



HAL
open science

Kac-Moody groups, hovels and Littelmann's paths

Stéphane Gaussent, Guy Rousseau

► **To cite this version:**

Stéphane Gaussent, Guy Rousseau. Kac-Moody groups, hovels and Littelmann's paths. 2007. hal-00137735v1

HAL Id: hal-00137735

<https://hal.science/hal-00137735v1>

Preprint submitted on 21 Mar 2007 (v1), last revised 13 Nov 2008 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Kac-Moody groups, hovels and Littelmann's paths

by

STÉPHANE GAUSSENT and GUY ROUSSEAU

Preliminary version.

Abstract. We give the definition of a kind of building \mathcal{I} for a symmetrizable Kac-Moody group over a field K endowed with a discrete valuation and with a residue field containing \mathbb{C} . Due to some bad properties, we call this \mathcal{I} a hovel. Nevertheless \mathcal{I} has some good properties, for example the existence of retractions with center a sector-germ. This enables us to generalize many results proved in the semi-simple case by S. Gaussent and P. Littelmann [Duke Math. J; 127 (2005), 35-88]. In particular, if $K = \mathbb{C}((t))$, the geodesic segments in \mathcal{I} , with a given special vertex as end point and a good image $\rho\pi$ under some retraction ρ , are parametrized by a Zariski open subset P of \mathbb{C}^N . This dimension N is maximum when $\rho\pi$ is a LS path and then P is closely related to some Mirković-Vilonen cycle.

Introduction

In the representation theory of a complex symmetrizable Kac-Moody algebra \mathfrak{g} , P. Littelmann [94; 95] introduced the path model. It gives a method to compute the multiplicities of a weight μ in an irreducible representation of highest weight λ (a dominant weight) by counting some “Lakshmibai-Seshadri” (or LS) paths of shape λ starting from 0 and ending in μ . When \mathfrak{g} is semi-simple and G is an algebraic group of Lie algebra the Langlands dual \mathfrak{g} of \mathfrak{g} , I. Mirković and K. Vilonen [00] gave a new interpretation of this multiplicity: it is the number of irreducible components (the MV cycles) in some subvariety X_λ^μ of the affine grassmannian $\mathcal{G} = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$.

S. Gaussent and P. Littelmann [05] gave a link between these two theories (when G is semi-simple). Actually the LS paths are drawn in a vector space V which is a Bruhat-Tits apartment A of G (over any non archimedean “local” field K , in particular $K = \mathbb{C}((t))$). So they replaced the LS paths of shape λ from 0 to μ by “LS galleries” of type λ from 0 to μ ; this gives a new “gallery model” for the representations of

2000 Mathematics Subject classification: 22E46 (primary), 20G05, 17B67, 22E65, 20E42, 51E24.

Keywords: Kac-Moody group, local field, building, path.

\mathfrak{g} . Moreover, if we consider the retraction of the Bruhat-Tits building \mathcal{I} of G over $K = \mathbb{C}((t))$ onto A with center some sector-germ $\mathfrak{S}_{-\infty}$ in A , then the image under ρ of a minimal gallery Γ of type λ starting from 0 in \mathcal{I} is a gallery γ in A of type λ which looks much like a LS gallery: it is “positively folded”. Conversely a positively folded gallery γ in A of type λ from 0 to μ is the image under ρ of many minimal galleries Γ in \mathcal{I} . These galleries are parametrized by a complex variety X_γ of dimension $\dim(\gamma)$ moreover γ is a LS gallery if and only if $\dim(\gamma)$ is maximal and then X_γ is isomorphic to an open of a MV cycle in X_λ^μ .

So, it was natural (and suggested to us by P. Littelmann) to try to generalize this in the Kac-Moody case. Actually G. Rousseau [06] had constructed some building for a Kac-Moody group over a “local” field K . But the apartments of this microaffine building are not the good ones to define LS paths inside. So we construct a new set \mathcal{I} associated to the Kac-Moody group G over $K = \mathbb{C}((t))$ (or more generally any local field K with residue field containing \mathbb{C}). The group $G(K)$ acts on \mathcal{I} , the apartments are the good ones for LS paths and $G(K)$ is transitive on them. Unfortunately any two points in \mathcal{I} are not always in a same apartment: this was already noticed (in a different language) by H. Garland [95], who told that Cartan decomposition is true only after some twist. Due to this bad property, \mathcal{I} is called a hovel. Moreover the system of walls in an apartment of \mathcal{I} is not discrete, so the notion of chamber in \mathcal{I} is unusual (but follows an idea of F. Bruhat and J. Tits [72]) and \mathcal{I} is not gallery-connected. So we have to come back to the path model instead of the gallery model.

Nevertheless, we get good generalizations of Gaussent-Littelmann’s results. The choice of a maximal torus in G and a system of positive roots gives us an apartment A in \mathcal{I} and a sector-germ $\mathfrak{S}_{-\infty}$ in A . But a sector-germ and a point in \mathcal{I} are always in a same apartment (this is equivalent to Iwasawa decomposition); so we get easily a retraction ρ of \mathcal{I} onto A with center $\mathfrak{S}_{-\infty}$. Following a definition of M. Kapovich and J. Millson [05], we call a Hecke path in A is a piecewise linear path π_1 in A which is positively folded along true walls (see 1.12 for a precise definition). Now an analogue of some results due to Kapovich-Millson or Gaussent-Littelmann in the semi-simple case may be proven:

Theorem A (3.3) *If π is a geodesic segment (of shape λ) in \mathcal{I} , then $\rho\pi$ is a Hecke path (of shape λ) in A .*

Conversely, any Hecke path π_1 in A is the image under ρ of a geodesic segment π in \mathcal{I} . But, if we want a finite dimensional variety of parameters, we can no longer look at segments with a given starting point but rather at segments with a given end. We get (when $K = \mathbb{C}((t))$):

Theorem B (3.4, 3.5) *Let π_1 be a Hecke path of shape λ in A with end point a special vertex y . Then there exist geodesic segments π in \mathcal{I} with end point y such that $\rho\pi = \pi_1$ and they are parametrized by a Zariski open subset $P(\pi_1, y)$ of \mathbb{C}^N .*

N is the so-called dual dimension of π_1 and it is maximal (among Hecke paths of shape λ with the same starting and ending points) if and only if π_1 is a LS path.

This result enables us to tell that $P(\pi_1, y)$ is a dense open subset of some Mirkovic-Vilonen cycle. In the semi-simple case this MV cycle is, up to isomorphism, the classical one associated to the dual path of π_1 .

In the first section, we recall some results on Kac-Moody groups and their affine apartments. Actually, in the literature one finds many kinds of Kac-Moody groups. We choose the minimal one, the most algebraic. But (in the second section), we will also have to use the maximal one which appears to be a formal completion of the one we choose and has better commutation relations. We give the definitions of LS paths, Hecke paths, dual dimension and codimension. We prove the characterization of LS paths as Hecke paths with maximal dual dimension (resp. minimal codimension).

The construction of the hovel \mathcal{I} is explained completely in section 2. The proofs are involved but we get all what is needed after: in particular the Iwasawa decomposition (2.8), the twin building structure of the residue of \mathcal{I} at some point x (2.16) and, for some "good" subsets Ω in apartments, the structure of their fixator G_Ω and the transitivity of G_Ω on apartments containing Ω (2.12 to 2.15).

We get in section 3 the results on paths explained above, in particular theorems A and B. The last theorem (3.8) tells that there is on \mathcal{I} a preorder relation which induces in each apartment the preorder given by the Tits cone.

We thank Peter Littelmann for his suggestion to look at these problems and for some interesting discussions.

§ 1. Kac-Moody groups

We recall here the main results on Kac-Moody groups and their affine apartments. A good reference is [Rémy-02], see also [Rousseau-06]. We give also a brief account and some new results on Littelmann's theory of paths [Littelmann-94-95-98]

Definitions 1.1.

1) A *Kac-Moody matrix* (or generalized Cartan matrix) is a square matrix $A = (a_{i,j})_{i,j \in I}$, with integer coefficients, indexed by a finite set I and verifying:

- (i) $a_{i,i} = 2 \quad \forall i \in I$,
- (ii) $a_{i,j} \leq 0 \quad \forall i \neq j$,
- (iii) $a_{i,j} = 0 \iff a_{j,i} = 0$.

2) A *root generating system* [Bardy-96] is a 5-tuple $\mathcal{S} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ made of a Kac-Moody matrix A indexed by I , of 2 dual free \mathbb{Z} -modules X (of characters) and Y (of cocharacters) of finite rank $rk(X)$, a family $(\alpha_i)_{i \in I}$ (of simple roots) in X and a family $(\alpha_i^\vee)_{i \in I}$ (of simple coroots) in Y . They have to verify the following compatibility condition: $a_{i,j} = \alpha_j(\alpha_i^\vee)$.

The *Langlands dual* of \mathcal{S} is $\mathcal{S}^\sim = ({}^tA, Y, X, (\alpha_i^\sim)_{i \in I}, (\alpha_i)_{i \in I})$, where tA is the transposed matrix of A .

3) The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}_{\mathcal{S}}$ is a complex Lie algebra generated by the standard Cartan subalgebra $\mathfrak{h} = Y \otimes \mathbb{C}$ and the Chevalley generators $(e_i)_{i \in I}, (f_i)_{i \in I}$; we shall not explain here the relations, see *e.g.* [Kac-90].

The adjoint action of \mathfrak{h} on \mathfrak{g} gives a gradation of \mathfrak{g} : $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$, where $\Delta \subset X \setminus \{0\} \subset \mathfrak{h}^*$ is the set of *roots* of \mathfrak{g} (with respect to \mathfrak{h}). For all $i \in I$, $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$. If $Q^+ = \sum_i \mathbb{N}\alpha_i$, $\Delta^+ = \Delta \cap Q^+$ and $\Delta^- = -\Delta^+$, one has $\Delta = \Delta^+ \sqcup \Delta^-$.

4) **N.B.** : For simplicity we shall assume in all the paper the following condition:

$F(\mathcal{S})$ The family $(\alpha_i)_{i \in I}$ is free in X and the family $(\alpha_i^\sim)_{i \in I}$ is free in Y .

See remark 1.5.

Starting in 2.10, we shall also assume the Lie algebra \mathfrak{g} symmetrizable, *i.e.* endowed with a non degenerate invariant \mathbb{C} -valued symmetric bilinear form.

1.2 Weyl group and real roots

Let $V = Y \otimes \mathbb{R} \subset \mathfrak{h}$; every element in X defines a linear form on this \mathbb{R} -vector space. For $i \in I$, the formula $r_i(v) = v - \alpha_i(v)\alpha_i^\sim$ defines an involution in V (or \mathfrak{h}), more precisely a reflection of hyperplane $\text{Ker}(\alpha_i)$.

The Weyl group W^v is the subgroup of $GL(V)$ generated by the set $\{r_i\}_{i \in I}$. One knows that it's a Coxeter group; it stabilizes the lattice Y of V . It also acts on X and stabilizes Δ .

One denotes by $\Phi = \Delta_{re}$ the set of real roots *i.e.* the roots which can be written as $\alpha = w(\alpha_i)$ with $w \in W^v$ and $i \in I$. This set Φ is infinite except in the classical case where A is a Cartan matrix and \mathfrak{g} is finite-dimensional (reductive). If $\alpha \in \Phi$, then $r_\alpha = w.r_i.w^{-1}$ is well determined by α , independently of the choice of w and of i such that $\alpha = w(\alpha_i)$. For $v \in V$ one has $r_\alpha(v) = v - \alpha(v)\alpha^\sim$, where the coroot $\alpha^\sim \in Y$ of α verifies $\alpha(\alpha^\sim) = 2$. Hence r_α is the reflection with respect to the hyperplane $M(\alpha) = \text{Ker}(\alpha)$ which is called the *wall* of α . The *half-apartment* associated to α is $D(\alpha) = \{v \in V \mid \alpha(v) \geq 0\}$.

The set Φ is a system of (real) roots in the sense of [Moody-Pianzola-95]. The set Δ is a system of roots in the sense of [Bardy-96]. The imaginary roots (those in $\Delta_{im} = \Delta \setminus \Phi$) will not be very much used here. We define $\Phi^\pm = \Phi \cap \Delta^\pm$.

A subset Ψ of Φ (or Δ) is said *closed* in Φ (or Δ) if: $\alpha, \beta \in \Psi, \alpha + \beta \in \Phi$ (or Δ) $\Rightarrow \alpha + \beta \in \Psi$. The subset Ψ is said *prenilpotent* if there exist $w, w' \in W^v$ such that $w\Psi \subset \Delta^+$ and $w'\Psi \subset \Delta^-$, then Ψ is finite and contained in the subset $w^{-1}(\Phi^+) \cap (w')^{-1}(\Phi^-)$ which is *nilpotent* (*i.e.* prenilpotent and closed).

One denotes by Q^\sim (resp. P^\sim, Q) the "*coroot-lattice*" (resp. "*coweight-lattice*", "*root-lattice*"), *i.e.* the subgroup of Y generated by the α_i^\sim (resp. $P^\sim = \{y \in Y \otimes \mathbb{Q} \mid \alpha_i(y) \in \mathbb{Z}, \forall i \in I\}$, $Q = \sum_i \mathbb{Z}\alpha_i$); one has $Q^\sim \subset Y \subset P^\sim$. Actually Q^\sim, P^\sim or Q is a lattice in V

if and only if the α_i generate V i.e. the α_i generate V^* i.e. $|I| = rk(X) = \dim(V)$. We define the set of *dominant weights* $X^+ = \{\chi \in X \mid \chi(\alpha_i) \geq 0, \forall i \in I\}$ and $X^- = -X^+$. Dually the set of *dominant coweights* is $Y^+ = \{\lambda \in Y \mid \lambda(\alpha_i) \geq 0, \forall i \in I\}$ and $Y^- = -Y^+$.

1.3 The Tits cone

The *positive fundamental chamber* $C_f^v = \{v \in V \mid \alpha_i(v) > 0 \quad \forall i \in I\}$ is a non empty open convex cone. Its closure $\overline{C_f^v}$ is the disjoint union of the *faces* $F^v(J) = \{v \in V \mid \alpha_i(v) = 0 \quad \forall i \in J; \alpha_i(v) > 0 \quad \forall i \notin J\}$ for $J \subset I$; one has $C_f^v = F^v(\emptyset)$ and $V_0 = F^v(I)$ is a vector subspace. These faces are called *vectorial* as they are convex cones with base point 0. One tells that the face $F^v(J)$ or the set J is *spherical* (or of *finite type*) if the matrix $A(J) = (a_{i,j})_{i,j \in J}$ is a Cartan matrix (in classical sense), i.e. if $W^v(J) = \langle r_i \mid i \in J \rangle$ is finite; it's true for the chamber C_f^v or its *panels* $F^v(\{i\}), \forall i \in I$.

The *Tits cone* is the union \mathcal{T} of the positive *closed-chambers* $w.\overline{C_f^v}$ for $w \in W^v$. Its interior is the *open-Tits-cone* \mathcal{T}^o , disjoint union of the (positive) *spherical faces* $w.F^v(J)$ for w in W^v and J spherical. Both \mathcal{T} , \mathcal{T}^o and their closure $\overline{\mathcal{T}}$ are convex cones stable under W^v . They may be defined as:

$$\begin{aligned} \mathcal{T} &= \{v \in V \mid \alpha(v) < 0 \text{ only for a finite number of } \alpha \in \Delta^+ \text{ (or } \Phi^+) \}, \\ \mathcal{T}^o &= \{v \in V \mid \alpha(v) \leq 0 \text{ only for a finite number of } \alpha \in \Delta^+ \text{ (or } \Phi^+) \}, \\ \overline{\mathcal{T}} &= \{v \in V \mid \alpha(v) \geq 0 \quad \forall \alpha \in \Delta_{im}^+ \}. \end{aligned}$$

The action of W^v on the positive chambers is simply transitive. The fixator or the stabilizer of $F^v(J)$ is $W^v(J)$.

We shall also consider the negative Tits cones $-\mathcal{T}$, $-\mathcal{T}^o$, $-\overline{\mathcal{T}}$ and all negative faces, chambers ... which are obtained by the change $\Omega \mapsto -\Omega$.

Actually, $\mathcal{T}^o \cap -\mathcal{T}^o = \emptyset$ except in the classical case (where $\mathcal{T}^o = -\mathcal{T}^o = V$) and $\mathcal{T} \cap -\mathcal{T} = \{v \in V \mid \alpha(v) = 0 \text{ for almost all } \alpha \in \Phi \text{ (or } \Delta) \}$ is reduced to $V_0 = \bigcap_{\alpha \in \Delta} \text{Ker}(\alpha)$ if no connected component of I is spherical.

1.4 The Kac-Moody groups

One considers the (split, complex) Kac-Moody group $G = G_S$ associated to the above root generating system as defined by Tits [1987], see [Rémy-02; chapitre 8]. It is actually an affine ind-algebraic-group [Kumar-02; 7.4.14].

For any field K containing \mathbb{C} , the group $G(K)$ of the points of G in K is generated by some subgroups:

- the fundamental torus $T_f(K)$ where $T = T_f = \text{Spec}(\mathbb{Z}[X])$, hence $T(K)$ is isomorphic to the group $(K^*)^n = (K^*) \otimes_{\mathbb{Z}} Y$ and the character (resp. cocharacter) group of T_f is X (resp. Y).

- root subgroups $U_\alpha(K)$ for $\alpha \in \Phi$, each isomorphic to the additive group $(K, +)$ by an isomorphism (of algebraic groups) x_α .

Actually we consider an isomorphism $x_\alpha : K \simeq \mathfrak{g}_\alpha \otimes K \rightarrow U_\alpha(K)$ where the additive group $\mathfrak{g}_\alpha \otimes K$ is identified with K by the choice of a Chevalley generator e_α of the 1-dimensional complex space \mathfrak{g}_α .

Let M be an \mathfrak{h} -diagonalizable \mathfrak{g} -module with weights in X , and where the action of each \mathfrak{g}_α is locally nilpotent (*e.g.* \mathfrak{g} itself); then $G(K)$ acts on $M \otimes K : T_f(K)$ acts via the character λ on $M_\lambda \otimes K$ and the action of $x_\alpha(a)$ for $a \in \mathfrak{g}_\alpha \otimes K$ is the exponential of the action of a .

1.5 About the freedom condition $F(\mathcal{S})$

This condition is used in 1.3 to obtain that C_f^v is non empty and in 1.13.2 below for the existence of ρ_{Φ^+} . Moreover, if it is not true, then W^v as defined in 1.2 could be more little than wanted (finite), and the roots of $\mathfrak{g}_\mathcal{S}$ could not be defined by the adjoint action of \mathfrak{h} .

But actually $F(\mathcal{S})$ is not necessary to define $\mathfrak{g}_\mathcal{S}$ or $G_\mathcal{S}$ [Rémy-02].

Kac and Kumar assume $F(\mathcal{S})$ (as also Littelmann) and a minimality condition for the rank of $X : rk(X) = |I| + \text{corank}(A)$. Kumar assumes moreover a simple-connectedness condition: $(SC) \quad \sum_{i \in I} \mathbb{Z}\alpha_i$ is cotorsion-free in Y .

For a root generating system \mathcal{S} , define $\mathcal{S}_{sc} = (A, X_{sc}, Y_{sc}, (\alpha_i)_{i \in I}, (\alpha_i^*)_{i \in I})$ by $X_{sc} = X \oplus \mathbb{Z}^I$, Y_{sc} the dual of X_{sc} and $\alpha_i^*(x + (n_j)_{j \in I}) = \alpha_i(x) + n_i$. The group $G_\mathcal{S}$ is a quotient of $G_{\mathcal{S}_{sc}}$ by a subtorus of the torus $T_{\mathcal{S}_{sc}}$, central in $G_{\mathcal{S}_{sc}}$.

Starting from \mathcal{S} verifying $F(\mathcal{S})$, then \mathcal{S}_{sc} verifies $F(\mathcal{S}_{sc})$ and (SC) . The group $G_{\mathcal{S}_{sc}}$ is the direct product of a torus and a group with the properties assumed by Kumar.

So, there is no trouble in using the results of [Kumar-02] for the groups we define.

1.6 Some commuting relations See [Rousseau-06; 1.5 and 1.6].

If $\{\alpha, \beta\}$ is a prenilpotent pair of roots in Φ (hence $\alpha \neq -\beta$), one denotes by $] \alpha, \beta [$ the finite set of the roots $\gamma = p\alpha + q\beta \in \Phi$ with p and q strictly positive integers and $[\alpha, \beta] =] \alpha, \beta [\cup \{\alpha, \beta\}$; we choose any total order on this set. Then, the product map: $\prod_{\gamma \in [\alpha, \beta]} U_\gamma(K) \rightarrow G(K)$ is a bijection onto the group $U_{[\alpha, \beta]}(K)$ generated by these $U_\gamma(K)$; it's actually an isomorphism of algebraic varieties. The commutator group $[U_\alpha(K), U_\beta(K)]$ is contained in $U_{] \alpha, \beta [}(K)$. More precisely, for $u, v \in K$, one has : $[x_\alpha(u), x_\beta(v)] = \prod x_\gamma(C_{p,q} u^p v^q)$ where the product is on the $\gamma = p\alpha + q\beta \in] \alpha, \beta [$ (in the fixed order) and the $C_{p,q}$ are integers.

The group $T_f(K)$ normalizes $U_\alpha(K)$: if $t \in T_f(K)$ and $u \in K$ one has $tx_\alpha(u)t^{-1} = x_\alpha(\alpha(t)u)$. The subgroup $G^{(\alpha)}$ of G generated by $U_\alpha, U_{-\alpha}$ and $T = T_f$ is, up to its center, isomorphic to PGL_2 . In particular for $\alpha \in \Phi$ and $u \in U_\alpha(K)$, the set $U_{-\alpha}(K)uU_{-\alpha}(K)$ contains a unique element $m(u)$ conjugating $U_\beta(K)$ to $U_{r_\alpha(\beta)}(K)$ for each $\beta \in \Phi$. Moreover for $v, v' \in K^*$:

$$\begin{aligned} m(x_\alpha(v)) &= m(x_{-\alpha}(v^{-1})) = x_{-\alpha}(-v^{-1})x_\alpha(v)x_{-\alpha}(-v^{-1}) = \\ &= \alpha^\vee(v)m(x_\alpha(1)) = m(x_\alpha(1))\alpha^\vee(v^{-1}) \end{aligned}$$

$$\begin{aligned} m(x_\alpha(v))x_\alpha(v')m(x_\alpha(v))^{-1} &= x_{-\alpha}(v^{-2}v') \\ m(x_\alpha(v))^2 &= \check{\alpha}(-1). \end{aligned}$$

If N is the normalizer of $T = T_f$ in G , then $N(K)$ is the group generated by $T(K) = T_f(K)$ and the $m(u)$ for all $\alpha \in \Phi$ and all $u \in U_\alpha(K)$.

