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Littlewood-Paley Theory and regularity issues in Boltzmann homogeneous equations

II. Non cutoff case and non maxwellian molecules

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Abstract

We use Littlewood-Paley theory for the analysis of regularization properties of weak solutions of the homogeneous Boltzmann equation. For non cutoff and non Maxwellian molecules, we show that such solutions are smoother than the initial data. In particular, our method applies to any weak solution, though we assume that it belongs to a weighted L^2 space.

1 Introduction

This is the second and final part of a work devoted to regularization properties of weak solutions to Boltzmann homogeneous equation, by using technics from Harmonic Analysis. We refer the reader to our first paper [4], where a very special case of collision cross sections was analyzed, namely non cutoff Maxwellian molecules. In this part, we wish to consider a larger class of collision sections, namely those corresponding to so called hard potentials.

More precisely, we shall consider smoothed versions of this case, see assumptions below for precise definitions

Since we have already given precise references on the framework considered herein in the first part [4], we shall be rather concise in this Introduction, but we refer to [7, 9, 10, 11]. We also mention the review [17] and also the recent one [2].

Let us just recall here that Boltzmann homogeneous equation reads as

$$\partial_t f(t, v) = Q(f, f)(t, v) \quad t \geq 0, \quad v \in \mathbb{R}^n, \quad (1.1)$$

where f is a positive function depending only (homogeneous framework) upon the two variables $t \geq 0$ (time) and $v \in \mathbb{R}^n$ (velocity) with $f(0, v) = f_0(v)$, where $n \geq 2$.

The initial datum $f_0 \neq 0$ is supposed to satisfy the usual "entropic" hypothesis, that is

$$f_0 \geq 0, \quad \int_{\mathbb{R}^n} f_0(v) \{1 + |v|^2 + \log(1 + f_0(v))\} dv < +\infty. \quad (1.2)$$

Boltzmann quadratic operator Q appearing on the right hand side of (1.1) depends on v as follows

$$Q(f, f) = \int_{\mathbb{R}^n} \int_{S^{n-1}} dv_* d\sigma B(v - v_*, \sigma) (f' f'_* - f f_*) d\sigma dv_*,$$

where $v_* \in \mathbb{R}^n$, $\sigma \in S^{n-1}$ (unit sphere of \mathbb{R}^n), $f = f(v)$, $f_* = f(v_*)$, $f' = f(v')$ and $f'_* = f(v'_*)$, and

$$v' \equiv \frac{v + v_*}{2} - \frac{|v - v_*| \sigma}{2}, \quad v'_* \equiv \frac{v + v_*}{2} + \frac{|v - v_*| \sigma}{2}$$

are the so called post (or pre) collisional velocities.

As in [4], we shall assume that the collision cross section $B(v - v_*, \sigma) > 0$ is given under the following multiplicative form

$$B(v - v_*, \sigma) = \Phi(|v - v_*|) b(\cos \theta), \quad \cos \theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle, \quad 0 \leq \theta \leq \frac{\pi}{2}. \quad (1.3)$$

and that it satisfies the following non cutoff assumption

$$\int_0^{\frac{\pi}{2}} \sin^n \theta b(\cos \theta) d\theta < +\infty \quad \text{and} \quad \sin^{n-2} \theta b(\cos \theta) \sim \frac{\kappa}{\theta^{1+\nu}} \quad \text{when} \quad \theta \rightarrow 0, \quad (1.4)$$

where $\kappa > 0$ and $0 < \nu < 1$ are fixed.

First part [4] of our work was concerned with the so called maxwellian case, corresponding to $\Phi \equiv 1$ (or constant) with the above notations.

Herein, the velocity part of the kernel, that is function Φ , shall be assumed to correspond to a smoothed version of the so called hard potentials, that is

$$\Phi(|v|) = (1 + |v|^2)^{\frac{\gamma}{2}}, \quad (1.5)$$

with the range of parameters $0 < \gamma \leq 1$; the real hard potentials case corresponds to the case $\Phi(|v|) = |v|^\gamma$.

We assume that a weak solution to Boltzmann (1.1) has already been constructed and that it satisfies the usual entropic estimate, for a fixed $T > 0$ (eventually $T = +\infty$), together with mass conservation, decrease of energy and entropy dissipation rate bounded, though we shall not use this last condition. However, we refer to final part of the paper. Such a weak solution shall be referred to as an entropic weak solution. It is then known that such a weak solution has then all moments with respect to velocity, for strictly positive time. Again, we refer for precise references to [4] and to the bibliography.

In this second part of our work, we are still interested in regularization properties of such solutions.

