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SHARP WELL-POSEDNESS RESULTS FOR THE GENERALIZED BENJAMIN-ONO EQUATION WITH HIGH NONLINEARITY

STÉPHANE VENTO

Abstract. We establish the local well-posedness of the generalized Benjamin-Ono equation $\partial_t u + \mathcal{H}\partial_x^2 u \pm u^k \partial_x u = 0$ in $H^s(\mathbb{R})$, $s > 1/2 - 1/k$ for $k \geq 12$ and without smallness assumption on the initial data. The condition $s > 1/2 - 1/k$ is known to be sharp since the solution map $u_0 \mapsto u$ is not of class $\mathcal{C}^{k+1}$ on $H^s(\mathbb{R})$ for $s < 1/2 - 1/k$. On the other hand, in the particular case of the cubic Benjamin-Ono equation, we prove the ill-posedness in $H^s(\mathbb{R})$, $s < 1/3$.

1. Introduction and statement of the results

1.1. **Introduction.** Our purpose in this paper is to study the initial value problem for the generalized Benjamin-Ono equation

$$\text{(gBO)} \qquad \begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u \pm u^k \partial_x u = 0, & x, t \in \mathbb{R}, \\ u(x, t = 0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $k \in \mathbb{N} \setminus \{0\}$, $\mathcal{H}$ is the Hilbert transform defined by

$$\mathcal{H}f(x) = \frac{1}{\pi}\mathrm{pv}\Big(\frac{1}{x} * u\Big)(x) = \mathcal{F}^{-1}\big(-i\ \mathrm{sgn}(\xi)\hat{f}(\xi)\big)(x)$$

and with initial data u_0 belonging to the Sobolev space $H^s(\mathbb{R}) = (1-\partial_x^2)^{-s/2}L^2(\mathbb{R})$.

The case $k = 1$ was deduced by T.B. Benjamin [1] and later by H. Ono [14] as a model in internal wave theory. The Cauchy problem for the Benjamin-Ono equation has been extensively studied. It has been proved in [16] that (BO) is globally well-posed (i.e. global existence, uniqueness and persistence of regularity of the solution) in $H^s(\mathbb{R})$ for $s \geq 3$, and then for $s \geq 3/2$ in [15] and [5]. Recently, T. Tao [17] proved the well-posedness of this equation for $s \geq 1$ by using a gauge transformation. More recently, combining a gauge transformation with a Bourgain's method, A.D. Ionescu and C.E. Kenig [4] shown that one could go down to $L^2(\mathbb{R})$, and this seems to be, in some sense, optimal. It is worth noticing that all these results have been obtained by compactness methods. On the other hand, L. Molinet, J.-C. Saut and N. Tzvetkov [10] proved that, for all $s \in \mathbb{R}$, the flow map $u_0 \mapsto u$ is not of class $\mathcal{C}^2$ from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$. Furthermore, building suitable families of approximate solutions, H. Koch and N. Tzvetkov proved in [9] that the flow map is not even uniformly continuous on bounded sets of $H^s(\mathbb{R})$, $s > 0$. As an important consequence of this, since a Picard iteration scheme would imply smooth dependance upon the initial data, one see that such a scheme cannot be used to get solutions in any space continuously embedded in $\mathcal{C}([0, T], H^s(\mathbb{R}))$.

Key words and phrases. NLS-like equations, Cauchy problem.

For higher nonlinearities, that is for $k \geq 2$, the picture is a little bit different. It turns out that one can get local well-posedness results throught a Picard iteration scheme but for small initial data only. This seems mainly due to the fact that the smoothing properties of the linear group $V(\cdot)$ associated to the linear (BO) equation is just sufficient to recover the lost derivative in the nonlinear term, but does not allow to get the required contraction factors. On the other hand, for large initial data, one can prove local well-posedness by compactness methods together with a gauge transformation. Unfortunately, this usually requires more smoothness on the initial data. We summurize now the known results about the Cauchy problem for (gBO) equations when $k \geq 2$.

In the case of the modified Benjamin-Ono equation ($k = 2$), C.E. Kenig and H. Takaoka [8] have recently obtained the global well-posedness in the energy space $H^{1/2}(\mathbb{R})$. This have been proved thanks to a localized gauge transformation combined with a L^2_{xT} estimate of the solution. This result is known to be sharp since the solution map $u_0 \mapsto u$ is not $\mathcal{C}^3$ in $H^s(\mathbb{R})$, $s < 1/2$ (see [12]).

For (gBO) with cubic nonlinearity ($k = 3$), the local well-posedness is known in $H^s(\mathbb{R})$, $s > 1/3$ for small initial data [12] but only in $H^s(\mathbb{R})$, $s > 3/4$, for large initial data. Moreover, the ill-posedness has been proved in $H^s(\mathbb{R})$, $s < 1/6$ [12]. In this paper, we show the ill-posedness of the cubic Benjamin-Ono equation in $H^s(\mathbb{R})$, $s < 1/3$, which turns out to be optimal according to the above results.

When $k \geq 4$, by a scaling argument, one can guess the best Sobolev space in which the Cauchy problem is locally well-posed, that is, the critical indice s_c such that (gBO) is well-posed in $H^s(\mathbb{R})$ for $s > s_c$ and ill-posed for $s < s_c$. Recall that if $u(x,t)$ is a solution of the equation then $u_\lambda(x,t) = \lambda^{1/k} u(\lambda x, \lambda^2 t)$ ($\lambda > 0$) solves (gBO) with initial data $u_\lambda(x,0)$ and moreover

$$\|u_\lambda(\cdot,0)\|_{\dot{H}^s} = \lambda^{s+\frac{1}{k}-\frac{1}{2}} \|u(\cdot,0)\|_{\dot{H}^s}.$$

Hence the $\dot{H}^s(\mathbb{R})$ norm is invariant if and only if $s = s_k = 1/2 - 1/k$ and one can conjecture that $s_c = s_k$.

In the case of small initial data, this limit have been reached by L. Molinet and F. Ribaud [12]. This result is almost sharp in the sense that the flow map $u_0 \mapsto u$ is not of class $\mathcal{C}^{k+1}$ from $H^s(\mathbb{R})$ to $\mathcal{C}([0,T], H^s(\mathbb{R}))$ at the origin when $s < s_k$, [11]. This lack of regularity is also described by H.A. Biagioni and F. Linares in [2] where they established, using solitary waves, that the flow map is not uniformly continuous in $\dot{H}^{s_k}(\mathbb{R})$, $k \geq 2$.

For large initial data, the local well-posedness of (gBO) is only known in $H^s(\mathbb{R})$, $s \geq 1/2$, whatever the value of k. This have been proved in [11] by using the gauge transformation

$$u \xmapsto{\mathcal{G}} P_+(e^{-i\int_{-\infty}^{x} u^k} u), \tag{1.1}$$

together with compactness methods. Note also that very recently, in the particular case $k = 4$, N. Burq and F. Planchon [3] derived the local well-posedness of (gBO) in the homogeneous space $\dot{H}^{1/4}(\mathbb{R})$.

In this paper, our aim is to improve the results obtained in [11] for large initial data. We show that for all $k \geq 12$, (gBO) is locally well-posed in $H^s(\mathbb{R})$, $s > s_k$. Our proofs follow those of [11] : we perform the gauge transformation $w = \mathcal{G}(u)$

of a smooth solution u of (gBO) and derive suitable estimates for w. The main interest of this transformation is to obtain an equation satisfied by w where the nonlinearity $u^k u_x$ is replaced by terms of the form $P_+(u^k P_- u_x)$ in which one can share derivatives on u with derivatives on u^k. Working in the surcritical case, this allows to get a contraction factor T^ν in our estimates. It is worth noticing that $\nu = \nu(s)$ verifies $\lim_{s\to s_k} \nu(s) = 0$, and this explains why our method fails in the critical case $s = s_k$. On the other hand, the restriction $k \geq 12$ appears when we estimate the integral term

$$P_+\Big(e^{-i\int_{-\infty}^x u^k} u \int_{-\infty}^x u^{k-2}\mathcal{H}u_{xx}\Big)$$

(see section 3.2). This term doesn't seem to have a "good structure" since the bad interaction

$$Q_j u \int_{-\infty}^x (P_j u)^{k-2}\mathcal{H}P_j u_{xx}$$

forbids the share of the antiderivative $\int_{-\infty}^x$ with other derivatives.

1.2. **Main results.** Our main results read as follows.

Theorem 1. *Let $k \geq 12$ and $u_0 \in H^s(\mathbb{R})$ with $s > 1/2 - 1/k$. Then there exist $T = T(s, k, \|u_0\|_{H^s}) > 0$ and a unique solution $u \in \mathcal{C}([0,T]; H^s(\mathbb{R}))$ of (gBO) such that*

$$\|D_x^{s+1/2}u\|_{L_x^\infty L_T^2} < \infty, \tag{1.2}$$

$$\|D_x^{s-1/4}u\|_{L_x^4 L_T^\infty} < \infty, \tag{1.3}$$

$$\|P_0 u\|_{L_x^2 L_T^\infty} < \infty. \tag{1.4}$$

Moreover, the flow map $u_0 \mapsto u$ is Lipschitz on every bounded set of $H^s(\mathbb{R})$.

As mentioned previously, these results are in some sense almost sharp. However, the critical case $s = s_k$ remains open. We will only consider the most difficult case, that is the lowest values for s. More precisely we will prove Theorem 1 for $s_k < s < 1/2$.

In the case $k = 3$, we have the following ill-posedness result.

