

JOINT SPECTRUM OF TRANSLATIONS ON $L^2_\omega(\mathbb{R}^2)$

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Abstract. In [6] we established a representation theorem for multipliers (bounded operators commuting with translations) operating on a Banach space E of functions on a locally compact abelian group G . This representation yields a symbol defined on an abstract set U_E of morphisms. In this paper we characterize the set U_E when $G = \mathbb{R}^k$, $E = L^2_\omega(\mathbb{R}^k)$, provided ω is given as a product of weights. More precisely, we show that U_E is completely determined by the joint spectrum of the generators of the algebra of translations on $L^2_\omega(\mathbb{R}^k)$.

1. INTRODUCTION

The aim of this paper is to obtain a representation for multipliers on a Banach spaces E of functions on \mathbb{R}^k .

Definition 1. *A bounded operator commuting with translations on a space of functions on a locally compact abelian group is called a multiplier.*

This kind of operators are often present in analysis and in physics and in particular in scattering theory and control theory. Such operators have been studied from the middle of the past century and there exists a lot of literature on this subject. A classical well-known result is the following

Theorem 1. *For every multiplier M on $L^p(\mathbb{R}^k)$, $1 \leq p < +\infty$ there exists a function $h_M \in L^\infty(\mathbb{R}^k)$ such that we have*

$$\widehat{Mf} = h_M \hat{f}, \forall f \in L^p(\mathbb{R}^k) \cap L^2(\mathbb{R}^k), \text{ and } \|h_M\|_\infty \leq \|M\|. \quad (1.1)$$

The function h_M is called the symbol of M . If E is not a L^p space, in a very general case every multiplier becomes the convolution by a quasimeasure. For more details about this result the reader may consult [2]. A Fourier transformation of a general quasimeasure cannot be defined and so it is complicated to obtain a result similar to (1.1) in the general case. However, in a precedent article [6] the author obtained a general representation theorem for the multipliers on a general

Banach spaces of functions E on a locally compact abelian group G . Denote by $C_c(\mathbb{R}^k)$ the space of the continuous compactly supported functions on \mathbb{R}^k . In the results established in [6] the set $U_E \subset \mathbb{C}^k$ defined below plays a crucial role.

Definition 2. We say that $z \in U_E \subset \mathbb{C}^k$ if we have

$$\left| \int_{\mathbb{R}^k} \phi(x) e^{-i\langle z, x \rangle} dx \right| \leq \|M_\phi\|, \quad \forall \phi \in C_c(\mathbb{R}^k),$$

where M_ϕ is the operator of convolution by ϕ on E and $\|\cdot\|$ denotes the norm of linear operator on E .

Denote by $\mathcal{M}(E)$ the algebra of the multipliers on E . We may consider U_E as a set of functionals on $C_c(\mathbb{R}^k)$ continuous in the topology induced from $\mathcal{M}(E)$. Several properties of the set U_E has been proved in [6]. In particular, U_E is not empty and in some cases U_E becomes the *optimal* (maximal) subset of \mathbb{C}^k so that we can define a symbol $h_M \in \mathcal{H}^\infty(\overset{\circ}{U}_E)$ of $M \in \mathcal{M}(E)$. More precisely, we have the following representation [6]

Theorem 2. For $M \in \mathcal{M}(E)$ there exists $h_M \in \mathcal{H}^\infty(\overset{\circ}{U}_E)$ (resp. $h_M \in L^\infty(U_E)$) if $\overset{\circ}{U}_E \neq \emptyset$ (resp. $\overset{\circ}{U}_E = \emptyset$) such that

$$\widehat{Mf}(z) = h_M(z) \hat{f}(z), \quad \forall z \in U_E, \quad \forall f \in C_c(\mathbb{R}^k).$$

Here

$$\widehat{Mf}(z) = \int_{\mathbb{R}^k} (Mf)(x) e^{-i\langle z, x \rangle} dx, \quad \forall z \in U_E.$$

The function h_M is called the symbol of M .

