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► **To cite this version:**

Henri Anciaux, Pascal Romon. Cyclic and ruled Lagrangian surfaces in complex Euclidean space. 2007. hal-00137761v1

**HAL Id: hal-00137761**

**<https://hal.science/hal-00137761v1>**

Preprint submitted on 21 Mar 2007 (v1), last revised 15 Apr 2008 (v2)

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# Cyclic and ruled Lagrangian surfaces in complex Euclidean space

Henri Anceaux and Pascal Romon

## Abstract

We study those Lagrangian surfaces in complex Euclidean space which are foliated by circles or by straight lines. The former, which we call *cyclic*, come in three types, each one being described by means of, respectively, a planar curve, a Legendrian curve of the 3-sphere or a Legendrian curve of the anti de Sitter 3-space. We also describe ruled Lagrangian surfaces. Finally we characterize those cyclic and ruled Lagrangian surfaces which are solutions to the self-similar equation of the Mean Curvature Flow.

Keywords: Lagrangian surfaces; circle foliation; Mean Curvature Flow.

2000 MSC: 53D12 (Primary) 53C42 (Secondary)

## Introduction

In this paper, we classify the Lagrangian surfaces of  $\mathbb{C}^2$  which are foliated either by round circles (henceforth called *cyclic* surfaces) or by straight lines (*ruled* surfaces). This completes a former paper of the authors together with Ildefonso Castro [ACR] in which all Lagrangian submanifolds of  $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ , with  $n \geq 3$ , which are foliated by round  $(n - 1)$ -spheres were characterized. The reason for the lower bound on the dimension was the following: since the submanifold is Lagrangian, any spherical leaf must be isotropic; when the dimension of this leaf is at least two, it spans a linear space which is itself Lagrangian. This observation simplifies the structure of the problem, roughly speaking by reducing the underlying group structure from  $SO(2n)$  to  $U(n)$ .

However this reduction no longer holds in dimension two, see for instance the Lagrangian cylinder  $\{(e^{it}, s) \in \mathbb{C}^2, (s, t) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}\}$ : this (Lagrangian) surface is foliated by circles which are contained in non-Lagrangian (actually complex) planes. Other examples are the Hopf tori studied by Pinkall in [P]. As expected the situation is richer in dimension two, and actually cyclic Lagrangian surfaces come in three families which are described in Theorem 1 and 2. In Section 1 we classify the cyclic Lagrangian surfaces when all the centers of the circles coincide. We call those surfaces *centered* cyclic. In Section 2 we treat the general case which amounts to adding a convenient translation term. In Section 3 we apply this characterization to finding self-similar cyclic surfaces, that is those surfaces which are solutions of the following elliptic PDE:

$$H + \lambda X^\perp = 0,$$

where  $H$  denotes the mean curvature vector of the surface and  $X^\perp$  the normal component of its position vector. The case of positive (resp. negative)  $\lambda$  corresponds to the case of a self-shrinking (resp. self-expanding) soliton of the Mean Curvature Flow (see [A]). Finally we give in the last Section a description of ruled Lagrangian surfaces (notably self-similar ones) using an analogous method, recovering more simply a known result from Blair [B].

Acknowledgment: the Authors wish to thank Ildefonso Castro for pointing out a better approach to type III surfaces due to Chen and himself [CC].

## 1 Centered cyclic Lagrangian surfaces

Let  $\Sigma$  a surface of  $\mathbb{R}^4$  foliated by circles with common center located at the origin of  $\mathbb{R}^4$ . Locally,  $\Sigma$  may be parametrized by the following immersion:

$$\begin{aligned} X : I \times \mathbb{R}/2\pi\mathbb{Z} &\rightarrow \mathbb{R}^4 \\ (s, t) &\mapsto r(s)(e_1(s) \cos t + e_2(s) \sin t), \end{aligned}$$

where  $r(s)$  is a positive function and  $(e_1(s), e_2(s))$  is an orthonormal basis of the plane containing our circle.

From now on we shall assume that  $\Sigma$  is Lagrangian with respect with some complex structure  $J$ . We will often identify  $\mathbb{R}^4$  with  $\mathbb{C}^2$  in such a way that  $J$  is the complex multiplication by  $i$ . Denote by  $K := \langle e_1, J e_2 \rangle = -\langle e_1, J e_2 \rangle$  the *Kähler angle* of the plane  $e_1 \wedge e_2$ . Note that the vanishing of  $K$  means that  $e_1 \wedge e_2$  is also Lagrangian. In this case the analysis done in [ACR] holds

and  $X$  takes the following form:  $X(s, t) = r(s)e^{i\phi(s)}(\cos t, \sin t)$  making use of the above identification.

Denoting by subscripts the partial derivatives (the prime corresponding also to the derivative in  $s$  for functions of just one variable), the Lagrangian assumption is equivalent to  $\langle X_s, JX_t \rangle = 0$ . Since

$$X_s = r'(e_1 \cos t + e_2 \sin t) + r(e'_1 \cos t + e'_2 \sin t)$$

$$X_t = r(e_2 \cos t - e_1 \sin t),$$

we see that

$$\begin{aligned} \langle X_s, JX_t \rangle &= rr'(K \cos^2 t + K \sin^2 t) \\ &\quad + r^2 (\cos^2 t \langle e'_1, Je_2 \rangle - \sin^2 t \langle e'_2, Je_1 \rangle) \\ &\quad + r^2 \cos t \sin t (\langle e'_2, Je_2 \rangle - \langle e'_1, Je_1 \rangle) \\ &= rr'K + \frac{r^2}{2} (\langle e'_1, Je_2 \rangle - \langle e'_2, Je_1 \rangle) \\ &\quad + \frac{r^2 \cos 2t}{2} (\langle e'_1, Je_2 \rangle + \langle e'_2, Je_1 \rangle) \\ &\quad + \frac{r^2 \sin 2t}{2} (\langle e'_2, Je_2 \rangle - \langle e'_1, Je_1 \rangle) \\ &= \frac{2rr'K + r^2 K'}{2} + \frac{r^2 \cos 2t}{2} (\langle e'_1, Je_2 \rangle + \langle e'_2, Je_1 \rangle) \\ &\quad + \frac{r^2 \sin 2t}{2} (\langle e'_2, Je_2 \rangle - \langle e'_1, Je_1 \rangle) \end{aligned}$$

which holds for all  $t$ . The vanishing of the constant term implies that  $r^2 K = C$  for some real constant  $C$ . If this constant vanishes, we recover the case  $K = 0$  mentioned above and treated in [ACR] (cf also [A]), so we may assume that  $C \neq 0$ . Thus  $r$  is completely determined by  $K$ , and both are non zero.