Theorem 2. *Let $k = 3$ and $s < 1/3$. There does not exist $T > 0$ such that the Cauchy problem (gBO) admits an unique local solution defined on the interval $[0,T]$ and such that the flow map $u_0 \mapsto u$ is of class $\mathcal{C}^4$ in a neighborhood of the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$.*

This result implies that we cannot solve (gBO) with $k = 3$ in $H^s(\mathbb{R})$, $s < 1/3$ by a contraction method on the Duhamel formulation. Recall that for small initial data [12], we have local well-posedness in $H^s(\mathbb{R})$ for $s > 1/3$. In view of this, we can conjecture that (gBO) is locally well-posed in $H^s(\mathbb{R})$, $s > 1/3$.

The remainder of this paper is organized as follows. In section 2, we first derive some linear estimates on the free evolution operator associated to (gBO) and we define our resolution space. Then we give some technical lemmas which will be used for nonlinear estimates. In section 3 we introduce the gauge transformation and derive the needed nonlinear estimates. The section 4 is devoted to the proof of Theorem 1. Finally we prove our ill-posedness result in the Appendix.

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1.3. **Notations.** For two positive numbers x, y, we write $x \lesssim y$ to mean that there exists a $C > 0$ which does not depend on x and y, and such that $x \leq Cy$. In the sequel, this constant may depend on s and k. We also use $\nu = \nu(s,k)$ to denote a positive power of T which may differ at each occurrence.

Our resolution space is constructed thanks to the space-time Lebesgue spaces $L^p_x L^q_T$ and $L^q_T L^p_x$ endowed for $T > 0$ and $1 \leq p, q \leq \infty$ with the norm

$$\|f\|_{L^p_x L^q_T} = \big\| \|f\|_{L^q_T([0;T])} \big\|_{L^p_x(\mathbb{R})} \quad \text{and} \quad \|f\|_{L^q_T L^p_x} = \big\| \|f\|_{L^p_x(\mathbb{R})} \big\|_{L^q_T([0;T])}.$$

When $p = q$ we simplify the notation by writing L^p_{xT}.

The well-known operators $\mathcal{F}$ (or $\hat{\cdot}$) and $\mathcal{F}^{-1}$ (or $\check{\cdot}$) are the Fourier operators defined by $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx$. The pseudo-differential operator D^α_x is defined by its Fourier symbol $|\xi|^\alpha$. Let P_+ and P_- be the Fourier projections to $[0, +\infty[$ and $]-\infty, 0]$. Thus one has

$$i\mathcal{H} = P_+ - P_-.$$

Let $\eta \in \mathcal{C}^\infty_0(\mathbb{R})$, $\eta \geq 0$, supp $\eta \subset \{1/2 \leq |\xi| \leq 2\}$ with $\sum_{-\infty}^{\infty} \eta(2^{-k}\xi) = 1$ for $\xi \neq 0$. We set $p(\xi) = \sum_{j \leq -3} \eta(2^{-j}\xi)$ and consider, for all $k \in \mathbb{Z}$, the operators Q_k and P_k respectively defined by

$$Q_k(f) = \mathcal{F}^{-1}(\eta(2^{-k}\xi)\hat{f}(\xi)) \quad \text{and} \quad P_k(f) = \mathcal{F}^{-1}(p(2^{-k}\xi)\hat{f}(\xi)).$$

Therefore we have the standard Littlewood-Paley decomposition

$$f = \sum_{j \in \mathbb{Z}} Q_j(f) = P_0(f) + \sum_{j \geq -2} Q_j(f) = P_0(f) + \tilde{P}(f). \tag{1.5}$$

We also need the operators

$$P_{\leq k} f = \sum_{j \leq k} Q_j f, \quad P_{\geq k} f = \sum_{j \geq k} Q_j f.$$

We finally introduce the operators $\tilde{P}_+ = P_+ \tilde{P}$ and $\tilde{P}_- = P_- \tilde{P}$ in order to obtain the smooth decomposition

$$f = \tilde{P}_-(f) + P_0(f) + \tilde{P}_+(f). \tag{1.6}$$

2. Linear estimates and technical lemmas

2.1. **Linear estimates and resolution space.** Recall that (gBO) is equivalent to its integral formulation

$$u(t) = V(t)u_0 \mp \frac{1}{k+1} \int_0^t V(t-\tau)\partial_x(u^{k+1})(\tau)d\tau, \tag{2.1}$$

where $V(t) = \mathcal{F}^{-1} e^{it\xi|\xi|}\mathcal{F}$ is the generator of the free evolution. Let us now gather the well-known estimates on the group $V(\cdot)$ in the following lemma.

Lemma 1. *Let $\varphi \in \mathcal{S}(\mathbb{R})$, then*

$$\|V(t)\varphi\|_{L^\infty_T L^2_x} \lesssim \|\varphi\|_{L^2}, \tag{2.2}$$

$$\|D^{1/2}_x V(t)\varphi\|_{L^\infty_x L^2_T} \lesssim \|\varphi\|_{L^2}, \tag{2.3}$$

$$\|D^{-1/4}_x V(t)\varphi\|_{L^4_x L^\infty_T} \lesssim \|\varphi\|_{L^2}. \tag{2.4}$$

Moreover, for $0 < T < 1$*, we have*

$$\|P_0 V(t)\varphi\|_{L^2_x L^\infty_T} \lesssim \|P_0\varphi\|_{L^2}. \tag{2.5}$$

The estimate (2.2) is straightforward whereas the proof of the Kato smoothing effect (2.3) and the maximal in time inequality (2.4) can be found in [6]. Estimate (2.5) has been proved in [7].

These estimates motivate the definition of our resolution space.

Definition 1. *For* $s_k < s < 1/2$*, we define the space* $X^s_T = \{u \in \mathcal{S}'(\mathbb{R}^2), \|u\|_{X^s_T} < \infty\}$ *where* $0 < T < 1$ *and*

$$\|u\|_{X^s_T} = \|u\|_{L^\infty_T H^s_x} + \|D^{s+1/2}_x u\|_{L^\infty_x L^2_T} + \|D^{s-1/4}_x u\|_{L^4_x L^\infty_T} + \|P_0 u\|_{L^2_x L^\infty_T}. \tag{2.6}$$

Thus lemma 1 implies immediately that for all $\varphi \in \mathcal{S}(\mathbb{R})$ and $0 < T < 1$,

$$\|V(t)\varphi\|_{X^s_T} \lesssim \|\varphi\|_{H^s}. \tag{2.7}$$

We now give some families of norms which are controlled by the X^s_T norm. This will be usefull to derive some nonlinear estimates in the sequel.

Definition 2. *A triplet* $(\alpha, p, q) \in \mathbb{R}\times[2,\infty]^2$ *is said to be 1-admissible if* $(\alpha, p, q) = (1/2, \infty, 2)$ *or*

$$4 \le p < \infty, \quad 2 < q \le \infty, \quad \frac{2}{p} + \frac{1}{q} \le \frac{1}{2}, \quad \alpha = \frac{1}{p} + \frac{2}{q} - \frac{1}{2}. \tag{2.8}$$

Proposition 1. *If* $(\alpha - s, p, q)$ *is 1-admissible, then for all* u *in* X^s_T*,*

$$\|D^\alpha_x u\|_{L^p_x L^q_T} \lesssim \|u\|_{X^s_T}. \tag{2.9}$$

Proof : The inequality

$$\|D^{s+1/2}_x u\|_{L^\infty_x L^2_T} \lesssim \|u\|_{X^s_T} \tag{2.10}$$

yields the result when $(\alpha, p, q) = (1/2, \infty, 2)$. Assume now $(\alpha, p, q) \neq (1/2, \infty, 2)$. Let $r \in [4; p]$. Then according to Sobolev embedding theorem,

$$\|D^{s+1/r-1/2}_x u\|_{L^r_x L^\infty_T} \lesssim \|D^{s-1/4}_x u\|_{L^4_x L^\infty_T} \lesssim \|u\|_{X^s_T}.$$

By interpolation with (2.10) we get for all $0 \le \theta \le 1$

$$\|D^{s+\frac{1}{2}-(1-\frac{1}{r})\theta}_x u\|_{L^{r/\theta}_x L^{2/(1-\theta)}_T} \lesssim \|u\|_{X^s_T}.$$

We deduce (2.9) by taking $\theta = r/p$ since the assumption $r \ge 4$ is equivalent to $\frac{2}{p} + \frac{1}{q} \le \frac{1}{2}$. □

We list now all the norms needed for the nonlinear estimates.

