Ordinary holomorphic webs of codimension one

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by Vincent CAVALIER and Daniel LEHMANN.

Abstract

To any d-web of codimension one on a holomorphic n-dimensional manifold M $(d > n)$, we associate an analytic subset S of M. We call **ordinary** the webs for which S has a dimension at most $n-1$ or is empty. This condition is generically satisfied.

We prove that the rank of a ordinary d-web has an upper-bound $\pi'(n,d)$ which, for $n \geq 3$, is strictly smaller than the bound $\pi(n,d)$ of Castelnuovo. This bound is optimal.

Setting $c(n,h) = \binom{n-1+h}{h}$, let k_0 be the integer such that $c(n,k_0) \leq d < c(n,k_0+1)$. The number $\pi'(n,d)$ is then equal

- to 0 for $d < c(n,2)$,
- and to $\sum_{h=1}^{k_0} \big(d - c(n,h)\big)$ for $d \geq c(n,2)$.

Moreover, if d is precisely equal to $c(n,k_0)$, we define a holomorphic connection on a holomorphic bundle $\mathcal{E}$ of rank $\pi'(n,d)$, such that the space of abelian relations is isomorphic to the space of holomorphic sections of $\mathcal{E}$ with vanishing covariant derivative : the curvature of this connection, which generalizes the Blaschke curvature, is then an obstruction for the rank of the web to reach the value $\pi'(n,d)$.

When $n = 2$, any web is ordinary, $\pi'(2,d) = \pi(2,d)$, any d may be written $c(2,k_0)$, and we recover the results given locally in [P][He1].

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1   Introduction

A holomorphic d-web W of codimension one on a n-dimensional holomorphic manifold M being given $(d > n)$, we denote by M_0 the open set in M where the web is locally defined by d holomorphic foliations $\mathcal{F}_i$ of codimension one, all non-singular and with tangent spaces to the leaves distinct at any point.

We shall say that the web is in **strong general position** if *any* subset of n leaves among the d leaves through a point of M_0 are in general position. If we assume only that *there exists* a subset of n leaves among the d leaves through a point of M_0 which are in general position (but not necessarily any n of them), we shall say that the web is in **weak general position**. Most of our results below will require only this weaker condition[1].

On an open set U in M_0 sufficiently small for the web to be defined as above by the data of d distinct local foliations $\mathcal{F}_i$ (such an open set is called "open set of distinguishability"), an abelian relation is the data of a family $(\omega_i)_i$ of holomorphic 1-forms ω_i, $(1 \leq i \leq d)$ such that

(i) each ω_i is closed,

(ii) the vector fields tangent to the local foliation $\mathcal{F}_i$ belong to the kernel of ω_i,

(iii) the sum $\sum_{i=1}^{d} \omega_i$ vanishes.

The set of abelian relations on U (resp. the set of germs of abelian relation at a point m of M_0) has the structure of a finite dimensional complex vector space, whose dimension is called the rank of the web on U (resp. at m). If the web is in strong general position, Hénaut proved in [He2] that its rank at a point does not depend on this point: abelian relations have then the structure of a local system of coefficients[2]. When we require only the web to be in weak general position, we shall call "rank of the web" the maximum[3] of the rank at each point of M_0.

When the web is in strong general position, its rank is always upper-bounded, after Chern ([C]), by the number $\pi(n, d)$ of Castelnuovo (the maximum of the arithmetical genus of an irreducible non-degenerate algebraic curve of degree d in the n-dimensional complex projective space $\mathbb{P}_n$). On the other hand, the rank of an algebraic d-web in $\mathbb{P}_n$ (i.e. the web whose leaves are the hyperplanes belonging to some algebraic curve Γ of degree d in the dual projective space $\mathbb{P}'_n$) is equal to the arithmetical genus of Γ: this is, after duality, a theorem coming back to Abel (see [CG]). Therefore, the bound $\pi(n, d)$ is optimal for webs in strong general position.

When $n = 2$, the obstruction for the web to have maximal rank $\pi(2, d) = (d-1)(d-2)/2$ is the Blaschke curvature, which has been defined by Blaschke-Dubourdieu ([B]) for $d = 3$, and generalized independently by Panzani ([P]) and Hénaut ([He1]) for any $d \geq 3$.

The first aim of this paper was to generalize this curvature to any $n \geq 2$. But some difficulties

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[1]More generally, if there exists ℓ and not more of the leaves through a point which are in general position ($\ell \leq n$), there exists locally a holomorphic foliation $\mathcal{G}$ of codimension ℓ, such that the web is locally the pullback of a d-web of codimension 1 on a ℓ-dimensional manifold transversal to $\mathcal{G}$. Then, many of the results below remain valid, after replacing n by ℓ.

[2]In [He3], he generalized this result in higher codimension.

[3]Notice that, in this case, the rank at a point is an upper-semicontinuous function of the point.

appear which do not exist in dimension 2. In fact, we wish to find a holomorphic vector bundle with a holomorphic connection such that the vector space of abelian relationsis isomorphic to the vector space of holomorphic sections with covariant derivative (so that the curvature of this connection will be an obstruction for the rank of the web to be maximal). Lemma 2-2 below implies that the projection $R_{k_0-2} \to R_{k_0-3}$ from the space of formal abelian relations at order $k_0 - 1$ into the space of formal abelian relations at order $k_0 - 2$ be an isomorphism of holomorphic vector bundles, at least above some everywhere dense open subset $M_0 \setminus S$ of M_0. Therefore :

- on one hand , d must be equal, for some $k_0 \geq 2$, to the dimension $c(n, k_0)$ of the vector space of all homogeneous polynomials of degree k_0 in n variables with scalar coefficients),

- on the other hand, some analytical subset S of M_0 attached to the web must have a dimension at most $n - 1$ or be empty (this assumption, is in fact generically satisfied ; the web is then said to be **ordinary**). If, exceptionally, S has dimension n, the web is said to be **extraordinary**.