The two remaining conditions are

$$\langle e'_1, Je_2 \rangle = -\langle e'_2, Je_1 \rangle, \quad \langle e'_1, Je_1 \rangle = \langle e'_2, Je_2 \rangle. \quad (1)$$

In order to make sense of these, we will identify  $\mathbb{R}^4$  with  $\mathbb{H}$ , in such a way that the complex structure is given by the left multiplication by the quaternion  $i$ . Then any element in  $SO(4)$  can be written as  $x \mapsto pxq^{-1}$  where  $p, q$  are two unit quaternions. Notice that right multiplication by  $q^{-1}$  corresponds exactly to the elements of  $SU(2)$ . Since  $SO(4)$  acts transitively on pairs of

orthonormal vectors, we may write  $e_1$  and  $e_2$  as the respective images of 1 and  $i$ , so that  $e_1 = pq^{-1}$  and  $e_2 = piq^{-1}$ . Note that  $(p, q)$  is not uniquely determined; rather we have a gauge freedom by right multiplication by  $e^{i\theta}$  on  $(p, q)$ . Finally, we may assume if needed that  $q(0)$  takes any prescribed value, since we consider surfaces up to  $U(2)$  congruence.

Then the conditions in (1) read as

$$\begin{cases} \langle p'q^{-1} - pq^{-1}q'q^{-1}, ipiq^{-1} \rangle + \langle p'iq^{-1} - piq^{-1}q'q^{-1}, ipq^{-1} \rangle = 0 \\ \langle p'q^{-1} - pq^{-1}q'q^{-1}, ipq^{-1} \rangle - \langle p'iq^{-1} - piq^{-1}q'q^{-1}, ipiq^{-1} \rangle = 0 \end{cases}$$

so, multiplying on the left by  $p^{-1}$  and the right by  $q$ , and further by  $i$  on the left in the second bracket,

$$\begin{cases} \langle p^{-1}p' - q^{-1}q', p^{-1}ipi \rangle - \langle p^{-1}p' + iq^{-1}q'i, p^{-1}ipi \rangle = 0 \\ \langle p^{-1}p' - q^{-1}q', p^{-1}ip \rangle - \langle p^{-1}p' + iq^{-1}q'i, p^{-1}ip \rangle = 0 \end{cases}$$

that is

$$\begin{cases} \langle q^{-1}q' + iq^{-1}q'i, p^{-1}ipi \rangle = 0 \\ \langle q^{-1}q' + iq^{-1}q'i, p^{-1}ip \rangle = 0. \end{cases}$$

Splitting and multiplying left and right by  $i$  in the second bracket yields

$$\begin{cases} \langle q^{-1}q', p^{-1}ipi \rangle - \langle q^{-1}q', ip^{-1}ip \rangle = 0 \\ \langle q^{-1}q', p^{-1}ip \rangle + \langle q^{-1}q', ip^{-1}ipi \rangle = 0 \end{cases}$$

$$\begin{cases} \langle q^{-1}q', p^{-1}ipi - ip^{-1}ip \rangle = 0 \\ \langle q^{-1}q', p^{-1}ip + ip^{-1}ipi \rangle = 0 \end{cases}$$

Writing  $p = p_0 + ip_1 + jp_2 + kp_3$  we have

$$u := p^{-1}ipi - ip^{-1}ip = 4(p_0p_2 + p_1p_3)j - 4(p_1p_2 - p_0p_3)k$$

and

$$v := p^{-1}ip + ip^{-1}ipi = -ui = 4(p_1p_2 - p_0p_3)j + 4(p_0p_2 + p_1p_3)k.$$

The two right-hand vectors  $u, v$  lie in  $\text{Span}(j, k)$  and are either linearly independent (over  $\mathbb{R}$ ) or both zero. So we have two cases:

- $u = v = 0$ , i.e.

$$0 = (p_0p_2 + p_1p_3)^2 + (p_1p_2 - p_0p_3)^2 = (p_0^2 + p_1^2)(p_2^2 + p_3^2)$$

hence  $p$  lies in  $\text{Span}(1, i)$  or  $\text{Span}(j, k)$  and conditions in (1) hold. Gauging  $p$  we may assume that  $p = 1$  or  $p = j$ , and the Kähler angle is then  $K = -1$  or  $K = +1$  respectively, so that the radius  $r$  remains constant (and we may as well assume  $r = 1$ ). This case corresponds to Hopf surfaces [P], i.e. inverse images of a curve by the Hopf fibration  $\mathbb{S}^3 \rightarrow \mathbb{S}^2$ . After a possible change of variable in  $t$  (replacing  $t$  by  $t + \varphi(s)$  for some function  $\varphi$ ), we may assume that the curve  $s \mapsto e_1(s)$  is Legendrian.

