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Regular holomorphic webs of codimension one

by Vincent CAVALIER and Daniel LEHMANN

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Abstract

Given a d -web of codimension one on a holomorphic n -dimensional manifold M_0 ($d > n$), we assume that, at any point of M_0 , the d hyperplanes tangent to the local foliations at a point of M_0 are distinct, and that there exists n of them in general position (but we do not require any n of them to be in general position). For such a web, we shall define some specific analytical subset S of M_0 which -generically- has dimension $\leq n - 1$ or is empty : in this case the web will be said “**regular**”; when -exceptionally- the set S is n -dimensional, the web will be said “**special**”.

We prove that the rank of regular d -webs has an upper-bound $\pi'(n, d)$ which, for $n \geq 3$, is strictly smaller than the bound $\pi(n, d)$ of Castelnuovo (the maximal arithmetical genus of an algebraic curve of degree d in the complex n -dimensional projective space \mathbb{P}_n).

Let $c(n, h)$ denote the dimension of the vector space of homogeneous polynomials of degree h in n variables. The number $\pi'(n, d)$ is then equal

- to 0 for $d < c(n, 2)$,
- and to $\sum_{h \geq 1} (d - c(n, h))^+$ for $d \geq c(n, 2)$, $(a)^+$ denoting the number $\sup(a, 0)$ for any $a \in \mathbb{Z}$.

For any regular d -web with $d = c(n, h)$ for some $h \geq 2$, we define a holomorphic connection on some holomorphic bundle \mathcal{E} of rank $\pi'(n, d)$ over $M_0 \setminus S$, such that the set of abelian relations inject into the set of sections of \mathcal{E} the covariant derivative of which vanishes: the curvature of this connection, which generalizes the Blaschke curvature, is then an obstruction for the rank of the web to have the maximal possible value $\pi'(n, d)$.

When $n = 2$, any web is regular and we recover the results of [He1].

Other examples are given. In particular, any affine regular d -web in dimension n has maximal rank $\pi'(n, d)$, hence the optimality of this bound.

Regular holomorphic webs of codimension one

Vincent Cavalier

Daniel Lehmann

1 Introduction

A holomorphic d -web of codimension one on a n -dimensional holomorphic manifold M being given ($d > n$), we denote by M_0 the open set in M where the web is locally defined by d holomorphic foliations \mathcal{F}_i of codimension one, all non-singular and with tangent spaces to the leaves distinct at any point.

We shall say that the web is in **strong general position** if *any* subset of n leaves among the d leaves through a point of M_0 are in general position. If we assume only that *there exists* a subset of n leaves among the d leaves through a point of M_0 which are in general position (but not necessarily any n of them), we shall say that the web is in **weak general position**. Most of our results below will require only this weaker condition¹.

On an open set U in M_0 sufficiently small for the web to be defined as above by the data of d distinct local foliations \mathcal{F}_i (such an open set is called “open set of distinguishability”), an abelian relation is the data of a family $(\omega_i)_i$ of holomorphic 1-forms ω_i , ($1 \leq i \leq d$) such that

- (i) each ω_i is closed,
- (ii) the vector fields tangent to the local foliation \mathcal{F}_i belong to the kernel of ω_i ,
- (iii) the sum $\sum_{i=1}^d \omega_i$ vanishes.

The set of abelian relations on U (resp. the set of germs of abelian relation at a point m of M_0) has the structure of a finite dimensional complex vector space, whose dimension is called the rank of the web on U (resp. at m). If the web is in strong general position, Hénaut proved in [He2] that its rank at a point does not depend on this point: abelian relations have then the structure of a local system of coefficients². When we require only the web to be in weak general position, we shall call “rank of the web” the maximum³ of the rank at each point of M_0 .

After Chern ([C]), the rank of a d -web of codimension one in a n -dimensional manifold is always upper-bounded, when the web is in strong general position, by the number $\pi(n, d)$ of Castelnuovo (the maximum of the arithmetical genus of an algebraic curve of degree d in the n -dimensional complex projective space \mathbb{P}_n). On the other hand, the rank of an algebraic d -web in \mathbb{P}_n (i.e. the web whose leaves are the hyperplanes belonging to some algebraic curve Γ of degree d in the dual projective space \mathbb{P}'_n) is equal to the arithmetical genus of Γ : this is, after duality, a theorem coming back to Abel. Therefore, the rank of a web may reach the bound $\pi(n, d)$.

When $n = 2$, the Blaschke curvature has been defined by Blaschke ([B]) for $d = 3$, and generalized by Hénaut ([He1]) for any $d \geq 3$: Hénaut proved in fact, at least locally

- that the set R_{d-4} of formal abelian relations at order $d - 3$ for a d -web in dimension 2 is a holomorphic vector bundle \mathcal{E} of rank $\pi(2, d) = (d - 1)(d - 2)/2$ on M_0 , as well as the set R_{d-3} of

¹More generally, if there exists ℓ and not more of the leaves through a point which are in general position ($\ell \leq n$), there exists locally a holomorphic foliation \mathcal{G} of codimension ℓ , such that the web is locally the pullback of a d -web of codimension 1 on a ℓ -dimensional manifold transversal to \mathcal{G} . Then, many of the results below remain valid, after replacing n by ℓ .

²In [He3], he generalized this result in higher codimension.