In [4], we have provided a very simple proof of C^∞ regularization property of weak solutions, for maxwellian molecules, that is when Φ is taken to be constant, so a case which is now excluded by our assumption (1.5).

As far as we have been able to check, it should be mentioned that the proof performed in [4], though extremely simple, does not (at least for us) adapt for non maxwellian molecules, and this is so from the first computations.

In this non maxwellian and non cutoff case, the up to date recent results about this regularization property question are due to Desvillettes and Wennberg [12], showing \mathcal{S} (in fact through weighted Sobolev spaces) regularity. However, the point is that, actually, Desvillettes and Wennberg show that, under suitable assumptions on the cross section, a solution in \mathcal{S} does exist, with f_0 satisfying (1.2).

Here, exactly as in [4], we wish to show the stronger result that *any entropic weak solution is smooth*. Up to an assumption of weighted L^2 bounds, we shall show that this is indeed the case. Thus, our result is strictly not comparable to [12]. A similar result was established in the context of Landau homogeneous equation by the second author [13].

Our arguments are still based on Littlewood-Paley theory. However, we need here commutators estimates, to take into account the fact that Φ is now really a non constant function. In particular, we shall need some results extracted from the first author's paper [1].

The bad point is that we need further integrability assumption with respect to variable v . This point is in fact easy to understand, see the remarks at the end of the paper.

Thus, according to [11], we shall furthermore assume that for some $t_0 > 0$,

$$f \in L^\infty([t_0, +\infty); L_\alpha^2(\mathbb{R}^n)), \text{ for all } \alpha \in \mathbb{R}. \quad (1.6)$$

Above $L_\alpha^2(\mathbb{R}^n)$ denotes weighted Lebesgue space, see the Appendix for the precise notations. It is still an open problem to show that any entropic weak solution (thus satisfying only the usual entropic bounds) enjoys automatically this L^2 integrability (1.6), even if we do take $t_0 = 0$ and an initial datum in the same class. This is in particular due to the lack of a good uniqueness result, and also to the fact that power of f , for an exponent less than 1 belongs to a worse Besov type space, see [15] for instance, and in view of the (quite optimal with respect to the index of regularity) functional properties of Boltzmann operator [1].

Our main result is given by

Theorem 1.1 *Let be given an initial datum f_0 and a collision cross section B such that (1.2), (1.3), (1.4) and (1.5) hold true. Let f be any entropic weak non negative solution of Boltzmann homogeneous equation (1.1). Furthermore, we assume that (1.6) holds true for some $t_0 > 0$. Then, for any $t > t_0$, for all $s, \alpha \in \mathbb{R}^+$, $f(t, \cdot)$ belongs to the weighted Besov-Sobolev space $B_{2,2,\alpha}^s(\mathbb{R}^n)$.*

□

In particular, it follows that for $t > t_0$, $f(t)$ belongs immediately to Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Plan of the paper: Section 2 is devoted to the proof of our main result. Then we make some final comments in Section 3. For convenience of the reader, we have again devoted a small Appendix to basic facts from Harmonic Analysis, and notations used herein, in particular weighted spaces.

2 Proof of the theorem

For any $j \in \mathbb{N}$, $k \in \mathbb{N}$, one has, using notations from [1] and the Appendix, $\langle \cdot, \cdot \rangle$ denoting the usual duality bracket,

$$\langle \psi_j p_k Q(g, f); \psi_j p_k f \rangle = \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv f \tau_{-v_*} \circ T^\Phi \circ [p_k \psi_j^2 p_k f]. \quad (2.7)$$

Above, we have introduced $g = f$ in order to show clearly on which functions we are going to perform fractional differentiation.

Furthermore, for any suitable test function, see [1] for more precisions,

$$T^\Phi \phi(v) = \int_{S^{n-1}} [\phi(v^+) - \phi(v)] b\left(\frac{v}{|v|}, \sigma\right) d\sigma \Phi(v),$$

τ_{v_*} denoting the usual translation.

In particular, when $\Phi \equiv 1$, corresponding to the Maxwellian case, let us recall that, see [3] for instance, that

$$v \mapsto T^1 \phi(v)$$

is adjoint to

$$f \mapsto Q_{Max}(\delta_{v_*=0}, f),$$

where Q_{Max} denotes Boltzmann operator in the case of Maxwellian molecules.