The problem of the characterization of U_E by the algebra of the translations is very interesting. Thus in the case when $G = \mathbb{R}$ and $E = L^2_\delta(\mathbb{R})$ is a weighted space we showed in [5] that $e^{-iU_E} = \sigma(S_1)$, where $\sigma(A)$ denotes the spectrum of the operator A and S_1 (resp. S_{-1}) is the translation by 1 (resp. by -1) on $L^2_\delta(\mathbb{R})$. Moreover,

$$U_E = \left\{ z \in \mathbb{C}; -\ln \rho(S_{-1}) \leq \operatorname{Im}(z) \leq \ln \rho(S_1) \right\},$$

where $\rho(B)$ denotes the spectral radius of B .

In order to consider other locally compact abelian groups G , as $\mathbb{R}^k, \mathbb{Z}^k$, we are going to search a link between

$$e^{-iU_E} = \{(e^{-iz_1}, \dots, e^{-iz_k}) : (z_1, \dots, z_k) \in U_E\}$$

and the joint spectrum of the algebra generated by translations. Recall that the joint spectrum of m operators $A_j, j = 1, \dots, m$, in an algebra \mathcal{A} is the set of complex numbers $(\mu_1, \dots, \mu_m) \in \mathbb{C}^m$ such that the operator

$$\sum_{j=1}^m (A_j - \mu_j I) J_j$$

is non invertible in \mathcal{A} for every $(J_1, \dots, J_m) \in (\mathcal{A})^m$. It is well known that the joint spectrum is included in the set $\prod_{j=1}^m \sigma(A_j)$ but in general this inclusion is strict. On the other hand, following [7], [4], the joint spectrum of a family of operators B_1, \dots, B_m in a commutative Banach algebra \mathcal{A} with unit coincides with the set

$$\{\chi(B_k), k = 1, \dots, m\}_{\chi \in \widehat{\mathcal{A}}},$$

where $\widehat{\mathcal{A}}$ denotes the set of the characters of \mathcal{A} .

It is easy to prove (see [6]) that U_E is a subset of the joint spectrum of the generators of the algebra generated by the translations. Motivated by the result for $G = \mathbb{R}$, we introduced in [6] the following conjecture

(U) *The set e^{-iU_E} coincides with the joint spectrum of the generators of the algebra generated by the translations.*

This conjecture is true for all discrete locally compact abelian groups G (see [6] for more details). The case of non discrete groups G is more complicated. In this work we deal with the case $G = \mathbb{R}^k$ assuming that $E = L^2_\omega(\mathbb{R}^k)$ is a weighted space defined below. As in [5], the main step in our argument is the construction of special sequence of functions $f_{1,m}, \dots, f_{k,m}$. Our construction is based on the spectrum of the semi-groups of contractions and so even in the case $G = \mathbb{R}$ we present in this work a simpler construction than that in [5]. On the other hand, to generalize the argument of Section 2 for more general weights on \mathbb{R}^k , a more fine analysis is needed since we cannot reduce the analysis to the examination of the spectrum of a semi-group.

We pass to the precise definitions of the weighted spaces. We say that δ is a weight on \mathbb{R} if δ is a measurable positive function on \mathbb{R} satisfying:

$$0 < \sup_{x \in \mathbb{R}} \operatorname{ess} \frac{\delta(x+y)}{\delta(x)} < +\infty, \forall y \in \mathbb{R}.$$

The space $L^2_\delta(\mathbb{R})$ is the set of all measurable functions f on \mathbb{R} such that $f\delta \in L^2(\mathbb{R})$.

The purpose of this paper is to prove the conjecture (U) in the case $G = \mathbb{R}^k$, $E = L^2_\omega(\mathbb{R}^k)$, provided ω is a product of weights. For simplicity of the exposition we assume $k = 2$ but our arguments works with trivial modifications in the case $k > 2$.

Let ω_1 and ω_2 be two weights on \mathbb{R} . Define a weight ω on \mathbb{R}^2 by

$$\omega(x, y) = \omega_1(x)\omega_2(y), \quad \forall(x, y) \in \mathbb{R}^2.$$

Set

$$L^2_\omega(\mathbb{R}^2) = \left\{ f : \mathbb{R}^2 \longrightarrow \mathbb{C}, \int_{\mathbb{R}^2} |f(x, y)|^2 \omega(x, y) dx dy < +\infty \right\}$$

and consider the operator

$$S_{a,b} : L^2_\omega(\mathbb{R}^2) \longrightarrow L^2_\omega(\mathbb{R}^2)$$

defined by

$$(S_{a,b}f)(x, y) = f(x - a, y - b), \quad a.e.$$

Definition 3. A bounded operator M on $L^2_\omega(\mathbb{R}^2)$ is called multiplier if