We proved, by the way, that whatever be d (not necessarily equal to some $c(n, k_0)$), the rank of all ordinary d-webs is at most equal to some bound $\pi'(n, d)$ which, for $n \geq 3$, is strictly smaller than the bound $\pi(n, d)$ of Castelnuovo.

For any d, $(d > n)$, denote by k_0 the integer (≥ 1) such that $c(n, k_0) \leq d < c(n, k_0 + 1)$, and set :

$$\pi'(n, d) \quad = \quad\quad\quad 0 \quad\quad\quad\quad \text{when } d < c(n, 2), \ (k_0 = 1),$$

$$= \quad \sum_{h=1}^{k_0} \big(d - c(n, h)\big) \quad \text{when } d \geq c(n, 2), \ (k_0 \geq 2).$$

The main results of this paper are then the two following :

Theorem 1.1 *The rank of any ordinary d-web on some n-dimensional manifold M_0 is at most equal to $\pi'(n, d)$. This bound is optimal.*

Theorem 1.2 *If $d = c(n, k_0)$, and if the d-web is ordinary, the space $\mathcal{E} = R_{k_0-3}$ of formal abelian relations at order $k_0 - 2$ is a holomorphic vector bundle of rank $\pi'(n, d)$ over $M_0 \setminus S$. There is a holomorphic connection ∇ on $\mathcal{E}$, such that the map $u \mapsto j^{k_0-2}u$ defines an isomorphism from the space of germs of abelian relations at a point of $M_0 \setminus S$ onto the vector space of germs of sections of $\mathcal{E}$ with vanishing covariant derivative: $\nabla(j^{k_0-2}u) = 0$. The curvature of this connection is therefore an obstruction for the rank of the web to reach the value $\pi'(n, d)$.*

For $n = 2$, it happens
- that S is always empty, so that all webs are ordinary,
- that the upper-bounds $\pi(2, d)$ and $\pi'(2, d)$ coincide,
- that any $d, d \geq 3$, may be written $d = c(2, k_0)$, with $k_0 = d - 1$.
Thus, we recover the results given locally in [P][He1].

For $n \geq 3$, the things are not so simple : S may be non-empty, $\pi'(n, d)$ is strictly smaller than $\pi(n, d)$, and not any d may be written $c(n, h)$. Moreover, the fact that $dim\,S$ must be equal to n for the rank of the web to reach the bound $\pi(n, d)$ is a criterium of practical interest.

In the last section, we give a second proof of theorem 1-1 in the particular case of webs which are in strong general position. We prove also that all ordinary affine d-webs in dimension n have rank $\pi'(n, d)$ (hence the optimality of this bound). We give finally significative examples of the various possible situations.

We thank A. Hénaut for very helpful conversations.

2   Notations and backgrounds

Most of the results in this section are well known or easy to prove, so that we shall omit their proof.

2.1 Some algebraic notations

Let $\mathbb{R}_h[X_1, \cdots, X_n]$ denote the vector space of homogeneous polynomials of degree h in n variables, with scalar coefficients (in fact, the field of scalars does not matter). Denote by $c(n, h)$ its dimension $\binom{h+n-1}{h}$. The monomials $X^L = (X_1)^{\lambda_1} \cdots (X_n)^{\lambda_n}$ make a basis, indexed by the set $\mathcal{P}(n, h)$ of the partitions $L = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of h, with $0 \le \lambda_i \le h$ for any $i = 1, \cdots, n$, and $\sum_{i=1}^{n} \lambda_i = h$. The number h will be also denoted by $|L|$, and is called the *height* of L.

Denoting by $(a)^+$ the number $sup \, (a, 0)$ for any real number a, remember ([GH]) that the number

$$\pi(n, d) = \sum_{h \ge 1} \left(d - h(n-1) - 1 \right)^+,$$

called *the bound of Castelnuovo*, is the maximum of the arithmetical genus of an algebraic curve of degree d in the complex projective space $\mathbb{P}_n$.

Define

$$\pi'(n, d) \quad = \qquad\qquad 0 \qquad\qquad \text{when } d < c(n, 2),$$

$$= \quad \textstyle\sum_{h \ge 1} \left(d - c(n, h) \right)^+ \quad \text{when } d \ge c(n, 2).$$

Lemma 2.1

    (i] *For $n \ge 3$, the inequality $\pi'(n, d) < \pi(n, d)$ holds.*

    (ii) *The equality $\pi'(2, d) = \pi(2, d)$ holds.*

2.2 Connections adapted to a differential operator :

Let $E \to V$ be a holomorphic vector bundle on a holomorphic manifold V. Remember the exact sequence of holomorphic vector bundles

$$0 \to T^*V \otimes E \to J^1 E \to E \to 0,$$

where $J^1 E$ denotes the holomorphic bundle of 1-jets of holomorphic sections of E, and T^*V the holomorphic bundle of 1-forms of type $(1, 0)$ on V. A C^∞-connection ∇ on E of type $(1, 0)$ (resp. a holomorphic connection if any) is a C^∞-splitting (resp. a holomorphic splitting if any) of this exact sequence :

$$0 \to T^*V \otimes E \xrightarrow{\beta} J^1 E \xrightarrow{\alpha} E \to 0.$$

If u is a C^∞ or holomorphic section of E, its covariant derivative is given by

$$\nabla u = \beta(j^1 u).$$

A holomorphic linear differential operator of order p on the holomorphic manifold V is a morphism of holomorphic vector bundles $D : J^p E \to F$, for some vector bundles E and F on V, to which we associate the map $\mathcal{D} : u \mapsto D(j^p u)$ from the sections of E into the sections of F. The kernel R of D is a subset of $J^p E$, which is the set of the formal solutions at order p of the equation $\mathcal{D}u = 0$. A sections u of E such that $\mathcal{D}u = 0$ is called a "solution" of this equation

The morphism D induces a morphism $j^1 D : J^1(J^p E) \to J^1 F$ whose restriction $\widetilde{D} : J^{p+1} E \to J^1 F$ to the sub-bundle $J^{p+1} E \subset J^1(J^p E)$ is called the first prolongation of D, and the sections u of E such that $\widetilde{D}(j^{p+1} u) = 0$ are also the solutions of the equation $\mathcal{D}u = 0$.