³Notice that, in this case, the rank at a point is an upper-semicontinuous function of the point.

formal abelian relations at order $d - 2$, the projection $R_{d-3} \rightarrow R_{d-4}$ being an isomorphism,

- that the ∞ -jets and the germs of abelian relations are completely defined by their projection onto \mathcal{E} , the projections $R_{k+1} \rightarrow R_k$ being injective for $k \geq d - 4$,

- and that there exists a unique holomorphic connection ∇ on \mathcal{E} such that a section of \mathcal{E} is the section $j^{d-3}u$ of some abelian relation u if and only if its covariant derivative $\nabla(j^{d-3}u)$ vanishes. The curvature of this connection generalizes the Blaschke curvature, and is an obstruction for the rank of the web to be maximal.

The first aim of this paper was to generalize the theory of Hénaut for $n \geq 2$. But we observed by the way

- that such a generalization is only possible for “regular” d -webs (see definition below), and for the values of d equal, for some $h \geq 2$, to the dimension $c(n, h)$ of the vector space of all homogeneous polynomials of degree h in n variables with scalar coefficients,

- that, whatever be d (not necessarily belonging to the previous sequence $(c(n, h))_h$), the rank of all regular d -webs is at most equal to some bound $\pi'(n, d)$ which, for $n \geq 3$, is strictly smaller than the bound $\pi(n, d)$ of Castelnuovo.

More precisely, we define, for any d -web which is in weak general position⁴, an analytical subset S of M_0 which has generically a dimension $\leq n - 1$ or is empty : in this case, the web will be said “**regular**”. Exceptionally, S may reach the dimension n : in this case, the web will be said “**special**”.

For any d , ($d > n$), denote by k_0 the integer (≥ 1) such that $c(n, k_0) \leq d < c(n, k_0 + 1)$, and set :

$$\begin{aligned} \pi'(n, d) &= 0 && \text{when } d < c(n, 2), (k_0 = 1), \\ &= \sum_{h=1}^{k_0} (d - c(n, h)) && \text{when } d \geq c(n, 2), (k_0 \geq 2). \end{aligned}$$

The main results of this paper are the two following :

Theorem 1.1 *The rank of any regular d -web on some n -dimensional manifold M_0 is upperbounded by $\pi'(n, d)$. This bound is optimal.*

Theorem 1.2 *If $d = c(n, k_0)$, and if the d -web is regular, there exists a holomorphic vector bundle \mathcal{E} of rank $\pi'(n, d)$ over $M_0 \setminus S$ and a holomorphic connection ∇ on \mathcal{E} , such that the vector space of germs of abelian relations at a point of $M_0 \setminus S$ inject into the vector space of germs of sections of \mathcal{E} which have a vanishing covariant derivative. This injection is a bijection when the rank of the web reaches the maximal possible value $\pi'(n, d)$: the curvature of this connection is then an obstruction for the rank of the web to reach this value $\pi'(n, d)$.*

For $n = 2$, we recover the results of [He1]: it happens in fact

- that S is always empty, so that all webs are regular,
- that the upper-bounds $\pi(2, d)$ and $\pi'(2, d)$ coincide, both being equal to $(d - 1)(d - 2)/2$,
- that any $d, d \geq 3$, may be written $d = c(2, k_0)$, with $k_0 = d - 1$.

For $n \geq 3$, S may be non-empty, $\pi'(n, d)$ is strictly smaller than $\pi(n, d)$, and not any d may be written $c(n, h)$. Moreover, the fact that $\dim S$ must be equal to n for the rank of the web to reach the bound $\pi(n, d)$ is a criterium of practical interest.

While we essentially generalize the method initiated by Hénaut ([He1]) in the case $n = 2$ (looking at the symbols of the successive prolongations of the differential operator the kernel of which is the space of abelian relations), we also give in the last section a shorter proof of theorem 1-1 in the particular case of webs which are in strong general position (using the affine web tangent to a web at a point).

⁴This property will be included once for all in the definition of a regular web.

Significative examples will be given in the last section. We observe in particular that all affine regular d -webs in dimension n have rank $\pi'(n, d)$ (hence the optimality of this bound), and are moreover algebraic. For $n = 3$ and $d = 6$, we also give an example on \mathbb{C}^3 , with coordinates (x, y, z) , depending on some holomorphic function ψ of the variable y . There is some scalar h such that the analytical set S has for equation $\psi(y) = h^2$. The web is then regular for ψ non identically equal to h^2 , S being even empty when ψ is constant ($\neq h^2$). We then prove that the web has rank 2 or 3 ($= \pi'(3, 6)$), according to the Blaschke curvature which vanishes when $(\psi')^2 + (h^2 - \psi)\psi''$ does, and in particular when ψ is a constant (different of h^2). In the special case $\psi \equiv h^2$, a fourth abelian relation appears, linearly independent of the three first ones, in such a way that the web has then rank $4 = \pi(3, 6)$.

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2 Notations and backgrounds

Most of the results in this section are well known or easy to prove, so that we shall omit their proof.

2.1 Some algebraic notations

Let $\mathbb{R}_h[X_1, \dots, X_n]$ denote the vector space of homogeneous polynomials of degree h in n variables, with scalar coefficients (in fact, the field of scalars does not matter). Denote by $c(n, h)$ its dimension $\binom{h+n-1}{h}$. The monomials $X^\Lambda = (X_1)^{\lambda_1} \dots (X_n)^{\lambda_n}$ make a basis, indexed by the set $\mathcal{P}(n, h)$ of the partitions $L = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of h , with $0 \leq \lambda_i \leq h$ for any $i = 1, \dots, n$, and $\sum_{i=1}^n \lambda_i = h$. The number h will be also denoted by $|L|$, and is called the *height* of L .