Lemma. [Rousseau-06] or [Rémy-02] *There exists an algebraic homomorphism ν^v from N onto W^v such that $\nu^v(m(u)) = r_i$ for $u \in U_{\pm\alpha_i}(K)$ and $\text{Ker}(\nu^v) = T_f$. As K is infinite, $N(K)$ is the normalizer of $T_f(K)$ in $G(K)$ and all maximal split subtori of $G(K)$ are conjugated to $T_f(K)$.*

The conjugacy action of N on T_f is given by ν^v where W^v acts on T_f through its action on X or Y .

1.7 Borel subgroups

The subgroup $U^+(K)$ of $G(K)$ is generated by the groups $U_\alpha(K)$ for $\alpha \in \Phi^+$; it is normalized by $T_f(K)$. We define the same way $U^-(K)$ and $U(\Psi)(K)$ for any set of positive roots in Φ .

The groups $B(K) = B^+(K) = T_f(K).U^+(K)$, $B^-(K) = T_f(K).U^-(K)$ are the *standard (positive, negative) Borel subgroups* of $G(K)$.

One has $U^+(K) \cap B^-(K) = U^-(K) \cap B^+(K) = \{1\}$; more generally:

Bruhat decompositions: $G(K) = U^+(K)N(K)U^+(K) = U^-(K)N(K)U^-(K)$

Moreover the maps from $N(K)$ onto $U^\pm(K) \backslash G(K) / U^\pm(K)$ are one to one.

Birkhoff decompositions: $G(K) = U^+(K)N(K)U^-(K) = U^-(K)N(K)U^+(K)$

Moreover the maps from $N(K)$ onto $U^\pm(K) \backslash G(K) / U^\mp(K)$ are one to one.

1.8 The affine apartment \mathbb{A} , its affine Weyl group W and its preorder relation

1) **Notation.** \mathbb{A} is V considered as an affine space.

The group W^v acts \mathbb{Z} -linearly on Y , hence it acts \mathbb{R} -linearly on $\mathbb{A} = V$. One has also an action of V by translations. Finally one obtains an affine action on \mathbb{A} of the semi-direct product $W_{\mathbb{R}} = W^v \rtimes V$.

For $\alpha \in \Phi$ and $k \in \mathbb{Z}$, $M(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k = 0\}$ is the *wall* associated to (α, k) , it is closed in \mathbb{A} . One has $M(\alpha, k) = M(-\alpha, -k)$.

For $\alpha \in X \setminus \{0\}$ and $k \in \mathbb{Z}$, we define $D(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k \geq 0\}$, it is closed in \mathbb{A} . When $\alpha \in \Phi$, we call $D(\alpha, k)$ the *half-apartment* associated to (α, k) and the set $D^\circ(\alpha, k) = D(\alpha, k) \setminus M(\alpha, k) = V \setminus D(-\alpha, -k)$ is the *open-half-apartment* associated to (α, k) .

The *reflection* associated to the wall $M(\alpha, k)$ is $r_{\alpha, k} : \mathbb{A} \rightarrow \mathbb{A}$ given by the formula:

$$r_{\alpha, k}(y) = r_\alpha(y) - k\check{\alpha}.$$

The group generated by the $r_{\alpha, k}$ is $W = W^v \rtimes Q^\sim \subset W_{\mathbb{R}}$.

The subgroup of $W_{\mathbb{R}}$ made of the elements stabilizing the set of walls is $W_P = W^v \times P^\vee$. One defines also $W_Y = W^v \times Y$ and one has :

$$W \subset W_Y \subset W_P \subset W_{\mathbb{R}} = W^v \times V.$$

2) **Definition.** The affine space \mathbb{A} together with its walls and its Tits cone is the *affine apartment* of G associated to T_f , its (*affine*) *Weyl group* $W = W(\mathbb{A})$ is generated by the reflections with respect to the walls.

As the Tits cone \mathcal{T} is convex, we can define a *preorder-relation* on \mathbb{A} given by $x \leq y \Leftrightarrow y - x \in \mathcal{T}$. This is a true (= antisymmetric) order relation only when Φ generates V and the Kac-Moody matrix A has no factor of finite type.

3) For $x \in \mathbb{A}$, the set Δ_x of all roots α such that $\alpha(x) \in \mathbb{Z}$ is a closed subsystem of roots of Δ in the sense of [Bardy-96; 5.1]. The associated Weyl group W_x^{min} is the subgroup of W generated by all the reflections associated to the walls containing x . It is isomorphic to its image W_x^v in W^v and is a Coxeter group, as shown in [loc. cit.; 5.1.12]. The canonical generators of W_x^v are the r_α for α simple in $\Phi_x^+ = \Delta_x \cap \Phi^+$.

The point x is *special* when $\Phi_x = \Delta_x \cap \Phi$ is equal to Φ *i.e.* when $W_x^v = W^v$.

1.9 Faces

The faces in \mathbb{A} are associated to the above systems of walls and half-apartments. But, as in [Bruhat-Tits-72], they no longer are subsets of \mathbb{A} , they are filters of subsets of \mathbb{A} .

1) **Filters.** See [Bruhat-Tits-72] or [Rousseau-04-06].

A *filter* in a set E is a non empty set F of non empty subsets of E , such that, if $S, S' \in F$ then $S \cap S' \in F$ and, if $S' \supset S \in F$ then $S' \in F$. If Z is a non empty subset of E , the set $F(Z)$ of subsets of E containing Z is a filter (usually identified with Z). If $E \subset E'$, to any filter F in E is associated the filter $F_{E'}$ in E' consisting of all subsets of E' containing a S in F ; one usually makes no difference between F and $F_{E'}$.

A filter F is said contained in another filter $F' : F \subset F'$ (resp. in a subset Z in $E : F \subset Z$) if and only if any set in F' (resp. if Z) is in F . The union of a family of filters in E is the filter consisting of subsets which are in all the filters. Note that these definitions are opposite to the natural ones.

A group Γ fixes (pointwise) (resp. stabilizes) a filter F , if and only if every γ in Γ fixes pointwise a $S \in F$ (resp. for all γ in Γ and all S in F , $\gamma S \in F$).

If E is a topological space, the closure of a filter F in E is the filter \overline{F} consisting of all subsets of E containing the closure of a set in F . If F is a subset of E containing an element x , the *germ* of F in x is the filter $germ_x(F)$ consisting of all subsets of E which are an intersection with F of a neighbourhood of x .

If E is a real affine space and $x \neq y \in E$, then the *segment-germ* $[x, y)$ is the germ of the segment $[x, y]$ in x .

All the definitions above for filters are compatible with the corresponding definitions for subsets and the identification of a subset Z with the filter $F(Z)$.

We say that a family \mathcal{F} of filters *generates* a filter Ω if a set S is in Ω if and only if it is in some filter $F \in \mathcal{F}$. If \mathcal{B} is a basis of the filter Ω , then the (filters canonically associated to the) sets in \mathcal{B} generate Ω .

The *enclosure* $cl(F)$ of a filter F of subsets of \mathbb{A} is the filter made of the subsets of \mathbb{A} containing a finite intersection of half-spaces $D(\alpha, k)$ in F , for $\alpha \in \Phi$ and $k \in \mathbb{Z}$; with this definition, the enclosure of a subset may not be a subset. For \mathcal{P} a non empty subset of $X \setminus \{0\}$, we define also the \mathcal{P} -*enclosure* $cl_{\mathcal{P}}(F)$ by the same definition, just replacing Φ by \mathcal{P} .

2) **Definitions.** A *face* F in the apartment \mathbb{A} is associated to a point $x \in \mathbb{A}$ and a vectorial face F^v in V ; it is called spherical, positive or negative according to the nature of F^v . More precisely a subset S of \mathbb{A} is an element of the face $F(x, F^v)$ if and only if it contains a finite intersection of open-half-apartments or walls containing $\Omega \cap (x + F^v)$ where Ω is an open neighborhood of x in \mathbb{A} . The enclosure of a face $F = F(x, F^v)$ is its closure: the closed-face \overline{F} ; it's the enclosure of the *local-face in x germ* $germ_x(x + F^v)$.

Actually, in the classical case where Φ is finite, this definition is still valid: $F(x, F^v)$ is a subset Z of \mathbb{A} (more precisely: is the filter of subsets containing a subset Z of \mathbb{A}), and this subset Z is a face in the sense of [Bruhat-Tits-72; §1] or [Brown-89; 6.1].

When x is special, the face $F(x, F^v)$ is equal to the local-face $germ_x(x + F^v)$ and thus determines F^v , but it's no longer true in general. Actually, if x is in no wall, this face is independent of F^v ; it may be the filter of all neighborhoods of x .

Note that the union of the faces $F(x, F^v)$ is not always the filter of neighborhoods of x ; it's contained in $(x + \mathcal{T}) \cup (x - \mathcal{T})$ if x is special.

1.10 Chambers, panels...

There is an order on the faces: the assertions " F is a face of F' ", " F' covers F " and " $F \leq F'$ " are by definition equivalent to $F \subset \overline{F'}$.

Any point $x \in \mathbb{A}$ is contained in a unique face $F(x, V_0)$ which is minimal (but seldom spherical); x is a *vertex* if and only if $F(x, V_0) = \{x\}$. When Φ generates V (*i.e.* $rk(X) = |I|$), a special point is a vertex, but the converse is not true.

The *dimension* of a face F is the smallest dimension of an affine space generated by a $S \in F$. The (unique) such affine space E of minimal dimension is the *support* of F . Any $S \in F$ contains a non empty open subset of E .

A *chamber* (or alcove) is a maximal face or equivalently a face such that all its elements contain a non empty open subset of A or a face of dimension $rk(X) = dim(\mathbb{A})$.

A *panel* is a face maximal among faces which are not chambers or equivalently a face of dimension $n - 1$. Its support is a wall.

So the set \mathcal{F} of faces of A , completely determines the set \mathcal{H} of walls.

A *wall of a chamber* C is the support M of a panel F covered by C . Two chambers are called *adjacent* (along F or M) if they cover a common panel (F of support M). But there may exist a chamber covering no panel hence having no wall. So \mathbb{A} is far from being “gallery-connected”.

1.11 Sectors

A *sector* in \mathbb{A} is a V -translate $\mathfrak{s} = x + C^v$ of a vectorial chamber $C^v = \pm w.C_f^v$ ($w \in W^v$), x is its *base point* and C^v its *direction*. Two sectors have the same direction if and only if they are conjugated by V -translation and if and only if their intersection contains another sector.

The *sector-germ* of a sector $\mathfrak{s} = x + C^v$ in \mathbb{A} is the filter \mathfrak{S} of subsets of \mathbb{A} consisting of the sets containing a V -translate of \mathfrak{s} , it is well determined by the direction C^v . So the set of translation classes of sectors in \mathbb{A} , the set of vectorial chambers in V and the set of sector-germs in \mathbb{A} are in canonical bijection.

The sector-germ associated to the positive (resp. negative) fundamental chamber C_f^v (resp. $-C_f^v$) is called the positive (resp. negative) fundamental germ-sector and is denoted by $\mathfrak{S}_{+\infty}$ (resp. $\mathfrak{S}_{-\infty}$).

A *sector-face* in \mathbb{A} is a V -translate $\mathfrak{f} = x + F^v$ of a vectorial face $F^v = \pm wF^v(J)$. The *sector-face-germ* of \mathfrak{f} is the filter \mathfrak{F} of subsets containing a translate \mathfrak{f}' of \mathfrak{f} by an element of F^v (i.e. $\mathfrak{f}' \subset \mathfrak{f}$). If F^v is spherical, then \mathfrak{f} and \mathfrak{F} are also called spherical. The sign of \mathfrak{f} and \mathfrak{F} is the sign of F^v .

1.12 Littelmann’s paths

1) **Definitions.** We consider piecewise linear continuous paths $\pi : [0, 1] \rightarrow \mathbb{A}$ such that each (existing) derived vector $\pi'(t)$ is in an orbit $W^v.\lambda$ of some $\lambda \in \overline{C_f^v}$ under the vectorial Weyl group W^v . Such a path is called a λ -*path*; it is increasing with respect to the preorder relation of 1.8. If $\pi(0)$, $\pi(1)$ and λ are in Y , we say that π is “in Y ”.

For any $t \neq 0$ (resp. $\neq 1$), we let $\pi'_-(t)$ (resp. $\pi'_+(t)$) denote the derivative of π in t from the left (resp. from the right). Further, we define $w_{\pm}(t) \in W^v$ to be the smallest element in its W_{λ}^v -class such that $\pi'_{\pm}(t) = w_{\pm}(t).\lambda$ (where W_{λ}^v is the fixator in W^v of λ). Moreover we denote by $\pi_-(t) = \pi(t) - [0, 1)\pi'_-(t) = [\pi(t), \pi(t - \varepsilon))$ (resp. $\pi_+(t) = \pi(t) + [0, 1)\pi'_+(t) = [\pi(t), \pi(t + \varepsilon))$) (for $\varepsilon > 0$ small) the positive (resp. negative) segment-germ of π at t (cf. 1.9.1).

The reverse path $\bar{\pi}$ defined by $\bar{\pi} = \pi(1 - t)$ has symmetric properties, it is a $(-\lambda)$ -path.

If, for all t , $w_+(t) \in W_{\pi(t)}^v.w_-(t)$, we shall say that π is a *billiard path*. This corresponds to what is told in [Kapovich-Millson-05; Lemma 4.4], but seems stronger than the definition given in [l.c. ; 2.5] which looks more like our definition of λ -path.

For any choices of $\lambda \in \overline{C_f^v}$, $\pi_0 \in \mathbb{A}$, $r \in \mathbb{N} \setminus \{0\}$ and sequences $\underline{\tau} = (\tau_1, \tau_2, \dots, \tau_r)$ of elements in W^v/W_{λ}^v and $\underline{a} = (a_0 = 0 < a_1 < a_2 < \dots < a_r = 1)$ of elements in \mathbb{R} , we

define a λ -path $\pi = \pi(\lambda, \pi_0, \underline{\tau}, \underline{a})$ by the formula:

$$\pi(t) = \pi_0 + \sum_{i=1}^{j-1} (a_i - a_{i-1})\tau_i(\lambda) + (t - a_{j-1})\tau_j(\lambda) \quad \text{for } a_{j-1} \leq t \leq a_j.$$

Any λ -path may be defined this way. We shall always assume $\tau_j \neq \tau_{j+1}$.

2) Definitions.

a) [Littelmann-94] A *Lakshmibai-Seshadri path* (or *LS path*) of shape $\lambda \in Y^+$ is a λ -path $\pi = \pi(\lambda, \pi_0, \underline{\tau}, \underline{a})$ starting in $\pi_0 \in Y$ and such that: for all $j = 1, \dots, r-1$, there is an a_j -chain from τ_j to τ_{j+1} i.e. a sequence of cosets in W^v/W_λ^v :

$$\sigma_{j,0} = \tau_j, \sigma_{j,1} = r_{\beta_{j,1}}\tau_j, \sigma_{j,2} = r_{\beta_{j,2}}r_{\beta_{j,1}}\tau_j, \dots, \sigma_{j,s_j} = r_{\beta_{j,s_j}} \dots r_{\beta_{j,1}}\tau_j = \tau_{j+1}$$

where $\beta_{j,1}, \dots, \beta_{j,s_j}$ are positive real roots such that, for all $i = 1, \dots, s_j$:

- (i) $\sigma_{j,i} < \sigma_{j,i-1}$, for the Bruhat Chevalley order on W^v/W_λ^v ,
- (ii) $a_j \cdot \beta_{j,i}(\sigma_{j,i}(\lambda)) \in \mathbb{Z}$,
- (iii) $\ell_\lambda(\sigma_{j,i}) = \ell_\lambda(\sigma_{j,i-1}) - 1$, where $\ell_\lambda(-)$ is the length in W^v/W_λ^v .

N.B. Actually Littelmann asks moreover the following condition

- (iv) π is normalized i.e. $\pi_0 = 0$.

b) [Kapovich-Millson-05; 3.27] A *Hecke path* of shape λ is a λ -path such that, for all $t \in [0, 1] \setminus \{0, 1\}$, $\pi'_+(t) \leq_{W_{\pi(t)}^v} \pi'_-(t)$ which means that there exists a $W_{\pi(t)}^v$ -chain from $\pi'_-(t)$ to $\pi'_+(t)$ i.e. finite sequences $(\xi_0 = \pi'_-(t), \xi_1, \dots, \xi_s = \pi'_+(t))$ of vectors in V and $(\beta_1, \dots, \beta_s)$ of positive real roots such that, for all $i = 1, \dots, s$:

- (v) $r_{\beta_i}(\xi_{i-1}) = \xi_i$,
- (vi) $\beta_i(\xi_{i-1}) < 0$,
- (vii) $r_{\beta_i} \in W_{\pi(t)}^v$ i.e. $\beta_i(\pi(t)) \in \mathbb{Z}$: $\pi(t)$ is in a wall of direction $\text{Ker}\beta_i$.

Remarks. 1) Conditions v) and vii) tell us that π is a billiard path. More precisely in $\pi(t)$ the path is folded by applying successive reflections along walls, the $M(\beta_i, -\beta_i(\pi(t)))$. Moreover condition vi) tells us that the path is "positively folded" (cf. [Gaussent-Littelmann-05]).

2) The definition of affine paths in [Littelmann-98] is a little different; in particular it is stable by concatenation.

3) Comparisons.

Let $\pi = \pi(\lambda, \pi_0, \underline{\tau}, \underline{a})$ be a λ -path. The conditions in b) are trivially verified for $t \neq a_1, \dots, a_{r-1}$. Suppose now $t = a_j$, $1 \leq j \leq r-1$. We compare the conditions above by taking $s = s_j$ (≥ 1), $\beta_i = \beta_{j,i}$. As $\pi'_-(t) = \tau_j(\lambda)$, the condition v) tells us that $\xi_i = \sigma_{j,i}(\lambda)$.

Lemma 1. *Conditions i) and vi) are equivalent. If they are satisfied (for all $i = 1, \dots, s$), then $w_+(t) < w_-(t)$ in the Bruhat-Chevalley order of W^v/W_λ^v .*

This is clear as $\sigma_{j,i} = r_{\beta_i}\sigma_{j,i-1}$ and $\xi_{i-1} = \sigma_{j,i-1}(\lambda)$.

Remark. When λ is in Y , the conditions i) and ii) tell us that $a_j \in \mathbb{Q}$ (as asked actually by Littelmann for LS paths).

Lemma 2. *Suppose that $\pi_0 \in Y$, $\lambda \in Y^+$ and that conditions ii) are verified for $1 \leq j' < j$ and $1 \leq i \leq s_{j'}$. Then the set of conditions ii) for $1 \leq i \leq s_j$ (and this j) is*

equivalent to the set of conditions vii) for $1 \leq i \leq s_j$ (and $t = a_j$).

If $\pi_0 \in Y$, $\lambda \in Y^+$ and conditions ii) (or vii)) are verified for all $1 \leq j \leq r-1$ and $1 \leq i \leq s_j$, then $\pi(1) \in Y$, hence π is in Y .

Proof. $\pi(a_j) = \pi_0 + \sum_{i=1}^j (a_i - a_{i-1})\tau_i(\lambda) = \pi_0 + a_j\tau_j(\lambda) + \sum_{i=1}^{j-1} a_i(\tau_i(\lambda) - \tau_{i+1}(\lambda))$ and (with the $\sigma_{j,i}$ above) :

$$a_j(\tau_{j+1}(\lambda) - \tau_j(\lambda)) = \sum_{i=1}^{s_j} a_j(\sigma_{j,i}(\lambda) - \sigma_{j,i-1}(\lambda)) = \sum_{i=1}^{s_j} a_j \cdot \langle \sigma_{j,i}(\lambda), \beta_{j,i} \rangle \beta_{j,i}^{\check{}}$$

hence the conditions (ii) for $1 \leq i \leq s_j$ tell that $a_j(\tau_{j+1}(\lambda) - \tau_j(\lambda)) \in Q^{\check{}} \subset Y$.

In particular, conditions (ii) for all i, j tell that $\pi(1) \in Y$.

The condition (ii) above for $i = 1, \dots, s_j$ may be written :

$$\mathbb{Z} \ni a_j \cdot \beta_{j,i}(\sigma_{j,i}(\lambda)) = a_j \langle r_{\beta_{j,i}} \dots r_{\beta_{j,1}} \tau_j(\lambda), \beta_{j,i} \rangle = \langle a_j \tau_j(\lambda), r_{\beta_{j,1}} \dots r_{\beta_{j,i}}(\beta_{j,i}) \rangle.$$

It's easy to verify that these conditions for all $i = 1, \dots, s_j$ mean that the roots $\beta_{j,i}$ verify $\langle a_j \tau_j(\lambda), \beta_{j,i} \rangle \in \mathbb{Z}$. If we assume (ii) for each $j' < j$ and $i \leq s_{j'}$, this is equivalent to $\beta_{j,i}(\pi(a_{j'})) \in \mathbb{Z}$. \square

4) Conclusion.

A LS path is a Hecke path in Y . The reverse path $\bar{\pi}$ has symmetric properties, it is a LS^- path. The reverse path of a Hecke path in Y has also symmetric properties, it is a Hecke $^-$ path in Y .

A Hecke path π of shape λ in Y is not far from being a LS path. Condition iii) only is missing. Actually by condition i) one has $s_j \leq \ell_\lambda(\tau_j) - \ell_\lambda(\tau_{j+1})$; so condition iii) is equivalent to $s_j = \ell_\lambda(\tau_j) - \ell_\lambda(\tau_{j+1})$. Hence π is a LS path if and only if the $W_{\pi(t)}^v$ -chains are of maximal lengths: see [Kapovich-Millson-05; prop. 3.24] which is more precise.

1.13 Statistics on paths

1) We define two statistics on λ -paths: The *dual dimension* of a λ -path π , denoted by $\text{ddim}(\pi)$ and the *codimension* of π , denoted by $\text{codim}(\pi)$ are the non negative integers:

$$\text{ddim}(\pi) = \sum_{t>0} \ell_{\pi(t)}(w_-(t)) \quad , \quad \text{codim}(\pi) = \sum_{t<1} \ell_{\pi(t)}(w_+(t)) \quad ,$$

where $\ell_{\pi(t)}(\cdot)$ is the relative length function in the Coxeter group W^v with respect to $W_{\pi(t)}^v$ defined as follows: $\ell_{\pi(t)}(w)$ is the number of walls $M(\alpha)$ for $\alpha \in \Phi_{\pi(t)}^+$ separating the fundamental chamber C_f^v from wC_f^v ; it coincides on $W_{\pi(t)}^v$ with the usual length.

It seems that the sums are infinite, but, actually, there are only a finite number of possible $w_-(t)$ or $\pi'_-(t) = w_-(t)\lambda$ (resp. $w_+(t)$ or $\pi'_+(t) = w_+(t)\lambda$). Moreover, for any t , $\ell_{\pi(t)}(w_-(t))$ (resp. $\ell_{\pi(t)}(w_+(t))$) is the number of roots $\beta \in \Phi_{\pi(t)}^+$ such that $\beta(\pi'_-(t)) < 0$ (resp. $\beta(\pi'_+(t)) < 0$). Hence $\text{ddim}(\pi)$ (resp. $\text{codim}(\pi)$) is the number of pairs $(t, M(\beta, k))$ of a $t > 0$ (resp. $t < 1$) and a wall associated to $\beta \in \Phi^+$ such that $\pi(t) = \bar{\pi}(1-t) \in M(\beta, k)$ and $\pi(t-\varepsilon) = \bar{\pi}(1-t+\varepsilon) \in D^\circ(\beta, k) = \mathbb{A} \setminus D(-\beta, -k)$ (resp. $\pi(t+\varepsilon) \notin D(\beta, k)$), for all small $\varepsilon > 0$; this number is clearly finite.