Corollary 1. *For* $u \in X^s_T$*, the following quantities are bounded by* $\|u\|_{X^s_T}$*.*

$$N_1 = \|u\|_{L^p_x L^\infty_T}, \quad 4 \le p \le (\tfrac{1}{2} - s)^{-1}, \quad N_2 = T^{-\nu}\|u\|_{L^{3k}_{xT}},$$
$$N_3 = T^{-\nu}\|u\|_{L^{k/(1-s)}_x L^{2k/s}_T}, \quad N_4 = T^{-\nu}\|u\|_{L^{k(\frac{1}{3}+s)^{-1}}_x L^{k(\frac{1}{3}-\frac{s}{2})^{-1}}_T},$$
$$N_5 = T^{-\nu}\|u\|_{L^{3k/4s}_x L^{k(\frac{1}{2}-\frac{2s}{3})^{-1}}_T}, \quad N_6 = T^{-\nu}\|u\|_{L^{k(1-\frac{s}{3})^{-1}}_x L^{6k/s}_T},$$
$$N_7 = T^{-\nu}\|u\|_{L^{k+1}_x L^{2k(k+1)}_T}, \quad N_8 = T^{-\nu}\|u\|_{L^{(k-1)(\frac{5}{6}-\frac{s}{3})^{-1}}_x L^{(k-1)(\frac{2s}{3}-\frac{1}{6})^{-1}}_T},$$
$$N_9 = \|D^{1-2s+6\varepsilon}_x u\|_{L^{(\frac{3}{2}-3s)^{-1}}_x L^{1/3\varepsilon}_T}, \quad N_{10} = \|D^s_x u\|_{L^6_{xT}},$$
$$N_{11} = \|D^{s+1/2-3\varepsilon}_x u\|_{L^{1/\varepsilon}_x L^{(\frac{1}{2}-2\varepsilon)^{-1}}_T}, \quad N_{12} = \|D^{1/2}_x u\|_{L^{3/s}_x L^{(\frac{1}{2}-\frac{2s}{3})^{-1}}_T},$$

where $\varepsilon, \nu > 0$ *are small enough.*

Proof :

(i) Let $4 \leq p \leq (\frac{1}{2}-s)^{-1}$. By separating low and high frequencies,

$$\|u\|_{L^p_x L^\infty_T} \lesssim \|P_0 u\|_{L^2_x L^\infty_T} + \|\tilde{P} D_x^{s+1/p-1/2} u\|_{L^p_x L^\infty_T} \lesssim \|u\|_{X^s_T}.$$

Here we used that $\tilde{P}$ is continuous on $L^p_x L^q_T$, $1 \leq p, q \leq \infty$, and the 1-admissibility of $(1/p - 1/2, p, \infty)$.

(ii)-(vii) We evaluate the norm of the form $N = \|u\|_{L^p_x L^q_T}$ with $p > 2$ and $q < \infty$. Fix $\delta > 0$ small enough so that $\alpha = s - s_k - 2\delta > 0$ and $\frac{1}{q} - \delta > 0$. Then using the previous decomposition, Bernstein and Hölder inequalities, we get

$$N \lesssim T^\nu \|P_0 u\|_{L^2_x L^\infty_T} + T^\nu \|\tilde{P} D_x^\alpha u\|_{L^p_x L_T^{(\frac{1}{q}-\delta)^{-1}}}.$$

One complete the proof by noticing that the triplet $(\alpha - s, p, (\frac{1}{q} - \delta)^{-1})$ is 1-admissible.

(viii) Following the same idea, we write

$$N_8 \lesssim T^\nu \|P_0 u\|_{L^2_x L^\infty_T} + T^\nu \|\tilde{P} D_x^{\frac{k}{k-1}(s - s_k - 2\frac{\delta}{k})} u\|_{L_x^{(k-1)(\frac{5}{6}-\frac{s}{3})^{-1}} L_T^{(k-1)(\frac{2s}{3}-\frac{1}{6}-\delta)^{-1}}}$$

for an appropriate $\delta > 0$. Once again, $(\frac{k}{k-1}(s - s_k - 2\frac{\delta}{k}) - s, (k-1)(\frac{5}{6} - \frac{s}{3})^{-1}, (k-1)(\frac{2s}{3} - \frac{1}{6} - \delta)^{-1})$ is 1-admissible.

(ix)-(xii) Note finally that the triplets $(1 - 3s + 6\varepsilon, (\frac{3}{2} - 3s)^{-1}, 1/3\varepsilon)$, $(0, 6, 6)$, $(1/2 - 3\varepsilon, 1/\varepsilon, (\frac{1}{2} - 2\varepsilon)^{-1})$ and $(1/2 - s, 3/s, (\frac{1}{2} - \frac{2s}{3})^{-1})$ are 1-admissible. □

We now turn to the non-homogenous estimates. Let us first recall the following result found in [11].

Lemma 2. *Let $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, $(\nu_1, \nu_2) \in \mathbb{R}^2_+$, and $1 \leq p_1, q_1, p_2, q_2 \leq \infty$ such that for all $\varphi \in \mathcal{S}(\mathbb{R})$,*

$$\|D_x^{\alpha_1} V(t)\varphi\|_{L^{p_1}_x L^{q_1}_T} \lesssim T^{\nu_1} \|\varphi\|_{L^2},$$
$$\|D_x^{\alpha_2} V(t)\varphi\|_{L^{p_2}_x L^{q_2}_T} \lesssim T^{\nu_2} \|\varphi\|_{L^2}.$$

Then for all $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\Big\| D_x^{\alpha_2} \int_0^t V(t-\tau) f(\tau) d\tau \Big\|_{L^\infty_T L^2_x} \lesssim T^{\nu_2} \|f\|_{L^{\bar{p}_2}_x L^{\bar{q}_2}_T}, \tag{2.11}$$

$$\Big\| D_x^{\alpha_1+\alpha_2} \int_0^t V(t-\tau) f(\tau) d\tau \Big\|_{L^{p_1}_x L^{q_1}_T} \lesssim T^{\nu_1+\nu_2} \|f\|_{L^{\bar{p}_2}_x L^{\bar{q}_2}_T} \tag{2.12}$$

provided $\min(p_1, q_1) > \max(\bar{p}_2, \bar{q}_2)$ or ($q_1 = \infty$ and $\bar{p}_2, \bar{q}_2 < \infty$), where $\bar{p}_2$ and $\bar{q}_2$ are defined by $1/\bar{p}_2 = 1 - 1/p_2$ and $1/\bar{q}_2 = 1 - 1/q_2$.

Using lemma 2 we infer the following result.

Lemma 3. *For all $f \in \mathcal{S}(\mathbb{R}^2)$, the quantity $\Big\| \int_0^t V(t-\tau) f(\tau) d\tau \Big\|_{X^s_T}$ can be estimated by*

$$\|f\|_{L_x^{(\frac{5}{6}+\frac{s}{3})^{-1}} L_T^{(\frac{5}{6}-\frac{2s}{3})^{-1}}}, \quad \|D_x^s f\|_{L^{6/5}_{xT}}, \quad \|D_x^{s-1/2} f\|_{L^1_x L^2_T}, \quad \|D_x^{s+1/4} f\|_{L_x^{4/3} L^1_T}. \tag{2.13}$$

Moreover,

$$\Big\| D_x^{s+1/2} \int_0^t V(t-\tau) f(\tau) d\tau \Big\|_{L^\infty_x L^2_T} \lesssim \|D_x^s f\|_{L^1_T L^2_x}. \tag{2.14}$$

Proof : (2.13) follows from (2.11)-(2.12) since the triplets $(s,(\frac{1}{6}-\frac{s}{3})^{-1},(\frac{1}{6}+\frac{2s}{3})^{-1})$, $(0,6,6)$, $(1/2,\infty,2)$ and $(-1/4,4,\infty)$ are 1-admissible. Inequality (2.14) is proved in [11], proposition 2.8. □

2.2. **Technical lemmas.** In this subsection, we recall some useful lemmas which allow to share derivatives of various expressions in $L^p_xL^q_T$ norms. One can find proofs of lemmas 4-8 in [11, 7].
Here f and g denote two elements of $\mathcal{S}(\mathbb{R})$.

Lemma 4. *If $\alpha>0$ and $1<p,q<\infty$, then*

$$\|D^\alpha_x(fg)\|_{L^p_xL^q_T}\lesssim\|f\|_{L^{p_1}_xL^{q_1}_T}\|D^\alpha_xg\|_{L^{p_2}_xL^{q_2}_T}+\|g\|_{L^{\tilde p_1}_xL^{\tilde q_1}_T}\|D^\alpha_xf\|_{L^{\tilde p_2}_xL^{\tilde q_2}_T}$$

where $1<p_1,p_2,q_2,\tilde p_1,\tilde p_2,\tilde q_2<\infty$, $1<q_1,\tilde q_1\le\infty$, $1/p_1+1/p_2=1/\tilde p_1+1/\tilde p_2=1/p$ and $1/q_1+1/q_2=1/\tilde q_1+1/\tilde q_2=1/q$.
Moreover the cases $(p_1,q_1)=(\infty,\infty)$ and $(\tilde p_1,\tilde q_1)=(\infty,\infty)$ are allowed.

Lemma 5. *If $0<\alpha<1$ and $1<p,q<\infty$ then*

$$\|D^\alpha_xF(f)\|_{L^p_xL^q_T}\lesssim\|F'(f)\|_{L^{p_1}_xL^{q_1}_T}\|D^\alpha_xf\|_{L^{p_2}_xL^{q_2}_T}$$

where $1<p_1,p_2,q_2<\infty$, $1<q_1\le\infty$, $1/p_1+1/p_2=1/p$ and $1/q_1+1/q_2=1/q$.

Lemma 6. *If $0<\alpha<1$, $0\le\beta<1-\alpha$ and $1<p,q<\infty$, then*

$$\|D^\beta_x([D^\alpha_x,f]g)\|_{L^p_xL^q_T}\lesssim\|g\|_{L^{p_1}_xL^{q_1}_T}\|D^{\alpha+\beta}_xf\|_{L^{p_2}_xL^{q_2}_T}$$

where $1<p_1,q_1,p_2,q_2<\infty$, $1/p_1+1/p_2=1/p$ and $1/q_1+1/q_2=1/q$.
Moreover, if $\beta>0$ then $q_1=\infty$ is allowed.