Assume that R is a holomorphic vector bundle above V. Then $J^1 R$ is a sub-vector bundle of $J^1(J^p E)$, as well as $J^{p+1} E$. The kernel R' of the morphism $\widetilde{D} : J^{p+1} E \to J^1 F$ (the space of formal

solutions at order $p+1$ of the equation $\mathcal{D}u = 0$) is then equal to the intersection $J^1 R \cap J^{p+1} E$ in $J^1(J^p E)$.

$$
\begin{array}{ccccccc}
J^1(J^p E) & = & & J^1(J^p E) & \xrightarrow{j^1 D} & J^1 F \\
& & & & & \\
\cup \uparrow & & & \cup \uparrow & & = \updownarrow \\
& & & & & \\
J^1 R & \hookleftarrow & R' & \hookrightarrow & J^{p+1} E & \xrightarrow{\widetilde{D}} & J^1 F \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R & = & R & \hookrightarrow & J^p E & \xrightarrow{D} & F \\
& & \downarrow & & & & \\
& & V & & & &
\end{array}
$$

Lemma 2.2 [4] *The two following assertions are equivalent :*

(i) The projection $R' \to R$ is an isomorphism of holomorphic vector bundles.

(ii) There is a holomorphic connection ∇ on $R \to V$ such that the map $u \mapsto j^p u$ is an isomorphism from the space of solutions u of the equation $\mathcal{D}u = 0$ into the space of sections of R with vanishing covariant derivative.

Moreover, such a connection ∇ is unique and defined as being the map $\alpha : R \to J^1 R$ equal to the composition of the inclusion $R' \hookrightarrow J^1 R$ with the inverse isomorphism $R \to R'$.

The connection ∇ above on R is said to be "completely adapted" to the differential operator $\mathcal{D}$.

2.3 The differential operator for abelian relations

Recall that a d-web on a n-dimensional holomorphic manifold M (in weak general position, with $d > n$) is defined by a n-dimensional analytical subspace W of the projectivized cotangent space $\mathbb{P}(T^* M)$, on the smooth part of which the canonical contact form induces a foliation $\widetilde{\mathcal{F}}$]. Let W_0 be the set of points in W over M_0 and $\pi_W : W_0 \to M_0$ be the d-fold corresponding covering. For any complex holomorphic vector bundle E of rank r over W_0, let $\pi_* E$ be the holomorphic bundle of rank $d \times r$ over M_0, whose fiber at a point $m \in M_0$ is defined by

$$
(\pi_* E)_m = \bigoplus_{\widetilde{m} \in (\pi_W)^{-1}(m)} E_{\widetilde{m}} \ .
$$

The sheaf of holomorphic sections of $\pi_* E$ is then the direct image of the sheaf of holomorphic sections of E by π_W. For instance, an element ω of $\pi_*(T^* \widetilde{\mathcal{F}})$ at m is a family (ω_i) of forms ω_i at m ($1 \le i \le d$), such that the kernel of ω_i contains the tangent space T_i to the local foliation $\mathcal{F}_i$ (and is therefore equal to it when ω_i is not zero).

Let $Tr : \pi_*(T^* \widetilde{\mathcal{F}}) \to T^* M_0$ be the morphism of holomorphic vector bundles given by

$$
Tr \ \omega \ = \ \sum_{i=1}^d \omega_i.
$$

Since the web is in weak general position, it is easy to check that Tr has constant rank n. Then, we define a holomorphic vector bundle A of rank $d - n$ over M_0 as the kernel of this morphism.

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[4]This lemma, following from the Spencer-Goldschmidt theory ([S]), is used by Hénaut in [He1] under an equivalent form.

Definition 2.3 *The vector bundle A of rank $d - n$ so defined will be called* the Blaschke bundle *of the web.*

Let us define similarly a holomorphic vector bundle B over M_0, of rank $(d-1)n(n-1)/2$ as the kernel of the morphism $Tr : \pi_*(\bigwedge^2 T^*W_0) \to \bigwedge^2 T^*M_0$ given by $Tr\ \varpi\ =\ \sum_{i=1}^d \varpi_i$.

Definition 2.4 *We call* abelian relation *of the web any holomorphic section u of the Blaschke bundle A, which is solution of the equation $\mathcal{D}u = 0$, the map $\mathcal{D}$ denoting the linear first order differential operator $(\omega_i) \mapsto (d\omega_i)$ from A to B.*

The differential operator $\mathcal{D}$ may still be seen as a linear morphism $D : J^1 A \to B$, and the kernel R_0 of this morphism is the space of "formal abelian relations at order one". Hence, a necessary condition for an abelian relation to exist above an open subset U of M_0 is that U belongs to the image of R_0 by the projection $J^1 A \to A$. We shall see that it generically not the case for $d < c(n, 2)$.

More generally, denote by $D_k : J^{k+1} A \to J^k B$ the kth prolongation of D. The kernel R_k of the morphism D_k is the space of formal abelian relations at order $k+1$. Abelian relations may still be seen as the holomorphic sections u of A such that $j^{k+1} u$ be in R_k. Let π_{k+1} denote the natural projection $R_{k+1} \to R_k$. Let $\sigma_{k+1} : S^{k+2}(T^* M_0) \otimes A \to S^{k+1}(T^* M_0) \otimes B$ be the symbol of D_{k+1}, g_{k+1} be the kernel of this symbol, and K_k its cokernel. After snake's lemma, there is a natural map $\partial_k : R_k \to K_k$ in such a way that we get a commutative diagram[5], with all lines and columns exact:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & R_k \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \partial_k \\
0 & \to & g_{k+1} & \to & S^{k+2}(T^* M_0) \otimes A & \overset{\sigma_{k+1}}{\to} & S^{k+1}(T^* M_0) \otimes B & \to & K_k & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & R_{k+1} & \to & J^{k+2} A & \overset{D_{k+1}}{\to} & J^{k+1} B & \to & coker\ D_{k+1} & \to 0 \\
& & \downarrow \pi_{k+1} & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & R_k & \to & J^{k+1} A & \overset{D_k}{\to} & J^k B & \to & coker\ D_k & \to 0 \\
& & \downarrow \partial_k & & \downarrow & & \downarrow & & \downarrow \\
& & K_k & & 0 & & 0 & & 0
\end{array}
$$

In this diagram, we allow k to take the value -1, with the convention $R_{-1} = A$, $J^{-1} B = 0$.