To be short, $\text{ddim}(\pi)$ is the number (with multiplicity) of walls positively leaved by the reverse path $\bar{\pi}$ (load-bearing walls for $\bar{\pi}$ as in [Gaussent-Littelmann-05]); and $\text{codim}(\pi)$ is the number (with multiplicity) of walls negatively leaved by π .

In the following, for $\beta \in \Phi^+$ and π a λ -path, we define $\text{pos}_\beta(\pi)$ (resp. $\text{neg}_\beta(\pi)$) as the number (with multiplicity) of walls of direction $\text{Ker}\beta$ leaved positively (resp. negatively) by π . Hence:

$$\text{ddim}(\pi) = \sum_{\beta>0} \text{pos}_\beta(\bar{\pi}) \quad \text{and} \quad \text{codim}(\pi) = \sum_{\beta>0} \text{neg}_\beta(\pi)$$

2) **Classical case.** Suppose the λ -path π in Y and denote $\nu = \pi(1) - \pi(0)$.

If Φ is finite, [Gaussent-Littelmann-05] suggests us to define the dimension of π as: $\text{dim}(\pi) = \sum_{\beta>0} \text{pos}_\beta(\pi)$ (so $\text{ddim}(\pi) = \text{dim}(\bar{\pi})$) and to prove (for Hecke paths) that $\text{dim}(\pi) \leq \rho(\lambda + \nu)$ where $\rho_{\Phi^+} = \rho$ is defined by $2\rho = \sum_{\beta>0} \beta$.

But actually, $\beta(\nu) = \text{pos}_\beta(\pi) - \text{pos}_\beta(\bar{\pi}) = \text{neg}_\beta(\bar{\pi}) - \text{neg}_\beta(\pi)$. So $\text{dim}(\pi) \leq \rho(\lambda + \nu)$ if and only if $\sum_{\beta>0} \text{pos}_\beta(\pi) \leq \rho(\lambda - \nu) + \sum_{\beta>0} \beta(\nu) = \rho(\lambda - \nu) + \sum_{\beta>0} \text{pos}_\beta(\pi) - \sum_{\beta>0} \text{pos}_\beta(\bar{\pi})$ if and only if $\text{ddim}(\pi) \leq \rho(\lambda - \nu)$.

From above we get, $\text{pos}_\beta(\pi) + \text{neg}_\beta(\pi) = \text{neg}_\beta(\bar{\pi}) + \text{pos}_\beta(\bar{\pi})$, and $\text{dim}(\pi) + \text{codim}(\pi) = \sum_{\beta>0} (\text{pos}_\beta(\pi) + \text{neg}_\beta(\pi)) = \sum_{\beta>0} (\text{neg}_\beta(\bar{\pi}) + \text{pos}_\beta(\bar{\pi}))$ is the number of pairs $(t, M(\beta, k))$ of a $t < 1$ (resp. $t > 0$) such that $\pi(t) \in M(\beta, k)$ and $\pi_+(t) \notin M(\beta, k)$ (resp. $\pi_-(t) \notin M(\beta, k)$). This expression is invariant if we change π to π_1 defined by: $\pi_1(t) = \pi(t)$ for $t \leq t_1$ and $\pi_1(t) = w\pi(t)$ for $t \geq t_1$, for some $t_1 \in [0, 1]$ and $w \in W_{\pi(t_1)}^{\text{min}}$. Moreover any billiard path of shape λ is obtained by a sequence of such transformations starting from the straight λ -path π_λ ($\pi_\lambda(t) = t\lambda$). So $\text{dim}(\pi) + \text{codim}(\pi) = \text{dim}(\pi_\lambda) + \text{codim}(\pi_\lambda) = \text{dim}(\pi_\lambda) = \sum_{\beta>0} \beta(\lambda) = \rho(2\lambda)$. So for billiard paths in Y , $\text{dim}(\pi) \leq \rho(\lambda + \nu)$ if and only if $\text{codim}(\pi) \geq \rho(\lambda - \nu)$.

1.14. We shall prove the inequalities $\text{codim}(\pi) \geq \rho(\lambda - \nu) \geq \text{ddim}(\pi)$ for Hecke paths in Y even if Φ is infinite. We choose $\rho_{\Phi^+} = \rho \in X$ such that $\rho(\alpha) = 1$ for all simple α . It is clear that $\lambda - \nu$ is a linear combination of coroots; so $\rho(\lambda - \nu)$ doesn't depend of the choice of ρ .

Proposition *Let π be a Hecke path of shape λ in Y and $\nu = \pi(1) - \pi(0)$. Then*

$$\text{ddim}(\pi) \leq \rho(\lambda - \nu) \leq \text{codim}(\pi) \quad \text{and} \quad \text{ddim}(\pi) + \text{codim}(\pi) = 2\rho(\lambda - \nu)$$

with equality if and only if π is a LS path.

Proof. By translation, we may (and shall often) suppose π normalized *i.e.* $\pi(0) = 0$.

1) We follow, more or less, the same strategy as in the proof of Proposition 4 in [Gaussent-Littelmann-05]. Particularly, we use the operators e_α and f_α (α simple) defined by Littelmann in [Littelmann-94; 1.2 and 1.3]. We do not recall the whole definition here, but note that when they exist, $e_\alpha\pi = \pi_1 * r_\alpha(\pi_2) * \pi_3$ (resp. $f_\alpha\pi =$

$\pi_1 * r_\alpha(\pi_2) * \pi_3$) for some well-defined cut of $\pi = \pi_1 * \pi_2 * \pi_3$. Further, $e_\alpha \pi(1) = \pi(1) + \check{\alpha}$ and $f_\alpha \pi(1) = \pi(1) - \check{\alpha}$. After a suitable reparametrization $e_\alpha \pi$ and $f_\alpha \pi$ are λ -paths in Y . More important, Littelmann obtains a characterization of LS paths using the operators. He proves [Littelmann-94; 5.6] that a λ -path π with $\pi(0) = 0$ is a LS path if, and only if, there exist some simple roots $\alpha_{i_1}, \dots, \alpha_{i_s}$ such that

$$e_{\alpha_{i_1}} \circ \dots \circ e_{\alpha_{i_s}}(\pi) = \pi_\lambda,$$

where for all $t \in [0, 1]$, $\pi_\lambda(t) = t\lambda$.

Nevertheless, we need an operator \tilde{e}_α to eventually transform Hecke paths into LS ones.

2) Definition Let π be a λ -path. Set $Q = \min\{\alpha(\pi([0, 1])) \cap \mathbb{Z}\}$, the minimal integral value attained by the function $\alpha(\pi(\cdot))$ and let q be the greatest number in $[0, 1]$ such that $\alpha(\pi([0, q])) \geq Q$. If $q < 1$ (i.e. if $Q > \min\{\alpha(\pi([0, 1]))\}$), let $\theta > q$ be such that

$$\alpha(\pi(q)) = \alpha(\pi(\theta)) = Q \quad \text{and} \quad \alpha(\pi(t)) < Q \text{ for } q < t < \theta.$$

We cut the path π into three parts in the following way. Let π_1, π_2 and π_3 be the paths defined by

$$\pi_1(t) = \pi(tq); \quad \pi_2(t) = \pi(q + t(\theta - q)) - \pi(q); \quad \pi_3(t) = \pi(\theta + t(1 - \theta)) - \pi(\theta)$$

for $t \in [0, 1]$. Then, by definition, $\pi = \pi_1 * \pi_2 * \pi_3$, where $*$ means the concatenation of paths as defined in [Littelmann-94; 1.1]. The path $\tilde{e}_\alpha \pi$ is equal to $\pi_1 * r_\alpha(\pi_2) * \pi_3$. After a suitable reparametrization $\tilde{e}_\alpha \pi$ is a λ -path in Y .

3) Lemma. *i) If π is a Hecke path in Y and $e_\alpha \pi$ (resp. $\tilde{e}_\alpha \pi$) is defined, then $\text{ddim}(e_\alpha \pi) = \text{ddim}(\pi) - 1$ and $\text{codim}(e_\alpha \pi) = \text{codim}(\pi) - 1$ (resp. $\text{ddim}(\tilde{e}_\alpha \pi) = \text{ddim}(\pi) + 1$ and $\text{codim}(\tilde{e}_\alpha \pi) = \text{codim}(\pi) - 1$), and, similarly, if $f_\alpha \pi$ is defined, then $\text{ddim}(f_\alpha \pi) = \text{ddim}(\pi) + 1$ and $\text{codim}(f_\alpha \pi) = \text{codim}(\pi) + 1$.*

ii) If π is a Hecke path in Y such that $\tilde{e}_\alpha \pi$ is defined, then $\tilde{e}_\alpha \pi$ is again a Hecke path in Y .

iii) If π is a Hecke path in Y such that $\tilde{e}_\alpha \pi$ is not defined but $e_\alpha \pi$ (resp. $f_\alpha \pi$) is, then $e_\alpha \pi$ (resp. $f_\alpha \pi$) is again a Hecke path in Y .

We postpone the proof of the Lemma.

4) End of proof of proposition 1.14. It is clear that $\text{ddim}(\pi_\lambda) = \text{codim}(\pi_\lambda) = 0$. As a corollary of i) and the characterization of LS paths, if π is a LS λ -path then $\text{ddim}(\pi) = \text{codim}(\pi) = \rho(\lambda - \nu)$. The other way round goes by induction on $\rho(\lambda - \nu)$. We suppose $\pi(0) = 0$. There is only one λ -path π such that $\pi(1) = \lambda$; it is π_λ . And in this case, $\text{ddim}(\pi_\lambda) = 0$.

If $\nu \neq \lambda$, then $w_+(0) \neq id$ and there exists α simple such that $e_\alpha \pi$ or $\tilde{e}_\alpha \pi$ is defined. If, for all β simple, $\tilde{e}_\beta \pi$ is not defined, then the claim follows immediately by

induction and by Lemma 1.13.5. Otherwise, we apply all possible operators \tilde{e}_β to π to end up with a Hecke path η such that $\eta(1) = \pi(1) = \nu$, $\text{ddim}(\eta) = \text{ddim}(\pi) + k$, $\text{codim}(\eta) = \text{codim}(\pi) - k$ ($k > 0$) and there still exists α such that $e_\alpha\eta$ is defined. But then, by induction, $\text{ddim}(\eta) - 1 = \text{ddim}(e_\alpha\eta) \leq \rho(\lambda - e_\alpha\eta(1)) = \rho(\lambda - \nu) - 1$, which implies that $\text{ddim}(\pi) < \rho(\lambda - \nu)$. Moreover $\text{codim}(\pi) + \text{ddim}(\pi) = \text{codim}(\eta) + \text{ddim}(\eta) = \text{codim}(e_\alpha\eta) + \text{ddim}(e_\alpha\eta) + 2 = 2\rho(\lambda - e_\alpha\eta(1)) + 2 = 2\rho(\lambda - \nu)$ (by induction).

Suppose now that $\text{ddim}(\pi) = \rho(\lambda - \nu) > 0$, then for dimension reasons, $\tilde{e}_\alpha\pi$ is never defined. But $e_\alpha\pi$ is and $\text{ddim}(e_\alpha\pi) = \rho(\lambda - \nu) - 1$. Repeating the same argument leads to a sequence of simple roots $\alpha_{i_1}, \dots, \alpha_{i_s}$ such that $e_{\alpha_{i_1}} \circ \dots \circ e_{\alpha_{i_s}}(\pi) = \pi_\lambda$, in other words π is a LS path. This proves the proposition. \square

Remark: This proof tells also that a Hecke path in Y is LS if and only if, for all simple roots $\alpha_j, \alpha_{i_1}, \dots, \alpha_{i_s}$, the minimum of $\alpha_j(e_{\alpha_{i_1}} \circ \dots \circ e_{\alpha_{i_s}}(\pi))$ is in \mathbb{Z} , cf. [Littelmann-95; 4.5].

5) **Proof of the lemma.** We suppose $\pi(0) = 0$. Let us start with the operator e_α and dual dimensions. The paths π and $e_\alpha\pi$ are cut into three parts, meaning that

$$\begin{cases} \pi(t) = \pi_1(t) & \text{and} & e_\alpha\pi(t) = \pi_1(t) & \text{if } 0 \leq t \leq 1/3 \\ \pi(t) = \pi_2(t) + \pi(1/3) & \text{and} & e_\alpha\pi(t) = r_\alpha(\pi_2)(t) + \pi_1(1/3) & \text{if } 1/3 \leq t \leq 2/3 \\ \pi(t) = \pi_3(t) + \pi(2/3) & \text{and} & e_\alpha\pi(t) = \pi_3(t) + e_\alpha\pi(2/3) & \text{if } 2/3 \leq t \leq 1 \end{cases}$$

For the first part of π , that is for $t \leq 1/3$, nothing happens. Because α is a simple root, if $1/3 < t \leq 2/3$, the relative position of $\pi_-(t)$ with respect to a wall $M(\beta, k)$ (with $\beta \neq \alpha$) is the same as the relative position of $e_\alpha\pi_-(t)$ with respect to $r_\alpha M(\beta, k)$. Further, if $2/3 < t \leq 1$, $e_\alpha\pi(t) = \pi(t) + \check{\alpha}$. So, again, the relative positions are the same. It remains to check the positions relatively to the walls $M(\alpha, -Q), M(\alpha, -Q - 2)$ at $t = 2/3$. But $2/3$ is the smallest real number t such that $\pi(t) \in M(\alpha, -Q)$, therefore $\pi_-(2/3) \notin D(-\alpha, Q)$, $e_\alpha\pi(2/3) \in M(-\alpha, Q + 2)$ and $e_\alpha\pi_-(2/3) \subset D(-\alpha, Q + 2)$. Therefore, $\text{ddim}(e_\alpha\pi) = \text{ddim}(\pi) - 1$.

For the formulas $\text{ddim}(\tilde{e}_\alpha\pi) = \text{ddim}(\pi) + 1$ and $\text{ddim}(f_\alpha\pi) = \text{ddim}(\pi) + 1$, the same kind of arguments shows that it suffices to look at the case $t = 2/3$ in the corresponding cuts of the path π . For the operator \tilde{e}_α , one has $\pi(2/3) = \tilde{e}_\alpha\pi(2/3) \in M(-\alpha, Q)$ and $\pi_-(2/3) \subset D(-\alpha, Q)$ whereas $\tilde{e}_\alpha\pi_-(2/3) \notin D(-\alpha, Q)$. This proves the formula for the operator \tilde{e}_α . And for the operator f_α one has: $\pi(2/3) \in M(-\alpha, Q + 1)$, $\pi_-(2/3) \subset D(-\alpha, Q + 1)$ whereas $f_\alpha\pi(2/3) \in D(-\alpha, Q - 1)$ and $f_\alpha\pi_-(2/3) \notin D(-\alpha, Q - 1)$. The proof of i) for the dual dimensions is then complete.

For the codimensions the same kinds of arguments show that it suffices to look at the case $t = 1/3$ and the root α . For the operator e_α , $e_\alpha\pi(1/3) = \pi(1/3) \in M(\alpha, -Q - 1)$, $\pi_+(1/3) \notin D(\alpha, -Q - 1)$, $e_\alpha\pi_+(1/3) \notin D(-\alpha, Q + 1)$; therefore $\text{codim}(e_\alpha\pi) = \text{codim}(\pi) - 1$. For the operator f_α (resp. \tilde{e}_α), $f_\alpha\pi(1/3) = \pi(1/3) \in M(\alpha, -Q)$ (resp.

$\tilde{e}_\alpha\pi(1/3) = \pi(1/3) \in M(\alpha, -Q)$, $\pi_+(1/3) \notin D(-\alpha, Q)$ and $f_\alpha\pi_+(1/3) \notin D(\alpha, -Q)$ (resp. $\pi_+(1/3) \notin D(\alpha, -Q)$ and $f_\alpha\pi_+(1/3) \notin D(-\alpha, Q)$), therefore $\text{codim}(f_\alpha\pi) = \text{codim}(\pi) + 1$ (resp. $\text{codim}(\tilde{e}_\alpha\pi) = \text{codim}(\pi) - 1$).

Concerning ii), using the same arguments again, one has to take only care of the places $t = 1/3$ and $t = 2/3$ in the path π . For $t = 1/3$, the $W_{\pi(1/3)}^v$ -chain for $\tilde{e}_\alpha\pi$ is obtained from the one for π just by adding $\xi_{s+1} = \tilde{e}_\alpha\pi'_+(1/3) = r_\alpha(\pi'_+(1/3))$ and $\beta_{s+1} = \alpha$; as $\alpha(\pi'_+(1/3)) < 0$ the conditions are verified. For $t = 2/3$, the $W_{\pi(2/3)}^v$ -chain for $\tilde{e}_\alpha\pi$ is obtained from the one for π just by adding $\xi_{-1} = \tilde{e}_\alpha\pi'_-(2/3) = r_\alpha(\pi'_-(2/3))$ and $\beta_0 = \alpha$; as $\alpha(\pi'_-(2/3)) > 0$ the conditions are verified (after a shift on the indices of the chain). Therefore, $\tilde{e}_\alpha\pi$ is a Hecke path and ii) is proved.

It remains to prove iii). Let's start with e_α . Once again, it amounts to check the values $t = 1/3$ and $t = 2/3$. The situation around the point $\pi(1/3)$ is the same as above. Because $\tilde{e}_\alpha\pi$ is not defined, $\alpha(\pi'_+(2/3)) \geq 0$. Let (ξ_0, \dots, ξ_s) , $(\beta_1, \dots, \beta_s)$ be the $W_{\pi(2/3)}^v$ -chain from $\pi'_-(2/3)$ to $\pi'_+(2/3)$. If $\alpha = \beta_u$, $1 \leq u \leq s$ (and u is minimum for this property) , then $(r_\alpha\xi_0, r_\alpha\xi_1, \dots, r_\alpha\xi_{u-1} = \xi_u, \xi_{u+1}, \dots, \xi_s)$, $(r_\alpha\beta_1, \dots, r_\alpha\beta_{u-1}, \beta_{u+1}, \dots, \beta_s)$ is the $W_{e_\alpha\pi(2/3)}^v$ -chain from $e_\alpha\pi'_-(2/3)$ to $e_\alpha\pi'_+(2/3)$. If no such u exists and $\alpha(\pi'_+(2/3)) > 0$ (resp. $\alpha(\pi'_+(2/3)) = 0$) this chain is $(r_\alpha\xi_0, \dots, r_\alpha\xi_s, \xi_{s+1} = \xi_s)$, $(r_\alpha\beta_1, \dots, r_\alpha\beta_s, \beta_{s+1} = \alpha)$ (resp. $(r_\alpha\xi_0, \dots, r_\alpha\xi_s = \xi_s)$, $(r_\alpha\beta_1, \dots, r_\alpha\beta_s)$). This proves that $e_\alpha\pi$ is still a Hecke path. The proof for $f_\alpha\pi$ goes an analogue way and is left to the reader ! \square

Corollary 1.15. *Let $y_0, y_1 \in Y$ and $\lambda \in Y^+$ fixed, then the number of Hecke paths π of shape λ starting in y_0 and ending in y_1 is finite.*

N.B. Using Littelmann's path model, it was already clear that the number of LS paths verifying the same conditions is finite, but our proof is purely combinatorial.

Proof. By 1.14, $\ell_{\pi(0)}(w_+(0)) \leq \text{codim}(\pi) \leq 2\rho(\lambda - \nu)$, with $\nu = y_1 - y_0$. As $\pi(0)$ is a special point, this means that there is a finite number of possible $w_+(0)$. So there is a finite number of possible $w_\pm(t)$, or $\sigma_{j,i}$, or $\beta_{j,i}$, or a_j satisfying (i) and (ii) of 1.12.2a. In conclusion the number of Hecke paths $\pi = \pi(\lambda, y_0, \underline{\tau}, \underline{a})$ is finite (perhaps = 0). \square

§ 2. The hovel

2.1. We suppose now the field K endowed with a discrete valuation ω , assumed normalized: $\omega(K^*) = \mathbb{Z}$. The ring of integers is \mathcal{O} ; we choose a uniformizing parameter ϖ , so $\omega(\varpi) = 1$, $\mathcal{O}^* = \mathcal{O} \setminus \varpi\mathcal{O}$ and the residue field is $\kappa = \mathcal{O}/\varpi\mathcal{O}$.

In this article, we suppose that κ contains \mathbb{C} , so, if it is complete for ω , $K = \kappa((\varpi))$ and $\mathcal{O} = \kappa[[\varpi]]$. We are actually interested in the case $\kappa = \mathbb{C}$. The general case needs more knowledge about G , it should appear in [Rousseau-?].

For $\alpha \in \Phi$, $u \in U_\alpha(K)$, we define: $\varphi_\alpha(u) = \omega(t)$, if $u = x_\alpha(t)$ with $t \in K$.

For all $\alpha \in \Phi$ and all $k \in \mathbb{R} \cup \{+\infty\}$, the set $U_{\alpha,k} = \varphi_{\alpha}^{-1}([k, +\infty])$ is a subgroup of $U_{\alpha}(K)$ and $U_{\alpha,\infty} = \{1\}$. See also [Rousseau-06; 2.2].

2.2 Action of $N(K)$ on \mathbb{A}

1) The group $T_f(K)$ acts on \mathbb{A} by translations:

if $t \in T_f(K)$, $\nu(t)$ is the element in V such that $\chi(\nu(t)) = -\omega(\chi(t))$, $\forall \chi \in X$.

This action is W^v -equivariant.

2) **Lemma.** *There exists an action ν of $N(K)$ on \mathbb{A} which induces the preceding one on $T_f(K)$ and such that for $n \in N(K)$, $\nu(n)$ is an affine map with associated linear map $\nu^v(n)$.*

This is a trivial consequence of the corresponding result 2.9.2 in [Rousseau-06].

3) The image of $N(K)$ in $\text{Aut}(\mathbb{A})$ is $\nu(N) = W_Y$. The kernel $H = \text{Ker}(\nu) \subset T_f(K)$ is $H = \mathcal{O}^* \otimes Y = T_f(\mathcal{O})$.

4) By construction $\nu(N(\mathbb{C}))$ fixes 0, so $\nu(m(x_{\alpha}(1)))$ is the reflection $r_{\alpha} = r_{\alpha,0}$ with respect to the wall $M(\alpha,0)$. Moreover $m(x_{\alpha}(u)) = \check{\alpha}(u)m(x_{\alpha}(1))$, hence the image $\nu(m(x_{\alpha}(u)))$ is the reflection $r_{\alpha,\omega(u)}$ with respect to the wall $M(\alpha,\omega(u))$, as by definition one has : $\alpha(\nu(\check{\alpha}(u))) = -\omega(\alpha(\check{\alpha}(u))) = -\omega(u^2) = -2\omega(u)$.