$$MS_{a,b} = S_{a,b}M, \quad \forall(a, b) \in \mathbb{R}^2.$$

Set $E = L^2_\omega(\mathbb{R}^2)$ and let \mathcal{M}_ω be the set of multipliers on $L^2_\omega(\mathbb{R}^2)$. Taking into account the properties of ω , it is obvious that

$$M_\phi : L^2_\omega(\mathbb{R}^2) \ni f \longrightarrow f * \phi \in L^2_\omega(\mathbb{R}^2), \quad \forall \phi \in C_c(\mathbb{R}^2)$$

is a multiplier. Denote $I_\omega = [-\ln \rho(S_{-1,0}), \ln \rho(S_{1,0})]$ and

$$\Omega_\omega = \{z \in \mathbb{C}, \operatorname{Im} z \in I_\omega\}.$$

Similarly, set $J_\omega = [-\ln \rho(S_{0,-1}), \ln \rho(S_{0,1})]$ and define

$$V_\omega = \{z \in \mathbb{C}, \operatorname{Im} z \in J_\omega\}.$$

We will prove the following

Theorem 3. 1) We have $U_E = \Omega_\omega \times V_\omega$.

2) The joint spectrum of $S_{1,0}$ and $S_{0,1}$ is the set

$$\begin{aligned} & \sigma(S_{1,0}) \times \sigma(S_{0,1}) = e^{-i\Omega_\omega} \times e^{-iV_\omega} \\ & = \left\{ z \in \mathbb{C}; \frac{1}{\rho(S_{-1,0})} \leq |z| \leq \rho(S_{1,0}) \right\} \times \left\{ z \in \mathbb{C}; \frac{1}{\rho(S_{0,-1})} \leq |z| \leq \rho(S_{0,1}) \right\}. \end{aligned}$$

From Theorem 3 and the result in [6] the following representation can be obtained directly.

Theorem 4. *Let $M \in \mathcal{M}_\omega$. For $(a, b) \in I_\omega \times J_\omega$, the function*

$$\mathbb{R}^2 \ni (x, y) \longrightarrow (Mf)(x, y)e^{xa+yb}$$

is in $L^2(\mathbb{R}^2)$. Moreover, there exists $h_{a,b} \in L^\infty(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} (Mf)(x, y)e^{-ix(t+ia)-iy(s+ib)} dx dy = h_{a,b}(s, t) \int_{\mathbb{R}^2} f(x, y)e^{-ix(t+ia)-iy(s+ib)} dx dy, \forall f \in C_c(\mathbb{R}^2)$$

and

$$\|h_{a,b}\|_\infty \leq C_\omega \|M\|, \forall (a, b) \in I_\omega \times J_\omega,$$

where C_ω is a constant depending only on ω .

If $\overset{\circ}{I}_\omega \neq \emptyset$ and $\overset{\circ}{J}_\omega \neq \emptyset$, then there exists a function $h \in \mathcal{H}^\infty(\overset{\circ}{\Omega}_\omega \times \overset{\circ}{V}_\omega)$ such that

$$\int_{\mathbb{R}^2} (Mf)(x, y)e^{-ix\mu-iy\nu} dx dy = h(\mu, \nu) \int_{\mathbb{R}^2} f(x, y)e^{-ix\mu-iy\nu} dx dy.$$

If $\overset{\circ}{I}_\omega = \{a\}$ and $\overset{\circ}{J}_\omega \neq \emptyset$, then there exists a function $h_a \in \mathcal{H}^\infty(\overset{\circ}{V}_\omega)$ such that

$$\int_{\mathbb{R}^2} (Mf)(x, y)e^{ax-iy\nu} dx dy = h_a(\nu) \int_{\mathbb{R}^2} f(x, y)e^{ax-iy\nu} dx dy, \forall \nu \in \overset{\circ}{V}_\omega.$$

If $\overset{\circ}{J}_\omega = \{b\}$ and $\overset{\circ}{I}_\omega \neq \emptyset$, then there exists a function $h_b \in \mathcal{H}^\infty(\overset{\circ}{\Omega}_\omega)$ such that

$$\int_{\mathbb{R}^2} (Mf)(x, y)e^{-ix\mu+by} dx dy = h_b(\mu) \int_{\mathbb{R}^2} f(x, y)e^{-ix\mu+by} dx dy, \forall \mu \in \overset{\circ}{\Omega}_\omega.$$