In the sequel,

the index i will run from 1 to d,

the indices $\alpha, \beta, \cdots$ will run from 1 to $n - 1$,

and the indices $\lambda, \mu, \cdots$ will run from 1 to n.

For any holomorphic function a and local coordinates x, a'_λ will denote the partial derivative $a'_\lambda = \frac{\partial a}{\partial x_\lambda}$. More generally, for any partition $L = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of $|L| = \sum_\mu \lambda_\mu$, ($L \in \mathcal{P}(n, |L|)$), we denote by $(a)'_L$ the corresponding higher order derivative $(a)'_L = \frac{\partial^{|L|} a}{(\partial x_1)^{\lambda_1} \cdots (\partial x_n)^{\lambda_n}}$.

3   Definition of the analytical set S

This definition is different according to the inequalities $d < c(n, 2)$ or $d \geq c(n, 2)$. However, in both cases, this set will satisfy to the

Lemma 3.1 *The set S is an analytical set which has generically a dimension $\leq n - 1$ or is empty.*

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[5]Notice that the R_k's, g_k's and K_k's are not necessarily vector bundles : exactitude has to be understood as the exactitude in the corresponding diagrams on the fibers at any point of M_0.

3.1 Definition of S in the case $n < d < c(n, 2)$:

For $n < d < c(n, 2)$, S will denote the subset of elements $m \in M_0$ such that the vector space $(R_0)_m = (\pi_0)^{-1}(A_m)$ has dimension at least 1. The proof of lemma 3-1 for $d < c(n, 2)$ will be given in the next section, after the description of the map π_0.

Proof of theorem 1-1 in the case $d < c(n, 2)$: *For $d < c(n, 2)$, all ordinary d-webs of codimension one have rank 0.*

The fact that all ordinary d-webs have rank 0 for $d < c(n, 2)$ on $M_0 \setminus S$ is a tautology, because of the definition of S. Therefore, they still have rank 0 on all of M_0, because of the semi-continuity of the rank at a point.

QED

3.2 Definition of S in the case $d \geq c(n, 2)$

Let η_i be an integrable 1-form defining the local foliation $\mathcal{F}_i$ of a d-web of codimension 1. For any integer $h \geq 1$, let $(\eta_i)^h$ be the hth symetric power of η_i in the space $S^h(T^* M_0)$ of homogeneous polynomials of degree h on the complex tangent tangent bundle $T M_0$. For any $m \in M_0$, let $r_h(m)$ be the dimension of the subspace $L_h(m)$ generated in $S^h(T_m^* M_0)$ by the $(\eta_i)^h(m)$'s with $1 \leq i \leq d$ (not depending on the choice of the η_i's). We have obviously the

Lemma 3.2 *The following inequality holds :*

$$r_h(m) \leq \min\big(d, c(n, h)\big).$$

In particular, $r_1 \equiv n$, since we assume $d > n$ and the web to be in weak general position. And $r_h \equiv d$ for $h > k_0$, where k_0 denotes the integer such that

$$c(n, k_0) \leq d < c(n, k_0 + 1).$$

Definition of S for $d \geq c(n, 2)$ (i.e. $k_0 \geq 2$) : Let S_h be the set of points $m \in M_0$ such that $r_h(m) < \min\big(d, c(n, h)\big)$, and set : $S = \bigcup_{h=2}^{k_0} S_h$.

Proof of lemma 3-1 for $d \geq c(n, 2)$: Let $(x_\lambda)_{1 \leq \lambda \leq n}$ be a system of holomorphic local coordinates near a point m_0 of M_0, and assume that the local foliation $\mathcal{F}_i$ is defined by $\sum_\lambda p_{i\lambda} dx_\lambda = 0$. Then $r_h(m)$ is the rank of the matrix

$$P_h = \big(\!\big(C_{i\ L}^{(h)}\big)\!\big)_{i, L}$$

of size $d \times c(n, h)$ at point m, where $1 \leq i \leq d$, and L runs through the set $\mathcal{P}(n, h)$ of the partitions $L = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of h (i.e. $\sum_{s=1}^n \lambda_s = h$), and where $C_{i\ L}^{(h)} = \prod_{s=1}^n p_{i\ \lambda_s}$. Thus, for $2 \leq h \leq k_0$, S_h is locally defined by the vanishing of all determinants of size $c(n, h)$ in P_h. Exceptionnaly, it may happen that all of these determinants are identically 0, so that S_h will have dimension n. But generically, these determinants will vanish on hypersurfaces or nowhere.

4 Computation of R_0

Locally, the d-web is defined over M_0 by a family of 1-forms $\eta_i = dx_n - \sum_\alpha p_{i\alpha}(x) \, dx_\alpha$, which we still may write $\eta_i = -\sum_\lambda p_{i\lambda}(x) \, dx_\lambda$ with the convention $p_{in} \equiv -1$.

Lemma 4.1 *The integrability conditions may be written locally :*

$$(p_{i\lambda})'_\mu - (p_{i\mu})'_\lambda + p_{i\mu}(p_{i\lambda})'_n - p_{i\lambda}(p_{i\mu})'_n \equiv 0 \quad \text{for all triples } (i, \lambda, \mu).$$

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Proof : Let $\eta = -\sum_\lambda p_\lambda(x)\, dx_\lambda$ be a holomorphic 1-form. Then

$$\eta \wedge d\eta = \sum_{\lambda < \mu < \nu} \left[p_\lambda \Big((p_\nu)'_\mu - (p_\mu)'_\nu \Big) + p_\mu \Big((p_\lambda)'_\nu - (p_\nu)'_\lambda \Big) + p_\nu \Big((p_\mu)'_\lambda - (p_\lambda)'_\mu \Big) \right] dx_\lambda \wedge dx_\mu \wedge dx_\nu.$$

Then, when $p_n \equiv -1$, we observe that the vanishing of all terms in $dx_\lambda \wedge dx_\mu \wedge dx_n$ implies the vanishing of all other terms, hence the lemma.