Lemma 7. *If $\alpha>0$, $\beta\ge0$ and $1<p,q<\infty$ then*

$$\|D^\alpha_xP_+(fP_-D^\beta_xg)\|_{L^p_xL^q_T}\lesssim\|D^{\gamma_1}_xf\|_{L^{p_1}_xL^{q_1}_T}\|D^{\gamma_2}_xg\|_{L^{p_2}_xL^{q_2}_T}$$

where $1<p_1,q_1,p_2,q_2<\infty$, $1/p_1+1/p_2=1/p$, $1/q_1+1/q_2=1/q$ and $\gamma_1\ge\alpha$, $\gamma_1+\gamma_2=\alpha+\beta$.

As in [11], we introduce the bilinear operator G defined by

$$G(f,g)=\mathcal{F}^{-1}\Big(\frac{1}{2}\int_{\mathbb{R}}\frac{\xi_1(\xi-\xi_1)}{i\xi}[\mathrm{sgn}(\xi_1)+\mathrm{sgn}(\xi-\xi_1)]\hat f(\xi_1)\hat g(\xi-\xi_1)d\xi_1\Big).$$

We easily verify that

$$G(f,f)=\partial_x^{-1}(f_x\mathcal{H}f_x)=\partial_x^{-1}(-i(P_+f_x)^2+i(P_-f_x)^2) \tag{2.15}$$

and

$$G(f,g)=\partial_x^{-1}(-iP_+f_xP_+g_x+iP_-f_xP_-g_x). \tag{2.16}$$

Lemma 8. *If $0\le\alpha\le1$ and $1<p,q<\infty$ then*

$$\|D^\alpha_xG(f,g)\|_{L^p_xL^q_T}\lesssim\|D^{\gamma_1}_xf\|_{L^{p_1}_xL^{q_1}_T}\|D^{\gamma_2}_xg\|_{L^{p_2}_xL^{q_2}_T}$$

where $0\le\gamma_1,\gamma_2\le1$, $\gamma_1+\gamma_2=\alpha+1$, $1<p_1,q_1,p_2,q_2<\infty$, $1/p_1+1/p_2=1/p$ and $1/q_1+1/q_2=1/q$.

We will also need the following lemma in order to treat low frequencies in the integral term.

Lemma 9. *If $\alpha \geq 0$ and $1 \leq p, q \leq \infty$ then*

$$\|P_0(fD_x^\alpha g)\|_{L_x^p L_T^q} \lesssim \|D_x^{\gamma_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\gamma_2} g\|_{L_x^{p_2} L_T^{q_2}} + \|P_0 f\|_{L_x^{\tilde{p}_1} L_T^{\tilde{q}_1}} \|D_x^\alpha P_0 g\|_{L_x^{\tilde{p}_2} L_T^{\tilde{q}_2}}$$

where $\gamma_1, \gamma_2 \geq 0$, $\alpha = \gamma_1 + \gamma_2$, $1 < p_i, q_i, \tilde{p}_i, \tilde{q}_i < \infty$, $1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2 = 1/p$ and $1/q_1 + 1/q_2 = 1/\tilde{q}_1 + 1/\tilde{q}_2 = 1/q$.

Proof : We split the product $fD_x^\alpha g$ as follows :

$$fD_x^\alpha g = P_+ f P_+ D_x^\alpha g + P_+ f P_- D_x^\alpha g + P_- f P_+ D_x^\alpha g + P_- f P_- D_x^\alpha g. \tag{2.17}$$

It is sufficient to consider the contribution of the first two terms. For the first one, we remark that

$$P_0[P_+ f P_+ (D_x^\alpha g)] = P_0[P_0(P_+ f) P_0(P_+ D_x^\alpha g)]$$

and thus using the continuity of P_0 on $L_x^p L_T^q$,

$$\begin{aligned} \|P_0[P_+ f P_+ (D_x^\alpha g)]\|_{L_x^p L_T^q} &\lesssim \|P_0(P_+ f) P_0(P_+ D_x^\alpha g)\|_{L_x^p L_T^q} \\ &\lesssim \|P_0 f\|_{L_x^{\tilde{p}_1} L_T^{\tilde{q}_1}} \|D_x^\alpha P_0 g\|_{L_x^{\tilde{p}_2} L_T^{\tilde{q}_2}}. \end{aligned}$$

For the second term in (2.17) we have typically contributions of the form $P_0[P_0(P_+ f) P_0(P_- D_x^\alpha g)]$ which are treated as above, and $P_0[\tilde{P}_+ f \tilde{P}_- D_x^\alpha g]$. Using decomposition (1.5), one can write

$$\begin{aligned} P_0(\tilde{P}_+ f \tilde{P}_- D_x^\alpha g) &= P_0\Big(\sum_{j\in\mathbb{Z}} Q_j(\tilde{P}_+ f) P_j(\tilde{P}_- D_x^\alpha g) + \sum_{j\in\mathbb{Z}} P_j(\tilde{P}_+ f) Q_j(\tilde{P}_- D_x^\alpha g)\Big) \\ &\quad + P_0\Big(\sum_{|p|\leq 2} \sum_{j\in\mathbb{Z}} Q_j(\tilde{P}_+ f) Q_{k-j}(\tilde{P}_- D_x^\alpha g)\Big). \end{aligned}$$

By a careful analysis of the various localisations, we get

$$P_0(\tilde{P}_+ f \tilde{P}_- D_x^\alpha g) = P_0\Big[\sum_{|p|\lesssim 1} \sum_{j\in\mathbb{Z}} Q_j(\tilde{P}_+ f) Q_{j+p}(\tilde{P}_- D_x^\alpha g)\Big].$$

Here we define the operators $Q_j^\lambda = 2^{-\lambda j} D_x^\lambda Q_j$. It follows that

$$P_0(\tilde{P}_+ f \tilde{P}_- D_x^\alpha g) = P_0\Big[\sum_{|p|\lesssim 1} \sum_{j\in\mathbb{Z}} Q_j^{-\gamma_1}(\tilde{P}_+ D_x^{\gamma_1} f) Q_{j+p}^{\gamma_1}(\tilde{P}_- D_x^{\gamma_2} g)\Big].$$

Thus using Cauchy-Schwarz and Hölder inequalities, and Littlewood-Paley theorem,

$$\begin{aligned} \|P_0(\tilde{P}_+ f \tilde{P}_- D_x^\alpha g)\|_{L_x^p L_T^q} &\lesssim \Big\| \sum_{|p|\lesssim 1} \Big[\Big(\sum_{j\in\mathbb{Z}} |Q_j^{-\gamma_1} \tilde{P}_+ D_x^{\gamma_1} f|^2\Big)^{1/2} \Big(\sum_{j\in\mathbb{Z}} |Q_j^{\gamma_1} \tilde{P}_- D_x^{\gamma_2} g|^2\Big)^{1/2}\Big]\Big\|_{L_x^p L_T^q} \\ &\lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}} |Q_j^{-\gamma_1} \tilde{P}_+ D_x^{\gamma_1} f|^2\Big)^{\frac{1}{2}}\Big\|_{L_x^{p_1} L_T^{q_1}} \Big\|\Big(\sum_{j\in\mathbb{Z}} |Q_j^{\gamma_1} \tilde{P}_- D_x^{\gamma_2} g|^2\Big)^{\frac{1}{2}}\Big\|_{L_x^{p_2} L_T^{q_2}} \\ &\lesssim \|D_x^{\gamma_1} f\|_{L_x^{p_1} L_T^{q_1}} \|D_x^{\gamma_2} g\|_{L_x^{p_2} L_T^{q_2}}. \square \end{aligned}$$

3. Nonlinear estimates

3.1. **Gauge transformation.** By a rescaling argument, it is sufficient to solve

$$u_t + \mathcal{H}u_{xx} = 2u^k u_x \tag{3.1}$$

(equation with minus sign in front of the nonlinearity could be treated in the same way). If $u \in \mathcal{C}([0,T]; H^\infty(\mathbb{R}))$ is a smooth solution, we define the gauge transformation[1]

$$w = P_+(e^{-iF}u), \quad F = F(u) = \int_{-\infty}^{x} u^k(y,t)dy. \tag{3.2}$$

The rest of this subsection is devoted to the proof of the following estimate.

Proposition 2. *Let be $k \geq 12$ and $s_k < s < 1/2$. Let $u \in \mathcal{C}([0,T]; H^\infty(\mathbb{R}))$ be a solution of the Cauchy problem associated to (3.1) with initial data $u_0 \in H^\infty(\mathbb{R})$. Then there exist $\nu = \nu(s,k) > 0$ and a positive nondecreasing polynomial function p_k such that*

$$\begin{aligned} \|u\|_{X_T^s} \lesssim\ & \|u_0\|_{H^s} + T^\nu p_k(\|u\|_{X_T^s})\|u\|_{X_T^s} \\ & +(\|u_0\|_{H^s}^k + T^\nu p_k(\|u\|_{X_T^s})\|u\|_{X_T^s})\|D_x^{s+1/2}w\|_{L_x^\infty L_T^2}. \end{aligned} \tag{3.3}$$

Proof : We start by splitting u according to (1.6). Then, using that $|P_+u| = |P_-u|$ (since u is real), we deduce

$$\|u\|_{X_T^s} \lesssim \|P_0 u\|_{X_T^s} + \|\tilde{P}_+ u\|_{X_T^s}. \tag{3.4}$$

For the low frequencies, we use the Duhamel formulation of (gBO), lemma 3 and (2.7) to get

$$\begin{aligned} \|P_0 u\|_{X_T^s} &\lesssim \|P_0 u_0\|_{H^s} + \|P_0 D_x^{1/2} u^{k+1}\|_{L_x^1 L_T^2} \\ &\lesssim \|u_0\|_{H^s} + \|u^{k+1}\|_{L_x^1 L_T^2} \\ &\lesssim \|u_0\|_{H^s} + T^\nu \|u\|^{k+1}_{L_x^{k+1} L_T^{2k(k+1)}} \\ &\lesssim \|u_0\|_{H^s} + T^\nu \|u\|_{X_T^s}. \end{aligned}$$