2. CONSTRUCTION OF A SEQUENCE $(f_m u_m)$

An important step in our proof is the following

Proposition 1. *For every fixed pair $(\mu, \nu) \in \Omega_\omega \times V_\omega$ and every $\phi \in C_c(\mathbb{R}^2)$, we have*

$$\left| \int_{\mathbb{R}^2} \phi(x, y)e^{-i\mu x - i\nu y} dx dy \right| \leq \|M_\phi\|.$$

Proof. In order to prove the proposition we need to construct a special sequence $(f_m u_m)_{m \in \mathbb{N}}$ of functions. First consider the group $\{S_{t,0}\}$, $t \in \mathbb{R}$, defined on $L^2_{\omega_1}(\mathbb{R})$. This group is clearly strongly continuous and we denote by A the generator of $S_{t,0} = e^{tA}$. We have the estimate

$$\|S_{t,0}\| \leq C e^{\alpha_1 |t|}, \forall t \in \mathbb{R}, \quad (2.1)$$

where $C > 0$ and $\alpha_1 \geq 0$ are constants. This follows from the fact that ω_1 is equivalent to the special weight ω_0 constructed in [1]. The details and the justifications of the construction of ω_0 are given in [5], [1]. For the convenience of the reader we expose below the main steps. Set $\gamma(x) = \ln(\omega_1(x))$ a.e. and

$$\omega_0(x) = \exp\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \gamma(x+u) du\right), \quad \forall x \in \mathbb{R}.$$

The weight ω_0 is well-defined and continuous (see [5], [1]). We have

$$\frac{\omega_0(x)}{\omega(x)} = \exp\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \gamma(x+u) - \gamma(x) du\right) \leq \exp\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} M_0 du\right) = e^{M_0} \quad a.e.,$$

where

$$M_0 = \sup_{x \in [-1,1]} \sup_{\text{ess}_{x \in \mathbb{R}}} |\gamma(x+t) - \gamma(x)|, \quad \forall t \in \mathbb{R}.$$

In the same way we get

$$\frac{\omega(x)}{\omega_0(x)} \leq e^{M_0} \quad a.e.$$

It follows that the weights ω_0 and ω_1 are equivalent and $L_{\omega_1}^2(\mathbb{R}^2) = L_{\omega_0}^2(\mathbb{R}^2)$.

For a bounded operator T on $L_{\omega_1}^2(\mathbb{R}^2)$, define

$$\|T\|_{\omega} := \sup_{f \in L_{\omega}^2(\mathbb{R}^2), f \neq 0} \frac{\|Tf\|_{\omega}}{\|f\|_{\omega}} \quad \text{and} \quad \|T\|_{\omega_0} := \sup_{f \in L_{\omega_0}^2(\mathbb{R}^2), f \neq 0} \frac{\|Tf\|_{\omega_0}}{\|f\|_{\omega_0}}.$$

We have

$$\|S_{t,0}\|_{\omega_0} = \sup_{x \in \mathbb{R}} \exp(\gamma_0(x+t) - \gamma_0(x)) \leq e^{M_0|t|}, \quad \forall t \in \mathbb{R}. \quad (2.2)$$

Set

$$\beta_{\omega} = \exp \int_{-\frac{1}{2}}^{\frac{1}{2}} \sup_{\text{ess}_{x \in \mathbb{R}}} (\gamma(x+u) - \gamma(x)) du.$$

Notice that

$$\beta_{\omega}^{-2} \|S_{t,0}\|_{\omega} \leq \|S_{t,0}\|_{\omega_0} \leq \beta_{\omega}^2 \|S_{t,0}\|_{\omega}$$

which leads to (2.1).