<div align="right">QED</div>

A section $(\omega_i)_i$ of A is locally given by the d functions f_i such that $\omega_i = f_i \left(\sum_\lambda p_{i\lambda}(x)\, dx_\lambda \right)$ satisfying to the identities

$$(E_\lambda) \qquad\qquad \sum_i p_{i\lambda} f_i \equiv 0 \qquad\quad \text{for any } \lambda,$$

hence, by derivation,

$$(E_{\lambda,\mu}) \qquad\quad \sum_i (p_{i\lambda} f_i)'_\mu \equiv 0 \qquad\quad \text{for any } \lambda, \mu.$$

Lemma 4.2 *For the family $(f_i)_i$ to define an abelian relation, it is necessary and sufficient that the identities be satisfied*

$$(F_{i\alpha}) \qquad\qquad\qquad (f_i)'_\alpha \equiv -\big(f_i p_{i\alpha}\big)'_n \;\; \text{for all pairs } (i, \alpha).$$

Proof : In fact, a holomorphic 1-form $f\big(\sum_\lambda p_\lambda(x)\, dx_\lambda\big)$ is closed iff $(f p_\lambda)'_\mu = (f p_\mu)'_\lambda$ for all pairs (λ, μ) such that $\lambda < \mu$. But, because of the integrability conditions, it is sufficient that this relation be satisfied when $\mu = n$, for it to be satisfied with all other μ's.

<div align="right">QED</div>

Lemma 4.3 *When the family $(f_i)_i$ defines an abelian relation, the identities $(E_{\lambda,\mu})$ and $(E_{\mu,\lambda})$ are the same.*

Proof : In fact, under the assumption, $(f_i p_{i\lambda})'_\mu \equiv (f_i p_{i\mu})'_\lambda$ for all pairs (λ, μ), hence the lemma by summation with respect to i.

<div align="right">QED</div>

The identity $(f_i)'_\alpha \equiv -\big(f_i p_{i\alpha}\big)'_n$ means that it is sufficient to know the $(f_i)'_n$ to know the other partial derivatives $(f_i)'_\alpha$ of a family $(f_i)_i$ defining an abelian relation.

Hence, writing $w_i = (f_i)'_n$, and combining $(E_{\lambda,\mu})$ and $(F_{i\alpha})$, we get :

Corollary 4.4 *The elements of R_0 above a given element $(f_i)_i$ in A map bijectively onto the solutions of the linear system Σ_0.*

$$(\widetilde{E}_{\lambda,\mu}) \qquad\qquad \sum_i p_{i\lambda} p_{i\mu} w_i \equiv \sum_i f_i \big[(p_{i\lambda})'_\mu - p_{i\lambda} (p_{i\lambda})'_n \big]$$

of $c(n,2)$ equations $(\widetilde{E}_{\lambda,\mu})$ with d unknown w_i.

Notice that the matrix of the system Σ_0 is the matrix $P_2 = \big((C_{i\,L}^{(2)})\big)_{i,L}$ seen in the previous section.

Proof of the lemma 3-1 in the case $d < (c(n,2)$ **:** Generically, the system Σ_0 has rank d. Hence S is defined locally by the vanishing of all characteristic determinants which are generically not all identically zero. If Σ_0 has a rank r smaller than d, the vanishing of all determinants of size $r+1, \cdots, d$ in the matrix P_2 have to be added to the vanishing of all charateristic determinants.

<div align="right">QED</div>

5   Computation of R_k $(k \geq 1)$:

For any pair of multi-indices of derivation $L = (\lambda_1, \cdots, \lambda_s, \cdots, \lambda_n)$, and $H = (h_1, h_2, \cdots, h_n)$, $L + H$ will denote the multi-index $(\lambda_1 + h_1, \lambda_2 + h_2, \cdots, \lambda_n + h_n)$. We define similarly $L - H$ if $\lambda_\mu \geq h_\mu$ for all μ's. For any λ, 1_λ will denote the muti-index with all λ_μ's equal to zero for $\mu \neq \lambda$ and $\lambda_\lambda = 1$. By definition the height $|L|$ of L is the sum $\sum_s \lambda_s$.

By derivation of the identities (E_λ), the elements of $J^k A$ are characterized by the identities

$$(E_{\lambda, L}) \qquad \sum_i (p_{i\lambda} f_i)'_L \equiv 0 \text{ for any } \lambda \text{ and for any multi-index } L \text{ of height } |L| \leq k.$$

Lemma 5.1 *If $(f_i)_i$ is an abelian relation, the relation $(E_{\lambda, L})$ remains unchanged by permutation of all the indices of $L \cup \{\lambda\}$.*

Proof : The left hand term of this identity is obviously symetric with respect to the indices of L. Thus, it is sufficient to prove that the identities $(E_{\lambda, \mu})$ and $(E_{\mu, \lambda})$ are the same, which we know already.
$$\text{QED}$$

The identity $(E_{\lambda, L})$ above will now be denoted by (E_H), where $H = L + 1_\lambda$.

Lemma 5.2
(i) *If $(f_i)_i$ is an abelian relation, all partial derivatives $(f_i)'_L$ may be written as a linear combination*

$$(\widetilde{F_{iL}}) \qquad\qquad \widetilde{(f_i)'_L} \equiv \sum_{k=0}^{|L|} D_{i\;L}^{(k)} \cdot (f_i)'_{n^k}$$

of f_i and of its partial derivatives $(f_i)'_{n^k} = \frac{\partial^{(k)} f_i}{(\partial x_n)^k}$ with respect to the only variable x_n, with coefficients $D_{i\;L}^{(k)}$ not depending on the f_i's.