We can then write (where $[\cdot, \cdot]$ denotes the usual commutator of two operators)

$$\langle \psi_j p_k Q(g, f); \psi_j p_k f \rangle = \mathcal{A} + \mathcal{B} + \mathcal{C},$$

where

$$\mathcal{A} = \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv f \tau_{-v_*} \circ T_{\Delta}^{\Phi} \circ \tau_{v_*} \{p_k \psi_j^2 p_k f\}, \quad (2.8)$$

$$\mathcal{B} = \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv f \tau_{-v_*} \circ T^1 \circ \{[\Phi, p_k] \tau_{v_*} \psi_j^2 p_k f\}, \quad (2.9)$$

and

$$\mathcal{C} = \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv f \tau_{-v_*} \circ T^1 (p_k \Phi \tau_{v_*} \psi_j^2 p_k f). \quad (2.10)$$

Above, T_{Δ}^{Φ} is defined as in [1] by

$$T_{\Delta}^{\Phi} \phi(v) = \int_{S^{n-1}} \phi(v^+) [\Phi(v^+) - \Phi(v)] b\left(\frac{v}{|v|}, \sigma\right) d\sigma$$

Using the fact that

$$\begin{aligned} \mathcal{C} &= \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv f \tau_{-v_*} \circ T^1 \circ (p_k \Phi \tau_{v_*} \psi_j^2 p_k f) \\ &= \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv (\tau_{v_*} f) T^1 (p_k \Phi \tau_{v_*} \psi_j^2 p_k f) \\ &= \int_{\mathbb{R}^n} dv_* g_* Q_{Max}(\delta_{w=0}, \tau_{v_*} f) p_k \Phi \tau_{v_*} \psi_j^2 p_k f \\ &= \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} d\xi \{Q_{Max}(\widehat{\delta_{w=0}}, \tau_{v_*} f)\} \psi_k(\xi) \{\Phi \tau_{v_*} \widehat{\psi_j^2 p_k f}\} \\ &= \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma b\left(\frac{\xi}{|\xi|}, \sigma\right) \left\{ e^{-iv_* \cdot \xi^+} \widehat{g}(\xi^+) - e^{-iv_* \cdot \xi} \widehat{f}(\xi) \right\} \psi_k(\xi) \left\{ \widehat{\Phi \tau_{v_*} \psi_j^2 p_k f} \right\}, \end{aligned}$$

it follows that

$$\mathcal{C} = \mathcal{D} + \mathcal{E}, \quad (2.11)$$

where

$$\mathcal{D} = \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \left\{ \psi_k(\xi) - \psi_k(\xi^+) \right\} e^{-iv_* \cdot \xi^+} \widehat{f(\xi^+)} \left\{ \widehat{\Phi \tau_{v_*} \psi_j^2 p_k f} \right\}, \quad (2.12)$$

$$\mathcal{E} = \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv p_k f \tau_{-v_*} \circ T^1 \circ (\Phi \tau_{v_*} \psi_j^2 p_k f), \quad (2.13)$$

by performing back the above computations.

We then write

$$\mathcal{E} = \mathcal{F} + \mathcal{G}, \quad (2.14)$$

where

$$\mathcal{F} = - \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv p_k f \tau_{-v_*} \circ T_{\Delta}^{\Phi} \tau_{v_*} \psi_j^2 p_k f, \quad (2.15)$$

and

$$\mathcal{G} = \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv p_k f \tau_{-v_*} \circ T^{\Phi} \circ \tau_{v_*} \psi_j^2 p_k f,$$

that is also

$$\mathcal{G} = \int_{\mathbb{R}^n} dv_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma g_* p_k f \Phi(v - v_*) b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) [(\psi_j^2 p_k f)' - (\psi_j^2 p_k f)]. \quad (2.16)$$

For this last term, one has

$$\mathcal{G} = \mathcal{H} + \mathcal{I}, \quad (2.17)$$

where

$$\mathcal{H} = \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) p_k f \Phi(v - v_*) (\psi_j' - \psi_j) (\psi_j p_k f)', \quad (2.18)$$

and

$$\mathcal{I} = \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) \psi_j p_k f \Phi(v - v_*) [(\psi_j p_k f)' - (\psi_j p_k f)]. \quad (2.19)$$

By using the simple identity $a(b - a) = -\frac{1}{2}(b - a)^2 + \frac{1}{2}(b^2 - a^2)$, it follows that

$$\mathcal{I} = \mathcal{J} - \mathcal{K}, \quad (2.20)$$

where

$$\mathcal{J} = \frac{1}{2} \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) \Phi(v - v_*) [(\psi_j p_k f)']^2 - (\psi_j p_k f)^2, \quad (2.21)$$

and

$$\mathcal{K} = \frac{1}{2} \int_{\mathbb{R}^n} dv_* g_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) \Phi(v - v_*) [(\psi_j p_k f)' - (\psi_j p_k f)]^2. \quad (2.22)$$

All in all, we have obtained, by applying operator $\psi_j p_k$ on Boltzmann equation and integrating against $\psi_j p_k f$, operations which are perfectly allowed even for entropic weak solutions, that is even without assumption (1.6), the following differential equality

$$\frac{d}{dt} \|\psi_j p_k f\|_{L^2}^2 + \mathcal{K} = \mathcal{A} + \mathcal{B} + \mathcal{D} + \mathcal{F} + \mathcal{H} + \mathcal{J}. \quad (2.23)$$

In the following, our task will be to found upper bounds on each term on the right hand side, while we shall look for a lower bound on \mathcal{K} .