Now we consider the second term in the right-hand side of (3.4). As mentioned in [11], $\tilde{P}_+ u$ satisfies the dispersive equation

$$\partial_t(\tilde{P}_+ u) + \mathcal{H}\partial_x^2(\tilde{P}_+ u) = \tilde{P}_+(e^{iF}u^k w_x) - \tilde{P}_+(e^{iF}u^k \partial_x P_-(e^{-iF}u)) + i\tilde{P}_+(u^{2k+1}).$$

Thus, according to lemma 3

$$\begin{aligned} \|\tilde{P}u\|_{X_T^s} \lesssim\ & \|V(t)u_0\|_{X_T^s} + \Big\| \int_0^t V(t-\tau)\tilde{P}_+(e^{iF}u^k w_x)(\tau)d\tau \Big\|_{X_T^s} \\ & +\|D_x^s u^{2k+1}\|_{L_{xT}^{6/5}} + \|\tilde{P}_+(e^{iF}u^k \partial_x P_-(e^{-iF}u))\|_{L_x^{(\frac{5}{6}+\frac{s}{3})^{-1}} L_T^{(\frac{5}{6}-\frac{2s}{3})^{-1}}} \\ \lesssim\ & \|u_0\|_{H^s} + A + B + C. \end{aligned}$$

Obviously,

$$B \lesssim \|u^{2k}\|_{L_{xT}^{3/2}} \|D_x^s u\|_{L_{xT}^6} \lesssim \|u\|_{L_{xT}^{3k}}^{2k} \|D_x^s u\|_{L_{xT}^6} \lesssim T^\nu \|u\|_{X_T^s}^{2k+1}.$$

[1] we can also set $F = \frac{1}{2}\int_{-\infty}^x u^k$ in the non-rescaled case $u_t + \mathcal{H}u_{xx} = u^k u_x$.

Term C has a structure $P_+(fP_-g_x)$ thus by lemma 7

$$\begin{aligned} C &\lesssim \|D_x^{1/2}(e^{iF}u^k)\|_{L_x^{6/5}L_T^3}\|D_x^{1/2}(e^{-iF}u)\|_{L_x^{3/s}L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}} \\ &\lesssim C_1C_2. \end{aligned}$$

Using lemmas 4-5, we infer

$$\begin{aligned} C_1 &\lesssim \|D_x^{1/2}u^k\|_{L_x^{6/5}L_T^3} + \|D_x^{1/2}e^{iF}\|_{L_x^{(\frac{1}{2}-s)^{-1}}L_T^{2/s}}\|u^k\|_{L_x^{(\frac{1}{3}+s)^{-1}}L_T^{(\frac{1}{3}-\frac{s}{2})^{-1}}} \\ &\lesssim \|u\|^{k-1}_{L_x^{(k-1)(\frac{5}{6}-\frac{s}{3})^{-1}}L_T^{(k-1)(\frac{2s}{3}-\frac{1}{6})^{-1}}}\|D_x^{1/2}u\|_{L_x^{3/s}L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}} \\ &\quad +\|D_x^{-1/2}(u^ke^{iF})\|_{L_x^{(\frac{1}{2}-s)^{-1}}L_T^{2/s}}\|u\|^k_{L_x^{k(\frac{1}{3}+s)^{-1}}L_T^{k(\frac{1}{3}-\frac{s}{2})^{-1}}} \\ &\lesssim T^\nu\|u\|^k_{X_T^s} + T^\nu\|u^k\|_{L_x^{1/(1-s)}L_T^{2/s}}\|u\|^k_{X_T^s} \\ &\lesssim T^\nu\|u\|^k_{X_T^s} + T^\nu\|u\|^{2k}_{X_T^s} \qquad (3.5) \end{aligned}$$

and in the same way

$$\begin{aligned} C_2 &\lesssim \|D_x^{1/2}u\|_{L_x^{3/s}L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}} + \|D_x^{1/2}e^{-iF}\|_{L_x^{(\frac{4s}{3}-\frac{1}{2})^{-1}}L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}}\|u\|_{L_x^{(\frac{1}{2}-s)^{-1}}L_T^\infty} \\ &\lesssim \|D_x^{1/2}u\|_{L_x^{3/s}L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}} + \|u^k\|_{L_x^{3/4s}L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}}\|u\|_{L_x^{(\frac{1}{2}-s)^{-1}}L_T^\infty} \\ &\lesssim \|u\|_{X_T^s} + T^\nu\|u\|^{k+1}_{X_T^s}. \qquad (3.6) \end{aligned}$$

Combining (3.5) and (3.6), C is bounded by

$$C \lesssim T^\nu(\|u\|^{k+1}_{X_T^s} + \|u\|^{2k+1}_{X_T^s} + \|u\|^{3k+1}_{X_T^s}) \lesssim T^\nu p_k(\|u\|_{X_T^s})\|u\|_{X_T^s}.$$

In order to study the contribution of A, we decompose $e^{iF}u^kw_x$ as

$$e^{iF}u^kw_x = D_x^{1/2}(e^{iF}u^k\mathcal{H}D_x^{1/2}w) - [D_x^{1/2}, e^{iF}u^k]\mathcal{H}D_x^{1/2}w.$$

Therefore, according to lemma 3, and using the fact that $\tilde{P}_+$ is continuous on $L_x^1L_T^2$,

$$\begin{aligned} A &\lesssim \|D_x^s(e^{iF}u^k\mathcal{H}D_x^{1/2}w)\|_{L_x^1L_T^2} + \|[D_x^{1/2}, e^{iF}u^k]\mathcal{H}D_x^{1/2}w\|_{L_x^{(\frac{5}{6}+\frac{s}{3})^{-1}}L_T^{(\frac{5}{6}-\frac{2s}{3})^{-1}}} \\ &\lesssim A_1 + A_2. \end{aligned}$$

Note that A_1 cannot be treated by lemma 4, so we use lemma A.13 in [7]. This leads to

$$\begin{aligned} A_1 &\lesssim \|D_x^s(e^{iF}u^k)\|_{L_x^{(1-\frac{s}{3})^{-1}}L_T^{3/2s}}\|D_x^{1/2}(e^{-iF}u)\|_{L_x^{3/s}L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}} + \|u^k\|_{L_x^1L_T^\infty}\|D_x^{s+1/2}w\|_{L_x^\infty L_T^2} \\ &\lesssim A_{11}C_2 + A_{12}^k\|D_x^{s+1/2}w\|_{L_x^\infty L_T^2}. \end{aligned}$$

By lemma 4 we bound the contribution of A_{11} by

$$\begin{aligned} A_{11} &\lesssim \|D_x^su^k\|_{L_x^{(1-\frac{s}{3})^{-1}}L_T^{3/2s}} + \|D_x^se^{iF}\|_{L_x^{3/2s}L_T^{6/s}}\|u^k\|_{L_x^{1/(1-s)}L_T^{2/s}} \\ &\lesssim \|u\|^{k-1}_{L_x^{(k-1)(\frac{5}{6}-\frac{s}{3})^{-1}}L_T^{(k-1)(\frac{2s}{3}-\frac{1}{6})^{-1}}}\|D_x^su\|_{L_{xT}^6} + \|u\|^k_{L_x^{k(1-\frac{s}{3})^{-1}}L_T^{6k/s}}\|u\|^k_{L_x^{k/(1-s)}L_T^{2k/s}} \\ &\lesssim T^\nu\|u\|^k_{X_T^s} + T^\nu\|u\|^{2k}_{X_T^s}. \end{aligned}$$

To treat $A_{12} = \|u\|_{L_x^k L_T^\infty}$ we use the Duhamel formulation of (gBO) and lemma 2,

$$\begin{aligned} A_{12} &\lesssim \|V(t)u_0\|_{L_x^k L_T^\infty} + \Big\| \int_0^t V(t-\tau)\partial_x u^{k+1}(\tau)d\tau \Big\|_{L_x^k L_T^\infty} \\ &\lesssim \|u_0\|_{H^s} + \|D_x^{s_k+1/2+3\varepsilon} u^{k+1}\|_{L_x^{(1-\varepsilon)^{-1}} L_T^{(\frac{1}{2}+2\varepsilon)^{-1}}} \end{aligned}$$

and setting $\varepsilon' = \frac{1}{3}(s-s_k) - \varepsilon > 0$ it follows that

$$\begin{aligned} &\|D_x^{s_k+1/2+3\varepsilon} u^{k+1}\|_{L_x^{(1-\varepsilon)^{-1}} L_T^{(\frac{1}{2}+2\varepsilon)^{-1}}} \\ &\lesssim \|D_x^{s+1/2-3\varepsilon'} u\|_{L_x^{1/\varepsilon'} L_T^{(\frac{1}{2}-2\varepsilon')^{-1}}} \|u^k\|_{L_x^{(1-\frac{1}{3}(s-s_k))^{-1}} L_T^{\frac{3}{2}(s-s_k)^{-1}}} \\ &\lesssim T^\nu \|u\|_{X_T^s} \|u\|^k_{L_x^{k(1-\frac{1}{3}(s-s_k))^{-1}} L_T^\infty} \\ &\lesssim T^\nu \|u\|^{k+1}_{X_T^s}. \end{aligned}$$

Finally, according to lemma 6 we write

$$\begin{aligned} A_2 &\lesssim \|D_x^{1/2}(e^{iF}u^k)\|_{L_x^{6/5} L_T^3} \|D_x^{1/2}(e^{-iF}u)\|_{L_x^{3/s} L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}} \\ &\lesssim C_1 C_2 \\ &\lesssim T^\nu p_k(\|u\|_{X_T^s})\|u\|_{X_T^s}, \end{aligned}$$

witch complete the proof of (3.3). □

3.2. **Estimate of $\|D_x^{s+1/2} w\|_{L_x^\infty L_T^2}$.** Now our aim is to estimate the term $\|D_x^{s+1/2} w\|_{L_x^\infty L_T^2}$ which appears in (3.3). More precisely we will prove the following proposition.