- $u, v$  are independent vectors and span  $j, k$ , thus forcing  $q^{-1}q'$  to lie in  $\text{Span}(1, i) \cap \text{Im}\mathbb{H}$ ; using gauge action, we may assume that  $q$  is constant, and up to congruence write  $q = 1$ . Reverting to complex coordinates,  $e_1 = p = \gamma_1 + \gamma_2 j \simeq (\gamma_1, \gamma_2) \in \mathbb{S}^3 \subset \mathbb{C}^2$  and  $e_2 = pi \simeq (i\gamma_1, -i\gamma_2)$ , while  $K = |\gamma_2|^2 - |\gamma_1|^2 \neq 0$ . We may normalize, assuming that  $K > 0$  (if  $K < 0$  pick the opposite orientation on the surface) and set  $(\alpha_1, \alpha_2) := \frac{1}{\sqrt{K}}(\gamma_1, \gamma_2)$ , so that

$$X = \frac{\sqrt{C}}{\sqrt{K}}(\gamma_1(s)e^{it}, \gamma_2(s)e^{-it}) = \sqrt{C}(\alpha_1 e^{it}, \alpha_2 e^{-it})$$

with  $|\alpha_1|^2 - |\alpha_2|^2 = -1$ , i.e.  $(\alpha_1, \alpha_2)$  lies in  $\mathbb{H}_1^3$ , the unit anti-De Sitter space. Again, up to a change in variable, we have a Legendrian curve for the indefinite metric in  $\mathbb{C}^{1,1}$ , i.e.  $\langle \alpha'_1, i\alpha_1 \rangle - \langle \alpha'_2, i\alpha_2 \rangle = 0$  (see [CLU] or [CC]).

Summing up, we have proved the following

**Theorem 1** *A centered cyclic Lagrangian surface may be locally parameterized, up to  $U(2)$  congruence, by one the following immersions:*

**Type I** (complex extensors):

$$\begin{aligned} X : I \times \mathbb{R}/2\pi\mathbb{Z} &\rightarrow \mathbb{C}^2 \\ (s, t) &\mapsto r(s)e^{i\phi(s)}(\cos t, \sin t) \end{aligned}$$

**Type II** (Hopf type):

$$\begin{aligned} X : I \times \mathbb{R}/2\pi\mathbb{Z} &\rightarrow \mathbb{C}^2 \\ (s, t) &\mapsto ce^{it}(\gamma_1(s), \gamma_2(s)) \end{aligned}$$

where  $\gamma = (\gamma_1, \gamma_2)$  is any Legendrian curve of  $\mathbb{S}^3$  and  $c$  is a real constant,

**Type III** (*De Sitter type*)

$$\begin{aligned} X : I \times \mathbb{R}/2\pi\mathbb{Z} &\rightarrow \mathbb{C}^2 \\ (s, t) &\mapsto c(\alpha_1(s)e^{it}, \alpha_2(s)e^{-it}) \end{aligned}$$

where  $\alpha = (\alpha_1, \alpha_2)$  is any Legendrian curve in the unit anti-De Sitter space  $\mathbb{H}_1^3$  and  $c$  is a real constant.

**Remark 1** *This analysis applies as well if we do not assume that  $X$  is an immersion but only has an isotropic image fibered by circles (we did not use the immersion hypothesis). So the same conclusion holds and will be used in the next Section.*

**Remark 2** *Type I surfaces are a particular case of a class of Lagrangian immersions which has been first described in [C1] where they were called complex extensors.*

**Remark 3** *In the type III case, it may happen that  $K$  is identically  $\pm 1$ . Then we fall back on type II with a Legendrian curve that actually reduces to a single point. The image of  $X$  is therefore a circle (lying in a complex plane).*

## 2 The general case

We now consider a surface  $\Sigma$  of  $\mathbb{R}^4$  which is foliated by circles. Locally,  $\Sigma$  may be parametrized by the following immersion (identifying as usual  $\mathbb{R}^4$  with  $\mathbb{C}^2$ ):

$$\begin{aligned} Y : I \times \mathbb{R}/2\pi\mathbb{Z} &\rightarrow \mathbb{C}^2 \\ (s, t) &\mapsto X(s, t) + V(s), \end{aligned}$$

where  $V(s)$  is a  $\mathbb{C}^2$ -valued function, and  $X(s, t) = r(s)(e_1(s) \cos t + e_2(s) \sin t)$  is a centered surface as in the previous section. Note that we do not assume *a priori* that  $X$  is Lagrangian nor that it is always an immersion.

As we have  $Y_t = X_t$  and  $Y_s = X_s + V'$ , the assumption that  $Y$  is Lagrangian leads to:

$$\begin{aligned} 0 &= \langle Y_s, JY_t \rangle = \langle X_s, JX_t \rangle + \langle V', JX_t \rangle \\ &= \langle X_s, JX_t \rangle + r \cos t \langle V', Je_2 \rangle - r \sin t \langle V', Je_1 \rangle \end{aligned}$$

Recall from the previous section that  $\langle X_s, JX_t \rangle$  contains only terms in  $\cos 2t$  and  $\sin 2t$  and a term independent from  $t$ . Thus the immersion  $Y$  is Lagrangian if and only if: (i)  $X$  is cyclic isotropic (cf Remark 1), and (ii)  $V'$  belongs to the symplectic orthogonal of  $\text{Span}(e_1, e_2)$ . Using Theorem 1 we infer:

- for type I surfaces,  $\text{Span}(e_1, e_2) = \text{Span}(e^{i\phi}(1, 0), e^{i\phi}(0, 1))$  is Lagrangian, so its symplectic orthogonal is itself; hence  $V'(s) = e^{i\phi(s)}(W_1(s), W_2(s))$  for some real-valued functions  $W_1, W_2$ ;
- for type II surfaces,  $\text{Span}(e_1, e_2)$  is a complex line, so its symplectic orthogonal is the same as its Riemannian orthogonal, which is  $\text{Span}((\bar{\gamma}_2, -\bar{\gamma}_1), (i\bar{\gamma}_2, -i\bar{\gamma}_1))$ ;  $V'$  is determined analogously;
- for type III surfaces, one can check that a basis of the symplectic orthogonal is  $(f_1, f_2)$ , where  $f_1 = (|\alpha_2|^2, \alpha_1\alpha_2)$ ,  $f_2 = (i|\alpha_2|^2, -i\alpha_1\alpha_2)$ .