(ii) *If $L = (\lambda_1, \cdots, \lambda_s, \cdots, \lambda_n)$, the coefficient $D_{i\;L}^{(|L|)}$ of highest order is equal to $(-1)^{|L|} \prod_{s=1}^{n} p_i^{\lambda_s}$, i.e. is equal to $(-1)^{|L|} C_{i\;L}^{(|L|)}$.*

Proof : We get the lemma by derivation of the identities $(F_{i\alpha})$ and an obvious induction on the height $|L|$ of L.
$$\text{QED}$$

The lemma above means that it is sufficient to know the $(f_i)'_{n^{k+1}}$ to know the other partial derivatives $(f_i)'_L$ in the $(k+1)$-jet of a family $(f_i)_i$ defining an abelian relation.

Hence, writing $w_i = (f_i)'_{n^{k+1}}$, and combining $(E_L$ and $(\widetilde{F_{iL}})$, we get :

Corollary 5.3 *The elements of R_k above a given element $a_0^{(k-1)}$ in R_{k-1} map bijectively onto the solutions of a linear system Σ_k of $c(n, k+2)$ equations $(\widetilde{E}_L)$ with d unknown w_i*

$$(\widetilde{E}_L) \qquad\qquad \sum_i C_{i\;L}^{(k+2)} w_i \equiv \Phi_L\big(a_0^{(k-1)}\big),$$

where L runs through the set $\mathcal{P}(n, k+2)$ of partitions of $k+2$, and where the second member $\Phi_L\big(a_0^{(k-1)}\big)$ depends only on $a_0^{(k-1)} \in R_{k-1}$. In particular, the symbol σ_k of D_k is defined by the matrix

$$P_{k+2} = ((C_{i\;L}^{|L|}))_{i, L},$$

$1 \leq i \leq d$, $|L| = k + 2$, *of size $d \times c(n, k+2)$.*

Theorem 5.4 *Assume the d-web to be ordinary. The map $\pi_k : R_k \to R_{k-1}$ is surjective for $k \le k_0 - 2$ and injective for $k > k_0 - 2$ above the open set $U = M_0 \setminus S$ of M_0. For $k \le k_0 - 2$, R_k is a holomorphic bundle of rank $\sum_{h=1}^{k+2} (d - c(n, h))$ over $M_0 \setminus S$.*

Proof : In fact, P_{k+2} is precisely the matrix of the system Σ_k. Thus, for $k \le k_0 - 2$ and off S_k, the space of solutions of Σ_k is an affine space of dimension $d - c(n, k + 2)$.

<div align="right">QED</div>

Proof of theorem 1-1 in the case $d > c(n, 2)$: *The rank of a ordinary d-web is at most equal to the number*

$$\pi'(n, d) = \sum_{h=1}^{k_0} \big(d - c(n, h) \big).$$

For $k > k_0 - 2$, the symbol σ_k is necessarily injective off S. In fact, the matrix defining this symbol in the corollary above contains the matrix defining σ_{k-1} when we choose the coordinates and the forms η_i's so that $p_{in} \equiv -1$. Hence, since $g_{k_0 - 2} = 0$ off S, $g_k = 0$ off S for all $k \ge k_0 - 2$. Consequently, above a given element in $R_{k_0 - 2}$, there exists at most one infinite jet of abelian relation, hence one germ of abelian relation since the framework is analytic. We deduce that the rank of the web is at most $\pi'(n, d)$ off S, hence everywhere (semi-continuity of the rank). We shall see in the next section that any ordinary affine web in strong general position has rank $\pi'(n, d)$, hence the optimality.

<div align="right">QED</div>

Proof of theorem 1-2 : When $d = c(n, k_0)$, $\mathcal{E} = R_{k_0 - 3}|_{M_0 \setminus S}$ is a vector bundle of rank $\pi'(n, d)$, and $\pi_{k_0 - 2} : R_{k_0 - 2} \to R_{k_0 - 3}$ is an isomorphism of vector bundles over $M_0 \setminus S$. Therefore, we just have to use lemma 2-2. In particular, for $d = c(n, k_0)$, the rank of a ordinary d-web is $\pi'(n, d)$ iff the curvature of the previous connexion vanishes.

6 Examples

6.1 Case $n = 2$:

We recover the results of [P][He1]). In fact, in this case :

- the set S is always empty (all determinants occuring in the computation of the symbols σ_k are determinants of Van-der-Monde for $k \le k_0 - 2$, vanishing nowhere on M_0); thus, all webs are ordinary.

- any d is equal to $c(2, d - 1)$,

- the rank $\sum_{h=1}^{d-2} \big(c(2, d - 1) - c(2, h) \big)$ of R_{d-4} is equal to $(d - 1)(d - 2)/2$.

6.2 Case $n = 3$, $d = 6$:

Use coordinates x, y, z on $\mathbb{C}^3$, with $n = 3$, and $c(n, 2) = 6$. Let a, b, c, e, h be five distinct complex numbers, all different of 0, and let ψ be some holomorphic function of y. Let W be the 6-web of codimension 1 on $\mathbb{C}^3$ defined by the 1-forms $\eta_i = dz - p_i dx - q_i dy$, $1 \le i \le 6$, where : $(p_1, q_1) = (0, 0)$, $(p_2, q_2) = (a, a^2)$, $(p_3, q_3) = (b, b^2)$, $(p_4, q_4) = (c, c^2)$, $(p_5, q_5) = (e, e^2)$ and $(p_6, q_6) = (h, \psi)$. The system Σ_0 is then equivalent to

$$\begin{pmatrix} a & b & c & e \\ a^2 & b^2 & c^2 & e^2 \\ a^3 & b^3 & c^3 & e^3 \\ a^4 & b^4 & c^4 & e^4 \end{pmatrix} \begin{pmatrix} w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_6.\psi' \end{pmatrix},$$

to which we add the equations $(h^2 - \psi)w_6 = 0$, and $w_1 + w_2 + \cdots + w_6 = 0$. The system is a system of Cramer off the locus S which is

  - the surface of equation $\psi(y) = h^2$ in general,

  - the empty set when ψ is any constant different of h^2,

  - all of $\mathbb{C}^3$ (the extraordinary case) for $\psi \equiv h^2$.