• **Upper bound on \mathcal{J}**

Since

$$\mathcal{J} = \frac{1}{2} \int_{\mathbb{R}^n} dv_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b\left(\frac{v-v_*}{|v-v_*|} \cdot \sigma\right) (\psi_j p_k f)^2 \Phi(v-v_*) \{g'_* - g_*\},$$

it follows that, using the results from [3], one may write, for a suitable kernel S

$$\mathcal{J} = C \int_{\mathbb{R}^n} dv (\psi_j p_k f)^2 S * g(v).$$

Since we have assumed all moments on f (and thus on g) bounded, we find

$$|\mathcal{J}| \lesssim 2^j \|\psi_j p_k f\|_{L^2}^2. \quad (2.24)$$

• **Upper bound on \mathcal{H}**

Firstly, we note immediately that

$$|\mathcal{H}| \lesssim \frac{1}{2^j} \int_{\mathbb{R}^n} dv_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b\left(\frac{v-v_*}{|v-v_*|} \cdot \sigma\right) g_* |p_k f| \Phi(v-v_*) |v-v_*| |\tilde{b}| |\psi_j p_k f|'$$

and thus similarly to [14] (see also [1]), we find

$$|\mathcal{H}| \lesssim \frac{1}{2^j} \|g\|_{L^1_{\gamma+1}} \|p_k f\|_{L^2} \|\psi_j p_k f\|_{L^2_{-\gamma-1}}.$$

It follows that

$$|\mathcal{H}| \lesssim \frac{1}{2^{j(\gamma+2)}} \|g\|_{L^1_{\gamma+1}} \|p_k f\|_{L^2} \|\psi_j p_k f\|_{L^2}. \quad (2.25)$$

• **Upper bound on \mathcal{F}**

Using notations from [1], one has

$$\mathcal{F} = - \langle Q_\Delta(g, p_k f); \psi_j^2 p_k f \rangle,$$

so that

$$|\mathcal{F}| \lesssim \|Q_\Delta(g, p_k f)\|_{L^2} \|\psi_j^2 p_k f\|_{L^2}.$$

Thus

$$|\mathcal{F}| \lesssim \|g\|_{L^1_\gamma} \|p_k f\|_{L^2_\gamma} \|\psi_j p_k f\|_{L^2}. \quad (2.26)$$

• **Upper bound on \mathcal{A}**

In the same way,

$$|\mathcal{A}| = | \langle Q_\Delta(g, f); p_k \psi_j^2 p_k f \rangle | \lesssim \|Q_\Delta(g, f)\|_{L^2} \|p_k \psi_j^2 p_k f\|_{L^2},$$

and thus

$$|\mathcal{A}| \lesssim \|g\|_{L^1_\gamma} \|f\|_{L^2_\gamma} \|\psi_j p_k f\|_{L^2}. \quad (2.27)$$

• **Upper bound on \mathcal{B}**

Similarly, again with notations from [1]

$$|\mathcal{B}| = | \langle B_k; \psi_j^2 p_k f \rangle |,$$

and thus

$$|\mathcal{B}| \lesssim \|g\|_{L^1} \|f\|_{L^2} \|\psi_j p_k f\|_{L^2}. \quad (2.28)$$

• **Estimate on \mathcal{D}**

Taking into account the fact that $|\xi^+|$ is bounded above and below by a constant times $|\xi|$ on the support of ψ_k , we can introduce another Littlewood-Paley partition \tilde{p}_k to get

$$\mathcal{D} = \int_{\mathbb{R}^n} dv_* \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) g_* \{ \widehat{\tau_{v_*} \tilde{p}_k f} \}(\xi^+) A_k^\xi \{ \widehat{\Phi \tau_{v_*} \psi_j^2 p_k f} \}$$

where $A_k^\xi \equiv \psi_k(\xi^+) - \psi_k(\xi)$. Since $|A_k^\xi| \lesssim \sin \frac{\theta}{2}$, it follows that