2.3 Wanted: parahoric subgroups

To define something like a building associated to $G(K)$ and the apartment \mathbb{A} of $T_f(K)$, we have to tell what will be the fixator \widehat{P}_x in $G(K)$ of a point x in \mathbb{A} *i.e.* the associated parahoric subgroup. This fixator \widehat{P}_x should contain the fixator \widehat{N}_x of x in $N(K)$ and the groups $U_{\alpha,k}$ for $x \in D(\alpha,k)$. When x is 0 (“origin” of \mathbb{A}), the group \widehat{P}_x should be $G(\mathcal{O})$, so that the orbit of 0 in the “building” will be the affine grassmannian $\mathcal{G} = G(K)/G(\mathcal{O})$, see 2.10.4.

But we have also to define and study parahoric subgroups associated to more general points or faces in \mathbb{A} and this will give us some trouble. Moreover the expected Bruhat decomposition for parahoric subgroups is actually false in general. So the “building” we can construct has bad properties, we shall call it an *hovel* (in french “measure”).

2.4 Definitions

1) Let Ω be a filter of subsets in \mathbb{A} . For $\alpha \in \Delta$, let $f_{\Omega}(\alpha) = \inf\{k \in \mathbb{Z} \mid \Omega \subset D(\alpha,k)\} = \inf\{k \in \mathbb{Z} \mid \alpha(\Omega) + k \subset [0, +\infty)\} \in \mathbb{N} \cup \{+\infty\}$, with this second equality f_{Ω} is defined on X .

The function f_{Ω} is concave [Bruhat-Tits-72]: on X , $f_{\Omega}(\alpha + \beta) \leq f_{\Omega}(\alpha) + f_{\Omega}(\beta)$ and $f_{\Omega}(0) = 0$; in particular $f_{\Omega}(\alpha) + f_{\Omega}(-\alpha) \geq 0$. We say that Ω is *narrow* (resp. *almost open*) if and only if $f_{\Omega}(\alpha) + f_{\Omega}(-\alpha) \in \{0, 1\}$ (resp. $\neq 0$), $\forall \alpha \in \Phi$. The filter Ω is almost open if and only if it is included in no wall, this is true for a chamber. A point or a face is narrow. Actually in the classical case, Ω is narrow if and only if it is included in the closure of a chamber.

2) We define $\mathfrak{g}_\Omega^1 = \mathfrak{h}_\mathcal{O} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha, \Omega}^1)$, where $\mathfrak{h}_\mathcal{O} = \mathfrak{h} \otimes_{\mathbb{C}} \mathcal{O}$, $\mathfrak{g}_{\alpha, \Omega}^1 = \mathfrak{g}_{\alpha, f_\Omega(\alpha)}$ and (in general) $\mathfrak{g}_{\alpha, k} = \mathfrak{g}_\alpha \otimes_{\mathbb{C}} \{t \in K \mid \omega(t) \geq k\}$. This is a sub- \mathcal{O} -Lie-algebra of $\mathfrak{g}_K = \mathfrak{g} \otimes_{\mathbb{C}} K$, but we shall use a slight modification of \mathfrak{g}_Ω^1 :

\mathfrak{g}_Ω is the sub- \mathcal{O} -Lie-algebra generated by $\mathfrak{h}_\mathcal{O}$ and the $\mathfrak{g}_{\alpha, \Omega}^1$ for $\alpha \in \Phi$. If $\mathfrak{g}_{\alpha, \Omega} = \mathfrak{g}_\Omega \cap (\mathfrak{g}_\alpha \otimes K)$, then $\mathfrak{g}_\Omega = \mathfrak{h}_\mathcal{O} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha, \Omega})$, and $\mathfrak{g}_{\alpha, \Omega} = \mathfrak{g}_{\alpha, \Omega}^1 = \mathfrak{g}_{\alpha, f_\Omega(\alpha)}$ for $\alpha \in \Phi$, $\mathfrak{g}_{\alpha, \Omega} \subset \mathfrak{g}_{\alpha, \Omega}^1$ for $\alpha \in \Delta^{im}$. The Lie algebra \mathfrak{g}_Ω depends only of the enclosure of Ω (not of Ω itself). This will be also true for the algebras and groups defined below in 2.4.3, 2.5 (except \widehat{N}_Ω and \widehat{W}_Ω), 2.6 and 2.7.

If Ω is bounded, then \mathfrak{g}_Ω^1 and \mathfrak{g}_Ω are lattices in \mathfrak{g}_K . If Ω is reduced to a special point, then $\mathfrak{g}_\Omega^1 = \mathfrak{g}_\Omega$.

3) We define U_Ω as the subgroup of $G(K)$ generated by the groups $U_{\alpha, \Omega} = U_{\alpha, f_\Omega(\alpha)}$ for $\alpha \in \Phi$, and $U_\Omega^\pm = U_\Omega \cap U^\pm(K)$.

For $\alpha \in \Phi$, $U_\Omega^{(\alpha)} (\subset U_\Omega)$ is generated by $U_{\alpha, \Omega}$ and $U_{-\alpha, \Omega}$; $N_\Omega^{(\alpha)} = N(K) \cap U_\Omega^{(\alpha)}$.

The group $N_\Omega^u (\subset N(K) \cap U_\Omega)$ is generated by all $N_\Omega^{(\alpha)}$ for $\alpha \in \Phi$.

All these algebras and groups are normalized by H . In particular one can define the groups $N_\Omega^{min} = H.N_\Omega^u$ and $P_\Omega^{min} = H.U_\Omega$.

4) Let M be a \mathfrak{g} -module of highest weight (resp. lowest weight) $\Lambda \in X$, then M is the sum of its weight spaces: $M = \oplus_{\lambda \in X} M_\lambda$. We define $M_\Omega = \oplus_{\lambda \in X} M_{\lambda, \Omega}$, where $M_{\lambda, \Omega} = M_{\lambda, f_\Omega(\lambda)}$ and (in general) $M_{\alpha, k} = M_\alpha \otimes_{\mathbb{C}} \{t \in K \mid \omega(t) \geq k\}$. It's a sub- \mathfrak{g}_Ω -module of $M \otimes K$, and a lattice when Ω is bounded.

If the module is integrable, then $\Lambda \in X^+$ (resp. $\Lambda \in X^-$) and $G(K)$ acts on $M \otimes K$. As we are in equal characteristic 0, it's clear that U_Ω stabilizes M_Ω .

Lemma 2.5. ($\alpha \in \Phi$) 1) $U_\Omega^{(\alpha)} = U_{\alpha, \Omega}.U_{-\alpha, \Omega}.N_\Omega^{(\alpha)} = U_{-\alpha, \Omega}.U_{\alpha, \Omega}.N_\Omega^{(\alpha)}$.

2) If $f_\Omega(\alpha) + f_\Omega(-\alpha) > 0$, then $N_\Omega^{(\alpha)} \subset H$.

If $f_\Omega(\alpha) = -f_\Omega(-\alpha) = k$, then $\nu(N_\Omega^{(\alpha)}) = r_{\alpha, k}$.

3) $N_\Omega^{(\alpha)}$ fixes Ω i.e. $\forall n \in N_\Omega^{(\alpha)}$, $\exists S \in \Omega$ pointwise fixed by $\nu(n)$.

Consequence. The group $W_\Omega^{min} = N_\Omega^{min}/H$ is isomorphic to its image W_Ω^v in W^v , it is generated by the reflections $r_{\alpha, k}$ for which $\Omega \subset M(\alpha, k)$ ($\alpha \in \Phi$, $k \in \mathbb{Z}$). The group N_Ω^{min} is included in the group \widehat{N}_Ω , fixator in $N(K)$ of Ω which normalizes H , U_Ω and P_Ω . The group $\widehat{W}_\Omega = \widehat{N}_\Omega/H$ is also isomorphic to a subgroup of W^v .

Proof. Parts 1) and 2) are proved by an easy calculus in SL_2 or PGL_2 ; one can also refer to [Bruhat-Tits-72; 6.4.7] where a more complicated result (non split case) is proved. Clearly 3) is a consequence of 2). \square

2.6 Maximal Kac-Moody groups

The commutation relations in $G(K)$ or $U^\pm(K)$ are complicated, so we shall work in larger groups which are easier to understand.

1) The *positively-maximal Kac-Moody algebra* associated to \mathfrak{g} is the Lie algebra $\widehat{\mathfrak{g}}^p = (\oplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha) \oplus \mathfrak{h} \oplus \widehat{\mathfrak{n}}^+$ where $\widehat{\mathfrak{n}}^+ = \prod_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ is the completion of $\mathfrak{n}^+ = \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$

[Kumar-02].

2) The *positively-maximal Kac-Moody group* G^{pmax} is defined in [Kumar-02] (under the name \mathcal{G}); it contains G as a subgroup (G is named \mathcal{G}^{min} by Kumar). For any closed subset Ψ of Δ^+ , G^{pmax} contains the pro-unipotent pro-group $U^{max}(\Psi)$ of Lie algebra $\widehat{\mathfrak{n}}(\Psi) = \prod_{\alpha \in \Psi} \mathfrak{g}_\alpha$; *i.e.* $U^{max}(\Psi)(K) = \prod_{\alpha \in \Psi} U_\alpha(K)$ where $U_\alpha(K)$ is isomorphic, via an isomorphism x_α , to $\mathfrak{g}_\alpha \otimes K$ (already defined when α is real).

One has the *Bruhat decomposition* $G^{pmax}(K) = \prod_{n \in N(K)} U^{max}(\Delta^+)(K)nU^{max}(\Delta^+)(K)$ and the *Birkhoff decomposition* $G^{pmax}(K) = \prod_{n \in N(K)} U^-(K)nU^{max}(\Delta^+)(K)$.

Moreover $U^-(K) \cap N(K)U^{max}(\Delta^+)(K) = N(K) \cap U^{max}(\Delta^+)(K) = \{1\}$.

N.B. In all the preceding or following notations a sign $+$ may replace (Ψ) when $\Psi = \Delta^+$.

3) The following subalgebras or subgroups associated to a filter Ω are also defined:

$\widehat{\mathfrak{g}}_\Omega^p = \mathfrak{n}_\Omega^- \oplus \mathfrak{h}_\Omega \oplus \widehat{\mathfrak{n}}_\Omega^+$, where $\mathfrak{n}_\Omega^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_{\alpha, \Omega}$ and $\widehat{\mathfrak{n}}_\Omega^+(\Psi) = \prod_{\alpha \in \Psi} \mathfrak{g}_{\alpha, \Omega}$.

$U_\Omega^{max}(\Psi) = \prod_{\alpha \in \Psi} U_{\alpha, \Omega}$, where $U_{\alpha, \Omega}$ is $x_\alpha(\mathfrak{g}_{\alpha, \Omega})$. As we are in equal characteristic zero, the Campbell-Hausdorff formula proves that this is a subgroup of $U^{max}(\Psi)(K)$.

$U_\Omega^{pm}(\Psi) = G(K) \cap U_\Omega^{max}(\Psi) = U^+(K) \cap U_\Omega^{max}(\Psi)$, as, by [Kumar-02; 7.4.3] $U^+(K) = G(K) \cap U^{max}(\Delta^+)(K)$.

4) Let α be a simple root, then by [Kumar-02; 6.1.2, 6.1.3], $U^{max+}(K) = U_\alpha(K) \times U^{max}(\Delta^+ \setminus \{\alpha\})(K)$. By the same proof, $U_\Omega^{max+} = U_{\alpha, \Omega} \times U_\Omega^{max}(\Delta^+ \setminus \{\alpha\})$.

And, intersecting with $G(K)$, one gets: $U_\Omega^{pm+} = U_{\alpha, \Omega} \times U_\Omega^{pm}(\Delta^+ \setminus \{\alpha\})$.

The groups $U_\Omega^{max}(\Delta^+ \setminus \{\alpha\})$ and $U_\Omega^{pm}(\Delta^+ \setminus \{\alpha\})$ above are normalized by $H.U_\Omega^{(\alpha)}$ and $U^{max}(\Delta^+ \setminus \{\alpha\})(K)$ is normalized by $G^{(\alpha)}(K) = \langle T_f(K), U_\alpha(K), U_{-\alpha}(K) \rangle$.

5) One has also to consider the *negatively-maximal Kac-Moody algebra* associated to \mathfrak{g} , $\widehat{\mathfrak{g}}^n = (\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha) \oplus \mathfrak{h} \oplus (\prod_{\alpha \in \Delta^-} \mathfrak{g}_\alpha)$ and the associated *negatively-maximal Kac-Moody group* G^{nmax} . More generally we use all what is obtained from 1,2,3,4 above by changing p to n and \pm to \mp .

Proposition 2.7 : *We have 3 subgroups of $G(K)$ associated to Ω and independent of the choice of Δ^+ in its W^v -conjugacy class:*

1) The group U_Ω (generated by all $U_{\alpha, \Omega}$) is equal to $U_\Omega = U_\Omega^- . U_\Omega^+ . N_\Omega^u = U_\Omega^+ . U_\Omega^- . N_\Omega^u$.

2) The group U_Ω^{pm} generated by U_Ω and U_Ω^{pm+} is equal to $U_\Omega^{pm} = U_\Omega^{pm+} . U_\Omega^- . N_\Omega^u$.

3) Symmetrically, $U_\Omega^{nm} = U_\Omega^{nm-} . U_\Omega^+ . N_\Omega^u$.

4) One has:

i) $U_\Omega \cap N(K) = N_\Omega^u$

ii) $U_\Omega^{pm} \cap N(K) = N_\Omega^u$

iii) $U_\Omega \cap (N(K).U^\pm(K)) = N_\Omega^u . U_\Omega^\pm$

iv) $U_\Omega^{pm} \cap (N(K).U^+(K)) = N_\Omega^u . U_\Omega^{pm+}$

v) $U_\Omega \cap U^\pm(K) = U_\Omega^\pm$

vi) $U_\Omega^{pm} \cap U^+(K) = U_\Omega^{pm+}$

and symmetrically for U_Ω^{nm} .

Remarks. 1) The group U_Ω^{++} generated by the $U_{\alpha, \Omega}$ for $\alpha \in \Phi^+$ is included in U_Ω^+ , itself included in U_Ω^{pm+} . The first inclusion may be strict even for Ω reduced to a special point and A of affine type. The equality $U_\Omega^+ = U_\Omega^{pm+}$ is equivalent to $U_\Omega = U_\Omega^{pm}$, it may

be false for Ω great (*e.g.* a negative sector). The situation should be better for Ω narrow. Actually we shall prove that $H.U_\Omega^{pmm} = H.U_\Omega^{nm}$ when Ω is reduced to a special point or is a spherical face (2.10.2). The problem then is to know if this group is generated by its intersections with the torus and the (real) root groups.

2) In the classical case of reductive groups, one has $G = G^{pmax} = G^{nmax}$ and $U_\Omega^{++} = U_\Omega^+ = U_\Omega^{max+} = U_\Omega^{pmm+}$, $U_\Omega^{--} = U_\Omega^- = U_\Omega^{max-} = U_\Omega^{nm-}$; moreover $U_\Omega (= U_\Omega^{pmm} = U_\Omega^{nm})$ is the same as the group defined in [Bruhat-Tits-72; 6.4.2, 6.4.9]. The group P_Ω^{min} is named P_Ω by Bruhat and Tits.

Proof. (after [Bruhat-Tits-72; 6.4.9])

a) Let $\mathcal{U} = U_\Omega^{pmm}(\Delta^+).U_\Omega^{nm}(\Delta^-).N_\Omega^u \subset G(K)$. By 2.6.4 and 2.5, for α simple, one has:

$$\begin{aligned} \mathcal{U} &= U_\Omega^{pmm}(\Delta^+ \setminus \{\alpha\}).U_\Omega^{nm}(\Delta^- \setminus \{-\alpha\}).U_{\alpha,\Omega}.U_{-\alpha,\Omega}.N_\Omega^u \\ &= U_\Omega^{pmm}(\Delta^+ \setminus \{\alpha\}).U_\Omega^{nm}(\Delta^- \setminus \{-\alpha\}).U_{-\alpha,\Omega}.U_{\alpha,\Omega}.N_\Omega^u \\ &= U_\Omega^{pmm}(\Delta^+ \setminus \{\alpha\}).U_{-\alpha,\Omega}.U_\Omega^{nm}(\Delta^- \setminus \{-\alpha\}).U_{\alpha,\Omega}.N_\Omega^u \\ &= U_\Omega^{pmm}(r_\alpha(\Delta^+)).U_\Omega^{nm}(r_\alpha(\Delta^-)).N_\Omega^u. \end{aligned}$$

So \mathcal{U} doesn't change when Δ^+ is changed by the Weyl group W^v .

b) Hence \mathcal{U} is stable by left multiplication by U_Ω^{pmm+} and all $U_{\alpha,\Omega}$ for $\alpha \in \Phi$. Moreover it contains them, so $\mathcal{U} \supset U_\Omega^{pmm} \supset U_\Omega$.

c) In $G^{pmax}(K)$, let's prove $\mathcal{U} \cap U^{max+}(K) = U_\Omega^{pmm+}$: if $xyz \in U^{max+}(K)$ with $x \in U_\Omega^{pmm+}$, $y \in U_\Omega^{nm-}$ and $z \in N_\Omega^u$, then $yz \in U^{max+}(K)$ and by the Birkhoff decomposition (2.6.2) one has $y = z = 1$.

d) So $U_\Omega^{pmm} \cap U^+(K) = U_\Omega^{pmm} \cap U^{max+}(K) = U_\Omega^{pmm+}$.

The group $U_\Omega(\Delta^+ \setminus \{\alpha\}) := U_\Omega \cap U_\Omega^{pmm}(\Delta^+ \setminus \{\alpha\}) = U_\Omega^+ \cap U_\Omega^{pmm}(\Delta^+ \setminus \{\alpha\})$ is normalized by $U_{\alpha,\Omega}$ and $U_{-\alpha,\Omega}$. By 2.6.4, $U_\Omega^+ = U_{\alpha,\Omega} \times U_\Omega(\Delta^+ \setminus \{\alpha\})$ and symmetrically for U_Ω^- .

e) Now we are able to argue as in a), b) above with a new \mathcal{U} where $U_\Omega^{pmm}(\Delta^+)$ is changed to U_Ω^+ and/or $U_\Omega^{nm}(\Delta^-)$ to U_Ω^- . This proves 1), 2) and 3).

f) Concerning 4), v) is by definition, vi) was proved in d). We prove now iv) and ii); iii) and i) are similar. Let $n \in N(K)$ and $v \in U^+(K)$ be such that $nv \in U_\Omega^{pmm}$. There exist $n' \in N_\Omega^u$, $u' \in U_\Omega^-$ and $v' \in U_\Omega^{pmm+}$ such that $nv = n'u'v'$. Now $n'^{-1}n = u'v'v^{-1}$ and by the Birkhoff decomposition $n = n' \in N_\Omega^u$, $v = u'v'$ so $u' = 1$ and $v = v' \in U_\Omega^{pmm+}$. \square

Proposition 2.8. "Iwasawa decomposition"

1) Suppose Ω narrow, then $G(K) = U^+(K).N(K).U_\Omega$.

2) Suppose moreover Ω almost open *i.e.* $\forall \alpha \in \Phi$, $f_\Omega(\alpha) + f_\Omega(-\alpha) \neq 0$ (hence = 1). Then the natural map from $W_Y = N(K)/H$ onto $U^+(K) \setminus N(K)/U_\Omega$ is one to one.

Remarks. i) We have also $G(K) = U^-(K).N(K).U_\Omega$,

$G^{pmax}(K) = U^{pmax+}(K).N(K).U_\Omega$ and $G^{nmax}(K) = U^{nmax-}(K).N(K).U_\Omega$.

ii) As a consequence when Ω is narrow every subgroup V of $G(K)$ containing U_Ω may be written $V = (V \cap (U^+(K).N(K))).U_\Omega$. If moreover $V \cap (U^+(K).N(K)) = U_V^+.N_V$ with $U_V^+ = V \cap U^+(K)$ and $N_V = V \cap N(K)$ normalizing U_Ω , then $V = U_V^+.N_V.U_\Omega^-$. We shall use this to (almost) identify U_Ω^{pmm} and U_Ω^{nm} (2.10.2).

Proof. The idea goes back to Steinberg. We follow [Bruhat-Tits-72; 7.3.1], see also [Kac-Peterson-85; 3.7] and [Garland-95; 1.6].

1) **Lemma.** Let α be in Φ , then $Z_\alpha := U_\alpha(K) \cdot \{1, r_\alpha\} \cdot T_f(K) \cdot U_\Omega^{(\alpha)}$ contains $G^{(\alpha)}(K)$.

Proof. By the Bruhat decomposition $G^{(\alpha)}(K) \subset U_\alpha(K) \cdot \{1, r_\alpha\} \cdot T_f(K) \cdot U_\alpha(K)$. So it's sufficient to prove that, for $m_\alpha \in N(K)$ such that $\nu^v(m_\alpha) = r_\alpha$ and $u \in U_\alpha(K)$, $m_\alpha u \in Z_\alpha$. If $\varphi_\alpha(u) \geq f_\Omega(\alpha)$, it's clear: $u \in U_{\alpha, \Omega}$. Otherwise $\varphi_\alpha(u) \leq f_\Omega(\alpha) - 1 \leq -f_\Omega(-\alpha)$ and $u = v' m v''$ with $\nu^v(m) = r_\alpha$, $v', v'' \in U_{-\alpha, -\varphi_\alpha(u)} \subset U_{-\alpha, \Omega}$. So $m_\alpha u = m_\alpha v' m v'' \in U_\alpha(K) \cdot T_f(K) \cdot U_{-\alpha, \Omega} \subset Z_\alpha$, and the lemma is proved.

2) The set $Z = U^+(K) \cdot N(K) \cdot U_\Omega$ is stable by left multiplication by $U^+(K)$ and $T_f(K)$. It remains to prove that it is stable by left multiplication by $U_{-\alpha}(K)$ for α a simple root. Let $U(\Phi^+ \setminus \{\alpha\})(K) = G(K) \cap U^{max}(\Delta^+ \setminus \{\alpha\})(K) \subset U^+(K)$, using the lemma 2.5 and 2.6, one gets:

$$\begin{aligned} U_{-\alpha}(K)Z &= U_{-\alpha}(K) \cdot U(\Phi^+ \setminus \{\alpha\})(K) \cdot U_\alpha(K) \cdot N(K) \cdot U_\Omega \\ &\subset U(\Phi^+ \setminus \{\alpha\})(K) \cdot G^{(\alpha)}(K) \cdot N(K) \cdot U_\Omega \\ &\subset U(\Phi^+ \setminus \{\alpha\})(K) \cdot U_\alpha(K) \cdot \{1, r_\alpha\} \cdot T_f(K) \cdot U_\Omega^{(\alpha)} \cdot N(K) \cdot U_\Omega \\ &\subset U^+(K) \cdot T_f(K) \cdot U_{-\alpha}(K) \cdot N(K) \cdot U_\Omega \cup U^+(K) \cdot T_f(K) \cdot r_\alpha \cdot U_{-\alpha}(K) \cdot U_\alpha(K) \cdot N(K) \cdot U_\Omega \\ &\subset U^+(K) \cdot T_f(K) \cdot U_{-\alpha}(K) \cdot N(K) \cdot U_\Omega. \end{aligned}$$

It remains to show that $U_{-\alpha}(K) \cdot N(K) \subset Z$. But $un = n \cdot n^{-1} un \in nU_\beta(K) \subset nU_{-\beta}(K) \cdot \{1, r_\beta\} \cdot T_f(K) \cdot U_\Omega^{(\beta)}$ with $\beta = -\nu^v(n^{-1}\alpha)$. So $un \in U_\alpha(K) \cdot n \cdot \{1, r_\beta\} \cdot T_f(K) \cdot U_\Omega^{(\beta)} \subset U_\alpha(K) \cdot N(K) \cdot U_\Omega$.