So we conclude this section by the

**Theorem 2** *A cyclic Lagrangian surface may be locally parameterized, up to  $U(2)$  congruence, by one the following immersions:*

### Type I

$$Y : I \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{C}^2$$

$$(s, t) \mapsto r(s)e^{i\phi(s)}(\cos t, \sin t) + \int_{s_0}^s e^{i\phi(u)}(W_1(u), W_2(u))du$$

where  $W_1, W_2$  are real valued; in particular when  $\phi$  is constant,  $Y$  stays within the Lagrangian plane  $\text{Span}(e^{i\phi}(1, 0), e^{i\phi}(0, 1))$ ;

### Type II

$$Y : I \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{C}^2$$

$$(s, t) \mapsto c(\gamma_1(s)e^{it}, \gamma_2(s)e^{it}) + \int_{s_0}^s W(u)(\bar{\gamma}_2(u), -\bar{\gamma}_1(u))du$$

where  $\gamma = (\gamma_1, \gamma_2)$  is any Legendrian curve of  $\mathbb{S}^3$ ,  $c$  is a real constant and  $W$  a complex valued function; in particular if  $\gamma$  is constant, then up to congruence, we may assume  $\gamma = (1, 0)$  and the immersion becomes  $Y(s, t) = (ce^{it}, V_2(s))$ . Thus the immersed surface is a Cartesian product of a circle with a planar curve;



### Type III

$$Y : I \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{C}^2$$

$$(s, t) \mapsto c(\alpha_1(s)e^{it}, \alpha_2(s)e^{-it}) + \int_{s_0}^s (W|\alpha_2|^2, \overline{W}\alpha_1\alpha_2)du$$

where  $\alpha = (\alpha_1, \alpha_2)$  is any Legendrian curve in the unit anti-De Sitter space  $\mathbb{H}_1^3$ ,  $c$  is a real constant and  $W$  a complex valued function.

**Remark 4** In the type II case, if the curve  $(\gamma_1, \gamma_2)$  is in addition regular, another basis of the orthogonal space to  $\text{Span}(e_1, e_2)$  is  $((\gamma'_1, \gamma'_2), (i\gamma'_1, i\gamma'_2))$ , so that the immersion may take the alternative form:

$$Y(s, t) = (\gamma_1(s)e^{it}, \gamma_2(s)e^{it}) + \int_{s_0}^s W(u)(\gamma'_1(u), \gamma'_2(u))du.$$

This is a particular case of Lagrangian immersions which have been recently described in [C2]. This alternative formula will also be useful in the next section.

## 3 Application to the self-similar equation

In this section we study the self-similar equation in the case of cyclic Lagrangian surfaces and prove the following:

**Theorem 3** *A Lagrangian cyclic surface of  $\mathbb{C}^2$  which is a soliton of the mean curvature flow, i.e. a solution to the self-similar equation*

$$H + \lambda X^\perp = 0,$$

for some non-vanishing number  $\lambda$  is locally congruent to an equivariant example described in [A] (in the terminology of the present article, a centered surface of type I) or to the Cartesian product  $\mathcal{C} \times \Gamma$  of a circle  $\mathcal{C}$  with a planar self-shrinking curve  $\Gamma$ . Such curves have been studied in detail in [AL].

*Proof.* The proof deals with the three cyclic cases separately: we first prove that a self-similar surface of type I must be centered, thus one of the examples of [A]; then we show that there no self-similar surfaces of type II except the Clifford torus  $\mathbb{S}^1 \times \mathbb{S}^1$  (which is also a type I surface) and the products of

curves. Finally we see that there are no self-similar surfaces of type III at all.

*Case 1: type I surfaces.*

A type I surface is parametrized by an immersion of the form:

$$\begin{aligned} X : I \times \mathbb{R}/2\pi\mathbb{Z} &\rightarrow \mathbb{C}^2 \\ (s, t) &\mapsto r(s)e^{i\phi(s)}(\cos t, \sin t) + \int_{s_0}^s e^{i\phi(u)}(W_1(u), W_2(u))du, \end{aligned}$$

where  $r(s) > 0$ . We shall use the following notations:  $\gamma(s) = r(s)e^{i\phi(s)}$ , and, as we assume that  $\gamma$  is parametrized by arclength, we also denote  $\gamma'(s) = e^{i\theta(s)}$ .

We start computing the first derivatives of the immersion:

$$X_s = \gamma'(\cos t, \sin t) + e^{i\phi}(W_1, W_2), \quad X_t = \gamma(-\sin t, \cos t),$$

from which we deduce the expression of the induced metric:

$$E = |X_s|^2 = 1 + |W|^2 + 2 \cos(\theta - \phi)(W_1 \cos t + W_2 \sin t),$$

$$F = \langle X_s, X_t \rangle = r \cos(\theta - \phi)(W_2 \cos t - W_1 \sin t), \quad G = |X_t|^2 = r^2,$$

and a basis of the normal space to the surface:

$$N_t = i\gamma(-\sin t, \cos t), \quad N_s = i\gamma'(\cos t, \sin t) + ie^{i\phi}(W_1, W_2).$$

We now compute the second derivatives of the immersion, in order to calculate the mean curvature vector:

$$X_{ss} = \gamma''(\cos t, \sin t) + i\phi' e^{i\phi}(W_1, W_2),$$

$$X_{st} = \gamma'(-\sin t, \cos t), \quad X_{tt} = \gamma(-\cos t, -\sin t).$$

This implies in particular that:

$$\langle X_{ss}, N_t \rangle = (W_2 \cos t - W_1 \sin t) \sin(\theta - \phi),$$

$$\langle X_{tt}, N_t \rangle = 0, \quad \langle X_{st}, N_t \rangle = r \sin(\theta - \phi).$$

On the other hand, we have:

$$\langle X, N_t \rangle = a \cos t + b \sin t,$$

where  $a := \langle i\gamma, \int_{s_0}^s W_1(u)e^{i\phi(u)}du \rangle$  and  $b := \langle i\gamma, \int_{s_0}^s W_2(u)e^{i\phi(u)}du \rangle$ .

We now assume that the immersion  $X$  is self-similar, so there exists a non-vanishing real number  $\lambda$  such that:

$$\langle H, N_t \rangle + \lambda \langle X, N_t \rangle = 0,$$

which is equivalent to

$$\langle X_{ss}, N_t \rangle G + \langle X_{tt}, N_t \rangle E - 2\langle X_{st}, N_t \rangle F = -2\lambda(EG - F^2)\langle X, N_t \rangle.$$

In the latter expression, the left hand side term is linear in  $\cos t$  and  $\sin t$  and the right hand side term is a polynomial of order 3. Linearizing the latter, we easily see that the coefficient of  $\cos 2t$  is  $aW_2 + bW_1$  and the one of  $\sin 2t$  is  $aW_1 - bW_2$ . So either  $W_1$  and  $W_2$  vanish, or  $a$  and  $b$  vanish. We are going to show that actually if  $a$  and  $b$  vanish, then so do  $W_1$  and  $W_2$ .