$$\begin{aligned} |\mathcal{D}| &\lesssim \int_{\mathbb{R}^n} dv_* \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma \tilde{b}\left(\frac{\xi}{|\xi|} \cdot \sigma\right) g_* \{ \widehat{\tau_{v_*} \tilde{p}_k f} \}(\xi^+) | \{ \widehat{\Phi \tau_{v_*} \psi_j^2 p_k f} \} | \\ &\lesssim \int_{\mathbb{R}^n} dv_* g_* \left\{ \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma | \widehat{\tau_{v_*} \tilde{p}_k f} \}(\xi^+) |^2 \tilde{b}(\cdot) \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\mathbb{R}^n} d\xi \int_{S^{n-1}} d\sigma | \widehat{\Phi \tau_{v_*} \psi_j^2 p_k f} \} |^2 \tilde{b}(\cdot) \right\}^{\frac{1}{2}}, \end{aligned}$$

where $\tilde{b}(\cdot) = \sin \frac{\theta}{2} \cdot b(\cdot)$.

Thus making the change of variables $\xi^+ \mapsto \xi$, we get

$$|\mathcal{D}| \lesssim \|g\|_{L^1_\lambda} \|\tilde{p}_k f\|_{L^2} 2^{j\gamma} \|\psi_j p_k f\|_{L^2}. \quad (2.29)$$

• **Lower bound on \mathcal{K}**

From Peetre's inequality, it follows that

$$\begin{aligned}
\mathcal{K} &\gtrsim \int_{\mathbb{R}^n} dv_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b\left(\frac{v-v_*}{|v-v_*|}, \sigma\right) g_* \langle v_* \rangle^{-\gamma} \langle v \rangle^\gamma \left\{ (\psi_j p_k f)' - (\psi_j p_k f) \right\}^2 \\
&\gtrsim \int_{\mathbb{R}^n} dv_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b\left(\frac{v-v_*}{|v-v_*|}, \sigma\right) g_* \langle v_* \rangle^{-\gamma} 2^{j\gamma} \tilde{\psi}_j^2(v) \left\{ (\psi_j p_k f)' - (\psi_j p_k f) \right\}^2 \\
&\gtrsim \int_{\mathbb{R}^n} dv_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b(\cdot) g_* \langle v_* \rangle^{-\gamma} 2^{j\gamma} \left\{ (\psi_j p_k f)' - (\psi_j p_k f) + [\tilde{\psi}_j - \tilde{\psi}_j'] (\psi_j p_k f) \right\}^2 \\
&\gtrsim \int_{\mathbb{R}^n} dv_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b(\cdot) g_* \langle v_* \rangle^{-\gamma} \left\{ (\psi_j p_k f)' - (\psi_j p_k f) \right\}^2 - c 2^{j\gamma} g_* \langle v_* \rangle^{-\gamma} |\tilde{\psi}_j - \tilde{\psi}_j'|^2 [(\psi_j p_k f)']^2 \\
&\gtrsim 2^{j\gamma} \|\psi_j p_k f\|_{H^{\frac{\nu}{2}}}^2 - C 2^{j(\gamma-2)} \int_{\mathbb{R}^n} dv_* \int_{\mathbb{R}^n} dv \int_{S^{n-1}} d\sigma b(\cdot) g_* \langle v_* \rangle^{-\gamma} |v-v_*|^2 \sin^2\left(\frac{\theta}{2}\right) |(\psi_j p_k f)'|^2,
\end{aligned}$$

using similar computations as those from [3]. Therefore, using one commutator, we find

$$\mathcal{K} \gtrsim 2^{j\gamma} 2^{k\nu} \|\psi_j p_k f\|_{L^2}^2 - C 2^{j(\gamma-2)} 2^{k(\nu-2)} \|p_k f\|_{L_\alpha^2}^2, \quad (2.30)$$

for all $\alpha \geq 0$.

• Differential inequality

Collecting all the above estimates, we have found that, setting $U_{j,k} = \|\psi_j p_k f\|_{L^2}^2$, one has

$$\begin{cases} \partial U_{j,k} + C 2^{j\gamma} 2^{k\nu} U_{j,k} \lesssim \\ 2^j U_{j,k} + 2^{-j(\gamma+2)} \|g\|_{L_{\gamma+1}^1} \|p_k f\|_{L^2} U_{j,k}^{\frac{1}{2}} + \|g\|_{L_\gamma^1} \|p_k f\|_{L_\gamma^2} U_{j,k}^{\frac{1}{2}} \\ + \|g\|_{L_\gamma^1} \|f\|_{L_\gamma^2} U_{j,k}^{\frac{1}{2}} + \|g\|_{L^1} \|f\|_{L^2} U_{j,k}^{\frac{1}{2}} \\ + \|g\|_{L_\gamma^1} \|\tilde{p}_k f\|_{L^2} 2^{j\gamma} U_{j,k}^{\frac{1}{2}} + 2^{j(\gamma-2)} 2^{k(\nu-2)} \|p_k f\|_{L_\alpha^2}^2. \end{cases} \quad (2.31)$$