3) With clear notations, suppose $n' \in U^+(K)nU_\Omega$. Then, using 2.7 and 2.5 one has: $n'^{-1}n \in U_\Omega n^{-1}U^+(K)n = U_\Omega \cdot U(n^{-1}\Phi^+)(K) \subset H \cdot U_\Omega(n^{-1}\Phi^-) \cdot U(n^{-1}\Phi^+)(K) \subset U(n^{-1}\Phi^-)(K) \cdot H \cdot U(n^{-1}\Phi^+)(K)$ and, by the Birkhoff decomposition, $n'^{-1}n \in H$. \square

2.9 The group \tilde{P}_Ω .

1) Clearly U_Ω^{max+} stabilizes $\hat{\mathfrak{g}}_\Omega^p$ and $G(K)$ stabilizes \mathfrak{g}_K ; so $U_\Omega^{pm+} = G(K) \cap U_\Omega^{max+}$ stabilizes $\mathfrak{g}_\Omega = \hat{\mathfrak{g}}_\Omega^p \cap \mathfrak{g}_K$. Finally U_Ω^{pm} and also U_Ω^{nm} (or H) stabilize \mathfrak{g}_Ω .

2) If M is a highest weight integrable \mathfrak{g} -module, then U_Ω^{max+} stabilizes M_Ω . The group U_Ω^{max-} stabilizes $\widehat{M}_\Omega = \prod_{\lambda \in X} M_{\lambda, \Omega}$ and $G(K)$ stabilizes $M \otimes K$. Finally U_Ω^{pm} and also U_Ω^{nm} (or H) stabilize M_Ω for every highest (or lowest) weight integrable \mathfrak{g} -module M .

3) **Definition.** The group \tilde{P}_Ω is the subgroup of elements in $G(K)$ stabilizing \mathfrak{g}_Ω and M_Ω for every highest (or lowest) weight integrable \mathfrak{g} -module M .

Hence \tilde{P}_Ω contains U_Ω^{pm} , U_Ω^{nm} , U_Ω and H .

We shall soon replace \tilde{P}_Ω by a subgroup P_Ω with more interesting properties.

We have $\tilde{P}_\Omega = (\tilde{P}_\Omega \cap U^+(K) \cdot N(K)) \cdot U_\Omega = (\tilde{P}_\Omega \cap U^-(K) \cdot N(K)) \cdot U_\Omega$ when Ω is narrow.

4) **Lemma.** Let \tilde{N}_Ω be $\tilde{P}_\Omega \cap N(K)$, then $\tilde{P}_\Omega \cap (U^+(K) \cdot N(K)) = U_\Omega^{pm+} \cdot \tilde{N}_\Omega$ and $\tilde{P}_\Omega \cap (U^-(K) \cdot N(K)) = U_\Omega^{nm-} \cdot \tilde{N}_\Omega$. Moreover \tilde{N}_Ω normalizes U_Ω and is the stabilizer (in

$N(K)$ for the action ν on \mathbb{A}) of the \mathcal{P} -enclosure $cl_{\mathcal{P}}(\Omega)$ of Ω , where $\mathcal{P} \subset X$ is the union of Φ and the set of all weights of \mathfrak{h} in all the modules M above.

Proof. a) Let $n \in N(K)$ and $u \in U^+(K)$ be such that $un \in \tilde{P}_{\Omega}$ and $w = \nu^v(n)$. For $\mathcal{M} = M_{\Omega}$ or \mathfrak{g}_{Ω} , $g \in \tilde{P}_{\Omega}$ and $\mu, \mu' \in \mathfrak{h}^*$, we define ${}_{\mu'}|g|_{\mu}$ as the restriction of g to \mathcal{M}_{μ} followed by the projection onto $\mathcal{M}_{\mu'}$ (with kernel $\oplus_{\mu'' \neq \mu'} \mathcal{M}_{\mu''}$). Now $\forall \mu$, ${}_{w\mu}|un|_{\mu} = {}_{w\mu}|n|_{\mu}$ and $n = \oplus_{\mu} {}_{w\mu}|n|_{\mu}$ (in a clear sense); so $n \in \tilde{N}_{\Omega}$. We have $\tilde{P}_{\Omega} \cap (U^+(K).N(K)) = (\tilde{P}_{\Omega} \cap U^+(K)).\tilde{N}_{\Omega}$ and it remains to determine \tilde{N}_{Ω} and $\tilde{P}_{\Omega} \cap U^+(K)$ (or $\tilde{P}_{\Omega} \cap U^-(K)$).

b) $\tilde{P}_{\Omega} \cap U^+(K) = U_{\Omega}^{pm+}$: an inclusion is already proved in 1) and 2) above. So let's take $u = \prod_{\alpha \in \Delta^+} u_{\alpha} \in U^{max+}(K)$ such that u stabilizes \mathfrak{g}_{Ω} (the order on the $u_{\alpha} \in U_{\alpha}(K)$ is chosen such that the height of α is increasing from right to left). We shall prove by induction that each u_{α} is in $U_{\alpha, \Omega}$. We may suppose $u_{\alpha'} \in U_{\alpha', \Omega}$ for $u_{\alpha'}$ on the right of u_{α} moreover, as $U_{\alpha', \Omega}$ stabilizes \mathfrak{g}_{Ω} , we may suppose these $u_{\alpha'}$ equal to 1. So $u = (\prod_{\beta \neq \alpha} u_{\beta}).u_{\alpha}$ where the u_{β} are in $U_{\beta}(K)$ and $ht(\beta) \geq ht(\alpha)$. But ${}_{\alpha}|u|_0 = {}_{\alpha}|u_{\alpha}|_0$ sends $\mathfrak{h}_{\mathcal{O}}$ into $\mathfrak{g}_{\alpha, \Omega}$, so $u_{\alpha} \in U_{\alpha, \Omega}$.

Now if $u \in \tilde{P}_{\Omega} \cap U^+(K)$ it is in $U^{max+}(K) \cap G(K)$ and stabilizes \mathfrak{g}_{Ω} ; by the above argument $u \in U_{\Omega}^{max+} \cap G(K) = U_{\Omega}^{pm+}$.

c) Let $n = n_0 t$, $n_0 \in N(\mathbb{C})$, $\nu^v(n) = w$ and $t \in T_f(K)$, then $nM_{\lambda, k} = M_{w\lambda, k + \omega(\lambda(t))}$. Look now at the action on \mathbb{A} : $nD(\lambda, k) = n_0 D(\lambda, k + \omega(\lambda(t))) = D(w\lambda, k + \omega(\lambda(t)))$. But \mathfrak{g}_{Ω} is generated by $\mathfrak{h}_{\mathcal{O}}$ and the $\mathfrak{g}_{\alpha, \Omega}$ for $\alpha \in \Phi$, so n is in \tilde{P}_{Ω} if and only if $\forall \lambda \in \mathcal{P}$ $f_{\Omega}(\lambda) + \omega(\lambda(t)) = f_{\Omega}(w\lambda)$ if and only if, $\forall \lambda \in \mathcal{P}$, $nD(\lambda, f_{\Omega}(\lambda)) = D(w\lambda, f_{\Omega}(w\lambda))$. This is exactly equivalent to the fact that n transforms a set in $cl_{\mathcal{P}}(\Omega)$ into another set in $cl_{\mathcal{P}}(\Omega)$. Moreover, as \tilde{N}_{Ω} stabilizes \mathfrak{g}_{Ω} , it normalizes U_{Ω} . \square

5) Actually $\mathcal{P} \supset \Phi \cup X^+ \cup X^-$, so it's easy to prove that each $\chi \in X$ is a positive linear combination of some $\lambda \in \mathcal{P}$ (use *e.g.* [Kac-90; 4.3]). Hence the intersections of a finite number of $D(\lambda, f_{\Omega}(\lambda))$ ($\lambda \in \mathcal{P}$) are closed non empty sets which may be supposed compact. So the intersection $c(\Omega)$ of all these $D(\lambda, f_{\Omega}(\lambda))$ is a non empty convex compact set. But \tilde{N}_{Ω} stabilizes $c(\Omega)$ and, as it acts affinely, it fixes a point x_{Ω} in $c(\Omega)$.

We know that $H \subset \tilde{N}_{\Omega}$ and $N_{\Omega}^u \subset \tilde{N}_{\Omega}$. So, to determine \tilde{N}_{Ω} , we only have to determine the subgroup $\tilde{W}_{\Omega} = \tilde{N}_{\Omega}/H = \nu(\tilde{N}_{\Omega})$ of $\widehat{W}_{x_{\Omega}}$; it contains $W_{\Omega}^{min} = \nu(N_{\Omega}^{min})$.

6) Suppose now Ω narrow, then $\tilde{P}_{\Omega} = U_{\Omega}^{pm+}.\tilde{N}_{\Omega}.U_{\Omega} = U_{\Omega}^{pm+}.U_{\Omega}.\tilde{N}_{\Omega} = U_{\Omega}^{pm+}.U_{\Omega}^-. \tilde{N}_{\Omega}$ and $\tilde{P}_{\Omega} = U_{\Omega}^{nm-}.\tilde{N}_{\Omega}.U_{\Omega} = U_{\Omega}^{nm-}.U_{\Omega}.\tilde{N}_{\Omega} = U_{\Omega}^{nm-}.U_{\Omega}^+.\tilde{N}_{\Omega}$. In particular \tilde{P}_{Ω} contains $H.U_{\Omega}^{pm}$ and $H.U_{\Omega}^{nm}$.

2.10 The (parahoric) group P_{Ω} and the "fixator" \hat{P}_{Ω} .

1) As \mathfrak{g}_{Ω} is generated by $\mathfrak{h}_{\mathcal{O}}$ and the $\mathfrak{g}_{\alpha, \Omega}$ for $\alpha \in \Phi$, the derived algebra of \mathfrak{g}_{Ω} is $\mathfrak{g}'_{\Omega} = (\sum_{\alpha \in \Phi} [\mathfrak{g}_{\alpha, \Omega}, \mathfrak{g}_{-\alpha, \Omega}]) \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha, \Omega})$. Let's look at the quotient algebra $\bar{\mathfrak{g}}_{\Omega} = \mathfrak{g}_{\Omega}/\varpi \mathfrak{g}_{\Omega} = (\mathfrak{h} \otimes \kappa) \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha, \Omega}/\varpi \mathfrak{g}_{\alpha, \Omega})$, its derived algebra is $\bar{\mathfrak{g}}'_{\Omega} = (\sum_{\alpha \in \Phi_{\Omega}} \kappa \alpha) \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha} \otimes \kappa)$,

where $\Phi_\Omega = \{\alpha \in \Phi \mid f_\Omega(\alpha) + f_\Omega(-\alpha) = 0\}$ is the set of the real roots α such that Ω is included in a wall of direction $\text{Ker } \alpha$.

From now on, we suppose \mathfrak{g} symmetrizable, so, the orthogonal $(\bar{\mathfrak{g}}'_\Omega)^\perp$ of $\bar{\mathfrak{g}}'_\Omega$ in $\bar{\mathfrak{g}}_\Omega$ is $\{x \in \mathfrak{h} \otimes \kappa \mid \alpha(x) = 0, \forall \alpha \in \Phi_\Omega\} = (\mathfrak{h} \otimes \kappa)^{W_\Omega^{min}}$, fixed point set of W_Ω^{min} in $\mathfrak{h} \otimes \kappa$.

The action of \tilde{P}_Ω (by inner automorphisms) is compatible with the invariant bilinear form; so \tilde{P}_Ω stabilizes \mathfrak{g}_Ω , $\bar{\mathfrak{g}}_\Omega$, $\bar{\mathfrak{g}}'_\Omega$ and $(\bar{\mathfrak{g}}'_\Omega)^\perp$.

2) **Definition.** P_Ω is the fixator of $(\bar{\mathfrak{g}}'_\Omega)^\perp$ for this action of \tilde{P}_Ω .

It's clear that P_Ω contains U_Ω , U_Ω^{pm+} , U_Ω^{nm-} and H . The group $N_\Omega = P_\Omega \cap N(K)$ contains N_Ω^{min} and is often equal to it. The quotient group $W_\Omega = N_\Omega/H$ contains W_Ω^{min} and is included in \tilde{W}_Ω ; it's isomorphic to the fixator in W^v of $(\mathfrak{h} \otimes \kappa)^{W_\Omega^{min}}$ and equal to the fixator in W of $(\mathbb{A})^{W_\Omega^{min}}$.

When for every set S in Ω , $\pm(S - x_\Omega)$ meets the open-Tits-cone (in particular when Ω is a spherical face), it is known that $W_\Omega = W_\Omega^{min}$.

When Ω is narrow $P_\Omega = U_\Omega^{pm+}.U_\Omega^-.N_\Omega = U_\Omega^{nm-}.U_\Omega^+.N_\Omega = U_\Omega^{pm}.N_\Omega = U_\Omega^{nm}.N_\Omega$.

In particular when Ω is a spherical face (or a special point) $N_\Omega = N_\Omega^{min}$ and $P_\Omega = H.U_\Omega^{pm} = H.U_\Omega^{nm}$ is called the *parahoric subgroup* associated to Ω .

3) The fixator \hat{N}_Ω (in $N(K)$ for the action ν) of Ω is actually the fixator of the *support* of Ω : the smallest affine subspace of \mathbb{A} generated by a set in Ω . Clearly $\text{supp}(\Omega) \subset \mathbb{A}^{W_\Omega}$, so $\hat{N}_\Omega \supset N_\Omega$. As \hat{N}_Ω normalizes all the groups previously defined, we may define the group $\hat{P}_\Omega = P_\Omega.\hat{N}_\Omega$. Clearly $\hat{P}_\Omega \cap N(K) = \hat{N}_\Omega$ and $\hat{P}_\Omega \supset U_\Omega^{pm+}, U_\Omega^{nm-}$.

When Ω is narrow $\hat{P}_\Omega = U_\Omega^{pm+}.U_\Omega^-. \hat{N}_\Omega = U_\Omega^{nm-}.U_\Omega^+. \hat{N}_\Omega$.

This group should be the fixator of Ω for the action of $G(K)$ on the "ugly-building" we shall build now. But this will be proved only for some Ω , see 2.12 to 2.14 below.

4) An explicit calculus: Suppose Ω reduced to the special point 0, origin of $V = \mathbb{A}$ chosen as in 2.2. Then in 2.4 one gets $f_0(\alpha) = 0, \forall \alpha$, $\mathfrak{g}_0^1 = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O} = \mathfrak{g}_0$ and $M_0 = M \otimes_{\mathbb{C}} \mathcal{O}$. Hence the definition of the ind-group structure of G [Kumar-02; 7.4.6 and 7.4.7] tells us that $\tilde{P}_0 \subset G(\mathcal{O})$. Moreover $\text{Lie}(G) = \mathfrak{g}$ and the highest or lowest weight modules are defined by morphisms of ind-varieties [*l.c.*; 7.4.E(6) and 7.4.13] so $\tilde{P}_0 = G(\mathcal{O})$. Now $\bar{\mathfrak{g}}_0 = \mathfrak{g} \otimes_{\mathbb{C}} \kappa$ and (as 0 is special) $\hat{N}_0 = N_0^{min}$, $(\bar{\mathfrak{g}}'_0)^\perp = \mathfrak{c} \otimes_{\mathbb{C}} \kappa$ where \mathfrak{c} is the center of \mathfrak{g} ; so $G(\mathcal{O}) = \tilde{P}_0 = P_0 = \hat{P}_0$ (= G_0 with the notation of 2.12).

5) In the classical case of reductive groups, W_Ω is always equal to W_Ω^{min} . If Ω is narrow (*i.e.* included in a closed-face), $P_\Omega = P_\Omega^{min}$ and \hat{P}_Ω are as defined by Bruhat and Tits (*cf.* Remark 2.7.2). In particular \hat{P}_x is the same as in Bruhat-Tits and the following definition gives the (pretty) Bruhat-Tits building.

2.11 Definitions.

The *hovel* (in french, measure) $\mathcal{I} = \mathcal{I}(G, K)$ of G over K is defined as the quotient of $G(K) \times \mathbb{A}$ by the relation:

$$(g, x) \sim (h, y) \iff \exists n \in N \text{ such that } y = \nu(n)x \text{ and } g^{-1}hn \in \hat{P}_x.$$

One proves easily [Bruhat-Tits-72; 7.4.1] that \sim is an equivalence relation. Moreover $\widehat{P}_x \cap N(K) = \widehat{N}_x$. So the map $x \mapsto cl(1, x)$ identifies \mathbb{A} with its image $A_f = A(T_f, K)$, the *apartment of T_f in $\mathcal{I}(G, K)$* .

The left action of $G(K)$ on $G(K) \times \mathbb{A}$ passes through the quotient to an action on \mathcal{I} . The *apartments* of \mathcal{I} are the $g.A_f$ for $g \in G(K)$.

The action of $N(K)$ on $\mathbb{A} = A_f$ is through ν ; in particular H fixes (pointwise) A_f . By construction the fixator of $x \in \mathbb{A}$ is \widehat{P}_x and, for $g \in G(K)$, one has $gx \in \mathbb{A} \Leftrightarrow g \in N(K) \widehat{P}_x$.

From the definition of the groups \widehat{P}_x , it's clear that, for $\alpha \in \Phi$ and $u \in K$, $x_\alpha(u)$ fixes $D(\alpha, \omega(u))$. Hence, for $k \in \mathbb{Z}$, the group $H.U_{\alpha, k}$ fixes $D(\alpha, k)$.

2.12 Good fixators

1) When $\Omega \subset \Omega' \subset \mathbb{A}$, then $\widehat{P}_\Omega \supset \widehat{P}_{\Omega'}$. As \widehat{P}_x is the fixator of $x \in \mathbb{A}$, \widehat{P}_Ω is included in the fixator G_Ω of Ω (for the action of $G(K)$ on $\mathcal{I} \supset \mathbb{A}$). Actually, when Ω is a set $G_\Omega = \cap_{x \in \Omega} \widehat{P}_x$, and when Ω is a filter $G_\Omega = \cup_{S \in \Omega} G_S$.

We have $G_\Omega \cap N(K) = \widehat{N}_\Omega$ and $G_\Omega \supset \widehat{P}_\Omega$ which contains $U_\Omega^{pm+}.U_\Omega^{nm-}.\widehat{N}_\Omega$ and $U_\Omega^{nm-}.U_\Omega^{pm+}.\widehat{N}_\Omega$.

2) For Ω a filter of subsets in \mathbb{A} , the subset of $G(K)$ consisting of the $g \in G(K)$ such that $g.\Omega \subset \mathbb{A}$ is: $G(\Omega \subset \mathbb{A}) = \cup_{S \in \Omega} (\cap_{x \in S} N(K).\widehat{P}_x)$:

$g.\Omega \subset \mathbb{A} \Leftrightarrow \exists S \in \Omega, g.S \subset \mathbb{A} \Leftrightarrow \exists S \in \Omega, \forall x \in S, gx \in \mathbb{A} \Leftrightarrow$ (by 2.11) $\exists S \in \Omega, \forall x \in S, g \in N(K).\widehat{P}_x$.

3) **Definition.** We say Ω in \mathbb{A} has a good fixator if $G_\Omega = \widehat{P}_\Omega = U_\Omega^{pm+}.U_\Omega^{nm-}.\widehat{N}_\Omega = U_\Omega^{nm-}.U_\Omega^{pm+}.\widehat{N}_\Omega$ and $G(\Omega \subset \mathbb{A}) = N(K).\widehat{P}_\Omega$.

By 2.7.a, this definition doesn't depend on the choice of Δ^+ in its $\pm W^v$ -conjugacy class and $N(K)$ permutes the filters with good fixators and the corresponding fixators.

By 2.10.3 and 2.11 a point has a good fixator.

In the classical case of reductive groups, every Ω has a good fixator and \widehat{P}_Ω is as defined by Bruhat and Tits [Bruhat-Tits-72; 7.1.8, 7.1.11, 7.4.8].

4) Suppose Ω has a good fixator or, more generally, that $G(\Omega \subset \mathbb{A}) = N(K).G_\Omega$. Then, if $g \in G(\Omega \subset \mathbb{A})$, there exists $n \in N(K)$ such that $g|_\Omega = n|_\Omega$. Moreover G_Ω is transitive on the apartments containing Ω : if $g.A_f \supset \Omega$, then $g^{-1}\Omega \subset \mathbb{A}$ and $g^{-1} = np \in N(K).G_\Omega$, so $g.A_f = p^{-1}n^{-1}.A_f = p^{-1}.A_f$. Hence G_Ω and all invariant subgroups of G_Ω do not depend of the particular choice of the apartment A_f containing Ω .

Proposition 2.13. 1) If Ω in \mathbb{A} has a good fixator, it's also true for its enclosure $\Omega' = cl(\Omega)$ (or any Ω' with $\Omega \subset \Omega' \subset cl(\Omega)$) and $\widehat{P}_\Omega = \widehat{N}_\Omega.\widehat{P}_{\Omega'}$, $N(K).\widehat{P}_\Omega = N(K).\widehat{P}_{\Omega'}$. In particular any apartment containing Ω contains $cl(\Omega)$.

Conversely, if $supp(\Omega) = \mathbb{A}$ or $supp(cl(\Omega)) = supp(\Omega)$ (hence $\widehat{N}_{cl(\Omega)} = \widehat{N}_\Omega$) and $cl(\Omega)$ has a good fixator, it's also true for Ω .

2) If a filter Ω in \mathbb{A} is generated by a family \mathcal{F} of filters with good fixators, then Ω has a good fixator $\widehat{P}_\Omega = \cup_{F \in \mathcal{F}} \widehat{P}_F$.

3) Suppose that the filter Ω in \mathbb{A} is the union of an increasing sequence $(F_i)_{i \in \mathbb{N}}$ of filters with good fixators and that, for some i , the space $\text{supp}(F_i)$ has a finite fixator W_0 in \widehat{W} , then Ω has a good fixator $\widehat{P}_\Omega = \cap_{i \in \mathbb{N}} \widehat{P}_{F_i}$.