We first write

$$\begin{aligned} a &= \langle i\gamma, \int_{s_0}^s W_1(u)e^{i\phi(u)}du \rangle = r \langle ie^{i\phi}, \int_{s_0}^s W_1(u)e^{i\phi(u)}du \rangle \\ &= r \left( -\sin \phi \left( \int_{s_0}^s W_1(u) \cos \phi(u)du \right) + \cos \phi \left( \int_{s_0}^s W_1(u) \sin \phi(u)du \right) \right) = 0. \end{aligned}$$

Thus the derivative of  $a/r$  with respect to  $s$  must vanish, which yields:

$$\begin{aligned} \phi' \left( -\cos \phi \left( \int_{s_0}^s W_1(u) \cos \phi(u)du \right) - \sin \phi \left( \int_{s_0}^s W_1(u) \sin \phi(u)du \right) \right) \\ + W_1(s)(-\sin \phi \cos \phi + \cos \phi \sin \phi) = 0. \end{aligned}$$

Now either  $\phi$  is (locally) constant, the curve  $\gamma$  is a straight line passing through the origin and the image of the immersion  $X$  is a piece of a plane (cf Example 2, page 5 of [ACR]), or one can find points around which  $\phi'$  does not vanish (locally again). In the latter case we get the following linear system:

$$\begin{cases} -\sin \phi \left( \int_{s_0}^s W_1(u) \cos \phi(u)du \right) + \cos \phi \left( \int_{s_0}^s W_1(u) \sin \phi(u)du \right) = 0 \\ -\cos \phi \left( \int_{s_0}^s W_1(u) \cos \phi(u)du \right) - \sin \phi \left( \int_{s_0}^s W_1(u) \sin \phi(u)du \right) = 0 \end{cases}$$

It follows that  $\int_{s_0}^s W_1(u) \cos \phi(u)du$  and  $\int_{s_0}^s W_1(u) \sin \phi(u)du$  must vanish except on an isolated set of points of  $I$ , which in turn implies the vanishing of

$W_1$ . Analogously it can be shown that  $W_2$  vanishes as well, so finally the surface must be centered. Lagrangian, self-similar, centered surfaces have been described in detail in [A]. We only recall here that such surfaces are obtained from planar curves  $\gamma$  which are solutions of the following equation:

$$k = \langle \gamma, N \rangle \left( \frac{1}{|\gamma|^2} - \lambda \right),$$

where  $k$  is the curvature of  $\gamma$  and  $N$  its unit normal vector. This equation admits a countable family of closed solutions which is parametrised by two relatively prime numbers  $p$  and  $q$  subject to the condition  $p/q \in (1/4, 1/2)$ ;  $p$  is the winding number of the curve and  $q$  is the number of maxima of its curvature. Except for the circles, none of these curves is embedded.

*Case 2: type II surfaces.*

As our discussion is local, we consider the two following cases: either the Legendrian curve  $\gamma$  is regular, or it reduces to a single point, and then the surface is a product of a circle with some plane curve. In the first case, the surface may be parametrized by an immersion of the form (cf Remark 4):

$$\begin{aligned} X : I \times \mathbb{R}/2\pi\mathbb{Z} &\rightarrow \mathbb{C}^2 \\ (s, t) &\mapsto c(\gamma_1(s)e^{it}, \gamma_2(s)e^{it}) + \int_{s_0}^s W(u)(\gamma'_1, \gamma'_2)du \end{aligned}$$

where  $(\gamma_1(s), \gamma_2(s))$  is some unit speed Legendrian curve of  $\mathbb{S}^3$ . Without loss of generality we fix  $c = 1$ . We start computing the first derivatives of the immersion:

$$X_s = (\gamma'_1 e^{it}, \gamma'_2 e^{it}) + W(\gamma'_1, \gamma'_2), \quad X_t = (i\gamma_1 e^{it}, i\gamma_2 e^{it}),$$

from which we deduce the expression of the induced metric:

$$E = |X_s|^2 = 1 + |W|^2 + 2\langle W, e^{it} \rangle, \quad F = \langle X_s, X_t \rangle = 0, \quad G = |X_t|^2 = 1,$$

and a basis of the normal space to the surface:

$$N_s = (i\gamma'_1 e^{it}, i\gamma'_2 e^{it}) + W(i\gamma'_1, i\gamma'_2), \quad N_t = (-\gamma_1 e^{it}, -\gamma_2 e^{it}).$$

We now compute the second derivatives of the immersion, in order to calculate the mean curvature vector:

$$\begin{aligned}
X_{ss} &= (\gamma_1'' e^{it}, \gamma_2'' e^{it}) + W(\gamma_1'', \gamma_2'') + W'(\gamma_1', \gamma_2'), \\
X_{st} &= (i\gamma_1' e^{it}, i\gamma_2' e^{it}), \quad X_{tt} = (-\gamma_1 e^{it}, -\gamma_2 e^{it}).
\end{aligned}$$

We first notice that

$$\langle X, N_t \rangle = -1 + a \cos t + b \sin t,$$

where  $a$  and  $b$  depend only on the variable  $s$ .

On the other hand,

$$\begin{aligned}
2\langle H, N_t \rangle &= \frac{\langle X_{ss}, N_t \rangle}{E} + \frac{\langle X_{tt}, N_t \rangle}{G} = \frac{\langle X_{st}, N_s \rangle}{E} + \frac{\langle X_{tt}, N_t \rangle}{G} \\
&= \frac{1 + \langle W, e^{it} \rangle}{1 + |W|^2 + 2\langle W, e^{it} \rangle} + 1 = \frac{2 + |W|^2 + 3\langle W, e^{it} \rangle}{1 + |W|^2 + 2\langle W, e^{it} \rangle}.
\end{aligned}$$

Thus the equation  $\langle H, N_t \rangle + \lambda \langle X, N_t \rangle = 0$  holds for some non-vanishing constant  $\lambda$  if and only if  $a$  and  $b$  vanish and  $|W| = 1$  or  $W = 0$ . In particular,  $\langle X, N_t \rangle = -1$ .