Denoting by $(a)^+$ the number $\sup(a, 0)$ for any real number a , remember ([GH]) that the number

$$\pi(n, d) = \sum_{h \geq 1} (d - h(n-1) - 1)^+,$$

called *the bound of Castelnuovo*, is the maximum of the arithmetical genus of an algebraic curve of degree d in the complex projective space \mathbb{P}_n .

Define

$$\begin{aligned} \pi'(n, d) &= 0 && \text{when } d < c(n, 2), \\ &= \sum_{h \geq 1} (d - c(n, h))^+ && \text{when } d \geq c(n, 2). \end{aligned}$$

Lemma 2.1

- (i) For $n \geq 3$, the inequality $\pi'(n, d) < \pi(n, d)$ holds.
- (ii) The equality $\pi'(2, d) = \pi(2, d)$ holds.

2.2 Connections adapted to a differential operator :

Let $E \rightarrow V$ be a holomorphic vector bundle on a holomorphic manifold V . Remember the exact sequence of holomorphic vector bundles

$$0 \rightarrow T^*V \otimes E \rightarrow J^1 E \rightarrow E \rightarrow 0.$$

A C^∞ -connection ∇ on E (resp. a holomorphic connection if any) is a C^∞ -splitting (resp. a holomorphic splitting if any) of this exact sequence :

$$0 \rightarrow T^*V \otimes E \xrightarrow{\beta} J^1 E \xrightarrow{\alpha} E \rightarrow 0.$$

If u is a C^∞ or holomorphic section of E , its covariant derivative is given by

$$\nabla u = \beta(j^1 u).$$

A (holomorphic) linear differential operator of order one on the holomorphic manifold V is a morphism of (holomorphic) vector bundles $D : J^1 E \rightarrow F$, for some vector bundles E and F on V , to which we associate the map $\mathcal{D} : u \mapsto D(j^1 u)$ from the sections of E into the sections of F . The kernel R of D is a subset of $J^1 E$, which is the set of the formal solutions at order one of the equation $\mathcal{D}u = 0$.

Lemma 2.2 *Given a connection ∇ on E , seen as a lift α of the projection $J^1 E \rightarrow E$, the two following assertions are equivalent :*

- (i) *The lift α takes values into R .*
- (ii) *The sections u of E with vanishing covariant derivative $\nabla u = 0$ are solutions of the equation $\mathcal{D}u = 0$.*

Such a connection on E will be said “adapted” to the differential operator \mathcal{D} . If, moreover, any solution of the equation $\mathcal{D}u = 0$ has a vanishing covariant derivative with respect to ∇ , the adapted connection ∇ will be said “completely adapted” to \mathcal{D} . In particular, we get

Lemma 2.3 *The two following assertions are equivalent :*

- (i) *The set R is a sub-vector bundle of $J^1 E$, and the restriction $\Phi : R \rightarrow E$ of the natural projection $J^1 E \rightarrow E$ is an isomorphism of holomorphic vector bundles,*
- (ii) *There is a holomorphic connection on E which is completely adapted to \mathcal{D} .*

Moreover, if such a completely adapted connection exists, it is necessary unique and defined by the composition of the inclusion $R \subset J^1 E$ with $\Phi^{-1} : E \xrightarrow{\cong} R$.

2.3 The differential operator for abelian relations

Let W be a d -web in weak general position on a holomorphic n -dimensional manifold M_0 , with $d > n$.

We define a holomorphic vector bundle A of rank $d - n$ over M_0 by defining the sheaf of its holomorphic sections. Let U be first an open set of distinguishability. Then the \mathcal{O}_U -module $\Gamma(U, A)$ is defined as the module of families $(\omega_i)_{1 \leq i \leq d}$ of holomorphic 1-forms ω_i on U such that

- (i) for any $i = 1, \dots, d$, the kernel of ω_i contains the tangent space T_i to \mathcal{F}_i (it is therefore equal to it when ω_i is not zero),
- (ii) the sum $\sum_{i=1}^d \omega_i$ is zero.

We define similarly the germs at a point, hence a sheaf which is easily seen to be a locally-free \mathcal{O}_{M_0} -module : it is therefore the sheaf of sections of a holomorphic vector bundle A .

Definition 2.4 *The vector bundle A of rank $d - n$ so defined will be called the Blaschke bundle of the web.*

Let us define in a similar way a holomorphic vector bundle B over M_0 , of rank $(d - 1)n(n - 1)/2$ by defining the sections belonging to $\Gamma(U, B)$ as the families $(\varpi_i)_{1 \leq i \leq d}$ of holomorphic 2-forms ϖ_i on U such that $\sum_{i=1}^d \varpi_i = 0$.

Definition 2.5 *We call abelian relation of the web any holomorphic section u of the Blaschke bundle A , which is solution of the equation $\mathcal{D}u = 0$, the map \mathcal{D} denoting the linear first order differential operator $(\omega_i) \mapsto (d\omega_i)$ from A to B .*

The differential operator \mathcal{D} may still be seen as a linear morphism $D : J^1 A \rightarrow B$, and the kernel R_0 of this morphism is the space of “formal abelian relations at order one”. Hence, a necessary condition for an abelian relation to exist above an open subset U of M_0 is that U belongs to the image of R_0 by the projection $J^1 A \rightarrow A$. We shall see that it generically not the case for $d < c(n, 2)$.