4) Let Ω and Ω' be two filters in \mathbb{A} with good fixators. Suppose there exist a finite number of positive, closed, vectorial chambers $\overline{C}_1^v, \dots, \overline{C}_n^v$ such that: $\Omega \subset \cup_{i=1,n} \Omega' + \overline{C}_i^v$ and $\Omega' \subset \cup_{i=1,n} \Omega - \overline{C}_i^v$ (or $\Omega \subset \cup_{i=1,n} \Omega' - \overline{C}_i^v$). Then $\Omega \cup \Omega'$ has a good fixator and $\widehat{P}_{\Omega \cup \Omega'} = \widehat{P}_\Omega \cap \widehat{P}_{\Omega'}$.

Remark. If in 4) above one has only the condition $\Omega \subset \cup_{i=1,n} \Omega' + \overline{C}_i^v$, the proof below tells that $G_{\Omega \cup \Omega'} = \widehat{P}_{\Omega \cup \Omega'} = U_{\Omega \cup \Omega'}^{pm+} \cdot U_{\Omega \cup \Omega'}^{nm-} \cdot \widehat{N}_{\Omega \cup \Omega'}$ and $(G_\Omega \cdot N(K)) \cap (G_{\Omega'} \cdot N(K)) = (G_\Omega \cap G_{\Omega'}) \cdot N(K)$; hence $G(\Omega \cup \Omega' \subset \mathbb{A}) = N(K) \cdot G_{\Omega \cup \Omega'}$.

Proof. 1) We know that $U_\Omega^{pm+} = U_{\Omega'}^{pm+} = U_{cl(\Omega)}^{pm+}$, $U_\Omega^{nm-} = U_{\Omega'}^{nm-} = U_{cl(\Omega)}^{nm-}$ and $\widehat{N}_{cl(\Omega)} = N(K) \cap G_{cl(\Omega)} \subset \widehat{N}_{cl(\Omega')} \subset \widehat{N}_\Omega$ (with equality when $\text{supp}(cl(\Omega)) = \text{supp}(\Omega)$); so the first and last sentences of 1) are clear. Then the second sentence is a consequence of 2.12.4

2) By definition, $G_\Omega = \cup_{F \in \mathcal{F}} G_F$, $U_\Omega^{pm+} = \cup_{F \in \mathcal{F}} U_F^{pm+}$, $U_\Omega^{nm-} = \cup_{F \in \mathcal{F}} U_F^{nm-}$, $\widehat{N}_\Omega = \cup_{F \in \mathcal{F}} \widehat{N}_F$ and $G(\Omega \subset \mathbb{A}) = \cup_{F \in \mathcal{F}} G(F \subset \mathbb{A})$; so 2) is clear.

3) By definition, $G_\Omega = \cap_{i \in \mathbb{N}} G_{F_i}$, $U_\Omega^{pm+} = \cap_{i \in \mathbb{N}} U_{F_i}^{pm+}$, $U_\Omega^{nm-} = \cap_{i \in \mathbb{N}} U_{F_i}^{nm-}$ and $G(\Omega \subset \mathbb{A}) = \cap_{i \in \mathbb{N}} G(F_i \subset \mathbb{A})$. We may suppose all $\text{supp}(F_i)$ have the same finite fixator W_0 , so $\widehat{N}_\Omega = \widehat{N}_{F_i} = W_0 \cdot H$.

If $g \in \cap_{i \in \mathbb{N}} G_{F_i} = \cap_{i \in \mathbb{N}} U_{F_i}^{pm+} \cdot U_{F_i}^{nm-} \cdot H \cdot W_0$, extracting a subsequence, there exists $n_0 \in N(K)$ such that $gn_0^{-1} \in \cap_{i \in \mathbb{N}} U_{F_i}^{pm+} \cdot U_{F_i}^{nm-} \cdot H$ and this is equal to $U_\Omega^{pm+} \cdot U_\Omega^{nm-} \cdot H$ as $U^\pm(K) \cap B^\mp(K) = \{1\}$ (1.7). So $G_\Omega = U_\Omega^{pm+} \cdot U_\Omega^{nm-} \cdot \widehat{N}_\Omega$.

If $g \in G(\Omega \subset \mathbb{A}) = \cap_{i \in \mathbb{N}} N(K)G_{F_i}$; then, for all i , $g \in w_i G_{F_i}$ for some $w_i \in \widehat{W}$, unique modulo W_0 as $G_{F_i} \cap N(K) = \widehat{N}_{F_i} = W_0 \cdot H$. Extracting a subsequence, we may suppose w_i independent of i , so $g \in w_i \cdot (\cap_j G_{F_j}) = w_i \cdot G_\Omega$ and $G(\Omega \subset \mathbb{A}) \subset N(K) \cdot G_\Omega$.

4) We only have to prove $G_\Omega \cap G_{\Omega'} = U_{\Omega \cup \Omega'}^{pm+} \cdot U_{\Omega \cup \Omega'}^{nm-} \cdot \widehat{N}_{\Omega \cup \Omega'}$ and $(G_\Omega \cdot N(K)) \cap (G_{\Omega'} \cdot N(K)) = (G_\Omega \cap G_{\Omega'}) \cdot N(K)$: the other equalities are similar. Some inclusions are clear, let's show the reverse inclusions; for this we shall only use the hypothesis $\Omega \subset \cup_{i=1,n} \Omega' + \overline{C}_i^v$. By induction we may suppose $\Omega \subset \Omega' + \overline{C}_1^v$. We may also suppose that \overline{C}_1^v is the closed positive fundamental chamber \overline{C}_f^v .

Let uvn be in $(G_\Omega \cdot N(K)) \cap (G_{\Omega'} \cdot N(K))$, with $u \in U_{\Omega'}^{pm+}$, $v \in U_{\Omega'}^{nm-}$ and $n \in N(K)$; we replace now Ω and Ω' by appropriate sets in these filters. Clearly $U_{\Omega'}^{pm+} \subset U_{\Omega' + \overline{C}_f^v}^{pm+} \subset U_{\Omega \cup \Omega'}^{pm+}$. So $\forall x \in \Omega$, $vn \in \widehat{P}_x \cdot N(K)$ and we may write $vn = v'_x u'_x n'_x$ with $v'_x \in U_x^{nm-}$, $u'_x \in U_x^{pm+}$ and $n'_x \in N(K)$. Hence $n'_x n^{-1} = (u'_x)^{-1} (v'_x)^{-1} v$ and, by Birkhoff (1.7), $n'_x = n$, $u'_x = 1$, $v'_x = v \in U_{\Omega' \cup \{x\}}^{nm-}$. Now we have $u \in U_{\Omega \cup \Omega'}^{pm+}$ and $v \in \cap_{x \in \Omega} U_{\Omega' \cup \{x\}}^{nm-} = U_{\Omega \cup \Omega'}^{nm-}$, so $uvn \in U_{\Omega \cup \Omega'}^{pm+} \cdot U_{\Omega \cup \Omega'}^{nm-} \cdot N(K)$.

When $uvn \in G_\Omega \cap G_{\Omega'}$, we have moreover $n \in \widehat{N}_\Omega$ and $n'_x \in \widehat{N}_x$. So $n = n'_x \in$

$\widehat{N}_{\Omega' \cup \{x\}}, \forall x \in \Omega$, and $n \in \widehat{N}_{\Omega \cup \Omega'}$, so $uvn \in U_{\Omega \cup \Omega'}^{pm+} \cdot U_{\Omega \cup \Omega'}^{nm-} \cdot \widehat{N}_{\Omega \cup \Omega'}$. \square

2.14 Examples of filters with good fixators

- 1) If $x \leq y$ in \mathbb{A} , then $\{x, y\}$, $[x, y]$ and $cl(\{x, y\})$ have good fixators and $G_{\{x, y\}} = G_{[x, y]}$.
- 2) A face in \mathbb{A} has a good fixator: the closed-face $\overline{F}(x, F^v)$ has the same support as $F(x, F^v)$ and is generated by the filters $cl(\{x, y_n\})$ where $y_n = x + \frac{1}{n}\xi$ and $\xi \in F^v$.
- 3) A sector in \mathbb{A} has a good fixator: $cl(x + C^v)$ is the increasing union of $cl(\{x, y_n\})$ where $y_n = x + n\xi$ and $\xi \in F^v$, and the sector $x + C^v$ has \mathbb{A} as support.
- 4) A sector-germ has a good fixator. The fixator of $\mathfrak{S}_{\pm\infty}$ is $B^\pm(K)$, as every element in $B^\pm(K)$ is a finite product of elements in groups $U_\alpha(K)$ for $\alpha \in \Phi^\pm$.

On the contrary $U^{max+}(K)$ is not the union of the U_Ω^{max+} for $\Omega \in \mathfrak{S}_{+\infty}$.

- 5) The apartment \mathbb{A} itself has a good fixator $\widehat{P}_\mathbb{A} = H$: \mathbb{A} is the increasing union of $cl(\{-n\xi, n\xi\})$ for $\xi \in C_f^v$.
- 6) For the same reasons a wall $M(\alpha, k)$ has a good fixator $U_{\alpha, k} \cdot U_{-\alpha, -k} \cdot \{1, r_{\alpha, k}\} \cdot H$.
- 7) Exercise: An half-apartment $D(\alpha, k)$ has a good fixator $HU_{\alpha, k}$. If $x_+ - x_- \in \mathcal{T}^o$, then $cl(\{F(x_-, F_-^v), F(x_+, F_+^v)\})$ has a good fixator for all vectorial faces F_ε^v .

2.15 Applications

- 1) The fixator (resp. stabilizer) of the apartment $\mathbb{A} = A_f = A(T_f, K)$ is H (resp. $N(K)$) by 2.14.5. In particular the maps $g \mapsto g \cdot \mathbb{A}$ and $g \mapsto g \cdot T_f \cdot g^{-1}$ give bijections $\{\text{apartments of } \mathcal{I}(G, K)\} \leftrightarrow G(K)/N(K) \leftrightarrow \{\text{maximal split tori of } G(K)\}$; the apartment associated to a torus T is denoted by $A(T, K)$.

Moreover, the action of $N(K)$ on \mathbb{A} preserves the affine structure of \mathbb{A} , its lattice of cocharacters Y , \mathcal{T} and \mathcal{T}^o . So any apartment A in $\mathcal{I}(G, K)$ is endowed with a canonical structure of real affine space, an affine action of a Weyl group $W(A)$, a lattice $Y(A)$ of cocharacter points, Tits cones and an "order-relation". More generally all structures in \mathbb{A} invariant under $N(K)$ are transferred to any apartment by the $G(K)$ -action: in an apartment the notions of (positive, negative, spherical) face, special point, cocharacter point, wall, sector, sector-germ or filter with good fixator are well defined (independently of the apartment containing them, as they all have good fixators).

When we speak of an isomorphism between apartments, we mean an affine isomorphism exchanging the walls and the Tits cones.

- 2) Let A_1, A_2 be two apartments and x, y be two points in $A_1 \cap A_2$. If $x \leq y$ in A_1 , then, by 2.12.4 and 2.14.1, there exists $g \in \widehat{P}_{cl(x, y)}$, such that $A_2 = g \cdot A_1$, hence $A_1 \cap A_2 \supset cl(x, y)$ and $x \leq y$ in A_2 . In particular the relation \leq is defined in the whole hovel $\mathcal{I}(G, K)$ (note that $x \leq y$ implies by definition that x and y are in a same apartment). We shall see below (3.8) that this relation is transitive, so it's a preorder-relation (reflexive, transitive, perhaps not antisymmetric).

The intersection of two apartments A_1, A_2 is *order-convex*: if $x, y \in A_1 \cap A_2$ and $x \leq y$, then the segment $[x, y]$ of A_1 is in $A_1 \cap A_2$ and equal to the corresponding

segment in A_2 . In particular any affine subspace of A_1 whose direction meets the open Tits cone $\mathcal{T}^\circ(A_1)$ and which is contained in $A_1 \cap A_2$ is also an affine subspace of A_2 .

3) For any face (or any narrow filter) F and any sector germ \mathfrak{S} in $\mathcal{I}(G, K)$ there exists an apartment A containing F and \mathfrak{S} : Using the $G(K)$ -action one may suppose $\mathfrak{S} = \mathfrak{S}_{\pm\infty}$. Now $F = g.F'$ with F' a face in \mathbb{A} . By the Iwasawa decomposition $g = unv$ with $u \in U^\pm(K) \subset G_{\mathfrak{S}}$, $n \in N(K)$ and $v \in U_{F'} \subset G_{F'}$. So $F = un.F' \subset un.\mathbb{A} = u.\mathbb{A}$ and $\mathfrak{S} \subset u.\mathbb{A}$.

By order-convexity, any apartment containing F and $\mathfrak{S} = \text{germ}(y + \overline{C^v})$ contains $F + C^v$ (and even $cl(F + C^v) \supset F + \overline{C^v}$, when F has a good fixator, by 2.13.1 and 2.13.4). In particular any apartment containing x and \mathfrak{S} contains the sector \mathfrak{s} of direction \mathfrak{S} and base point x . By 2.14.3 and 2.12.4 two such apartments are conjugated by $G_{\mathfrak{s}}$.

4) If $\Omega_1 = F(x, F_1^v)$ is a face of base point x and Ω_2 a narrow filter containing x , there exists an apartment A containing both of them: in an apartment A_1 containing Ω_1 we choose a vectorial chamber C^v such that $\overline{C^v} \supset F_1^v$; now an apartment A containing Ω_2 and the germ of the sector $x + C^v$ contains Ω_1 and Ω_2 . If moreover Ω_2 is also a face, then $G_{\Omega_1 \cup \Omega_2}$ is transitive on the apartments containing Ω_1 and Ω_2 by 2.14.2, remark 2.13 and 2.12.4. Actually one can prove that $\Omega_1 \cup \Omega_2$ has a good fixator when the faces Ω_1 and Ω_2 are of opposite signs or if one of them is spherical.

If $C = F(x, C^v)$ is a chamber (in \mathbb{A}) and $M(\alpha, k)$ one of its walls (with $C \subset D(\alpha, k)$), then $U_{\alpha, k} = U_{\alpha, C}$ is transitive on the chambers C' adjacent to C along $M(\alpha, k)$: this is a consequence of 2.14.2 and 2.12.4 as G_C may be written $U_{\alpha, C}.U_C^{pm}(\Phi^+ \setminus \{\alpha\}).U_C^{nm-}.H$ and, in this decomposition, all factors but $U_{\alpha, C}$ fix the chamber C'_0 in \mathbb{A} adjacent to C along $M(\alpha, k)$. In particular any such chamber C' and $D(\alpha, k)$ are contained in a same apartment.

5) Retraction with respect to a sector-germ

Let \mathfrak{S} be a sector-germ in an apartment A of $\mathcal{I}(G, K)$.

For $x \in \mathcal{I}(G, K)$, choose an apartment A' containing \mathfrak{S} and x . As \mathfrak{S} has a good fixator, there exists a g in $\widehat{P}_{\mathfrak{S}}$ such that $A = g.A'$. If g and g' are two such elements, then $g^{-1}g'$ induces an automorphism of A' fixing the sector germ \mathfrak{S} , hence this automorphism is the identity: the map $A' \rightarrow A$, $y \mapsto g.y$ is unique. Moreover $\mathfrak{S} \cup \{x\}$ has a good fixator (2.13.4), so $\widehat{P}_{\mathfrak{S} \cup \{x\}}$ is transitive on the possible apartments A' : the point $g.x$ doesn't depend of the choice of A' . So one may define $\rho_{A, \mathfrak{S}}(x) = g.x$.

The map $\rho = \rho_{A, \mathfrak{S}} : \mathcal{I} \rightarrow A$ is a retraction of \mathcal{I} onto A . It depends only on A and \mathfrak{S} and is called *the retraction of \mathcal{I} onto A with center \mathfrak{S}* .

It is clear that, up to canonical isomorphisms, $\rho_{A, \mathfrak{S}}$ depends only on \mathfrak{S} . We denote $\rho_{\pm\infty} = \rho_{\mathbb{A}, \mathfrak{S}_{\pm\infty}}$.

6) A segment-germ $[x, y)$ for $x \neq y$ in an apartment A (cf. 1.9.1) is a narrow filter. When $x \leq y$ (resp. $y \leq x$) its enclosure is a positive (resp. negative) closed-face and $[x, y)$ has a good fixator (2.14.1 and 2.13.3); we say that $[x, y)$ is *positive* (resp. *negative*)

and that $[x, y]$ and $[x, y)$ are *generic*.

For any sector-germ \mathfrak{S} and any segment-germ $[x, y)$ there exists an apartment containing \mathfrak{S} and $[x, y)$ *i.e.* containing $[x, z]$ for some $z \in [x, y] \setminus \{x\}$.

A segment $[x, y]$ in an apartment is compact and, for $z \in [x, y]$, $[z, x) \cup [z, y)$ gives open neighbourhoods of z . So, if \mathfrak{S} is a sector-germ, there exist an integer n , points $x_0 = x, x_1, \dots, x_n = y \in [x, y]$ and apartments A_1, A_2, \dots, A_n such that A_i contains \mathfrak{S} and $[x_{i-1}, x_i]$. As a consequence, for all apartment A' containing \mathfrak{S} , $\rho_{A', \mathfrak{S}}([x, y])$ is the piecewise linear path $[\rho x_0, \rho x_1] \cup [\rho x_1, \rho x_2] \cup \dots \cup [\rho x_{n-1}, \rho x_n]$.

We shall give a better description of this piecewise linear path when $x \leq y$ in the next §.

7) The fixator of some spherical sector-face-germ $\mathfrak{F} = \text{germ}(x + F^v)$ contains clearly the group $P(F^\mu)$ associated in [Rousseau-06] to a microaffine face $F^\mu = F^v \times \mathcal{F}$ (for some \mathcal{F} containing x); and it was proved in [*l.c.*; 3.5] that $G = P(F^\mu).N.P(E^\mu)$ for any microaffine faces F^μ and E^μ of the same sign. Actually, using 2.8 above, the proof of this result is still correct if only one among E^v and F^v is spherical and the signs of E^v and F^v may be opposite. So any two sector-face-germs in \mathcal{I} are contained in a same apartment, if at least one of these sector-face-germs is spherical. For an abstract definition of affine hovels, this property would be a good substitute to axioms (A3) and (A4) in Tits' definition of affine buildings [Tits-86], see also [Ronan-89; appendix 3].

2.16 Residue buildings

1) Let us denote the set of all positive (resp. negative) segment-germs $[x, y)$ with $x < y$ (resp. $y < x$) by \mathcal{I}_x^+ (resp. \mathcal{I}_x^-). The set \mathcal{I}_x^+ (resp. \mathcal{I}_x^-) can be given two structures of building. An apartment \mathfrak{a}_+ in \mathcal{I}_x^+ (resp. \mathfrak{a}_- in \mathcal{I}_x^-) is the intersection $A \cap \mathcal{I}_x^+$ (resp. $A \cap \mathcal{I}_x^-$) of an apartment A of \mathcal{I} containing x with \mathcal{I}_x^+ (resp. \mathcal{I}_x^-) (or more precisely the set of all $[x, y)$ for $y \in A$ and $x < y$ (resp. $x > y$)). Now on any apartment \mathfrak{a}_\pm , one can put two Coxeter complex structures:

- the restricted one modeled on (W_x^{\min}, S_x) , where W_x^{\min} is the subgroup of W generated by all reflections with respect to true walls passing through x , and where $S_x = \{s_H \mid H \text{ is a wall of } F(x, C_f^v) \text{ containing } x\}$. We restrain the action of W_x^{\min} on \mathfrak{a}_\pm . The faces of this structure are the faces $F(x, F^v)$ with F^v positive (resp. negative) (or more precisely, the set of all segment-germs $[x, y)$ contained in $F(x, F^v)$).

- the unrestricted one modeled on (W^v, S^v) , where we force x to be a special point and consider the faces $\text{germ}_x(x + F^v)$ (local-face in x) with F^v positive (resp. negative) (note that $\text{germ}_x(x + F^v) \subset F(x, F^v)$). So we add new (ghost) walls $M(\alpha, k)$ for $\alpha \in \Phi$ and $k \in \mathbb{R} \setminus \mathbb{Z}$, $\alpha(x) + k = 0$.

Let us prove that these data satisfy the two other axioms of building (as in [Brown-89; IV.1] or [Rémy-02; 2.4.1]). We focus on the positive case, the negative one is obtained the same way. In both Coxeter structures, 2.15.4) shows that, given two faces F_1 and F_2 , there exists an apartment containing x and both of them. Further, the group $G_{F_1 \cup F_2}$

is transitive on the apartments containing $F_1 \cup F_2$. Hence, if A and A' are of that kind, there exists an element $g \in G_{F_1 \cup F_2}$ such that $A' = g \cdot A$ which also gives an isomorphism $\mathfrak{a}_+ \simeq \mathfrak{a}'_+$ fixing $F_1 \cup F_2$.

Note that the unrestricted building structure can be thick only when it coincides with the restricted one, *i.e.* when x is special. The buildings \mathcal{I}_x^\pm may be spherical for the restricted structures, as W_x^{min} may be finite when x is not special.

2) We consider now \mathcal{I}_x^\pm endowed with the unrestricted structure. On the set of chambers $Ch(\mathcal{I}_x^\epsilon)$ of \mathcal{I}_x^ϵ (for a sign $\epsilon \in \{+, -\}$), we have a distance $d_\epsilon : Ch(\mathcal{I}_x^\pm) \times Ch(\mathcal{I}_x^\pm) \rightarrow W^v$ defined as follows, *cf.* [Rémy-02; 2.2 to 2.4]:

If $c, c' \in Ch(\mathcal{I}_x^\epsilon)$, choose an apartment A containing c, c' and a chamber $c_0 = germ_x(x + \epsilon C_0)$ in $\mathfrak{a}_\epsilon = A \cap \mathcal{I}_x^\epsilon$; identify (A, C_0) to $(A_f, \epsilon C_f^v)$; this enables us to identify $W^v(A)$ to $W^v = W^v(A_f)$. Now, if $c = wc_0$ and $c' = w'c_0$ for some $w, w' \in W^v$, then $d_\epsilon(c, c') = w^{-1}w'$. Note that if we choose $c_0 = c$, then $c' = d_\epsilon(c, c').c$.