We leave to the reader the easy task to check that if  $W$  vanishes, the immersion is self-similar if and only if  $\langle \gamma'', J\gamma' \rangle$  vanishes, that is the curve  $\gamma$  has vanishing curvature and thus is a great circle. The corresponding Lagrangian surface is the Clifford torus  $\frac{1}{\sqrt{2}}\mathbb{S}^1 \times \frac{1}{\sqrt{2}}\mathbb{S}^1$ . So we assume in the remainder that  $|W| = 1$ . In particular, we may write  $W = e^{i\phi}$  and  $W' = i\phi' e^{i\phi}$ , where  $\phi$  is some real function of the variable  $s$ .

We want to look at the other scalar equation  $\langle H, N_s \rangle + \lambda \langle X, N_s \rangle = 0$ , so we compute

$$\begin{aligned}
\langle X_{ss}, N_s \rangle &= \langle \gamma'', J\gamma' \rangle (1 + |W|^2) + \langle iW, W' \rangle + 2\langle W, e^{-it} \rangle \langle \gamma'', J\gamma' \rangle + \langle W', e^{it} \rangle, \\
\langle X_{tt}, N_s \rangle &= 0.
\end{aligned}$$

Hence we get

$$\begin{aligned}
2\langle H, N_s \rangle &= \frac{\langle X_{ss}, N_s \rangle}{E} + \frac{\langle X_{tt}, N_s \rangle}{G} = \frac{\langle X_{ss}, N_s \rangle}{1 + |W|^2 + 2\langle W, e^{it} \rangle} + 0 \\
&= \frac{\langle \gamma'', J\gamma' \rangle(1 + |W|^2) + \langle iW, W' \rangle + 2\langle W, e^{-it} \rangle \langle \gamma'', J\gamma' \rangle + \langle W', e^{it} \rangle}{1 + |W|^2 + 2\langle W, e^{it} \rangle} \\
&= \frac{2\langle \gamma'', J\gamma' \rangle + \phi' + 2\cos(\phi + t)\langle \gamma'', J\gamma' \rangle - \phi' \sin(\phi - t)}{2 + 2\cos(\phi - t)} \\
&= \frac{2\langle \gamma'', J\gamma' \rangle + \phi' + (2\langle \gamma'', J\gamma' \rangle \cos \phi - \phi' \sin \phi) \cos t}{2(1 + \cos \phi \cos t + \sin \phi \sin t)} \\
&\quad + \frac{(-2\langle \gamma'', J\gamma' \rangle \sin \phi + \phi' \cos \phi) \sin t}{2(1 + \cos \phi \cos t + \sin \phi \sin t)}
\end{aligned}$$

On the other hand, it is easy to see that  $\langle X, N_s \rangle$  takes the form  $a \cos t + b \sin t + c$ , where  $a$ ,  $b$  and  $c$  depend on the variable  $s$ . This forces  $\langle H, N_s \rangle$  to take a simpler form and implies that

$$\frac{2\langle \gamma'', J\gamma' \rangle + \phi'}{2} = \frac{2\langle \gamma'', J\gamma' \rangle \cos \phi - \phi' \sin \phi}{2 \cos \phi} = \frac{-2\langle \gamma'', J\gamma' \rangle \sin \phi + \phi' \cos \phi}{2 \sin \phi}$$

In particular we have  $\phi' = -\phi' \tan \phi$ , which implies that  $\phi$  is constant. It follows that  $W$  is constant as well, so  $\langle \gamma'', J\gamma' \rangle$  vanishes and the curve  $\gamma$  has vanishing curvature, so it is a great circle of the unit sphere. There is no loss of generality to assume that  $\gamma(s) = \frac{1}{\sqrt{2}}(e^{is}, e^{-is})$ . Now the immersion takes the following, explicit form:

$$X(s, t) = \frac{1}{\sqrt{2}}(e^{i(s+t)}, e^{i(-s+t)}) + \frac{1}{\sqrt{2}}(-ie^{i(\phi+s)}, ie^{i(\phi-s)}).$$

But in this case it is easy to check  $\langle X, N_t \rangle = -1$  does not hold, so we conclude that there is no self-similar type II surface with a regular curve  $\gamma$  and non-vanishing  $W$ .

It remains to treat the case of a Cartesian product  $\mathcal{C} \times \Gamma$  of a circle  $\mathcal{C}$  with an arbitrary curve  $\Gamma$ . It is straightforward that such a product is self-similar if and only if both curves are solutions of the equation

$$k + \lambda \langle X, N \rangle = 0$$

for the same  $\lambda$ . A circle of radius  $r$  is trivially self-similar for  $\lambda = r^2$ . The other self-similar curves with positive  $\lambda$  (self-shrinking curves) have been described in [AL]. Except for the circles, none of them is embedded.

*Case 3: type III surfaces.*

A type III surface is parametrized by an immersion of the form:

$$\begin{aligned} X : I \times \mathbb{R}/2\pi\mathbb{Z} &\rightarrow \mathbb{R}^4 \\ (s, t) &\mapsto c(\alpha_1(s)e^{it}, \alpha_2(s)e^{-it}) + \int_{s_0}^s (W|\alpha_2|^2, \overline{W}\alpha_1\alpha_2)du, \end{aligned}$$

where  $(\alpha_1, \alpha_2)$  is a Legendrian curve of  $\mathbb{H}_1^3$ , in particular we may assume that

$$|\alpha_1'|^2 - |\alpha_2'|^2 = 1 \quad \langle \alpha_1', i\alpha \rangle - \langle \alpha_2', i\alpha_2 \rangle = 0.$$

Observe that this implies the following identity (cf [CC]):  $|\alpha_1| = |\alpha_2|$ . Again, without loss of generality we fix  $c = 1$ .