More generally, denote by $D_k : J^{k+1} A \rightarrow J^k B$ the k^{th} prolongation of D . The kernel R_k of the morphism D_k is the space of formal abelian relations at order $k+1$. Abelian relations may still be seen as the holomorphic sections u of A such that $j^{k+1} u$ be in R_k . Let π_{k+1} denote the natural projection $R_{k+1} \rightarrow R_k$. Let $\sigma_{k+1} : S^{k+2}(T^* M_0) \otimes A \rightarrow S^{k+1}(T^* M_0) \otimes B$ be the symbol of D_{k+1} , g_{k+1} be the kernel of this symbol, and K_k its cokernel. After snake’s lemma, there is a natural map $\partial_k : R_k \rightarrow K_k$ in such a way that we get a commutative diagram⁵, with all lines and columns exact:

$$\begin{array}{ccccccccccc}
& & 0 & & 0 & & 0 & & R_k & & \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \partial_k & & \\
0 & \rightarrow & g_{k+1} & \rightarrow & S^{k+2}(T^* M_0) \otimes A & \xrightarrow{\sigma_{k+1}} & S^{k+1}(T^* M_0) \otimes B & \rightarrow & K_k & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_{k+1} & \rightarrow & J^{k+2} A & \xrightarrow{D_{k+1}} & J^{k+1} B & \rightarrow & \text{coker } D_{k+1} & \rightarrow & 0 \\
& & \downarrow \pi_{k+1} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_k & \rightarrow & J^{k+1} A & \xrightarrow{D_k} & J^k B & \rightarrow & \text{coker } D_k & \rightarrow & 0 \\
& & \downarrow \partial_k & & \downarrow & & \downarrow & & \downarrow & & \\
& & K_k & & 0 & & 0 & & 0 & &
\end{array}$$

In this diagram, we allow k to take the value -1 , with the convention $R_{-1} = A$, $J^{-1} B = 0$.

In the sequel,

the index i will run from 1 to d ,

the indices α, β, \dots will run from 1 to $n-1$,

and the indices λ, μ, \dots will run from 1 to n .

For any holomorphic function a and local coordinates x , a'_λ will denote the partial derivative $a'_\lambda = \frac{\partial a}{\partial x_\lambda}$. More generally, for any partition $L = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of $|L| = \sum \lambda_\mu$, ($L \in \mathcal{P}(n, |L|)$), we denote by $(a)'_L$ the corresponding higher order derivative $(a)'_L = \frac{\partial^{|L|} a}{(\partial x_1)^{\lambda_1} \dots (\partial x_n)^{\lambda_n}}$.

3 Definition of the analytical set S

This definition is different according to the inequalities $d < c(n, 2)$ or $d \geq c(n, 2)$. However, in both cases, this set will satisfy to the

Lemma 3.1 *The set S is an analytical set which has generically a dimension $\leq n-1$ or is empty.*

3.1 Definition of S in the case $n < d < c(n, 2)$:

For $n < d < c(n, 2)$, S will denote the subset of elements $m \in M_0$ such that the vector space $(R_0)_m = (\pi_0)^{-1}(A_m)$ has dimension at least 1. The proof of lemma 3-1 for $d < c(n, 2)$ will be given in the next section, after the description of the map π_0 .

Proof of theorem 1-1 in the case $d < c(n, 2)$: *For $d < c(n, 2)$, all regular d -webs of codimension one have rank 0.*

⁵Notice that the R_k ’s, g_k ’s and K_k ’s are not necessarily vector bundles : exactitude has to be understood as the exactitude in the corresponding diagrams on the fibers at any point of M_0 .

The fact that all regular d -webs have rank 0 for $d < c(n, 2)$ on $M_0 \setminus S$ is a tautology, because of the definition of S . Therefore, they still have rank 0 on all of M_0 , because of the semi-continuity of the rank at a point.

QED

3.2 Definition of S in the case $d \geq c(n, 2)$

Let η_i be an integrable 1-form defining the local foliation \mathcal{F}_i of a d -web of codimension 1. For any integer $h \geq 1$, let $(\eta_i)^h$ be the h^{th} symmetric power of η_i in the space $S^h(T^*M_0)$ of homogeneous polynomials of degree h on the complex tangent bundle TM_0 . For any $m \in M_0$, let $r_h(m)$ be the dimension of the subspace $L_h(m)$ generated in $S^h(T_m^*M_0)$ by the $(\eta_i)^h(m)$'s with $1 \leq i \leq d$ (not depending on the choice of the η_i 's). We have obviously the

Lemma 3.2 *The following inequality holds :*

$$r_h(m) \leq \min(d, c(n, h)).$$

In particular, $r_1 \equiv n$, since we assume $d > n$ and the web to be in weak general position. And $r_h \equiv d$ for $h > k_0$, where k_0 denotes the integer such that

$$c(n, k_0) \leq d < c(n, k_0 + 1).$$

Definition of S for $d \geq c(n, 2)$ (i.e. $k_0 \geq 2$) : Let S_h be the set of points $m \in M_0$ such that $r_h(m) < \min(d, c(n, h))$, and set $S = \bigcup_{h=2}^{k_0} S_h$.

Proof of lemma 3-1 for $d \geq c(n, 2)$: Let $(x_\lambda)_{1 \leq \lambda \leq n}$ be a system of holomorphic local coordinates near a point m_0 of M_0 , and assume that the local foliation \mathcal{F}_i is defined by $\sum_\lambda p_{i\lambda} dx_\lambda = 0$. Then $r_h(m)$ is the rank of the matrix

$$P_h = ((C_i^{(h)}))_{i,L}$$

of size $d \times c(n, h)$ at point m , where $1 \leq i \leq d$, and L runs through the set $\mathcal{P}(n, h)$ of the partitions $L = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of h (i.e. $\sum_{s=1}^n \lambda_s = h$), and where $C_i^{(h)} = \prod_{s=1}^n p_{i \lambda_s}$. Thus, for $2 \leq h \leq k_0$, S_h is locally defined by the vanishing of all determinants of size $c(n, h)$ in P_h . Exceptionnally, it may happen that all of these determinants are identically 0, so that S_h will have dimension n . But generically, these determinants will vanish on hypersurfaces or nowhere.