Proposition 3. *Let $k \geq 12$ and $s_k < s < 1/2$. For all solution $u \in \mathcal{C}([0,T]; H^\infty(\mathbb{R}))$ of (3.1) with initial data $u_0 \in H^\infty(\mathbb{R})$, we have the following bound,*

$$\|D_x^{s+1/2} w\|_{L_x^\infty L_T^2} \lesssim p_k(\|u_0\|_{H^s})\|u_0\|_{H^s} + T^\nu p_k(\|u\|_{X_T^s})\|u\|_{X_T^s} \tag{3.7}$$

where p_k is a positive nondecreasing polynomial function.

Proof : Following [11], we see that w satisfies the equation

$$\begin{aligned} w_t + \mathcal{H} w_{xx} &= P_+[2e^{-iF}(-ku^k P_- u_x - iP_- u_{xx})] \\ &\quad -ik(k-1)P_+\Big(e^{-iF} u \int_{-\infty}^x u^{k-2} u_x \mathcal{H} u_x\Big). \end{aligned} \tag{3.8}$$

Thus using the Duhamel formulation of (3.8) and lemma 3 we infer

$$\begin{aligned} \|D_x^{s+1/2} w\|_{L_x^\infty L_T^2} &\lesssim \|D_x^{s+1/2} V(t)w(0)\|_{L_x^\infty L_T^2} \\ &\quad + \|P_+[2e^{-iF}(-ku^k P_- u_x - iP_- u_{xx})]\|_{L_x^{(\frac{5}{6}+\frac{s}{3})^{-1}} L_T^{(\frac{5}{6}-\frac{2s}{3})^{-1}}} \\ &\quad + \Big\| D_x^{s+1/2} \int_0^t V(t-\tau) P_+\Big(e^{-iF} u \int_{-\infty}^x u^{k-2} u_x \mathcal{H} u_x\Big) d\tau \Big\|_{L_x^\infty L_T^2}. \end{aligned} \tag{3.9}$$

The first term of right-hand side can be bounded by

$$\begin{aligned}
\|D_x^{s+1/2}V(t)w(0)\|_{L_x^\infty L_T^2} &\lesssim \|e^{-iF(u_0)}u_0\|_{H^s} \\
&\lesssim \|u_0\|_{L^2} + \|e^{-iF(u_0)}\|_{L^\infty}\|D_x^s u_0\|_{L^2} + \|D_x^s e^{-iF(u_0)}\|_{L^{1/s}}\|u_0\|_{L^{(\frac{1}{2}-s)^{-1}}} \\
&\lesssim \|u_0\|_{H^s} + \|D^{s-1}(e^{-iF(u_0)}u_0^k)\|_{L^{1/s}}\|u_0\|_{H^s} \\
&\lesssim \|u_0\|_{H^s}(1+\|u_0^k\|_{L^1}) \\
&\lesssim \|u_0\|_{H^s}(1+\|u_0\|_{H^s}^k).
\end{aligned}$$

On the other hand, according to lemma 7, we see that

$$\begin{aligned}
&\|P_+[2e^{-iF}(-ku^kP_-u_x - iP_-u_{xx})]\|_{L_x^{(\frac{5}{6}+\frac{s}{3})^{-1}}L_T^{(\frac{5}{6}-\frac{2s}{3})^{-1}}} \\
&\lesssim \|D_x^{1/2}(e^{-iF}u^k)\|_{L_x^{6/5}L_T^3}\|D_x^{1/2}u\|_{L_x^{3/s}L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}} \\
&\lesssim C_1\|u\|_{X_T^s} \\
&\lesssim T^\nu\|u\|_{X_T^s}^{k+1} + T^\nu\|u\|_{X_T^s}^{2k+1}
\end{aligned}$$

Thus it remains to estimate the integral term in (3.9), that is, the last one. For this purpose, we split it as

$$\begin{aligned}
\int_{-\infty}^x u^{k-2}u_x\mathcal{H}u_x &= P_0\int_{-\infty}^x u^{k-2}u_x\mathcal{H}u_x + \tilde{P}_+\int_{-\infty}^x u^{k-2}u_x\mathcal{H}u_x + \tilde{P}_-\int_{-\infty}^x u^{k-2}u_x\mathcal{H}u_x \\
&= I + II + III.
\end{aligned}$$

By symmetry, it will be enough to consider the contributions of I and II.

Contribution of I

Using a commutator operator, we decompose

$$D_x^s(e^{-iF}uI) = D_x^s(e^{-iF}u)I + [D_x^s, I]e^{-iF}u.$$

Therefore thanks to lemma 3 we obtain

$$\begin{aligned}
&\Big\|D_x^{s+1/2}\int_0^t V(t-\tau)P_+\Big(e^{-iF}u\int_{-\infty}^x P_0(u^{k-2}u_x\mathcal{H}u_x)\Big)d\tau\Big\|_{L_x^\infty L_T^2} \\
&\lesssim \Big\|D_x^s(e^{-iF}u)\int_{-\infty}^x P_0(u^{k-2}u_x\mathcal{H}u_x)\Big\|_{L_T^1L_x^2} \\
&\qquad + \Big\|D_x^{1/4}\Big[D_x^s, \int_{-\infty}^x P_0(u^{k-2}u_x\mathcal{H}u_x)\Big]e^{-iF}u\Big\|_{L_x^{4/3}L_T^1} \\
&\lesssim D + E.
\end{aligned}$$

The contribution of D is treated as follows.

$$\begin{aligned}
D &\lesssim \|D_x^s(e^{-iF}u)\|_{L_T^\infty L_x^2}\Big\|\int_{-\infty}^x P_0(u^{k-2}u_x\mathcal{H}u_x)\Big\|_{L_T^1L_x^\infty} \\
&\lesssim (\|D_x^s u\|_{L_T^\infty L_x^2} + \|D_x^s e^{-iF}\|_{L_T^\infty L_x^{1/s}}\|u\|_{L_T^\infty L_x^{(\frac{1}{2}-s)^{-1}}})\|P_0(u^{k-2}\partial_x G(u,u))\|_{L_{xT}^1} \\
&\lesssim T^\nu\|u\|_{L_T^\infty H_x^s}(1+\|u\|_{L_T^\infty H_x^{s_k}}^k)\|P_0(u^{k-2}\partial_x G(u,u))\|_{L_x^1L_T^{1/(1-\varepsilon)}} \\
&\lesssim T^\nu\|u\|_{X_T^s}(1+\|u\|_{X_T^s}^k)\|P_0(u^{k-2}\partial_x G(u,u))\|_{L_x^1L_T^{1/(1-\varepsilon)}}.
\end{aligned}$$

The low frequencies term is estimated with lemma 9. We get

$$\begin{aligned}
&\|P_0(u^{k-2}\partial_x G(u,u))\|_{L^1_x L^{1/(1-\varepsilon)}_T} \\
&\lesssim \|D_x^{1-2s+6\varepsilon}u^{k-2}\|_{L^{1/(1-2\varepsilon)}_x L^{1/3\varepsilon}_T}\|D_x^{2s-6\varepsilon}G(u,u)\|_{L^{1/2\varepsilon}_x L^{1/(1-4\varepsilon)}_T} \\
&\quad +\|P_0 u^{k-2}\|_{L_x^{(1-\frac{2s}{3})^{-1}} L_T^{(\frac{4s}{3}-\varepsilon)^{-1}}}\|P_0\partial_x G(u,u)\|_{L_x^{3/2s} L_T^{(1-\frac{4s}{3})^{-1}}} \\
&\lesssim \|u^{k-3}\|_{L_x^{(3s-\frac{1}{2}-2\varepsilon)^{-1}} L_T^{\infty}}\|D_x^{1-2s+6\varepsilon}u\|_{L_x^{(\frac{3}{2}-3s)^{-1}} L_T^{1/3\varepsilon}}\|D_x^{s+1/2-3\varepsilon}u\|^2_{L_x^{1/\varepsilon} L_T^{(\frac{1}{2}-2\varepsilon)^{-1}}} \\
&\quad +\|u\|^{k-2}_{L_x^{k-2} L_T^{\infty}}\|D_x^{1/2}u\|^2_{L_x^{3/s} L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}} \\
&\lesssim \|u\|^k_{X^s_T}.
\end{aligned}$$

Note that in order to bound the norm $N_9 = \|D_x^{1-2s+6\varepsilon}u\|_{L_x^{(\frac{3}{2}-3s)^{-1}} L_T^{1/3\varepsilon}}$, we have to impose $k \geq 12$. Indeed, for $\varepsilon > 0$ small enough, the triplet $(1-3s+6\varepsilon, (\frac{3}{2}-3s)^{-1}, 1/3\varepsilon)$ is 1-admissible if and only if

$$\Big(\frac{3}{2}-3s\Big)^{-1} \geq 4 \quad \text{and} \quad 2\Big(\frac{3}{2}-3s\Big)+3\varepsilon \leq \frac{1}{2}$$

if and only if $s > 5/12 = 1/2 - 1/12$.