• Iteration- First step

By assumption, for all $t \geq t_0$, $U_{j,k} \lesssim \frac{1}{2^{j\beta}}$, for all $\beta \geq 0$, $\|p_k f\|_{L_\alpha^2} \lesssim C$ and $\|g\|_{L_\alpha^1} \lesssim C$. It follows that we found

$$\partial_t U_{j,k} + 2^{j\gamma} 2^{k\nu} U_{j,k} \lesssim 2^{j(\gamma-2)}.$$

Thus, it follows from (2.31) that for $t \geq t_1 > t_0$,

$$U_{j,k} \lesssim 2^{j(\gamma-3)} 2^{-k\nu}.$$

Since we have also

$$U_{j,k} \lesssim 2^{-j\alpha},$$

it follows that, for any $\varepsilon > 0$ small, any $\alpha \geq 0$

$$U_{j,k} \lesssim 2^{-j\alpha} 2^{-k(\nu-\varepsilon)}.$$

Thus, we have obtained that for any $\varepsilon > 0$, small, $f \in B_{2,\infty,\alpha}^{\frac{\nu}{2}-\varepsilon}$ (α referring to the weight). These bounds were obtained by using punctual (in j and k) estimates. But, if we take into account that we have also summability, then we can relax the parameter ε , and we get in fact that $f \in B_{2,2,\alpha}^{\frac{\nu}{2}}$, for all $t \geq t_1 > t_0$.

• **Iteration- Second step**

We now want to improve the index of regularity. For this purpose, we need to work back on the terms \mathcal{A} and \mathcal{B} , from which we deduce the two estimates appearing on the third line of (2.31). In order to improve these two estimates, the simplest way is to use the results from [1]. Then, in view of the regularity obtained in the first step, we obtain immediately that

$$|\mathcal{A}| \lesssim \|g\|_{W_\alpha^{1,\frac{\nu}{2}}} \|f\|_{B_{2,2,\alpha}^{\frac{\nu}{2}}} 2^{-k\frac{\nu}{2}} 2^{-j\alpha} \|\psi_j p_k f\|_{L^2} \lesssim 2^{-k\frac{\nu}{2}} 2^{-j\alpha} \|\psi_j p_k f\|_{L^2} \quad (2.32)$$

(for all big α). Similarly, taking into account the results on T^1 from [1] and the fact that there is a commutator appearing in \mathcal{B} , we get

$$|\mathcal{B}| \lesssim 2^{-j\alpha} 2^{-k(1-\nu)} \|\psi_j p_k f\|_{L^2}. \quad (2.33)$$

Replacing the two estimates on the third line of (2.31) by the estimates obtained in (2.32) and (2.33), we get this time from (2.31)

$$\partial_t U_{j,k} + 2^{j\gamma} 2^{k\nu} U_{j,k} \lesssim 2^{j(\gamma-2)} 2^{-k\nu}$$

and by iterating, we get

$$U_{j,k} \lesssim 2^{-j\alpha} 2^{-k(2\nu-\varepsilon)}$$

and finally $f \in B_{2,2,\alpha}^\nu$ for all $\alpha \geq 0$, by the same type of arguments.

In conclusion, we have passed from the regularity index $\frac{\nu}{2}$ to the regularity index ν .

We can now bootstrap this new index of regularity, by using it to again get improved estimates on \mathcal{A} and \mathcal{B} . That is, we get estimates similar to (2.32) and (2.33) but with ν replaced by 2ν . This concludes the proof.

3 Final comments

We wish to finish on some remarks connected in particular with assumption (1.6).

1) First of all, we assumed that $\nu \in (0, 1)$. This is only for convenience, since we have used results from [1]. The range $\nu \in [1, 2)$ is in fact available, see [5]. Thus, our main result can be also extended to this case. We have also considered a smoothed version of the

kinetic kernel. It should be certainly possible to consider truly the real case $|v|^\gamma$, by using in particular technics from [1] and [13].

2) Next, what about relaxing assumption (1.6)? Then, note that adding the assumption of boundedness on entropy dissipation rate (which is in fact part of the definition of an entropic weak solution), we can assume that

$$\int_0^T \| \langle v \rangle^{\frac{\gamma}{2}} \sqrt{f}(s) \|_{H^{\frac{n}{2}}}^2 ds < +\infty. \quad (3.34)$$

From the books quoted in the bibliography, in particular [15], we get

$$\int_0^T \| \langle v \rangle^\gamma f(s) \|_{H_{p_1}^s} ds < +\infty, \quad (3.35)$$

where $p_1 = \frac{n}{n - \frac{\nu}{2}} > 1$, but $p_1 < 2$.