4 Computation of R_0

Locally, the d -web is defined over M_0 by a family of 1-forms $\eta_i = dx_n - \sum_\alpha p_{i\alpha}(x) dx_\alpha$, which we still may write $\eta_i = -\sum_\lambda p_{i\lambda}(x) dx_\lambda$ with the convention $p_{i_n} \equiv -1$.

Lemma 4.1 *The integrability conditions may be written locally :*

$$(p_{i\lambda})'_\mu - (p_{i\mu})'_\lambda + p_{i\mu}(p_{i\lambda})'_n - p_{i\lambda}(p_{i\mu})'_n \equiv 0 \quad \text{for all triples } (i, \lambda, \mu).$$

Proof : Let $\eta = -\sum_\lambda p_\lambda(x) dx_\lambda$ be a holomorphic 1-form. Then

$$\eta \wedge d\eta = \sum_{\lambda < \mu < \nu} \left[p_\lambda \left((p_\nu)'_\mu - (p_\mu)'_\nu \right) + p_\mu \left((p_\lambda)'_\nu - (p_\nu)'_\lambda \right) + p_\nu \left((p_\mu)'_\lambda - (p_\lambda)'_\mu \right) \right] dx_\lambda \wedge dx_\mu \wedge dx_\nu.$$

Then, when $p_n \equiv -1$, we observe that the vanishing of all terms in $dx_\lambda \wedge dx_\mu \wedge dx_n$ implies the vanishing of all other terms, hence the lemma.

QED

A section $(\omega_i)_i$ of A is locally given by the d functions f_i such that $\omega_i = f_i (\sum_{\lambda} p_{i\lambda}(x) dx_{\lambda})$ satisfying to the identities

$$(E_{\lambda}) \quad \sum_i p_{i\lambda} f_i \equiv 0 \quad \text{for any } \lambda,$$

hence, by derivation,

$$(E_{\lambda,\mu}) \quad \sum_i (p_{i\lambda} f_i)'_{\mu} \equiv 0 \quad \text{for any } \lambda, \mu.$$

Lemma 4.2 *For the family $(f_i)_i$ to define an abelian relation, it is necessary and sufficient that the identities be satisfied*

$$(F_{i\alpha}) \quad (f_i)'_{\alpha} \equiv -(f_i p_{i\alpha})'_n \quad \text{for all pairs } (i, \alpha).$$

Proof : In fact, a holomorphic 1-form $f(\sum_{\lambda} p_{\lambda}(x) dx_{\lambda})$ is closed iff $(fp_{\lambda})'_{\mu} = (fp_{\mu})'_{\lambda}$ for all pairs (λ, μ) such that $\lambda < \mu$. But, because of the integrability conditions, it is sufficient that this relation be satisfied when $\mu = n$, for it to be satisfied with all other μ 's.

QED

Lemma 4.3 *When the family $(f_i)_i$ defines an abelian relation, the identities $(E_{\lambda,\mu})$ and $(E_{\mu,\lambda})$ are the same.*

Proof : In fact, under the assumption, $(f_i p_{i\lambda})'_{\mu} \equiv (f_i p_{i\mu})'_{\lambda}$ for all pairs (λ, μ) , hence the lemma by summation with respect to i .

QED

The identity $(f_i)'_{\alpha} \equiv -(f_i p_{i\alpha})'_n$ means that it is sufficient for knowing the $(f_i)'_n$ for knowing the other partial derivatives $(f_i)'_{\alpha}$ of a family $(f_i)_i$ defining an abelian relation.

Hence, writing $w_i = (f_i)'_n$, and combining $(E_{\lambda,\mu})$ and $(F_{i\alpha})$, we get :

Corollary 4.4 *The elements of R_0 above a given element $(f_i)_i$ in A map bijectively onto the solutions of the linear system Σ_0 .*

$$(\tilde{E}_{\lambda,\mu}) \quad \sum_i p_{i\lambda} p_{i\mu} w_i \equiv \sum_i f_i [(p_{i\lambda})'_{\mu} - p_{i\lambda} (p_{i\lambda})'_n]$$

of $c(n, 2)$ equations $(\tilde{E}_{\lambda,\mu})$ with d unknown w_i .

Notice that the matrix of the system Σ_0 is the matrix $P_2 = ((C_i^{(2)}))_{i,L}$ seen in the previous section.

Proof of the lemma 3-1 in the case $d < c(n, 2)$: Generically, the system Σ_0 has rank d . Hence S is defined locally by the vanishing of all characteristic determinants which are generically not all identically zero. If Σ_0 has a rank r smaller than d , the vanishing of all determinants of size $r+1, \dots, d$ in the matrix P_2 have to be added to the vanishing of all characteristic determinants.

QED

5 Computation of R_k ($k \geq 1$) :

For any pair of multi-indices of derivation $L = (\lambda_1, \dots, \lambda_s, \dots, \lambda_n)$, and $H = (h_1, h_2, \dots, h_n)$, $L + H$ will denote the multi-index $(\lambda_1 + h_1, \lambda_2 + h_2, \dots, \lambda_n + h_n)$. We define similarly $L - H$ if $\lambda_{\mu} \geq h_{\mu}$ for

all μ 's. For any λ , 1_λ will denote the multi-index with all λ_μ 's equal to zero for $\mu \neq \lambda$ and $\lambda_\lambda = 1$. By definition the height $|L|$ of L is the sum $\sum_s \lambda_s$.