To bound E by lemma 6,

$$\begin{aligned}
E &\lesssim T^\nu \Big\| D_x^{1/4}\Big[D_x^s, \int_{-\infty}^x P_0(u^{k-2}u_x\mathcal{H}u_x)\Big]e^{-iF}u\Big\|_{L_x^{4/3} L_T^{1/(1-\varepsilon)}} \\
&\lesssim T^\nu \|D_x^{s-3/4}P_0(u^{k-2}\partial_x G(u,u))\|_{L_x^{(s+\frac{1}{4})^{-1}} L_T^{1/(1-\varepsilon)}}\|u\|_{L_x^{(\frac{1}{2}-s)^{-1}} L_T^{\infty}} \\
&\lesssim T^\nu \|P_0(u^{k-2}\partial_x G(u,u))\|_{L^1_x L^{1/(1-\varepsilon)}_T}\|u\|_{X^s_T} \\
&\lesssim T^\nu \|u\|^{k+1}_{X^s_T}.
\end{aligned}$$

Contribution of II

We split the term II into

$$II = II_1 + II_2 + II_3$$

with

$$\begin{aligned}
II_1 &= \int_{-\infty}^x \tilde{P}_+(u^{k-2}P_-(u_x\mathcal{H}u_x)), \\
II_2 &= \int_{-\infty}^x \tilde{P}_+(P_+(u^{k-2})P_+(u_x\mathcal{H}u_x)), \\
II_3 &= \int_{-\infty}^x \tilde{P}_+(P_-(u^{k-2})P_+(u_x\mathcal{H}u_x)).
\end{aligned}$$

Contribution of II_1

The treatment of II_1 is similar to the one of I. We write

$$D_x^s(e^{-iF}uII_1) = D_x^s(e^{-iF}u)II_1 + [D_x^s, II_1]e^{-iF}u$$

and thus

$$\begin{aligned}
&\Big\|D_x^{s+1/2}\int_0^t V(t-\tau)P_+\Big(e^{-iF}u\int_{-\infty}^x \tilde{P}_+(u^{k-2}P_-(u_x\mathcal{H}u_x))\Big)d\tau\Big\|_{L_x^\infty L_T^2}\\
\lesssim\ &\Big\|D_x^s(e^{-iF}u)\int_{-\infty}^x \tilde{P}_+(u^{k-2}P_-(u_x\mathcal{H}u_x))\Big\|_{L_T^1L_x^2}\\
&+\Big\|D_x^{1/4}\Big[D_x^s,\int_{-\infty}^x \tilde{P}_+(u^{k-2}P_-(u_x\mathcal{H}u_x))\Big]e^{-iF}u\Big\|_{L_x^{4/3}L_T^1}\\
\lesssim\ &D'+E'.
\end{aligned}$$

We first bound D' as

$$\begin{aligned}
D' &\lesssim \|D_x^s(e^{-iF}u)\|_{L_T^\infty L_x^2}\Big\|\int_{-\infty}^x \tilde{P}_+(u^{k-2}P_-(u_x\mathcal{H}u_x))\Big\|_{L_T^1L_x^\infty}\\
&\lesssim \|u\|_{X_T^s}(1+\|u\|_{X_T^s}^k)\|\tilde{P}_+(u^{k-2}P_-\partial_xG(u,u))\|_{L_T^1L_x^{1/(1-\varepsilon)}}\\
&\lesssim T^\nu\|u\|_{X_T^s}(1+\|u\|_{X_T^s}^k)\|\tilde{P}_+(u^{k-2}P_-\partial_xG(u,u))\|_{L_{xT}^{1/(1-\varepsilon)}}
\end{aligned}$$

and using lemma 7, we get

$$\begin{aligned}
&\|\tilde{P}_+(u^{k-2}P_-\partial_xG(u,u))\|_{L_{xT}^{1/(1-\varepsilon)}}\\
&\lesssim \|D_x^{1-2s+6\varepsilon}u^{k-2}\|_{L_x^{1/(1-3\varepsilon)}L_T^{1/3\varepsilon}}\|D_x^{2s-6\varepsilon}G(u,u)\|_{L_x^{1/2\varepsilon}L_T^{1/(1-4\varepsilon)}}\\
&\lesssim \|u^{k-3}\|_{L_x^{(3s-\frac12-3\varepsilon)^{-1}}L_T^\infty}\|D_x^{1-2s+6\varepsilon}u\|_{L_x^{(\frac32-3s)-1}L_T^{1/3\varepsilon}}\|D_x^{s+1/2-3\varepsilon}u\|^2_{L_x^{1/\varepsilon}L_T^{(\frac12-2\varepsilon)-1}}\\
&\lesssim \|u\|_{X_T^s}^k.
\end{aligned}$$

Next, E' is estimated as follows

$$\begin{aligned}
E' &\lesssim T^\nu\Big\|D_x^{1/4}\Big[D_x^s,\int_{-\infty}^x \tilde{P}_+(u^{k-2}P_-(u_x\mathcal{H}u_x))\Big]e^{-iF}u\Big\|_{L_x^{4/3}L_T^{1/(1-\varepsilon)}}\\
&\lesssim T^\nu\|D_x^{s-3/4}\tilde{P}_+(u^{k-2}P_-\partial_xG(u,u))\|_{L_x^{(s+\frac14)^{-1}}L_T^{1/(1-\varepsilon)}}\|u\|_{L_x^{(\frac12-s)^{-1}}L_T^\infty}\\
&\lesssim T^\nu\|\tilde{P}_+(u^{k-2}P_-\partial_xG(u,u))\|_{L_{xT}^{1/(1-\varepsilon)}}\|u\|_{X_T^s}\\
&\lesssim T^\nu\|u\|_{X_T^s}^{k+1}.
\end{aligned}$$

Contribution of II_2

A decomposition of u^{k-2} into low and high frequencies, an integration by parts, and formulas (2.15-2.16) give

(3.10)

$$\begin{aligned}
II_2 &= \int_{-\infty}^x \tilde{P}_+[P_+P_{\le-4}u^{k-2}P_+P_{\ge-3}\partial_xG(u,u)]+\int_{-\infty}^x \tilde{P}_+[P_+P_{\ge-3}u^{k-2}P_+\partial_xG(u,u)]\\
&= \tilde{P}_+[P_+P_{\le-4}u^{k-2}P_+P_{\ge-3}G(u,u)]-i\tilde{P}_+G(P_{\le-4}u^{k-2},P_{\ge-3}\partial_x^{-1}G(u,u))\\
&\quad+i\tilde{P}_+G(P_{\ge-3}\partial_x^{-1}u^{k-2},G(u,u)).
\end{aligned}$$

We bound the first term by

$$\begin{aligned}
&\Big\|D_x^{s+1/2}\int_0^t V(t-\tau)P_+(e^{-iF}u\tilde{P}_+[P_+P_{\leq -4}u^{k-2}P_+P_{\geq -3}G(u,u)])(\tau)d\tau\Big\|_{X_T^s}\\
&\lesssim \|u\tilde{P}_+[P_+P_{\leq -4}u^{k-2}P_+P_{\geq -3}G(u,u)]\|_{L_x^{(\frac{5}{6}+\frac{s}{3})^{-1}}L_T^{(\frac{5}{6}-\frac{2s}{3})^{-1}}}\\
&\lesssim \|u\|_{L_x^{(k-1)(\frac{5}{6}-\frac{s}{3})^{-1}}L_T^{(k-1)(\frac{2s}{3}-\frac{1}{6})^{-1}}}\\
&\quad\times\|P_+P_{\leq -4}u^{k-2}P_+P_{\geq -3}G(u,u)\|_{L_x^{(k-1)[k(\frac{5}{6}+\frac{s}{3})-\frac{5}{3}]^{-1}}L_T^{(k-1)[k(\frac{5}{6}-\frac{2s}{3})-\frac{2}{3}]^{-1}}}\\
&\lesssim \|u\|_{L_x^{(k-1)(\frac{5}{6}-\frac{s}{3})^{-1}}L_T^{(k-1)(\frac{2s}{3}-\frac{1}{6})^{-1}}}\\
&\quad\times\|u^{k-2}\|_{L_x^{\frac{k-1}{k-2}(\frac{5}{6}-\frac{s}{3})^{-1}}L_T^{\frac{k-1}{k-2}(\frac{2s}{3}-\frac{1}{6})^{-1}}}\|G(u,u)\|_{L_x^{3/2s}L_T^{(1-\frac{4s}{3})^{-1}}}\\
&\lesssim \|u\|^{k-1}_{L_x^{(k-1)(\frac{5}{6}-\frac{s}{3})^{-1}}L_T^{(k-1)(\frac{2s}{3}-\frac{1}{6})^{-1}}}\|D_x^{1/2}u\|^2_{L_x^{3/s}L_T^{(\frac{1}{2}-\frac{2s}{3})^{-1}}}\\
&\lesssim T^\nu\|u\|^{k+1}_{X_T^s}. \qquad (3.11)
\end{aligned}$$

The other terms in (3.10) are teated in the same way via lemma 8.