Now, we define a codistance

$$d_x^* : (Ch(\mathcal{I}_x^+) \times Ch(\mathcal{I}_x^-)) \cup (Ch(\mathcal{I}_x^-) \times Ch(\mathcal{I}_x^+)) \rightarrow W^v$$

in the following way. If $(c, e) \in Ch(\mathcal{I}_x^\epsilon) \times Ch(\mathcal{I}_x^{-\epsilon})$, by 2.15.4, there exists an apartment A containing x, c and e , unique up to isomorphism. If $c' = germ_x(x + C')$ is a chamber in A , we denote the chamber opposite to c' in A by $-c'$, *i.e.* $-c' = germ_x(x - C')$. Choose a chamber $c_0 = germ_x(x \pm \epsilon C_0)$ in $\mathfrak{a}_{\pm\epsilon} = A \cap \mathcal{I}_x^{\pm\epsilon}$ and identify (A, C_0) to (A_f, C_f^v) . If $c = \pm w_1.c_0$ and $e = \mp w_2.c_0$, the codistance between c and e is then $d_x^*(c, e) = d_{-\epsilon}(-c, e) = d_\epsilon(c, -e) = w_1^{-1}w_2$. It does not depend on the choices. Further, it satisfies the axioms of twinning as given in [Tits-92; 2.2], see also [Rémy-02; 2.5.1]:

Indeed, the first axiom (Tw1) is fulfilled: $d_x^*(e, c) = w_2^{-1}w_1 = d_x^*(c, e)^{-1}$. Let now $c \in Ch(\mathcal{I}_x^\epsilon)$ and $e, e' \in Ch(\mathcal{I}_x^{-\epsilon})$ be chambers such that $d_x^*(c, e) = w$ and $d_{-\epsilon}(e, e') = s \in S^v$ with $\ell(ws) = \ell(w) - 1$. Let A be an apartment containing x, c and e and choose $c_0 = -c$; since $\ell(ws) = \ell(w) - 1$, the wall H generated by the panel of e of type $\{s\}$ separates the latter from $-c$. In other words, c and e are on the same side of H . Therefore, by 2.15.4) there exists an apartment A' containing c, e and e' . In this apartment $e = w.(-c)$ and $e' = (ws w^{-1}).e$, so $e' = ws.(-c)$ and $d_x^*(c, e') = ws$. This is the second axiom (Tw2).

To check the third axiom (Tw3), let again c and e be two chambers such that $d_x^*(c, e) = w$, and let $s \in S^v$. In an apartment A containing c and e , the chamber h adjacent to e along the panel of type $\{s\}$ satisfies $d_{-\epsilon}(e, h) = s$ and $d_x^*(c, h) = ws$.

Conclusion, \mathcal{I}_x^+ and \mathcal{I}_x^- , with the unrestricted structures, form a twined pair of buildings. With analogue arguments, one shows that it is still true if \mathcal{I}_x^+ and \mathcal{I}_x^- are endowed with the restricted structures.

An apartment A of \mathcal{I} containing x gives a twin apartment $\mathfrak{a} = \mathfrak{a}_+ \cup \mathfrak{a}_-$, where $\mathfrak{a}_\pm = A \cap \mathcal{I}_x^\pm$. If c_0 is a chamber in \mathfrak{a} , there is (as in any twin building) a retraction ρ

of center c_0 of \mathcal{I}_x^+ onto \mathfrak{a}_+ and of \mathcal{I}_x^- onto \mathfrak{a}_- ; it preserves the distances or codistances to c_0 .

§3. Segments in the hovel and Littelmann's paths

The field K is still as explained in 2.1. Note however that, in the classical case where G is a split reductive group, all what follows is true for any field K endowed with a discrete valuation: we just have to use the Bruhat-Tits building instead of the hovel constructed in §2.

3.1. We consider a negative sector germ \mathfrak{S} and denote by ρ the retraction of center \mathfrak{S} without telling on which apartment A (containing \mathfrak{S}) ρ maps \mathcal{I} , as ρ doesn't depend on A up to canonical isomorphisms. Actually we identify any pair (A, \mathfrak{S}) of an apartment A containing \mathfrak{S} with the fundamental pair $(A_f = \mathbb{A}, \mathfrak{S}_{-\infty})$, this is well determined up to translation.

We consider 2 points x, y in the hovel with $x \leq y$. The segment $[x, y]$ is the image of the path $\pi : [0, 1] \rightarrow \mathcal{I}$ defined by $\pi(t) = x + t(y - x)$ in any apartment containing x and y (2.15.2). As each segment in $[x, y]$ has a good fixator, the derivative $\pi'(t)$ is independent of the apartment containing a neighbourhood of $\pi(t)$ in $[x, \pi(t)]$ or $[\pi(t), y]$, up to the Weyl group W^v .

We saw in 2.15.5 that the image $\rho\pi$ is a piecewise linear continuous path in A . By the preceding results, it's clear that there is a λ in the fundamental closed-chamber $\overline{C_f^v}$ such that $\rho\pi'(t) = w_t \cdot \lambda$ for each $t \in [0, 1]$ (different from $\pi^{-1}(x_i)$ for x_i as in 2.15.6) and some $w_t \in W^v$ (chosen minimal with this property). Hence $\rho\pi$ is a λ -path (in particular the map ρ is increasing with respect to the "preorder" of 2.15.2) and may be described as $\pi(\lambda, \rho\pi(0), \underline{\tau}, \underline{a})$. We shall prove that it is a Hecke path and often a LS path.

3.2. We choose some $t \in]0, 1[= [0, 1] \setminus \{0, 1\}$ and we set $z = \rho\pi(t)$. We denote by $\rho\pi'_-$ (resp. $\rho\pi'_+$) the left (resp. right) derivative of $\rho\pi$ in t and w_- (resp. w_+) the minimal element in W^v such that $\rho\pi'_- = w_- \lambda$ (resp. $\rho\pi'_+ = w_+ \lambda$).

Proposition. $\rho\pi'_+ \leq_{W_z^v} \rho\pi'_-$ (cf. 1.12.2b) and $w_+ \leq w_-$ in the Bruhat-Chevalley order of W^v/W_λ^v . More precisely there exist $s \in \mathbb{N}$ and a sequence β_1, \dots, β_s of positive real roots such that:

- for $1 \leq i \leq s$, there exists a wall of direction $\text{Ker}\beta_i$ containing $z = \rho\pi(t)$,
- if one defines $\xi_0 = \rho\pi'_-$, $\xi_1 = r_{\beta_1} \cdot \xi_0$, \dots , $\xi_s = r_{\beta_s} \dots r_{\beta_1} \cdot \xi_0$, one has $\xi_s = \rho\pi'_+$ and $\beta_k(\xi_{k-1}) < 0$ for $1 \leq k \leq s$,
- if one defines $\sigma_0 = w_-$, $\sigma_1 = r_{\beta_1} \cdot w_-$, \dots , $\sigma_s = r_{\beta_s} \dots r_{\beta_1} \cdot w_-$, then $\rho\pi'_+ = \sigma_s \lambda$ and, for $1 \leq k \leq s$, $\sigma_k < \sigma_{k-1}$ in the Bruhat-Chevalley order of W^v/W_λ^v ,
- there exists in $\mathfrak{a}_+ = A \cap \mathcal{I}_z^+$ an (unrestricted) gallery $\delta = (c_0, c'_1, \dots, c'_n)$ from $c_0 = \text{germ}_z(z + C_f^v)$ to $c'_n = \text{germ}_z(z + w_+ C_f^v) \supset z + [0, 1]\rho\pi'_+$, the type of which

is associated to a (given) reduced decomposition of w_- . The panels along which this gallery is folded (actually positively folded: see the proof) are successively the walls $z + \text{Ker}\beta_1, \dots, z + \text{Ker}\beta_s$.

Proof. Let A^0 be an apartment containing $[x, y]$; set $\pi_- = [\pi(t), x]$ and $\pi_+ = [\pi(t), y]$. By 2.15.3 and 2.15.6 there exist apartments A^+ and A^- containing the sector \mathfrak{s} (of direction \mathfrak{S} and base point $\pi(t)$) and respectively π_+ and π_- . We choose A^- for the image A of ρ , so $\pi(t) = \rho\pi(t) = z$ and $\pi_- = \rho\pi_- = z - [0, 1)\pi'_-$.

As A^0 and A^- contain π_- , there exists a $g \in \widehat{P}_{\pi_-}$ such that $A^0 = g.A^-$. In the decomposition $\widehat{P}_{\pi_-} = U_{\pi_-}^{nm-} . U_{\pi_-}^{pm+} . \widehat{N}_{\pi_-}$ the group \widehat{N}_{π_-} fixes π_- and stabilizes A^- so one has $A^0 = u^- u^+ A^-$ with $u^- \in U_{\pi_-}^{nm-}$ and $u^+ \in U_{\pi_-}^{pm+}$. Let's consider the apartment $A^1 = (u^-)^{-1} A^0 = u^+ A^-$; it contains $\pi_- = (u^-)^{-1}(\pi_-)$ and $\pi_+^1 = (u^-)^{-1}(\pi_+)$, which are opposite segment germs; moreover $\rho(\pi_+^1) = \rho(\pi_+)$. On the other side A^1 contains the chamber $C_0 = F(z, C_f^v)$ in A^- , which is opposite to \mathfrak{s} . We replace in the following π_+ by π_+^1 and A^0 by A^1 .

In A^- , $\pi_- \in w_- \mathfrak{s}$, hence the germ opposite to π_- is in $w_- C_0$. So, in A^1 , $\pi_+^1 \in w_-^1 C_0$, where w_-^1 corresponds to w_- in the identification of $W^v(A^-)$ and $W^v(A^1)$ via u^+ .

We choose in $\mathfrak{a}_+^1 = A^1 \cap \mathcal{I}_z^+$ (apartment of \mathcal{I}_z^+ endowed with the unrestricted building structure) a minimal gallery $m = (c_0, c_1, \dots, c_n)$ between $c_0 = \text{germ}_z(z + C_f^v)$ and $c_n \supset \pi_+^1 = z + [0, 1)w_-^1 \lambda$ of type $\tau = (i_1, \dots, i_n)$, $i_j \in I$; hence $w_-^1 = r_{i_1} \dots r_{i_n}$ is a reduced decomposition. The restriction of ρ to the residue twin building $(\mathcal{I}_z^+, \mathcal{I}_z^-, d_z^*)$ (with the unrestricted structure) preserves the codistance to $\bar{\mathfrak{s}} = \text{germ}_z(\mathfrak{s})$, which is a chamber in \mathcal{I}_z^- . Therefore, it is the retraction $\rho_z : \mathcal{I}_z^+ \rightarrow \mathfrak{a}_+^- = A^- \cap \mathcal{I}_z^+$ of centre $\bar{\mathfrak{s}}$. We have $\rho\pi_+^1 = \rho_z\pi_+^1$.

The retracted gallery $\delta = \rho_z(m) = (c_0, c'_1 = \rho_z(c_1), \dots, c'_n = \rho_z(c_n))$ in $A = A^-$ is a positively folded gallery, meaning that $\rho_z(c_j) = \rho_z(c_{j+1})$ implies that this chamber is on the positive side of the wall H_j spanned by the panel of type $\{i_j\}$ of $\rho_z(c_j)$ (note that H_j is a wall for the unrestricted structure). Otherwise, suppose that $\rho_z(c_j) = \rho_z(c_{j+1})$ is on the negative side of H_j . Then, because $\bar{\mathfrak{s}}$ is the opposite fundamental chamber in z , it is always on the negative side of H_j . Further, let \mathfrak{a} be a twin apartment containing $\bar{\mathfrak{s}}$ and c_j , as the retraction preserves the codistance to $\bar{\mathfrak{s}}$, we also have that $\bar{\mathfrak{s}}$ and c_j are on the same side of the wall spanned by the panel of type $\{i_j\}$ of c_j in \mathfrak{a} . Therefore 2.15.4) proves that, modifying the latter if needed, we can assume that c_{j+1} is still in \mathfrak{a} . But, on one side, $c_j \neq c_{j+1}$ then, computing in \mathfrak{a} , $\ell(d_z^*(\bar{\mathfrak{s}}, c_{j+1})) = \ell(d_z^*(\bar{\mathfrak{s}}, c_j)) - 1$; on the other side, $\ell(d_z^*(\bar{\mathfrak{s}}, c_{j+1})) = \ell(d_z^*(\bar{\mathfrak{s}}, \rho_z c_{j+1})) = \ell(d_z^*(\bar{\mathfrak{s}}, \rho_z c_j)) = \ell(d_z^*(\bar{\mathfrak{s}}, c_j))$. Contradiction!

If the wall H_j^1 separating c_j from c_{j+1} in \mathfrak{a}_+^1 is a ghost wall *i.e.* not a true wall (for the restricted structure), then the enclosure of c_j in the hovel contains c_{j+1} and there is an apartment of \mathcal{I} containing \mathfrak{s} , c_j and c_{j+1} , so $\rho_z(c_j) \neq \rho_z(c_{j+1})$.

Let us now denote by j_1, \dots, j_s the indices such that $c'_j = \rho_z(c_j) = \rho_z(c_{j+1}) = c'_{j+1}$. For any $k \in \{1, \dots, s\}$, H_{j_k} is a true wall spanned by the panel $\rho(H_{j_k}^1 \cap c_j)$ and we

denote the positive real root associated with H_{j_k} by β_k (i.e. H_{j_k} is of direction $\text{Ker}\beta_k$). Actually, the gallery δ is obtained from the minimal gallery $\delta^0 = (c_0^0 = c_0, c_1^0, \dots, c_n^0)$ of type (i_1, \dots, i_n) beginning in c_0 , ending in $c_n^0 = w_-(c_0)$ and staying inside A^- by applying successive (positive) foldings along the walls associated to the indices $\{j_1, \dots, j_s\}$, starting with H_{j_1} , then folding along $H_{j_2} \dots$. At each step, one gets a positively folded gallery $\delta^k = (c_0^k = c_0, c_1^k, \dots, c_n^k)$ ending closer and closer to the chamber c_0 .

Let's denote $\xi_0 = \pi'_- = \rho\pi'_- = w_-\lambda$ (in A_-) and $\xi_k = r_{\beta_k} \cdots r_{\beta_1} \xi_0$. As $\rho_z(c_n) \supset \rho\pi_+ = z + [0, 1)\rho\pi'_+$ and $\rho\pi'_+ \in W^v\rho\pi'_-$ one has $\xi_s = \rho\pi'_+$, and more generally, $z + [0, 1)\xi_k \subset c_n^k$. As δ^0 is a minimal gallery from c_0 to $z + [0, 1)\pi'_-$, c_{j+1}^0, \dots, c_n^0 and $z + [0, 1)\pi'_-$ are on the same side of any wall separating c_0^0 from c_{j+1}^0 ; in particular $(c_{j_k+1}^k, \dots, c_n^k)$ is a minimal gallery, entirely on the same side of H_{j_k} and $z + [0, 1)\xi_k \not\subset H_{j_k}$. But $c_{j_k}^k = \rho_z(c_{j_k}) = \rho_z(c_{j_k+1}) = c_{j_k+1}^k$ and we saw that this chamber is on the positive side of the wall H_{j_k} (of direction $\text{Ker}\beta_k$). So $c_{j_k+1}^k, \dots, c_n^k$ are on the positive side of H_{j_k} ; this means that $\beta_k(\xi_k) > 0$ i.e. $\beta_k(\xi_{k-1}) < 0$. Hence the sequences $(\xi_0, \xi_1, \dots, \xi_s)$ and $(\beta_1, \dots, \beta_s)$ give a W_z^v -chain from $\pi'_- = \rho\pi'_-$ to $\rho\pi'_+$. This proves the proposition, in view of lemma 1.12.3.1. \square

Theorem 3.3. *Let $\pi = [x, y]$ be a segment in an apartment A' with $x < y$ and ρ the retraction of \mathcal{I} with center the fundamental sector-germ $\mathfrak{S}_{-\infty}$ onto an apartment A . Then the retracted segment $\rho\pi$ is a Hecke path in A .*

If moreover x and y are cocharacter points (i.e. $x, y \in Y(A')$), then $\rho\pi$ is a Hecke path in $Y(A)$.

Proof. The path $\rho\pi$ is Hecke by proposition 3.2 and definition 1.12.2. If $x, y \in Y(A')$, calculating in A' , $\lambda = W^v(y - x) \cap \overline{C_f^v}$ is in $Y^+(A')$. Moreover, by 2.15.1, $\rho(Y(A')) \subset Y(A)$, so $\rho\pi$ is in $Y(A)$. \square

3.4 Segments retracting on a given Hecke path

We look now to all segments $\pi = [x, y]$ from some $x \in \mathcal{I}$ to some $y \in \mathcal{I}$ (i.e. $\pi(t) = x + t(y - x)$) whose retraction $\rho\pi$ is a given Hecke path π_1 in A . There are too many of them, so, in this subsection, we fix $y = \pi(1)$.

More precisely, let $y \in \mathcal{I}$ and π_1 a Hecke path in A with $\pi_1(1) = \rho(y)$, we define $S(\pi_1, y)$ as the set of segments $\pi = [x, y]$ in \mathcal{I} such that $\pi_1 = \rho\pi$.

Theorem. *$S(\pi_1, y)$ is non empty and is parametrized by exactly $N = \text{ddim}(\pi_1)$ parameters. More precisely the set $P(\pi_1, y)$ of parameters is a Zariski open subset of κ^N containing κ^{*N} .*

Proof. We shall prove that, for $t \in [0, 1]$, the segments $\pi^t : [t, 1] \rightarrow \mathcal{I}$ with $\pi^t(1) = y$, retracting to $\pi_1^t = \pi_1|_{[t, 1]}$ are given by $\sum_{t' > t} \ell_{\pi_1(t)}(w_-(t))$ parameters. It's clear for $t = 1$. Suppose the result true for some t . So $\pi(t)$ is given and we shall prove now that the number of parameters for the choice of the segment-germ $\pi_-(t)$ of origin $\pi(t)$ is $\ell_{\pi_1(t)}(w_-(t))$. By this result and the arguments in 1.13.1 $\pi_-(t)$ determines $\pi|_{[t', t]}$ where

t' ($< t$) is 0 or the next number in $[0, 1]$ such that $\overline{\pi_1}$ leaves positively a wall in t' . So the proof will be finished.

Now $\pi(t)$ is given and we look to the choice of the segment-germ $\pi_-(t)$. We choose A in order that it contains \mathfrak{S} and $\pi_+(t)$; so $\pi_+(t) = \rho\pi_+(t)$ and we set $z = \pi(t)$. There is an apartment A^- containing \mathfrak{S} and $\pi_-(t)$, so ρ is an isomorphism from A^- to A . The (unrestricted) chamber $c_0^- = \text{germ}_z(z - C_f^v)$ (cf. 2.16) is in $A \cap A^-$. We choose a reduced decomposition $w_-(t) = r_{i_1} \dots r_{i_n}$ in W^v . So there is a minimal (unrestricted) gallery $m^- = (c_0^-, c_1^-, \dots, c_n^-)$ of type (i_1, \dots, i_n) from c_0^- to $\pi_-(t)$. Clearly $\pi_-(t)$ is entirely determined by the gallery m^- , so it seems to depend on $n = \ell(w_-(t))$ parameters in κ . But actually if the wall separating c_{j-1}^- from c_j^- (or ρc_{j-1}^- from ρc_j^-) is a ghost wall *i.e.* not a true (= restricted) wall, the chamber c_j^- is determined by c_{j-1}^- ; whereas if this wall is true (= $M(\alpha, k)$ for some $\alpha \in \Phi^-$), then the choice of c_j^- depends on a parameter in $U_{\alpha, k}^\times = U_{\alpha, k} / U_{\alpha, > k} \simeq \kappa$ (cf. 2.15.4). So the true number of parameters is $\ell_{\pi_1(t)}(w_-(t))$.

There is still a condition of which we haven't taken care till now: $\pi_-(t)$ and $\pi_+(t)$ should be opposite. We shall prove now that, removing at most one value for each parameter, this last condition is fulfilled. Hence the first part of the theorem will be proved.

As π_1 is a Hecke path, we can use the first part of 1.12.3: in particular conditions i), v), vi) and vii) of 1.12.2 are verified for some roots β_1, \dots, β_s . Let $\delta^0 = (c_0, c_1^0, \dots, c_n^0)$ be the minimal gallery of type (i_1, \dots, i_n) in $\mathfrak{a}_+ = A \cap \mathcal{I}_x^+$ starting from $c_0 = \text{germ}_z(z + C_f^v)$; its end $c_n^0 = w_- c_0$ contains $z + [0, 1)\pi'_-(t)$. We shall fold this gallery progressively. As $r_{\beta_1} w_- < w_-$, the wall $z + \text{Ker}\beta_1$ separates c_0 from c_n^0 : it is the wall between some adjacent chambers c_{j-1}^0 and c_j^0 . We define $\delta^1 = (c_0, c_1^1 = c_1^0, \dots, c_{j-1}^1 = c_{j-1}^0, c_j^1 = r_{\beta_1} c_j^0, \dots, c_n^1 = r_{\beta_1} c_n^0)$, so $c_{j-1}^1 = c_{j-1}^0$ and $c_n^1 = r_{\beta_1} w_- c_0$. But $r_{\beta_2} r_{\beta_1} w_- < r_{\beta_1} w_-$, so the wall $z + \text{Ker}\beta_2$ separates c_0 from c_n^1 : it is the wall between some adjacent chambers c_{k-1}^1 and c_k^1 . We define $\delta^2 = (c_0, c_1^2 = c_1^1, \dots, c_{k-1}^2 = c_{k-1}^1, c_k^2 = r_{\beta_2} c_k^1, \dots, c_n^2 = r_{\beta_2} c_n^1)$. At the end of this procedure we get a gallery $\delta^s = (c_0, c_1^s, \dots, c_n^s)$ of type (i_1, \dots, i_n) in A starting from c_0 and ending with $c_n^s = w_+(t)c_n^0 \supset \pi_+(t)$. Moreover this gallery is positively folded along true walls.

As $\pi'_+(t) \in W^v \pi'_-(t)$, to prove that $\pi_-(t) = z - [0, 1)\pi'_-(t)$ and $\pi_+(t) = z + [0, 1)\pi'_+(t)$ are opposite segment-germs it is sufficient to prove that c_n^s and c_n^- are opposite chambers. For this we prove that, except perhaps for one choice of each parameter, c_j^s and c_j^- are opposite for $0 \leq j \leq n$. This is true for $j = 0$. Suppose c_{j-1}^- opposite to c_{j-1}^s . Then c_j^- (resp. c_j^s) is adjacent to c_{j-1}^- (resp. c_{j-1}^s) along an (unrestricted) panel of type i_j . If the wall containing these 2 panels is not true (=restricted), then c_j^- and c_j^s are automatically opposite. Now, if this wall is true, by 2.16 and the general properties of twin buildings (see [Rémy-02; 2.5.1]) among the chambers adjacent (or equal) to c_{j-1}^- along the panel of type i_j , there is one and only one chamber not opposite to c_j^s . Hence

all but (perhaps) one choice for c_j^- is opposite to c_j^s ; and the corresponding parameter has to be chosen in κ or κ^* .

We know now that $S(\pi_1, y)$ is non empty. Let's look closer to the set of parameters. Choose $\pi \in S(\pi_1, y)$ and the gallery (c_0^-, \dots, c_n^-) as above in A^- . We choose the apartment A containing \mathfrak{S} and a chamber $c_n \supset \pi_+(t)$ opposite to c_n^- . Using the same properties of twin buildings as above, we find a gallery $\delta = (c_0, c_1, \dots, c_n)$ of type (i_1, \dots, i_n) in A , folded only along true walls, and such that, for all j , c_j and c_j^- are opposite. In particular c_0 is as defined above. So, using δ instead of δ^s , the path π is as studied in the preceding paragraph. Moreover the number of possibilities for δ is finite. So the set of parameters for $S(\pi_1, y)$ will be a finite union of subsets of κ^N , each being a product of factors either κ or κ^* . \square

Corollary 3.5. *Suppose π_1 is a Hecke path in $Y(A)$. Then the number $\text{ddim}(\pi_1)$ of parameters for $S(\pi_1, y)$ is at most $\rho_{\Phi^+}(\lambda - \pi_1(1) + \pi_1(0))$, with equality if and only if π_1 is a LS path.*

Proof. This is a simple consequence of proposition 1.14 and theorem 3.4. \square

3.6 Algebraic structure of $S(\pi_1, y)$ and Mirković-Vilonen cycles

To simplify the notations we suppose $y = 0$ in A and (as before) $\lambda \in Y^+$. We suppose moreover $K = \mathbb{C}((\varpi))$.