By derivation of the identities (E_λ) , the elements of $J^k A$ are characterized by the identities

$$(E_{\lambda,L}) \quad \sum_i (p_{i\lambda} f_i)'_L \equiv 0 \text{ for any } \lambda \text{ and for any multi-index } L \text{ of height } |L| \leq k.$$

Lemma 5.1 *If $(f_i)_i$ is an abelian relation, the relation $(E_{\lambda,L})$ remains unchanged by permutation of all the indices of $L \cup \{\lambda\}$.*

Proof: The left hand term of this identity is obviously symmetric with respect to the indices of L . Thus, it is sufficient to prove that the identities $(E_{\lambda,\mu})$ and $(E_{\mu,\lambda})$ are the same, which we know already. QED

The identity $(E_{\lambda,L})$ above will now be denoted by (E_H) , where $H = L + 1_\lambda$.

Lemma 5.2

(i) *If $(f_i)_i$ is an abelian relation, all partial derivatives $(f_i)'_L$ may be written as a linear combination*

$$(\widetilde{F}_{iL}) \quad (\widetilde{f_i}'_L) \equiv \sum_{k=0}^{|L|} D_i^{(k)} \cdot (f_i)'_{n^k}$$

of f_i and of its partial derivatives $(f_i)'_{n^k} = \frac{\partial^{(k)} f_i}{(\partial x_n)^k}$ with respect to the only variable x_n , with coefficients $D_i^{(k)}$ not depending on the f_i 's.

(ii) *If $L = (\lambda_1, \dots, \lambda_s, \dots, \lambda_n)$, the coefficient $D_i^{(|L|)}$ of highest order is equal to $(-1)^{|L|} \prod_{s=1}^n p_i^{\lambda_s}$, i.e. is equal to $(-1)^{|L|} C_i^{(|L|)}$.*

Proof: We get the lemma by derivation of the identities $(F_{i\alpha})$ and an obvious induction on the height $|L|$ of L . QED

The lemma above means that it is sufficient to know the $(f_i)'_{n^{k+1}}$ to know the other partial derivatives $(f_i)'_L$ in the $(k+1)$ -jet of a family $(f_i)_i$ defining an abelian relation.

Hence, writing $w_i = (f_i)'_{n^{k+1}}$, and combining (E_L) and (\widetilde{F}_{iL}) , we get :

Corollary 5.3 *The elements of R_k above a given element $a_0^{(k-1)}$ in R_{k-1} map bijectively onto the solutions of a linear system Σ_k of $c(n, k+2)$ equations (\widetilde{E}_L) with d unknown w_i*

$$(\widetilde{E}_L) \quad \sum_i C_i^{(k+2)} w_i \equiv \Phi_L(a_0^{(k-1)}),$$

where L runs through the set $\mathcal{P}(n, k+2)$ of partitions of $k+2$, and where the second member $\Phi_L(a_0^{(k-1)})$ depends only on $a_0^{(k-1)} \in R_{k-1}$. In particular, the symbol σ_k of D_k is defined by the matrix

$$P_{k+2} = ((C_i^{(|L|)})_{i,L}),$$

$1 \leq i \leq d$, $|L| = k+2$, of size $d \times c(n, k+2)$.

Theorem 5.4 *Assume the d -web to be regular. The map $\pi_k : R_k \rightarrow R_{k-1}$ is surjective for $k \leq k_0 - 2$ and injective for $k > k_0 - 2$ above the open set $U = M_0 \setminus S$ of M_0 . For $k \leq k_0 - 2$, R_k is a holomorphic bundle of rank $\sum_{h=1}^{k+2} (d - c(n, h))$ over $M_0 \setminus S$.*

Proof : In fact, P_{k+2} is precisely the matrix of the system Σ_k . Thus, for $k \leq k_0 - 2$ and off S_k , the space of solutions of Σ_k is an affine space of dimension $d - c(n, k + 2)$.

QED

Proof of theorem 1-1 in the case $d > c(n, 2)$: *The rank of a regular d -web is at most equal to the number*

$$\pi'(n, d) = \sum_{h=1}^{k_0} (d - c(n, h)).$$

For $k > k_0 - 2$, the symbol σ_k is necessarily injective off S . In fact, the matrix defining this symbol in the corollary above contains the matrix defining σ_{k-1} when we choose the coordinates and the forms η_i 's so that $p_{in} \equiv -1$. Hence, since $g_{k_0-2} = 0$ off S , $g_k = 0$ off S for all $k \geq k_0 - 2$. Consequently, above a given element in R_{k_0-2} , there exists at most one infinite jet of abelian relation, hence one germ of abelian relation since the framework is analytic. We deduce that the rank of the web is at most $\pi'(n, d)$ off S , hence everywhere (semi-continuity of the rank). We shall see in the next section that any regular affine web in strong general position has rank $\pi'(n, d)$, hence the optimality.

QED

Proof of theorem 1-2 : When $d = c(n, k_0)$, $\mathcal{E} = R_{k_0-3}|_{M_0 \setminus S}$ is a vector bundle of rank $\pi'(n, d)$, and $\pi_{k_0-2} : R_{k_0-2} \rightarrow R_{k_0-3}$ is an isomorphism of vector bundles over $M_0 \setminus S$. Since R_{k_0-2} , which is equal to the intersection $J^1 R_{k_0-3} \cap J^{k_0-1} A$, is included into $J^1 R_{k_0-3}$, the previous isomorphism defines, after lemma 2-3, a natural holomorphic connection ∇ on \mathcal{E} , such that the map $u \mapsto j^{k_0-2} u$ from the sections of A into the sections of $J^{k_0-2} A$ induces an injective map from abelian relations on $M_0 \setminus S$ into the set of sections of \mathcal{E} with vanishing covariant derivative. In particular, for $d = c(n, k_0)$, the rank of a regular d -web is $\pi'(n, d)$ iff the curvature of the previous connexion vanishes: in this case, every section of \mathcal{E} with vanishing covariant derivative is conversely the $(k_0 - 2)$ -jet of an abelian relation, since then the both spaces of germs at a point (abelian relations, and sections of \mathcal{E} with vanishing covariant derivative) have the same dimension $\pi'(n, d)$.