Let $\mu \in \mathbb{C}$ be such that $e^{\mu} \in \sigma(S_{1,0})$. We denote by $\langle, \rangle_{\omega_i}$, $i = 1, 2$, the scalar product in $L_{\omega_i}^2(\mathbb{R})$, $i = 1, 2$, and by $\|\cdot\|_{\omega_i}$, $i = 1, 2$, the norm in these spaces. Next we denote by $\langle, \rangle_{\omega}$ and $\|\cdot\|_{\omega}$ the scalar product and the norm in the space

$L^2_\omega(\mathbb{R}^2)$. First, our aim is to prove that there exists a sequence $(f_m)_{m \in \mathbb{N}}$ of functions of $L^2_{\omega_1}(\mathbb{R})$ such that

$$\lim_{m \rightarrow +\infty} \langle (e^{At} - e^{\mu t})f_m, f_m \rangle_{\omega_1} = 0, \quad \forall t \in \mathbb{R}, \quad (2.3)$$

$$\|f_m\|_{\omega_1} = 1, \quad \forall m \in \mathbb{N}. \quad (2.4)$$

Since $e^{\sigma(A)} \subset \sigma(S_{1,0})$, we have to deal with two cases:

- (i) $\mu \in \sigma(A)$,
- (ii) $\mu \notin \sigma(A)$.

In the case (i) we have $\mu \in \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$, where $\sigma_p(A)$ is the point spectrum, $\sigma_c(A)$ is the continuous spectrum and $\sigma_r(A)$ is the residual spectrum of A . If we have

$$\mu \in \sigma_p(A) \cup \sigma_c(A),$$

it is easy to see that there exists a sequence $(f_m) \in L^2_{\omega_1}(\mathbb{R})$ such that

$$\|(A - \mu)f_m\|_{\omega_1} \xrightarrow{m \rightarrow +\infty} 0, \quad \|f_m\|_{\omega_1} = 1, \quad \forall m \in \mathbb{N}.$$

Then the equality

$$(e^{At} - e^{\mu t})f_m = \left(\int_0^t e^{\mu(t-s)} e^{As} ds \right) (A - \mu)f_m,$$

yields

$$\|(e^{At} - e^{\mu t})f_m\|_{\omega_1} \xrightarrow{m \rightarrow +\infty} 0, \quad \forall t \in \mathbb{R}$$

and we deduce (2.3). If $\mu \notin \sigma_p(A) \cup \sigma_c(A)$, we have $\mu \in \sigma_r(A)$ and

$$\overline{\text{Ran}(A - \mu I)} \neq L^2_{\omega_1}(\mathbb{R}),$$

where $\text{Ran}(A - \mu I)$ denotes the range of the operator $A - \mu I$. Therefore there exists $h \in D(A^*)$, $\|h\|_{\omega_1} = 1$, such that

$$\langle f, (A^* - \bar{\mu})h \rangle_{\omega_1} = 0, \quad \forall f \in D(A).$$

This implies $(A^* - \bar{\mu})h = 0$ and we take $f = h$. Then

$$\begin{aligned} \langle (e^{At} - e^{\mu t})f, f \rangle_{\omega_1} &= \langle f, (e^{A^*t} - e^{\bar{\mu}t})f \rangle_{\omega_1} \\ &= \left\langle f, \left(\int_0^t e^{\bar{\mu}(t-s)} e^{A^*s} ds \right) (A^* - \bar{\mu})f \right\rangle_{\omega_1} = 0. \end{aligned}$$

In this case we set

$$f_m = f, \quad \forall m \in \mathbb{N}$$

and we get again (2.3).

The case (ii) is more difficult. First, we observe that if $\mu \notin \sigma(A)$, we have $e^\mu \notin e^{\sigma(A)}$, hence $e^\mu \in \sigma(e^A) \setminus e^{\sigma(A)}$. Let ω_0 be the special weight equivalent to ω_1 (see the beginning of the section). Without loss of generality, we can consider $\{S_{t,0}\}$ and A as operators on $L^2_{\omega_0}(\mathbb{R}^2)$. The spectrum of $S_{t,0}$ (resp. A) operating on $L^2_{\omega_1}(\mathbb{R}^2)$ is the same as the spectrum of $S_{t,0}$ (resp. A) operating on $L^2_{\omega_0}(\mathbb{R}^2)$. Then, taking into account (2.2), we are in situation to apply in $L^2_{\omega_0}(\mathbb{R}^2)$ the results for the spectrum of the semi-groups in a Hilbert space (see [2], [3]). Following these results, there exists a sequence of integers n_k converging to ∞ such that

$$\|(A - (\mu + 2\pi i n_k))^{-1}\|_{\omega_0} \geq \frac{2}{k}, \quad \forall k \in \mathbb{N}.$$