To simplify the exposition, let's forget about integrability w.r.t. time t . Then it follows from Sobolev embedding that $\langle v \rangle^\gamma f \in L^{p_2}$, where $p_2 = \frac{n}{n-\nu}$. Of course $p_2 > 1$, but we note that $p_2 \geq 2$ iff $\nu \geq \frac{n}{2}$. In particular, in dimension $n = 2$, this is the case iff $\nu \geq 1$, while in dimension 3, this is the case iff $\nu \geq \frac{3}{2}$. In conclusion when ν is really very close to 2, then this L^2 bound is available.

In conclusion, in dimension $n = 2$ or $n = 3$, it should be certainly possible (with some extra work) to relax assumption (1.6) and get our result.

We also note, that having in mind [11], small power of f should have good regularity.

These small remarks explain also the fact that Landau equation, corresponding to a version of Boltzmann equation with $\nu = 2$ is much more easy to deal with, see for instance [13].

3) Finally, as regards the non homogeneous version of Boltzmann equation, let us note our work in progress [6], where we show regularization properties, for solutions satisfying very weak assumptions. This is has to be compared to the non homogeneous Landau equation [8], where initial assumptions are quite strong.

4 Appendix: Littlewood Paley decomposition.

This Appendix is devoted to Littlewood-Paley decomposition and some links with Sobolev type spaces, see the books of Runst, Sickel and Triebel [16, 15].

We fix once for all a collection $\{\psi_k = \psi_k(\xi)\}_{k \in \mathbb{N}}$ of smooth functions such that

$$\text{supp } \psi_0 \subset \{\xi \in \mathbb{R}^N, |\xi| \leq 2\},$$

$\text{supp } \psi_k \subset \{\xi \in \mathbb{R}^N, 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$ for all $k \geq 1$,

and

$$\sum_{k=0}^{+\infty} \psi_k(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^N.$$

To simplify some computations, all functions ψ_k , for $k \geq 1$, are constructed from a single one $\psi \geq 0$, i.e. we are given ψ such that $\text{supp } \psi \subset \{\xi \in \mathbb{R}^N, \frac{1}{2} \leq |\xi| \leq 2\}$, $\psi > 0$ if $\frac{1}{\sqrt{2}} \leq |\xi| \leq \sqrt{2}$ such that $\psi_k(\xi) \equiv \psi(\frac{\xi}{2^k})$, for all $k \geq 1$ and $\xi \in \mathbb{R}^N$.

Then, Littlewood-Paley projection operators p_k , for $k \geq 0$, are defined by

$$\widehat{p_k f}(\xi) = \psi_k(\xi) \hat{f}(\xi),$$

yielding

$$f = \sum_{k=0}^{+\infty} p_k f \text{ for all } f \in \mathcal{S}'.$$

By construction, we can find a new collection $\{\tilde{\psi}_k = \tilde{\psi}_k(\xi)\}_{k \in \mathbb{N}}$ of smooth functions such that $\text{supp } \tilde{\psi}_0 \subset \{\xi \in \mathbb{R}^N, |\xi| \leq 4\}$, $\text{supp } \tilde{\psi}_k \subset \{\xi \in \mathbb{R}^N, 2^{k-2} \leq |\xi| \leq 2^{k+2}\}$ for all $k \geq 1$, and such that $\psi_k \tilde{\psi}_k = \psi_k$, for all integer k .

As before, corresponding operator \tilde{p}_k , for $k \geq 0$, are defined as

$$\widehat{\tilde{p}_k f}(\xi) = \tilde{\psi}_k(\xi) \hat{f}(\xi).$$

All these functions $\tilde{\psi}_k$, for $k \geq 1$, are constructed from a single one $\tilde{\psi} \geq 0$, i.e. we take $\tilde{\psi}$ such that $\text{supp } \tilde{\psi} \subset \{\xi \in \mathbb{R}^N, \frac{1}{2^2} \leq |\xi| \leq 2^2\}$, $\tilde{\psi} > 0$ if $\frac{1}{2} \leq |\xi| \leq 2$, such that $\tilde{\psi}_k(\xi) \equiv \tilde{\psi}(\frac{\xi}{2^k})$, for all $k \geq 1$ and $\xi \in \mathbb{R}^N$.