QED

6 Examples

6.1 Case $n = 2$:

We recover the results of A. Hénaut ([He1]). In fact, in this case :

- the set S is always empty (all determinants occuring in the computation of the symbols σ_k are determinants of Van-der-Monde for $k \leq k_0 - 2$, vanishing nowhere on M_0); thus, all webs are regular.
- any d is equal to $c(2, d - 1)$,
- the rank $\sum_{h=1}^{d-2} (c(2, d - 1) - c(2, h))$ of R_{d-4} is equal to $(d - 1)(d - 2)/2$.

6.2 Case $n = 3, d = 6$:

Use coordinates x, y, z on \mathbb{C}^3 , with $n = 3$, and $c(n, 2) = 6$. Let a, b, c, e, h be five distinct complex numbers, all different of 0, and let ψ be some holomorphic function of y . Let W be the 6-web of codimension 1 on \mathbb{C}^3 defined by the 1-forms $\eta_i = dz - p_i dx - q_i dy$, $1 \leq i \leq 6$, where : $(p_1, q_1) = (0, 0)$, $(p_2, q_2) = (a, a^2)$, $(p_3, q_3) = (b, b^2)$, $(p_4, q_4) = (c, c^2)$, $(p_5, q_5) = (e, e^2)$ and

$(p_6, q_6) = (h, \psi)$. The system Σ_0 is then equivalent to

$$\begin{pmatrix} a & b & c & e \\ a^2 & b^2 & c^2 & e^2 \\ a^3 & b^3 & c^3 & e^3 \\ a^4 & b^4 & c^4 & e^4 \end{pmatrix} \begin{pmatrix} w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_6 \cdot \psi' \end{pmatrix},$$

to which we add the equations $(h^2 - \psi)w_6 = 0$, and $w_1 + w_2 + \dots + w_6 = 0$. The system is a system of Cramer off the locus S which is

- the surface of equation $\psi(y) = h^2$ in general,
- the empty set when ψ is any constant different of h^2 ,
- all of \mathbb{C}^3 (the special case) for $\psi \equiv h^2$.

Let's precise the connection and its curvature for $\psi \neq h^2$ in the two first cases (the regular case).

A section (f_i) of A is defined by (f_4, f_5, f_6) , since (f_1, f_2, f_3) can be deduced using the equations $\sum_i f_i = 0$, $\sum_i p_i f_i = 0$ and $\sum_i q_i f_i = 0$. Setting $\Delta = abce(b-a)(c-a)(e-a)(c-b)(e-b)(e-c)(h^2 - \psi)$, $K = \frac{\psi'}{\Delta(h^2 - \psi)}$, $\Delta_4 = abe((b-a)(e-a)(e-b))$, and $\Delta_5 = abc((b-a)(c-a)(c-b))$, we get :
 $w_4 = -K\Delta_4$, $w_5 = K\Delta_5$, $w_6 = 0$, hence $u_4 = cK\Delta_4$, $u_5 = -eK\Delta_5$, $u_6 = 0$ and $v_4 = c^2K\Delta_4$,
 $u_5 = -e^2K\Delta_5$, $u_6 = 0$. With respect to the trivialization $(\sigma_4, \sigma_5, \sigma_6)$ of A given by
 $\sigma_4 = (f_4 \equiv 1, f_5 \equiv 0, f_6 \equiv 0)$, $\sigma_5 = (f_4 \equiv 0, f_5 \equiv 1, f_6 \equiv 0)$ and $\sigma_6 = (f_4 \equiv 0, f_5 \equiv 0, f_6 \equiv 1)$, the matrix of the connection is :

$$\omega = \frac{\psi'}{\Delta(h^2 - \psi)} \begin{pmatrix} 0 & 0 & \Delta_4 \eta_4 \\ 0 & 0 & -\Delta_5 \eta_5 \\ 0 & 0 & 0 \end{pmatrix},$$

hence the curvature

$$\Omega = \frac{1}{\Delta} \left(\frac{\psi'}{h^2 - \psi} \right)' \begin{pmatrix} 0 & 0 & \Delta_4(dy \wedge dz + c dx \wedge dy) \\ 0 & 0 & -\Delta_5(dy \wedge dz + e dx \wedge dy) \\ 0 & 0 & 0 \end{pmatrix}.$$

We observe that σ_4 and σ_5 are linearly independant abelian relations. We knew it already since the first integrals $(z - cx - c^2y)$ and $(z - ex - e^2y)$ of \mathcal{F}_4 and \mathcal{F}_5 respectively are linear combinations of the first integrals z , $(z - ax - a^2y)$ and $(z - bx - b^2y)$ of \mathcal{F}_1 , \mathcal{F}_2 and \mathcal{F}_3 . Thus, when $h^2 - \psi$ does not vanish, the rank of the 6-web is at least 2, and has the maximum possible value $\pi'(3, 6) = 3$ in the regular case if and only if $\frac{\psi'}{h^2 - \psi}$ is constant, that is if there exists two scalar constant C and D ($C \neq 0$), such that

$$\psi(y) = h^2 + Ce^{Dy},$$

in particular for $\psi = \text{constant}$ (case $D = 0$). For given C and D as above, $K = -D/\Delta$, and $\sigma_6 - K\Delta_4\sigma_4 + K\Delta_5\sigma_5$ is an abelian relation (the function ψ occurs in the computation of f_1 , f_2 and f_3 from $f_4 = -K\Delta_4$, $f_5 = K\Delta_5$ and $f_6 = 1$).