Consequently, the equivalence of the norms corresponding to ω_1 and ω_0 shows that there exists a sequence (g_{n_k}) such that $\|g_{n_k}\|_{\omega_1} = 1$ and

$$\|(A - (\mu + 2\pi i n_k))^{-1}g_{n_k}\|_{\omega_1} \geq \frac{C}{k},$$

where $C > 0$ is a constant independent on k and we set

$$f_{n_k} = \frac{(A - (\mu + 2\pi i n_k)I)^{-1}g_{n_k}}{\|(A - (\mu + 2\pi i n_k)I)^{-1}g_{n_k}\|_{\omega_1}}.$$

This implies

$$\begin{aligned} (e^{At} - e^{\mu+2\pi i n_k t})f_{n_k} &= (e^{At} - e^{\mu t})f_{n_k} = \\ &\left(\int_0^t e^{(\mu+2\pi i n_k)(t-s)} e^{As} ds \right) (A - (\mu + 2\pi i n_k))f_{n_k} \end{aligned}$$

and we deduce

$$\|(e^{At} - e^{\mu t})f_{n_k}\| \xrightarrow[k \rightarrow +\infty]{} 0.$$

In this case we set $f_m = f_{n_m}$ and we arrange again the properties (2.3) and (2.4).

Remark. We cannot apply directly the argument concerning the case (ii) to $L^2_{\omega}(\mathbb{R}^2)$ since for the group $S_{t,0}$ in $L^2_{\omega}(\mathbb{R}^2)$ we may have $C > 1$ in the estimate (2.1). On the other hand, by a standard argument we may reduce the analysis of $S_{t,0}$ in $L^2_{\omega_0}(\mathbb{R}^2)$ to a contraction semi-group.

Let $\nu \in \mathbb{C}$ be such that $e^\nu \in \sigma(S_{0,1})$. Repeating the above argument, we construct a sequence (u_m) satisfying properties (2.3) and (2.4) replacing A by the generator B of the semi-group $\{S_{0,t}\}_{t \in \mathbb{R}}$ and μ by ν . Now, we write

$$\left\langle \int_{\mathbb{R}^2} \phi(t, s) \left(e^{\mu t + \nu s} - e^{At + Bs} \right) f_m u_m dt ds, f_m u_m \right\rangle_{\omega}$$

$$\begin{aligned}
&= \int_{\mathbb{R}} \langle u_m e^{\nu s} \int_{\mathbb{R}} \langle \phi(t, s)(e^{\mu t} - e^{At})f_m, f_m \rangle_{\omega_1} dt, u_m \rangle_{\omega_2} ds \\
&+ \int_{\mathbb{R}} \langle \int_{\mathbb{R}} \langle \phi(t, s)(e^{\nu s} - e^{Bs})u_m, u_m \rangle_{\omega_2} ds e^{At} f_m, f_m \rangle_{\omega_1} dt \\
&= \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

We have

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \langle \phi(t, s)(e^{\mu t} - e^{At})f_m, f_m \rangle_{\omega_1} dt \right| \\
&\leq \|\phi\|_{\infty} \int_{\text{supp}\phi(\cdot, s)} | \langle (e^{\mu t} - e^{At})f_m, f_m \rangle_{\omega_1} | dt.
\end{aligned}$$

The property

$$\langle (e^{\mu t} - e^{At})f_m, f_m \rangle_{\omega_1} \xrightarrow{m \rightarrow +\infty} 0, \forall t \in \mathbb{R}.$$

implies

$$\mathcal{I}_1 \xrightarrow{m \rightarrow +\infty} 0.$$

In the same way we get

$$\mathcal{I}_2 \xrightarrow{m \rightarrow +\infty} 0.$$

It is clear that

$$| \langle f_m u_m, f_m u_m \rangle_{\omega} | = 1, \forall m \in \mathbb{N}.$$

Consequently, for every $\phi \in C_c(\mathbb{R}^2)$ we get

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \phi(t, s) e^{\mu t + \nu s} dt ds \right| \\
&= \left| \langle \int_{\mathbb{R}^2} \phi(t, s) e^{\mu t + \nu s} f_m u_m dt ds, f_m u_m \rangle_{\omega} \right| \\
&\leq \mathcal{I}_1 + \mathcal{I}_2 + \left| \langle \int_{\mathbb{R}^2} \phi(t, s) e^{At + Bs} f_m u_m dt ds, f_m u_m \rangle_{\omega} \right|
\end{aligned}$$

and we conclude that

$$\left| \int_{\mathbb{R}^2} \phi(t, s) e^{\mu t + \nu s} dt ds \right| \leq \|M_{\phi}\|.$$

This completes the proof of Proposition 1. \square

3. PROOFS OF THE CONJECTURE (U)

For $E = L^2_\omega(\mathbb{R}^2)$ denote U_E by U_ω . We pass to the proof of the main theorem of the paper.