Note that, for any integer k

$$p_k \tilde{p}_k = p_k. \tag{4.36}$$

Moreover, using Plancherel formula, it follows that

$$\int_v f(v) p_k \text{ (resp. } \tilde{p}_k) g(v) dv = \int_v p_k \text{ (resp. } \tilde{p}_k) f(v) g(v) dv, \text{ for all } f, g \in \mathcal{S}'. \tag{4.37}$$

For all $f \in L^1$, one has Bernstein's inequality:

$$\|p_k f\|_{L^2} \leq C 2^{\frac{Nk}{2}} \|p_k f\|_{L^1}, \tag{4.38}$$

where C is a constant depending on the function $\tilde{\psi}$.

We set for any $v \in \mathbb{R}^n$, $\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}$. Lebesgue weighted spaces L^α_α , $\alpha \in \mathbb{R}$, are

defined as the spaces of those functions $f = f(v)$ such that $\langle v \rangle f \in L^p$. We denote the corresponding norm by $\|\cdot\|_{L_\alpha^p}$.

Thanks to this decomposition, weighted space L_α^1 satisfies

$$\forall \alpha > 0, \quad \|f\|_{L_\alpha^1} \sim \sum_{j=0}^{\infty} 2^{js} \|\psi_j f\|_{L^1}.$$

More generally, usual weighted Sobolev-Besov spaces can be described by the following important result, see for instance the results quoted in the books [16, 15], last index α referring to the weight

$$\|f\|_{B_{p,q,\alpha}^s}^q \simeq \sum_{k=0}^{+\infty} 2^{jq_s} \|p_k f\|_{L_\alpha^p}^q \simeq \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} 2^{kq_s} 2^{jq_\alpha} \|\psi_j p_k f\|_{L^p}^q.$$

References

- [1] ALEXANDRE.R Integral kernel estimates for a linear singular operator linked with Boltzmann equation. Part I: Small singularities $0 < \nu < 1$. *Indiana Univ. J. Maths*, **55-6**, (2007).
- [2] ALEXANDRE.R *Boltzmann equation and singular kernels*. Lecture Notes. Given during C.H.K.E, Shanghai, Dec. 2006. In preparation.
- [3] ALEXANDRE.R, DESVILLETES.L, VILLANI.C, WENBERG.B Entropy dissipation and long range interactions. *Arch. Rat. Mech. Anal.* **152-4** (2000) 327-355.
- [4] ALEXANDRE.R, ELSAFADI.M Littlewood-Paley decomposition and regularity issues in Boltzmann homogeneous equations.I. Non cutoff and Maxwell cases. *Math. Meth. Models in Applied Sci.*, **15-6** (2005).
- [5] ALEXANDRE.R, HE.L Integral kernel estimates for a linear singular operator linked with Boltzmann equation. Part II: High singularities $1 \leq \nu < 2$. Work in preparation.
- [6] ALEXANDRE.R, MORIMOTO.Y, UKAI.S, XU.C.-Y, YANG.T Regularization properties of non homogeneous kinetic equations and applications. Work in progress.
- [7] CERCIGNANI.C *The Boltzmann equation and its applications*. (Springer-Verlag, New-York, 1988).

- [8] CHEN.Y, DESVILLETES.L, HE.L Smoothing effects for classical solutions of the full Landau equation. Preprint (2006).
- [9] DESVILLETES.L About the regularization properties of the non cut-off Kac equation. *Com. Math. Phys.* **168** (1995) 417-440.
- [10] DESVILLETES.L Regularization properties of the 2-dimensional non radially symmetric non cutoff spatially homogeneous Boltzmann equation for Maxwellian molecules. *Transport Theory Stat Phys.* **26-3** (1997) 341-357.
- [11] DESVILLETES.L, MOUHOT.C About L^p estimates for the spatially homogeneous Boltzmann equation. Preprint (2003).
- [12] DESVILLETES.L, WENBERG.B Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff. *Comm. P.D.E* **29** (2004) 133-155.
- [13] EL SAFADI.M Smoothness of weak solutions of the spatially homogeneous Landau equation. *Analysis and Applications.* **5-1** (2007) 29-49.
- [14] MOUHOT.C, VILLANI.C Regularity theory for the homogeneous Boltzmann equation with angular cutoff. *Arch. Rational Mech. Anal.* **173** (2004) 169-212.
- [15] RUNST.T, SICKEL. W *Sobolev spaces of fractional order, Nemytskij operators and Non linear Pde.* (De Gruyter, New York, 1996).
- [16] TRIEBEL.H *Theory of function spaces.* (Birkhauser Verlag, Basel and al., 1983).
- [17] VILLANI.C *A review of mathematical topics in collisional kinetic theory.* Handbook of Fluid Mechanics (Ed. S. Friedlander, D.Serre, 2002).