The special case $\psi \equiv h^2$ will be seen in the next subsection.

6.3 Regular affine webs and optimality of the bound $\pi'(n, d)$:

For any pair (n, d) with $d > n$, give d linear forms (l_1, l_2, \dots, l_d) . Let \mathcal{F}_i be the foliation defined in \mathbb{C}^n by the parallel hyperplanes $l_i = \text{constant}$, and W be the d -web defined by these d foliations. Let k_0 be the integer such that $c(n, k_0) \leq d < c(n, k_0 + 1)$. Let \mathbb{P}'_{n-1} denote the hyperplane at infinity of the n -dimensional projective space $\mathbb{P}_n = \mathbb{C}^n \amalg \mathbb{P}_{n-1}$: the parallel hyperplanes of the pencil $l_i = \text{constant}$ meet at infinity along a hyperplane of \mathbb{P}_{n-1} , i.e. define an element $[l_i]$ of the dual projective space \mathbb{P}'_{n-1} .

Remark 6.1 The web W on \mathbb{C}^n extends to a web on \mathbb{P}_n (with singularities) which is algebraic. It is in fact dual to the union of d straight lines in the dual projective space \mathbb{P}'_n .

Definition 6.2 Such a web on \mathbb{P}_n will be said an “**affine d -web**”, and it is said regular if its restriction to \mathbb{C}^n is regular.

Lemma 6.3 The two following properties are equivalent :

- (i) The affine d -web above is regular.
- (ii) For any h , ($1 \leq h \leq k_0$), there exists $c(n, h)$ points among the d points $[l_i]$, which do not belong to a same algebraic (reducible or not) hypersurface of degree h in \mathbb{P}'_{n-1} .

Proof : For $d \geq c(n, h)$, ($h \geq 1$), assume that the matrix $((C_{iL}^{(h)}))_{i,L}$, $1 \leq i \leq d$, $|L| = h$, of size $d \times c(n, h)$ has rank $< c(n, h)$. This means that the determinant of any square sub-matrix of size $c(n, h)$ vanishes. Saying that the determinant of the sub-matrix given for instance by the $c(n, h)$ first l_i 's vanishes means precisely that $[l_1], [l_2], \dots, [l_{c(n,h)}]$ belong to some hypersurface of degree h in \mathbb{P}'_{n-1} (may be reducible), and same thing for any other subset of $c(n, h)$ indices i . Hence the lemma. QED

Theorem 6.4

- (i) If the d points $[l_i]$ are in general position in \mathbb{P}'_{n-1} (i.e. if any n of the d linear forms l_i are linearly independant), the above affine d -web has a rank $\geq \pi'(n, d)$.
- (ii) It has exactly rank $\pi'(n, d)$ iff it is regular.

Proof : For any h , consider the vector space L_h generated by the h^{th} symmetric products $(l_i)^h$ of the l_i 's, and denote by r_h the dimension of this vector space. The rank of an affine web in strong general position is $\sum_{h=1}^{k_0} (d - r_h)$ (see Trépreau [T], section 2). Hence we have only to prove that $r_h \leq c(n, h)$ in general, and $r_h = c(n, h)$ in the regular case. But this is obvious after the previous lemma, since the dimension of L_h is exactly the rank of the matrix $P_h = ((C_{iL}^{(h)}))_{i,L}$ with $i \leq d$, $|L| = h$ in the linear system Σ_{h-2} of corollary 5-3. QED

Corollary 6.5 The bound $\pi'(n, d)$ for the rank of a regular web is optimal.

Proof : Remember that a algebraic hypersurface of degree h in \mathbb{P}'_{n-1} is defined in general by the data of $c(n, h) - 1$ of its points : thus, the property (ii) of lemma 6-3 above is generically satisfied, so that there exists regular affine d -webs in dimension n for any (n, d) . QED

For instance, an affine 6-web on \mathbb{P}_3 in strong general position will be special of rank 4 ($= \pi(3, 6)$) or regular of rank 3 ($= \pi'(3, 6)$), according to the fact that the six points $[l_i]$ belong or not to a same conic⁶ of \mathbb{P}'_2 : when ψ is constant in the example $(p_1, q_1) = (0, 0)$, $(p_2, q_2) = (a, a^2)$, $(p_3, q_3) = (b, b^2)$, $(p_4, q_4) = (c, c^2)$, $(p_5, q_5) = (e, e^2)$ and $(p_6, q_6) = (h, \psi)$ of the previous subsection, the five first points belong to the conic of equation $q = p^2$, hence the dichotomy according to the fact that ψ is equal or different of h^2 .

New proof of the theorem 1-1 for $d \geq c(n, 2)$ in the particular case of regular webs which are in strong general position :

The rank of a web in strong general position is upper-bounded by the rank of the “tangent affine web” at a point (see again [T]). Since the regularity of a web near a point is equivalent to the regularity

⁶In the special case, it is amazing to deduce the fourth abelian relation from the theorem of the hexagon (Pascal).

of the tangent affine web at that point, theorem 1-1 is also a corollary of the previous theorem 6-4, when the regular web is in strong general position.

QED

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