We start computing the first derivatives of the immersion:

$$X_s = (\alpha_1' e^{it}, \alpha_2' e^{-it}) + (W|\alpha_2|^2, \overline{W}\alpha_1\alpha_2), \quad X_t = (i\alpha_1 e^{it}, -i\alpha_2 e^{-it}),$$

from which we deduce the expression of the induced metric:

$$\begin{aligned} E &= 1 + 2|\alpha_1|^2 + |W|^2(1 + |\alpha_1|^2)(1 + 2|\alpha_1|^2) + |\alpha_2|^2 \langle \alpha_1' \bar{\alpha}_1, W e^{-it} \rangle + |\alpha_1|^2 \langle \alpha_2' \bar{\alpha}_2, \overline{W} e^{it} \rangle, \\ F &= -2(1 + |\alpha_1|^2)(\text{Im}(\alpha_1 \overline{W}) \cos t + \text{Re}(\alpha_1 \overline{W}) \sin t), \\ G &= 1 + 2|\alpha_1|^2, \end{aligned}$$

and a basis of the normal space to the surface:

$$\begin{aligned} N_s &= (i\alpha_1' e^{it}, i\alpha_2' e^{-it}) + (iW|\alpha_2|^2, i\overline{W}\alpha_1\alpha_2), \\ N_t &= (-\alpha_1 e^{it}, \alpha_2 e^{-it}). \end{aligned}$$

We now compute the second derivatives of the immersion:

$$X_{ss} = (\alpha_1'' e^{it}, \alpha_2'' e^{-it}) + \frac{d}{ds}(W|\alpha_2|^2, \overline{W}\alpha_1\alpha_2),$$

$$X_{st} = (i\alpha_1' e^{it}, -i\alpha_2' e^{-it}), \quad X_{tt} = (-\alpha_1 e^{it}, -\alpha_2 e^{-it});$$

and some coefficients of the second fundamental form:

$$\langle X_{tt}, N_t \rangle = |\alpha_1|^2 - |\alpha_2|^2 = -1, \quad \langle X_{st}, N_t \rangle = 2\langle i\alpha_1', \alpha_1 \rangle;$$

Moreover it is easy to see that the coefficient  $\langle X_{ss}, N_t \rangle$  is an affine function of  $\cos t$  and  $\sin t$ .

On the other hand, we have

$$\langle X, N_t \rangle = 1 + a \cos t + b \sin t,$$

where  $a$  and  $b$  depend only on the variable  $s$ .

We now assume that  $\langle H, N_t \rangle + \lambda \langle X, N_t \rangle = 0$  holds for some non-vanishing constant  $\lambda$ ; this implies

$$G \langle X_{ss}, N_t \rangle + E \langle X_{tt}, N_t \rangle - 2F \langle X_{st}, N_t \rangle = -\lambda(1 + a \cos t + b \sin t)(EG - F^2).$$

In the latter expression, the left hand side term is linear in  $\cos t$  and  $\sin t$  and the right hand side is a polynomial of degree 3. Linearizing the latter and introducing the notation  $F = \tilde{a} \cos t + \tilde{b} \sin t$ , we easily see that the coefficient of  $\cos 3t$  is (up to a multiplicative constant)  $a\tilde{a}^2 - 2b\tilde{a}\tilde{b}$  and the one of  $\sin 3t$  is  $2a\tilde{a}\tilde{b} - b\tilde{b}^2$ . Thus we get the following system:

$$\begin{cases} a\tilde{a}^2 - 2b\tilde{a}\tilde{b} = 0 \\ 2a\tilde{a}\tilde{b} - b\tilde{b}^2 = 0 \end{cases}$$

This implies that either  $\tilde{a}$  and  $\tilde{b}$  vanish, or  $a$  and  $b$  vanish. However, in the latter case, the coefficient of  $\cos 2t$  is (up to a multiplicative constant)  $\tilde{a}^2 + \tilde{b}^2$  and the one of  $\sin 2t$  is  $2\tilde{a}\tilde{b}$  thus again  $\tilde{a}$  and  $\tilde{b}$  must vanish. We conclude by observing that

$$\tilde{b} + i\tilde{a} = -2(1 + |\alpha_1|^2)\alpha_1 \overline{W},$$

so either  $\alpha_1$  or  $W$  vanishes. In the first case, as the discussion is local we may assume that  $\alpha_1$  vanishes identically, and the immersion takes the form

$$Y(s, t) = (V_1(s), \alpha_2 e^{-it}),$$

so the immersed surface is a product of curves; this case was already treated in the former section (type II surfaces).

To complete the proof it remains to show that a centered type III surface cannot be self-similar. We shall use the following result which can be found in [CC] (Proposition 2.1, page 3 and Corollary 3.5, page 9) and that we state here in accordance to our own notations:

**Proposition 1** ([CC]) *Let  $\gamma = (\gamma_1, \gamma_2)$  be unit speed Legendrian curve in  $\mathbb{S}^3$  and  $\alpha = (\alpha_1, \alpha_2)$  a unit speed Legendrian curve in  $\mathbb{H}_1^3$ .*



Then the following immersion

$$\begin{aligned} X : I_1 \times I_2 &\rightarrow \mathbb{C}^2 \\ (s, t) &\mapsto (\alpha_1(s)\gamma_1(t), \alpha_2(s)\gamma_2(t)), \end{aligned}$$

is conformal and Lagrangian; moreover, its mean curvature vector is given by

$$H = e^{-2u}(k_\alpha JX_s + k_\gamma JX_t),$$

where  $e^{-2u}$  is the conformal factor and  $k_\alpha$  and  $k_\gamma$  are the curvature functions of  $\alpha$  and  $\gamma$  respectively.