Contribution of II_3

In order to share the derivative on $G(u,u)$ in II_3 with lemma 7, we first integrate by parts

$$II_3=\tilde{P}_+[P_-u^{k-2}P_+G(u,u)]-\tilde{P}_+\Big[\int_{-\infty}^x P_-\partial_x u^{k-2}P_+G(u,u)\Big].$$

Then we see that the first term can be estimated exactly as (3.11). Finally for the last term in the previous equality we repeat the proof for the contribution of II_1. □

4. Proof of Theorem 1

In this section we briefly recall the standard arguments which yield well-posedness for (gBO) ; we refer the reader to [11] for details. We choose $k\geq 12$ and $s_k<s<1/2$.

We start by taking a sequence $(u_0^n)_n$ in $H^\infty(\mathbb{R})$ such that $u_0^n\to u_0$ in $H^s(\mathbb{R})$ and $\|u_0^n\|_{H^s}\leq\|u_0\|_{H^s}$. Now let $u_n\in H^\infty(\mathbb{R})$ be the solutions of (gBO) with initial data u_0^n. Then bounds (3.3) and (3.7) imply the *a priori* estimate

$$\|u_n\|_{X_T^s}\lesssim p_k(\|u_0^n\|_{H^s})\|u_0^n\|_{H^s}+T^\nu p_k(\|u_n\|_{X_T^s})\|u_n\|_{X_T^s}. \tag{4.1}$$

This allows us to obtain the existence of a $T>0$ small enough and a solution $u\in X_T^s$ of (gBO).

Using the integral equation (2.1) and (3.3)-(3.7) it follows that for all $0<t_1<t_2<T$,

$$\begin{aligned}
\|u(t_1)-u(t_2)\|_{H^s}&\lesssim \sup_{t\in[t_1,t_2]}\|u(t)-u(t_1)\|\\
&\lesssim \|u(t)-u(t_1)\|_{L^\infty([t_1,t_2];H^s)}+\Big\|\int_{t_1}^t V(t-\tau)\partial_x(u^{k+1})(\tau)d\tau\Big\|_{L^\infty([t_1,t_2];H^s)}\\
&\lesssim o(1).
\end{aligned}$$

This shows that $u\in\mathcal{C}([0,T];H^s(\mathbb{R}))$.

We now turn to the proof of the uniqueness and the dependance of the solution upon the data. In this purpose we must establish the estimate

$$\|u_1-u_2\|_{X_T^s} \lesssim p_k(\|u_{0,1}\|_{H^s}+\|u_{0,2}\|_{H^s})\|u_{0,1}-u_{0,2}\|_{H^s}+T^\nu p_k(\|u_1\|_{X_T^s}+\|u_2\|_{X_T^s})\|u_1-u_2\|_{X_T^s}$$

for u_1, u_2 two solutions of (gBO) associated to initial data $u_{0,1}$ and $u_{0,2}$ respectively. We process exactly as in section 3 with the gauge transformation

$$w = w_1 - w_2, \quad w_j = P_+(e^{-iF_j}u_j), \quad F_j = \int_{-\infty}^x u_j^k(y,t)dy.$$

The main new ingredient to use is the estimate

$$|e^{i\int_{-\infty}^x f_1} - e^{i\int_{-\infty}^x f_2}| \lesssim \|f_1 - f_2\|_{L^1}$$

for any real functions f_1, f_2 as explained in [11].

Appendix

This subsection is devoted to the proof of theorem 2. As in [12, 13, 10], it is a consequence of the following result.

Lemma 10. *Let $s < 1/3$. Then there exists a sequence of functions $\{h_N\} \subset H^s(\mathbb{R})$ such that for all $T > 0$,*

$$\|h_N\|_{H^s} \lesssim 1$$

$$\lim_{N\to+\infty} \sup_{[0,T]} \Big\| \int_0^t V(t-s)\partial_x((V(s)h_N)^4)ds \Big\|_{H^s} = +\infty \tag{A-1}$$

We show first that lemma 10 implies the result.
Suppose that theorem 2 fails. Since the flow-map $\varphi \mapsto u(\varphi)$ is of class $\mathcal{C}^4$ at the origin, we have the relation

$$F(u,\varphi) := u(\varphi) - V(t)\varphi + \int_0^t V(t-s)u^3(s)\partial_x u(s)ds = 0$$

which together with the implicit function theorem yields

$$v(t,x) := \frac{\partial^3 u}{\partial \varphi^3}(t,x,0)[h_N,...,h_N] = 3!\int_0^t V(t-s)\partial_x((V(s)h_N)^4)ds.$$

Hence

$$\sup_{[0,T]} \|v(t)\|_{H^s} \lesssim \|h_N\|_{H^s} \lesssim 1,$$

witch contradicts (A-1).

Proof of lemma 10

For each integer N, we define the function h_N though its Fourier transform by

$$\widehat{h_N}(\xi) = \alpha^{-1/2}N^{-s}\big(\chi_1(\xi) + \chi_2(\xi)\big)$$

where $\chi_1 = \chi_{[N,N+\alpha]}$, $\chi_2(\xi) = \chi_1(-\xi)$ and $\alpha = N^{-\theta}$, $\theta > 0$ to be chosen later. Observe that h_N is a real valued function since $\widehat{h_N}$ is even. Moreover, an obvious

calculation yields $\|h_N\|_{H^s} \simeq 1$.
We now want to estimate $|\hat{v}|$:

$$\begin{aligned}
\hat{v}(\xi_0,t) &\simeq \xi_0 e^{ip(\xi_0)t}\int_0^t e^{-ip(\xi_0)s}\mathcal{F}_x((V(s)h_N)^4)ds \\
&\simeq \alpha^{-2}N^{-4s}\xi_0 e^{ip(\xi_0)t}\sum_{n=0}^{4}\int_0^t e^{-ip(\xi_0)s}\big(e^{ip(\xi_0)s}\chi_1\big)^{*n} * \big(e^{ip(\xi_0)s}\chi_2\big)^{*(4-n)}ds \\
&:= \sum_{n=0}^{4} v_n
\end{aligned}$$

where we defined $p(\xi)=\xi|\xi|$ and $f^{*n}=f*...*f$. The function v_4 is supported in $[4N,4N+4\alpha]$ which is disjoined with the supports of v_n, $n=0,1,2,3$. Consequently,

$$\begin{aligned}
&\hat{v}(\xi_0,t)\chi_{[4N,4N+4\alpha]}(\xi_0) \\
&\simeq \alpha^{-2}N^{-4s}\xi_0 e^{ip(\xi_0)t}\int_0^t\int_{\mathbb{R}^3} e^{-ip(\xi_0)s}e^{ip(\xi_0-\xi_1)s}e^{ip(\xi_1-\xi_2)s}e^{ip(\xi_2-\xi_3)s}e^{ip(\xi_3)s} \\
&\qquad \times\chi_1(\xi_0-\xi_1)\chi_1(\xi_1-\xi_2)\chi_1(\xi_2-\xi_3)\chi_1(\xi_3)d\xi_1 d\xi_2 d\xi_3 ds \\
&\simeq \alpha^{-2}N^{-4s}\xi_0 e^{ip(\xi_0)t}\int_{\mathbb{R}^3}\frac{e^{itP(\xi_0,\xi_1,\xi_2,\xi_3)}-1}{P(\xi_0,\xi_1,\xi_2,\xi_3)} \\
&\qquad \times\chi_1(\xi_0-\xi_1)\chi_1(\xi_1-\xi_2)\chi_1(\xi_2-\xi_3)\chi_1(\xi_3)d\xi_1 d\xi_2 d\xi_3
\end{aligned}$$

with $P(\xi_0,\xi_1,\xi_2,\xi_3)=-2\sum_{j=1}^{3}\xi_j(\xi_{j-1}-\xi_j)$. For $(\xi_{j-1}-\xi_j)$ and ξ_3 in $[4N,4N+4\alpha]$, we have

$$P(\xi_0,\xi_1,\xi_2,\xi_3)\simeq N^2 \quad \text{and} \quad \Big|\frac{e^{itP(\xi_0,\xi_1,\xi_2,\xi_3)}-1}{P(\xi_0,\xi_1,\xi_2,\xi_3)}\Big| = |t|+O(N^2)\gtrsim 1,$$

witch yields

$$|\hat{v}(\xi_0,t)|\chi_{[4N,4N+4\alpha]}(\xi_0)\gtrsim \alpha^{-2}N^{-4s}|\xi_0|\chi_1^{*4}(\xi_0).$$

By straightforward calculations,

$$\chi_1^{*4}(\xi)\simeq\int_{\mathbb{R}}\exp\big(-i\xi_1(4N+2\alpha-\xi)\big)\Big(\frac{\sin i\alpha\xi_1/2}{\xi_1}\Big)^4 d\xi_1$$

and hence $\chi_1^{*4}(4N+2\alpha)\simeq\alpha^3$. By a continuity argument, $\chi_1^{*4}(\xi)\simeq\alpha^3$ for all $\xi\in[4N,4N+4\alpha]$. This proves that

$$|\hat{v}(\xi_0,t)|\chi_{[4N,4N+4\alpha]}(\xi_0)\gtrsim \alpha N^{-4s+1}\chi_{[4N,4N+4\alpha]}(\xi_0)$$

and finally

$$\|v\|_{H^s}\gtrsim \alpha N^{-4s+1}\Big(\int_{4N}^{4N+4\alpha}(1+|\xi|^2)^s d\xi\Big)^{1/2}\gtrsim \alpha N^{-4s+1}N^s\alpha^{1/2}\gtrsim N^{-3s+1-3\theta/2}.$$

Since $s<1/3$, we can choose $\theta>0$ such that $-3s+1-3\theta/2>0$ and it follows that $\|v\|_{H^s}\to+\infty$. $\square$

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