Let's precise the connection and its curvature for $\psi \neq h^2$ in the two first cases (the ordinary case).

A section (f_i) of A is defined by (f_4, f_5, f_6), since (f_1, f_2, f_3) can be deduced using the equations $\sum_i f_i = 0$, $\sum_i p_i f_i = 0$ and $\sum_i q_i f_i = 0$. Setting $\Delta = abce(b-a)(c-a)(e-a)(c-b)(e-b)(e-c)(h^2-\psi)$, $K = \frac{\psi'}{\Delta(h^2-\psi)}$, $\Delta_4 = abe((b-a)(e-a)(e-b))$, and $\Delta_5 = abc((b-a)(c-a)(c-b))$, we get :
$w_4 = -K\Delta_4, w_5 = K\Delta_5, w_6 = 0$, hence $u_4 = cK\Delta_4, u_5 = -eK\Delta_5, u_6 = 0$ and $v_4 = c^2 K\Delta_4$, $u_5 = -e^2 K\Delta_5$, $u_6 = 0$. With respect to the trivialization $(\sigma_4, \sigma_5, \sigma_6)$ of A given by
$\sigma_4 = (f_4 \equiv 1, f_5 \equiv 0, f_6 \equiv 0)$, $\sigma_5 = (f_4 \equiv 0, f_5 \equiv 1, f_6 \equiv 0)$ and $\sigma_6 = (f_4 \equiv 0, f_5 \equiv 0, f_6 \equiv 1)$, the matrix of the connection is :

$$\omega = \frac{\psi'}{\Delta(h^2 - \psi)} \begin{pmatrix} 0 & 0 & \Delta_4\,\eta_4 \\ 0 & 0 & -\Delta_5\,\eta_5 \\ 0 & 0 & 0 \end{pmatrix},$$

hence the curvature

$$\Omega = \frac{1}{\Delta}\left(\frac{\psi'}{h^2 - \psi}\right)' \begin{pmatrix} 0 & 0 & \Delta_4(dy \wedge dz + c\, dx \wedge dy) \\ 0 & 0 & -\Delta_5(dy \wedge dz + e\, dx \wedge dy) \\ 0 & 0 & 0 \end{pmatrix}.$$

We observe that σ_4 and σ_5 are linearly independant abelian relations. We knew it already since the first integrals $(z - cx - c^2y)$ and $(z - ex - e^2y)$ of $\mathcal{F}_4$ and $\mathcal{F}_5$ respectively are linear combinations of the first integrals z, $(z - ax - a^2y)$ and $(z - bx - b^2y)$ of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$. Thus, when $h^2 - \psi$ does not vanish, the rank of the 6-web is at least 2, and has the maximum possible value $\pi'(3, 6) = 3$ in the ordinary case if and only if $\frac{\psi'}{h^2-\psi}$ is constant, that is if there exists two scalar constant C and D $(C \neq 0)$, such that

$$\psi(y) = h^2 + Ce^{Dy},$$

in particular for $\psi = $ constant (case $D = 0$). For given C and D as above, $K = -D/\Delta$, and $\sigma_6 - K\Delta_4\sigma_4 + K\Delta_5\sigma_5$ is an abelian relation (the function ψ occurs in the computation of f_1, f_2 and f_3 from $f_4 = -K\Delta_4$, $f_5 = K\Delta_5$ and $f_6 = 1$).

The extraordinary case $\psi \equiv h^2$ will be seen in the next subsection.

6.3 Ordinary affine webs and optimality of the bound $\pi'(n, d)$:

For any pair (n, d) with $d > n$, give d linear forms $(l_1, l_2, \cdots, l_d)$. Let $\mathcal{F}_i$ be the foliation defined in $\mathbb{C}^n$ by the parallel hyperplanes $l_i = constant$, and W be the d-web defined by these d foliations. Let k_0 be the integer such that $c(n, k_0) \leq d < c(n, k_0 + 1)$. Let $\mathbb{P}'_{n-1}$ denote the hyperplane at infinity of the n-dimensional projective space $\mathbb{P}_n = \mathbb{C}^n \coprod \mathbb{P}_{n-1}$: the parallel hyperplanes of the pencil $l_i = constant$ meet at infinity along a hyperplane of $\mathbb{P}_{n-1}$, i.e. define an element $[l_i]$ of the dual projective space $\mathbb{P}'_{n-1}$.

Remark 6.1 The web W on $\mathbb{C}^n$ extends to a web on $\mathbb{P}_n$ (with singularities) which is algebraic. It is in fact dual to the union of d straight lines in the dual projective space $\mathbb{P}'_n$.

Definition 6.2 *Such a web on $\mathbb{P}_n$ will be said an* **affine d-web**, *and it is said ordinary if its restriction to $\mathbb{C}^n$ is ordinary.*

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Lemma 6.3 *The two following properties are equivalent :*

(i) The affine d-web above is ordinary.

(ii) For any h, $(1 \leq h \leq k_0)$, there exists $c(n, h)$ points among the d points $[l_i]$, which do not belong to a same algebraic (reducible or not) hypersurface of degree h in $\mathbb{P}'_{n-1}$.

Proof : For $d \geq c(n, h)$, $(h \geq 1)$, assume that the matrix $((C^{(h)}_{i\ L}))_{i,L}$, $1 \leq i \leq d$, $|L| = h$, of size $d \times c(n, h)$ has rank $< c(n, h)$. This means that the determinant of any square sub-matrix of size $c(n, h)$ vanishes. Saying that the determinant of the sub-matrix given for instance by the $c(n, h)$ first l_i's vanishes means precisely that $[l_1], [l_2], \cdots, [l_{c(n,h)}]$ belong to some hypersurface of degree h in $\mathbb{P}'_{n-1}$ (may be reducible), and same thing for any other subset of $c(n, h)$ indices i. Hence the lemma.

