\rangle}{G} = \frac{\langle X_{st}, N_s\rangle}{E} = \frac{1+x}{(t+x)^2+y^2}.$$

On another hand, we have:

$$\langle X, N_t\rangle = \langle i\gamma(s), \int_{s_0}^{s} \alpha(u)(\gamma_1'(u), \gamma_2'(u))du\rangle,$$

which clearly does not depend on t. So the equation

$$\langle H, N_t\rangle + \lambda\langle X, N_t\rangle = 0$$

can never hold for such an immersion, so the only self-similar ruled Lagrangian surfaces are products $\mathcal{L} \times \Gamma$. It is then straightforward to see that the curve Γ must be a solution of the self-similar equation.

Remark 5 *We recover the Lagrangian helicoid of [B] by taking* $\gamma(s) = (k + il)(\cos s, \sin s)$, *where* k *and* l *are two constants such that* $k^2 + l^2 = 1$. *In the notation of [B], we have* $x(s) = G/2$ *and* $y(s) = -A$.

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Henri Anciaux
PUC-Rio, Mathematics Department
rua Marquês de São Vincente, 225
Gávea, Rio de Janeiro, Brazil
henri@mat.puc-rio.br

Pascal Romon
Université de Marne-la-Vallée
5, bd Descartes, Champs-sur-Marne
77454 Marne-la-Vallée cedex 2, France
pascal.romon@univ-mlv.fr