1) The set \mathcal{G}_λ of segments π in \mathcal{I} of shape λ and ending in 0 may be identified with the set of its starting points $\pi(0)$ i.e. with $G_0.(-\lambda)$. For some $\nu \in Y$ let's define $\mathcal{G}_{\lambda, \nu}$ as the subset of \mathcal{G}_λ consisting of the segments π with $\rho(\pi(0)) = -\nu$. So $\mathcal{G}_{\lambda, \nu}$ is identified with $U^-(K).(-\nu) \cap G_0.(-\lambda)$; as $-\lambda \in G(K).0$, $\mathcal{G}_{\lambda, \nu}$ is thus a subset of the affine grassmannian $\mathcal{G} = G(K)/G(\mathcal{O})$, cf. 2.3 and 2.10.4. We shall look to the algebraic structure of $\mathcal{G}_{\lambda, \nu}$ as inherited from $U^-(K)$.

By 3.3 and 1.15, $\mathcal{G}_{\lambda, \nu}$ is the finite (disjoint) union of the subsets $S(\pi_1, 0)$ for π_1 a Hecke path of shape λ in A from $-\nu$ to 0.

2) We describe now better the parameters for $S(\pi_1, 0)$ found in 3.4.

Let $0 < t_1 < \dots < t_m \leq 1$ be the values of t such that $n_i = \ell_{\pi_1(t_i)}(w_-(t_i)) > 0$ and $t_0 = 0, t_{m+1} = 1$. For $1 \leq i \leq m$ given, there exist negative roots $\alpha_{i,j}$ and integers $k_{i,j}$, $1 \leq j \leq n_i$, such that $M(\alpha_{i,n_i}, k_{i,n_i}), \dots, M(\alpha_{i,1}, k_{i,1})$ are the true walls successively crossed by a minimal gallery from $c_0^- = \text{germ}_{\pi_1(t_i)}(\pi_1(t_i) - C_f^v)$ to $\pi_{1-}(t_i)$. Now the parameters corresponding to $M(\alpha_{i,j}, k_{i,j})$ are the $x_{i,j}(a) = x_{\alpha_{i,j}}(a\varpi^{k_{i,j}}) \in U_{\alpha_{i,j}, k_{i,j}} \setminus U_{\alpha_{i,j}, k_{i,j}+1}$ for $a \in \mathbb{C}$.

Its clear from the proof of theorem 3.4 that, for some $(a_{i,j}) \in P(\pi_1, 0) \subset \mathbb{C}^N$: $\pi(0) = (\prod_{i \geq 1; j \leq n_i} x_{i,j}(a_{i,j})).(-\nu)$ where the product is taken in lexicographical order from right to left.

More generally, for $t_{i_0-1} \leq t \leq t_{i_0}$, $\pi(t) = (\prod_{i \geq i_0; j \leq n_i} x_{i,j}(a_{i,j})).(\pi_1(t))$.

We thus define a map

$$\mu : \mathbb{C}^N \supset P(\pi_1, 0) \rightarrow U^-(K) \quad , \quad (a_{i,j})_{i \leq m; j \leq n_i} \mapsto \prod_{i \leq m; j \leq n_i} x_{i,j}(a_{i,j})$$

such that the composition $\bar{\mu} = proj \circ \mu : \mathbb{C}^N \supset P(\pi_1, 0) \rightarrow U^-(K)/U^-(K)_{-\nu}$ is injective. Moreover $U^-(K) \subset U^{nmax-}(K) = \prod_{\alpha \in \Delta^-} U_\alpha(K) = \prod_{\alpha \in \Delta^-} \mathbb{C}((\varpi))$ and, as μ involves finitely many groups $U_{\alpha,k}$ with $\alpha \in \Phi^-$, there exists y in Y such that the image of μ is in $U^-(K)_y \subset U^{nmax-}(K)_y = \prod_{\alpha \in \Delta^-} U_{\alpha, -\alpha(y)} = \prod_{\alpha \in \Delta^-} \mathbb{C}[[\varpi]]$. This last group has the structure of a pro-group in the sense of [Kumar-02] and the map μ is clearly a morphism for this algebraic structure.

So $S(\pi_1, 0)$ is the image under a bijective morphism of a quasi-affine irreducible variety of dimension $\text{ddim}(\pi_1)$ in $\mathcal{G}_{\lambda, \nu}$.

3) Hence $\mathcal{G}_{\lambda, \nu}$ is a finite (disjoint!) union of sets $S(\pi_1, 0)$ each in bijection with a quasi-affine irreducible variety $P(\pi_1, 0)$ and these sets are indexed by the Hecke paths π_1 of shape λ from $-\nu$ to 0 in A . The maximum dimension of these varieties is $\rho_{\Phi^+}(\lambda - \nu)$, and the varieties of maximum dimension correspond to LS paths from $-\nu$ to 0 in A . The Mirković-Vilonen cycle inside $\mathcal{G}_{\lambda, \nu}$ should be the closure of a set $S(\pi_1, 0)$ (for π_1 a LS path) and $P(\pi_1, 0)$ would be isomorphic to a dense open subvariety of this cycle.

This is true in the classical case of reductive groups and these cycles in $\mathcal{G}_{\lambda, \nu}$ are dense in the classical Mirković-Vilonen cycles (*cf.* [Mirković-Vilonen-00]) corresponding to $-\lambda$ and $-\nu$ and described using the reverses of the paths above *cf.* [Gaussent-Littelmann-05]. More details may be given in this case, *cf.* [Baumann-Gaussent-06].

3.7 Another characterization of LS paths

Suppose π_1 is a Hecke path of shape λ in the apartment A . For each t , $0 < t < 1$, let $w_-(t)$ (resp. $w_+(t)$) be the minimal element in W^v such that $\pi'_{1-}(t) = w_-(t)\lambda$ (resp. $\pi'_{1+}(t) = w_+(t)\lambda$). By 3.4 and 3.2 there exists an unrestricted gallery $\delta_t = (d_0, \dots, d_n)$ in $\mathfrak{a}_+ = A \cap \mathcal{I}_{\pi_1(t)}^+$, of type (i_1, \dots, i_n) associated to a (given) reduced decomposition of $w_-(t)$, starting from $d_0 = c_0 = germ_{\pi_1(t)}(\pi_1(t) + C_f^v)$ and ending in $d_n \supset \pi_1(t) + [0, 1)\pi'_{1+}(t)$; moreover this gallery may be taken positively folded along true walls. For each t , we choose such a gallery and we set $\tilde{\pi}_1 = (\pi_1, (\delta_t)_{0 < t < 1})$.

The gallery δ_t is minimal for almost all t (when $\pi'_{1-}(t) = \pi'_{1+}(t)$). Let's define $neg(\delta_t)$ as the number of the unrestricted walls H_j (containing the panel of type i_j in d_j or d_{j-1}) which are true walls and separate d_j from d_0 . Actually, as δ_t is positively folded, such an H_j separates d_j from d_{j-1} *i.e.* $d_j \neq d_{j-1}$.

We define the *codimension* of $\tilde{\pi}_1$ as :

$$\text{codim}(\tilde{\pi}_1) = \ell_{\pi(0)}(w_+(0)) + \sum_{0 < t < 1} neg(\delta_t)$$

By the same arguments as in 1.13.1 $\text{codim}(\tilde{\pi}_1)$ is a non negative integer; actually $\text{codim}(\tilde{\pi}_1) \leq \ell_{\pi(0)}(w_+(0)) + \text{ddim}(\pi_1) - \ell_{\pi(1)}(w_-(1))$.

Proposition. *Let π_1 be a Hecke path in Y . For each choice of $\tilde{\pi}_1$, $\text{codim}(\tilde{\pi}_1) \geq \text{codim}(\pi_1)$. And π_1 is a LS path if and only if there is equality for (at least) one choice of $\tilde{\pi}_1$.*

Remark. So $\text{codim}(\tilde{\pi}_1) \geq \text{codim}(\pi_1) \geq \rho_{\Phi^+}(\lambda - \nu) \geq \text{ddim}(\pi_1)$, with equalities if and only if π_1 is a LS-path (for good choices of $\tilde{\pi}_1$).

Proof. It is clear that any true wall H separating d_0 from $\pi_{1+}(t) = \pi_1(t) + [0, 1)\pi'_{1+}(t)$ is among the walls H_j , and, if j is chosen maximal for this property, H separates d_0 from d_j . So $\ell_{\pi_1(t)}(w_+(t)) \leq \text{neg}(\delta_t)$ for $0 < t < 1$ and $\text{codim}(\tilde{\pi}_1) \geq \text{codim}(\pi_1)$.

Suppose $\ell_{\pi_1(t)}(w_+(t)) = \text{neg}(\delta_t)$, then every true wall H separating d_0 from $\pi_{1+}(t)$ is leaved negatively once and only once by δ_t ; in particular δ_t cannot be negatively folded along such a wall and cannot cross it positively. Moreover δ_t cannot leave negatively any other true wall. As, by hypothesis, δ_t may only be folded along a true wall, this gallery remains inside the (unrestricted) enclosure of d_0 and $\pi_{1+}(t)$. The number of foldings of δ_t is $s = \ell_{\pi_1(t)}(w_-(t)) - \ell_{\pi_1(t)}(w_+(t))$ and δ_t is positively folded. One can now argue as at the end of the proof of proposition 3.2. One gets so positive roots β_1, \dots, β_s such that conditions (i) and (ii) of 1.12.2 are fulfilled; condition (iii) is then a consequence of $s = \ell_{\pi_1(t)}(w_-(t)) - \ell_{\pi_1(t)}(w_+(t))$ and π_1 is a LS path.

Conversely, if π_1 is LS, the construction of δ_t as in theorem 3.4 may be done using a set $(\beta_1, \dots, \beta_s)$ of positive roots with $s = \ell_{\pi_1(t)}(w_-(t)) - \ell_{\pi_1(t)}(w_+(t))$. This gallery δ_t is folded exactly s times (positively and along true walls), its length is $n = \ell_{\pi_1(t)}(w_-(t))$; so, once we get rid of the stutterings, we get a minimal gallery δ_t^{ns} from c_0 to $\pi_{1+}(t) = \pi_1(t) + [0, 1)w_+(t)\lambda$. Hence, as the foldings were positive, $\text{neg}(\delta_t) = \text{neg}(\delta_t^{ns}) = \ell_{\pi_1(t)}(w_+(t))$, so $\text{codim}(\tilde{\pi}_1) = \text{codim}(\pi_1)$. \square

Theorem 3.8. *In the hovel \mathcal{I} , the relation \leq (defined in 2.15.2) is a preorder relation. More precisely if three different points x, y and z in \mathcal{I} are such that $x \leq y$ and $y \leq z$ then $x \leq z$ and, in particular, x and z are in a same apartment.*

Remark. This result precises the structure of the hovel \mathcal{I} . It is a generalization of Lemme 7.3.6 in [Bruhat-Tits-72]. It may also be seen as a generalization of the Cartan decomposition proved by Garland for p-adic loop groups [Garland-95], even if it is weaker than this decomposition in the affine case. As Garland tells, the Cartan decomposition is true only after some twisting; this is more or less equivalent to the fact that not any two points in \mathcal{I} are in a same apartment.

Lemma. *In the above situation, there exists an apartment A containing x and $[y, z)$. Moreover this apartment is unique up to isomorphism.*

Proof. In an apartment A_1 containing x and y , there exists a vectorial chamber C^v such that $x \in y + \overline{C^v}$. Moreover, there exists an apartment A containing $\mathfrak{S} = \text{germ}(y + \overline{C^v})$

and $[y, z]$; this apartment also contains $y + \overline{C^v} \ni x$ (cf. 2.15.3). The uniqueness is a consequence of 2.14.2, 2.14.4, remark 2.13 and 2.12.4. \square

Proof of the theorem.

1) For $z' \in [y, z[$ such that $x \leq z'$, we choose an apartment A containing $[z', x)$ and $[z', z)$ (2.15.4); this apartment has an associated system of real roots $\Phi(A)$ and we define the finite set $\Phi(z')$ of the roots $\alpha \in \Phi(A)$ such that $\alpha(z') > \alpha(x)$ and $\alpha(z') > \alpha(z_1)$ for some $x_1 \in [x, z'] \cap A$ and some $z_1 \in [z, z'] \cap A$. As $[z', x)$ and $[z', z)$ are generic, 2.15.4 proves that $\Phi(z')$ depends, up to isomorphism, only on $[z', x)$ and $[z', z)$ not on A .

By the above lemma (mutatis mutandis), there exists an apartment $A_{z'}$ containing z and $[z', x)$ and this apartment is unique up to isomorphism. We define $N_{z'}$ as the finite number of walls (in $A_{z'}$) of direction $\text{Ker}\alpha$ for some $\alpha \in \Phi(z')$ and separating z' from z (in a strict sense). We shall argue by induction on $(|\Phi(y)|, N_y)$ (with lexicographical order).

2) By the lemma, there is an apartment A_1 containing x and $[y, z)$. We choose a vectorial chamber C^v in A_1 such that its associated system of positive roots $\Phi^+(C^v)$ contains the roots $\alpha \in \Phi(A_1)$ such that $\alpha(y) > \alpha(x)$ or $\alpha(y) = \alpha(x)$ and $\alpha(z_1) > \alpha(y)$ (for some $z_1 \in [y, z] \cap A_1$); in particular $[x, y] \subset y - \overline{C^v}$. Now if $\alpha \in \Phi^+(C^v)$ is such that $\alpha(z_1) < \alpha(y)$ (for some $z_1 \in [y, z] \cap A_1$) then $\alpha(y) > \alpha(x)$; hence $\Phi(y)$ (calculated in A_1) is the set of roots $\alpha \in \Phi^+(C^v)$ such that $\alpha(z_1) < \alpha(y)$ (for some $z_1 \in [y, z] \cap A_1$).

Let \mathfrak{S} be the sector-germ associated to $-C^v$ in A_1 and ρ the retraction of center \mathfrak{S} onto A_1 .

3) Suppose $z_1 \in]y, z]$ is such that no wall (in A_y or any apartment containing z_1 and $[y, x)$) of direction $\text{Ker}\alpha$ for some $\alpha \in \Phi(y)$ separates y from z_1 . We shall prove that the enclosure of \mathfrak{S} and z_1 contains y and x , so there is an apartment containing x, y, z_1 and \mathfrak{S} . Hence the theorem is true if $z_1 = z$ and this gives the beginning of the induction when $\Phi(y) = \emptyset$ or $N_y = 0$.

As in 2.15.6) we get a sequence $y_0 = y, y_1, \dots, y_n = z_1 \in [y, z_1]$ and apartments A_1, A_2, \dots, A_n such that A_i contains \mathfrak{S} and $[y_{i-1}, y_i]$. The characterization of $\Phi(y)$ in 2) above and the hypothesis on the walls prove that y is in the enclosure of y_1 and \mathfrak{S} , then x is also in this enclosure. So A_2 , which contains y_1 and \mathfrak{S} contains also y and x . We can replace y_1 by y_2 and A_1 by A_2 ; by induction on n we obtain the result of 3).

4) We choose for $z_1 \in]y, z]$ the point verifying the hypothesis of 3) which is the nearest to z , it exists as $\Phi(y)$ and N_y are finite. We may (and do) suppose $z_1 \neq z$. We choose for A_1 the apartment containing x, y, z_1 and the \mathfrak{S} explained in 3). An apartment A_2 containing \mathfrak{S} and $[z_1, z)$ is sent isomorphically by ρ onto A_1 . This enables us to identify $\Phi(z_1)$ with the set $\Phi'(z_1)$ of the roots $\alpha \in \Phi(A_1)$ such that $\alpha(z_1) > \alpha(x)$ (hence $\alpha \in \Phi^+(C^v)$) and $\alpha(z_1) > \alpha(\rho z_2)$ (for some $z_2 \in [z_1, z]$ near z_1). By Proposition 3.2, $\rho([z_1, z]) = z_1 + [0, 1)w^+\lambda$, $[y, z_1] = y + [0, 1)w^-\lambda$ for some $\lambda \in \overline{C^v}$ and $w^+, w^- \in W^v$ such that $w^+ \leq w^-$. In particular for $\alpha \in \Phi^+(C^v)$, $\alpha(z_1) > \alpha(\rho z_2)$ means $\alpha(w^+\lambda) < 0$,

so $\Phi'(z_1) \subset \{\alpha \in \Phi^+(C^v) \mid \alpha(w^+\lambda) < 0\}$ and (as w^+ is chosen minimal) this set is of cardinal $\ell(w^+)$. Now we saw in 2) that $\Phi(y) = \{\alpha \in \Phi^+(C^v) \mid \alpha(w^-\lambda) < 0\}$. Hence, as $w^+ \leq w^-$, $|\Phi'(z_1)| \leq \ell(w^+) \leq \ell(w^-) \leq |\Phi(y)|$.

If $|\Phi'(z_1)| < |\Phi(y)|$ the expected result is true by induction. Otherwise, the 4 numbers above are equal; in particular, as $w^+ \leq w^-$, one has $w^+ = w^-$ and $\Phi'(z_1) = \Phi(y)$.

We consider the segment $[y, z]$ as a linear path $\pi : [0, 1] \rightarrow [y, z]$, $\pi(0) = y$, $\pi(1) = z$, $z_1 = \pi(t_1)$ and $\pi'(t) \in W^v\lambda$, $\forall t$. The number N_{z_1} is calculated in an apartment A_{z_1} containing z and $[z_1, x]$ using $\Phi(z_1)$ and $[z_1, z]$. We may suppose A_{z_1} also containing $germ_{z_1}(x - C^v)$; then there is an isomorphism from $(A_{z_1}, \Phi(z_1), [z_1, z])$ to $(A_1, \Phi'(z_1), [z_1, Z])$ where $Z = z_1 + (1 - t_1)w^+\lambda$, so N_{z_1} may be calculated with this last triple. Arguing the same way, we see that N_y may be calculated with the triple $(A_1, \Phi(y), [z_1, Z'])$ with $Z' = y + w^-\lambda$. Actually $z_1 = y + t_1.w^-\lambda$ and we saw that $\Phi'(z_1) = \Phi(y)$ and $w^+\lambda = w^-\lambda$, so $Z' = Z$. Moreover, by the choice of z_1 , there is a wall of direction $\text{Ker}\alpha$ for some $\alpha \in \Phi(y)$ containing z_1 . So $N_{z_1} < N_y$ and the expected result is true by induction. \square

References

Nicole BARDY

[96] *Systèmes de racines infinis*, Mémoire Soc. Math. France (N.S.) **65** (1996).

Pierre BAUMANN and Stéphane GAUSSENT

[06] On Mirkovic-Vilonen cycles and crystals combinatorics. preprint (6/2006).

Kenneth S. BROWN

[89] *Buildings*, Springer (1989).

François BRUHAT and Jacques TITS

[72] Groupes réductifs sur un corps local I, Données radicielles valuées, *Publ. Math. I.H.É.S.* **41** (1972), 5-251.

Howard GARLAND

[95] A Cartan decomposition for p-adic loop groups, *Math. Ann.* **302** (1995), 151-175.

Stéphane GAUSSENT and Peter LITTELMANN

[05] LS -galleries, the path model and MV -cycles, *Duke Math. J.* **127** (2005), 35-88.

Victor G. KAC

[90] *Infinite dimensional Lie algebras*, third edition, Cambridge University Press (1990).

Victor G. KAC and Dale H. PETERSON

- [85] Defining relations of certain infinite dimensional groups, in *Élie Cartan et les mathématiques d'aujourd'hui, Lyon (1984)*, Astérisque n° hors série (1985), 165-208.

Misha KAPOVICH and John J. MILLSON

- [05] A path model for geodesics in euclidean buildings and its applications to representation theory, preprint (2005), 75pp.

Shrawan KUMAR

- [02] *Kac-Moody groups, their flag varieties and representation theory*, Progress in Math. **204** Birkhäuser (2002).

Peter LITTELMANN

- [94] A Littlewood-Richardson rule for symmetrizable Kac-Moody Lie algebras, *Inventiones Math.* **116** (1994), 329-346.
- [95] Paths and root operators in representation theory, *Annals of Math.* **142** (1995), 499-525.
- [98] The path model, the quantum Frobenius map and standard monomial theory, in *Algebraic groups and their representations (Cambridge, 1997)*, R. Carter and J. Saxl editors, (Kluwer, Dordrecht, 1998), 175-212.

Ivan MIRKOVIĆ and Kari VILONEN

- [00] Perverse sheaves on affine Grassmannians and Langlands duality, *Math. Res. Lett.* **7** (2000), 13-24.

Robert MOODY and Arturo PIANZOLA

- [95] *Lie algebras with triangular decompositions*, (Wiley-Interscience, New York, 1995).

Bertrand RÉMY

- [02] *Groupes de Kac-Moody déployés et presque déployés*, Astérisque **277** (2002).

Mark A. RONAN

- [89] *Lectures on buildings*, Perspectives in Math. **7** Academic Press (1989).

Guy ROUSSEAU

- [04] *Euclidean buildings*, preprint (2004) 27 pp. In Géométries à courbure négative ou nulle, groupes discrets et rigidité, École d'été, Grenoble, 14 juin-2 juillet 2004. (<http://hal.ccsd.cnrs.fr/ccsd-00094363>)
- [06] Groupes de Kac-Moody déployés sur un corps local, immeubles microaffines, *Compositio Mathematica* **142** (2006), 501-528.
- [?] Groupes de Kac-Moody déployés sur un corps local 2, mesures ordonnées, en préparation.

Jacques TITS

- [86] Immeubles de type affine, in *Buildings and the geometry of diagrams, Como (1984)*, L.A. Rosati éditeur, Lecture notes in Math. **1181** (Springer, Berlin, 1986), 159-190.

- [87] Uniqueness and presentation of Kac-Moody groups over fields, *J. of Algebra* **105** (1987), 542-573.
- [92] Twin buildings and groups of Kac-Moody type, in *Groups combinatorics and geometry, Durham (1990)*, M. Liebeck and J. Saxl editors, London Math. Soc. lecture note **165**, Cambridge U. Press (1992), 249-286.

Stéphane GAUSSENT

gaussent@iecn.u-nancy.fr

Guy ROUSSEAU

rousseau@iecn.u-nancy.fr

Institut Élie Cartan
Unité Mixte de Recherche 7502
Nancy-Université, CNRS, INRIA
B.P 239
54506 Vandœuvre lès Nancy Cedex
FRANCE