Proof of Theorem 3. Recall that

$$\sigma(S_{1,0}) = \left\{ z \in \mathbb{C}; \frac{1}{\rho(S_{-1,0})} \leq |z| \leq \rho(S_{1,0}) \right\} = e^{-i\Omega_\omega}$$

and

$$\sigma(S_{0,1}) = \left\{ z \in \mathbb{C}; \frac{1}{\rho(S_{0,-1})} \leq |z| \leq \rho(S_{0,1}) \right\} = e^{-iV_\omega}$$

(see [5]). Taking into account these results and applying directly Proposition 1, we get

$$\Omega_\omega \times V_\omega \subset U_\omega.$$

Let \mathcal{G} be the set of the continuous morphisms from \mathbb{R}^2 into \mathbb{C}^* . More precisely

$$\theta : \mathbb{R}^2 \longrightarrow \mathbb{C}^*$$

is an element of \mathcal{G} if and only if θ is continuous and

$$\theta((x_1, x_2), (y_1, y_2)) = \theta(x_1, x_2)\theta(y_1, y_2), \quad \forall (x_1, x_2) \in \mathbb{R}^2, \quad \forall (y_1, y_2) \in \mathbb{R}^2.$$

Set

$$\mathcal{U}_\omega = \left\{ \theta \in \mathcal{G} : \left| \int_{\mathbb{R}^2} \phi(x, y)\theta(x, y)^{-1} dx dy \right| \leq \|M_\phi\|, \quad \forall \phi \in C_c(\mathbb{R}^2) \right\}.$$

There exists a trivial isomorphism between \mathcal{U}_ω and U_ω . Indeed,

$$\theta \in \mathcal{U}_\omega \Leftrightarrow \theta(x, y) = e^{i(z_1 x + z_2 y)}, \quad \text{for some } (z_1, z_2) \in U_\omega. \quad (3.1)$$

Notice that it was proved in [6] that if $\theta \in \mathcal{U}_\omega$, then there exists a character Γ_θ on the algebra of the translations such that

$$\Gamma_\theta(S_{x,y}) = \theta(x, y), \quad \forall (x, y) \in \mathbb{R}^2. \quad (3.2)$$

From the theory of commutative Banach algebras with unit, it follows that the joint spectrum of $S_{1,0}$ and $S_{0,1}$ is the set of pairs $(\chi(S_{1,0}), \chi(S_{0,1}))$, where χ runs over the set of all characters on the algebra of the translations ([7]). It is clear that the pair

$$(\theta(1, 0), \theta(0, 1)) = (\Gamma_\theta(S_{1,0}), \Gamma_\theta(S_{0,1}))$$

is in the joint spectrum of $S_{1,0}$ and $S_{0,1}$. Hence, the property (3.1) implies that

$$e^{-i\Omega_\omega} \times e^{-iV_\omega} = \sigma(S_{1,0}) \times \sigma(S_{0,1})$$

is a subset of the joint spectrum of $S_{1,0}$ and $S_{0,1}$. Since the joint spectrum of two operators is a subset of the product of their spectra, we conclude that the joint spectrum of $S_{1,0}$ and $S_{0,1}$ is

$$e^{-i\Omega_\omega} \times e^{-iV_\omega} = \sigma(S_{1,0}) \times \sigma(S_{0,1}).$$

This completes the proof of 2).

If $z = (z_1, z_2) \in U_\omega$, by (3.1) and (3.2) we get

$$(e^{-iz_1}, e^{-iz_2}) \in \sigma(S_{1,0}) \times \sigma(S_{1,0}) = e^{-i\Omega_\omega} \times e^{-iV_\omega}$$

and hence

$$U_\omega \subset \Omega_\omega \times V_\omega.$$

We conclude that

$$U_\omega = \Omega_\omega \times V_\omega$$

and the proof of 1) is complete.

□

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