By taking  $\gamma(t) = (\frac{1}{\sqrt{2}}e^{it}, \frac{1}{\sqrt{2}}e^{-it})$  in the immersion above, we recover exactly a centered type III immersion. In particular,  $\gamma$  is a great circle of  $\mathbb{S}^3$  and has vanishing curvature. This implies that  $\langle H, JX_t \rangle$  vanishes. On the other hand, we calculate

$$\langle X, JX_t \rangle = \frac{1}{2} \langle (\alpha_1 e^{it}, \alpha_2 e^{-it}), (-\alpha_1 e^{it}, \alpha_2 e^{-it}) \rangle = \frac{1}{2} (-|\alpha_1|^2 + |\alpha_2|^2) = \frac{1}{2}.$$

So we deduce that this immersion  $X$  cannot be solution of the self-similar equation  $H + \lambda X^\perp = 0$ .

## 4 Ruled Lagrangian surfaces

Let  $\Sigma$  be a smooth, ruled, Lagrangian surface. If the rulings are parallel, it is straightforward that  $\Sigma$  is a Cartesian product  $\mathcal{L} \times \Gamma$  of some straight line  $\mathcal{L}$ , and some planar curve  $\Gamma$ , such that  $\mathcal{L} \subset P_1$  and  $\Gamma \subset P_2$ , where  $P_1$  and  $P_2$  are two orthogonal, complex planes. So from now on, our discussion being local, we shall assume that the rulings are not parallel.

Locally,  $\Sigma$  may be parametrized by the following immersion:

$$\begin{aligned} X : I \times \mathbb{R} &\rightarrow \mathbb{C}^2 \\ (s, t) &\mapsto \gamma(s)t + V(s), \end{aligned}$$

where  $\gamma(s)$  is a unit speed curve of  $\mathbb{S}^3$  and  $V(s) \in \mathbb{C}^2$ , and we have

$$X_s = \gamma' t + V', \quad X_t = \gamma.$$

We now claim that, without loss of generality, we may assume that  $X_s$  and  $X_t$  are orthogonal: to see this, we reparametrize the surface by  $\hat{X}(s, t) = X(s, t) + \gamma(s)T(s)$ , where  $T(s)$  is some real function. We observe that the images of  $X$  and  $\hat{X}$  are the same. Then we compute

$$\hat{X}_s = X_s + \gamma'T + \gamma T', \quad \hat{X}_t = X_t = \gamma.$$

Thus by choosing  $T$  such that  $T' = -\langle X_s, X_t \rangle$ , we may reparametrize our surface such that  $\hat{X}_s$  and  $\hat{X}_t$  are orthogonal.

The Lagrangian assumption amounts to

$$\omega(X_s, X_t) = \langle X_s, JX_t \rangle = t\langle \gamma', J\gamma \rangle + \langle V', J\gamma \rangle = 0.$$

The vanishing of  $\langle \gamma', J\gamma \rangle$  means that the curve  $\gamma$  is Legendrian. It follows that  $(\gamma, J\gamma, \gamma', J\gamma')$  is an orthonormal basis of  $\mathbb{R}^4 \simeq \mathbb{C}^2$ . Thus the conditions  $\langle V', J\gamma \rangle = 0$  and  $\langle X_s, X_t \rangle = \langle V', \gamma \rangle = 0$  imply that there exists a planar curve  $\alpha(s) = x(s) + iy(s)$  such that  $V' = x\gamma' + yJ\gamma' = \alpha\gamma'$ . So we have shown the first part of the following:

**Theorem 4** *A smooth, ruled, Lagrangian surface of  $\mathbb{C}^2$  is either the Cartesian product  $\mathcal{L} \times \Gamma$  of a straight line  $\mathcal{L}$  with a planar curve  $\Gamma$  or may be locally parametrized by an immersion of the form*

$$\begin{aligned} X : I \times \mathbb{R} &\rightarrow \mathbb{C}^2 \\ (s, t) &\mapsto \gamma(s)t + \int_{s_0}^s \alpha(u)(\gamma'_1(u), \gamma'_2(u))du, \end{aligned}$$

where  $\gamma = (\gamma_1, \gamma_2)$  is some Legendrian curve of  $\mathbb{S}^3$  and  $\alpha = (x, y)$  is some planar curve. Moreover, the only self-similar ruled Lagrangian surfaces of  $\mathbb{C}^2$  are Cartesian products  $\mathcal{L} \times \Gamma$ , where  $\mathcal{L}$  is a straight line and  $\Gamma$  is a planar self-similar curve (cf [AL]).

*Proof.* We have already characterized ruled Lagrangian surfaces, so it remains to study the self-similar equation.

We start computing the first derivatives of the immersion:

$$X_t(s, t) = \gamma(s), \quad X_s(s, t) = (1 + \alpha(s))(\gamma'_1(s), \gamma'_2(s))$$

from which we deduce the expression of the induced metric:

$$E = (t + x)^2 + y^2, \quad G = 1, \quad F = 0,$$

and a basis of the normal bundle:

$$N_t = i\gamma(s), \quad N_s = i(1 + \alpha(s))(\gamma'_1(s), \gamma'_2(s)).$$

We now compute some second derivatives:

$$X_{st} = \gamma'(s), \quad X_{tt} = 0.$$

We deduce that

$$2\langle H, N_t \rangle = \frac{\langle X_{ss}, N_t \rangle}{E} + \frac{\langle X_{tt}, N_t \rangle}{G} = \frac{\langle X_{st}, N_s \rangle}{E} = \frac{1+x}{(t+x)^2 + y^2}.$$

On another hand, we have:

$$\langle X, N_t \rangle = \langle i\gamma(s), \int_{s_0}^s \alpha(u)(\gamma'_1(u), \gamma'_2(u))du \rangle,$$

which clearly does not depend on  $t$ . So the equation

$$\langle H, N_t \rangle + \lambda \langle X, N_t \rangle = 0$$

can never hold for such an immersion, so the only self-similar ruled Lagrangian surfaces are products  $\mathcal{L} \times \Gamma$ . It is then straightforward to see that the curve  $\Gamma$  must be a solution of the self-similar equation.

**Remark 5** We recover the Lagrangian helicoid of [B] by taking  $\gamma(s) = (k + il)(\cos s, \sin s)$ , where  $k$  and  $l$  are two constants such that  $k^2 + l^2 = 1$ . In the notation of [B], we have  $x(s) = G/2$  and  $y(s) = -A$ .

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