<div align="right">QED</div>

Theorem 6.4

(i) If the d points $[l_i]$ are in general position in $\mathbb{P}'_{n-1}$ (i.e. if any n of the d linear forms l_i are linearly independant), the above affine d-web has a rank $\geq \pi'(n, d)$.

(ii) It has exactly rank $\pi'(n, d)$ iff it is ordinary.

Proof : For any h, consider the vector space L_h generated by the hth symetric products $(l_i)^h$ of the l_i's, and denote by r_h the dimension of this vector space. The rank of an affine web in strong general position is $\sum_{h=1}^{k_0}(d - r_h)$ (see Trépreau [T], section 2). Hence we have only to prove that $r_h \leq c(n, h)$ in general, and $r_h = c(n, h)$ in the ordinary case. But this is obvious after the previous lemma, since the dimension of L_h is exactly the rank of the matrix $P_h = ((C^{(h)}_{iL}))_{i,L}$ with $i \leq d, |L| = h$ in the linear system Σ_{h-2} of corollary 5-3.

<div align="right">QED</div>

Corollary 6.5 *The bound $\pi'(n, d)$ for the rank of a ordinary web is optimal.*

Proof : Remember that a algebraic hypersurface of degree h in $\mathbb{P}'_{n-1}$ is defined in general by the data of $c(n, h) - 1$ of its points : thus, the property (ii) of lemma 6-3 above is generically satisfied, so that there exists ordinary affine d-webs in dimension n for any (n, d).

<div align="right">QED</div>

For instance, an affine 6-web on $\mathbb{P}_3$ in strong general position will be extraordinary of rank 4 $\left(= \pi(3, 6)\right)$ or ordinary of rank 3 $\left(= \pi'(3, 6)\right)$, according to the fact that the six points $[l_i]$ belong or not to a same conic[6] of $\mathbb{P}'_2$: when ψ is constant in the example $(p_1, q_1) = (0, 0)$, $(p_2, q_2) = (a, a^2)$, $(p_3, q_3) = (b, b^2)$, $(p_4, q_4) = (c, c^2)$, $(p_5, q_5) = (e, e^2)$ and $(p_6, q_6) = (h, \psi)$ of the previous subsection, the five first points belong to the conic of equation $q = p^2$, hence the dichotomy according to the fact that ψ is equal or different of h^2.

New proof of the theorem 1-1 for $d \geq c(n, 2)$ in the particular case of ordinary webs which are in strong general position :

The rank of a web in strong general position is upper-bounded by the rank of the "tangent affine web" at a point (see again [T]). Since the ordinaryity of a web near a point is equivalent to the ordinaryity of the tangent affine web at that point, theorem 1-1 is also a corollary of the previous theorem 6-4, when the ordinary web is in strong general position.

<div align="right">QED</div>

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[6]In the extraordinary case, it is amazing to deduce the fourth abelian relation from the theorem of the hexagon (Pascal).

6.4 Another example of ordinary web of maximal rank 26 for $n = 3$, $d = 15$

The following example has been given to us by J.V. Pereira and L. Pirio ([PP]).

In $\mathbb{C}^3$ with coordinates x, y, z, take the 15-web defined by the ten pencils of planes, 4 pencils of quadratic cones or cylinders, and a last pencil of quadrics, respectively defined by the first integals u_i ($1 \leq i \leq 15$) :

$u_1 = x$, $u_2 = y$, $u_3 = z$, $u_4 = \frac{z}{z-y}$, $u_5 = \frac{z}{z-x}$, $u_6 = \frac{y}{y-x}$, $u_7 = \frac{1-z}{y-z}$, $u_8 = \frac{1-z}{z-x}$, $u_9 = \frac{1-y}{y-x}$, $u_{10} = \frac{x-y}{z-y}$,

$u_{11} = \frac{z(1-y)}{z-y}$, $u_{12} = \frac{z(1-x)}{z-x}$, $u_{13} = \frac{y(1-x)}{y-x}$, $u_{14} = \frac{z(x-y)}{x(z-y)}$,

$u_{15} = \frac{(1-z)(y-x)}{(1-x)(y-z)}$.

Denote by $m_X = [1;0;0;0]$, $m_Y = [0;1;0;0]$, $m_Z = [0;0;1;0]$ and $m_T = [0;0;0;1]$ the edges of the standard projective frame, and by $\Omega_X = [0;1;1;1]$, $\Omega_Y = [1;0;1;1]$, $\Omega_Z = [1;1;0;1]$ and $\Omega_T = [1;1;1;0]$ the barycenters of the faces of the previous tetrahedron. The foliations 7, 8, 9 and 10 are respectively the foliations given by the pencil of planes through the line $m_X \Omega_X$ (resp. $m_Y \Omega_Y$, $m_Z \Omega_Z$ and $m_T \Omega_T$.

The foliation 11 is the pencil of the quadratic cylinders of summit m_X having for basis the conics through Ω_X, m_Y, m_Z, m_T in the plane $X = 0$. We get similarly the foliations 12, 13, and 14 by permutation of the letters X, Y, Z, T.

The 5-subweb $(1, 2, 3, 10, 14)$ is then the 5-web of cones of summit m_T over the Bol's 5-web defined by the 4 points m_X, m_Y, m_Z, Ω_T in the plane at infinity $T = 0$. We can do the same with the other 5-subwebs $(2, 3, 4, 7, 11)$, $(1, 3, 5, 8, 12)$ and $(1, 2, 6, 9, 13)$.

L. Pirio checked, using Mapple, that this 15-web is ordinary. Since $15 = c(3, 4)$, this web has a curvature, and we could theoretically check that this curvature vanishes. In fact, Pereira and Pirio exhibited directly 26 independant abelian relations : since $\pi'(3, 15) = 26$, the rank of the web is exactly 26 (while $\pi(3, 15) = 42$).

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