

JOINT SPECTRUM OF TRANSLATIONS ON $L^2_\omega(\mathbb{R}^2)$

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Abstract. In [5] we established a representation theorem for multipliers (bounded operators commuting with translations) operating on a Banach space E of functions on a locally compact abelian group G . This representation yields a symbol defined on an abstract set \mathcal{U}_E of morphisms. In this paper we characterize the set \mathcal{U}_E when $E = L^2_\omega(\mathbb{R}^k)$, where ω is a weight on \mathbb{R}^k . More precisely, we show that \mathcal{U}_E is isomorphic to the joint spectrum of the translations on $L^2_\omega(\mathbb{R}^k)$.

1. INTRODUCTION

The aim of this paper is to obtain a representation for multipliers on a Banach spaces E of functions on \mathbb{R}^k . The multipliers are bounded operators commuting with the translations on E . This kind of operators are often present in analysis and in physics and in particular in scattering theory and control theory. Such operators have been studied from the middle of the past century and there exists a lot of literature on this subject. A classical well-known result is the following

Theorem 1. *For every multiplier on $L^p(\mathbb{R}^k)$, $1 \leq p < +\infty$ there exists a function $h_M \in L^\infty(\mathbb{R}^k)$ such that we have*

$$\widehat{Mf} = h_M \hat{f}, \forall f \in L^p(\mathbb{R}^k) \cap L^2(\mathbb{R}^k), \text{ and } \|h_M\|_\infty \leq \|M\|. \quad (1.1)$$

The function h_M is called the symbol of M . If E is not a L^p space, in a very general case every multiplier becomes the convolution by a quasimeasure. For more details about this result the reader may consult [2]. A Fourier transformation of a general quasimeasure can not be defined and so it is complicated to obtain a result similar to (1.1) in the general case. However, in a precedent article [5] the author obtained a general representation theorem for the multipliers (bounded operators commuting with the translations) on a general Banach spaces of functions E on a locally compact abelian group G . Denote by $C_c(\mathbb{R}^k)$ the space of the continuous functions on \mathbb{R}^k . In [5] a set $U_E \subset \mathbb{C}^k$ plays a crucial role.

Definition 1. We define $U_E \subset \mathbb{C}^k$ so that for every $z \in U_E$ we have

$$\left| \int_{\mathbb{R}^k} \phi(x) e^{-i\langle z, x \rangle} dx \right| \leq \|M_\phi\|, \quad \forall \phi \in C_c(\mathbb{R}^k),$$

where M_ϕ is the operator of convolution by ϕ on E .

Denote by $\mathcal{M}(E)$ the algebra of the multipliers on E . When the group G is \mathbb{R}^k the representation established in [5] is given by the following

Theorem 2. For $M \in \mathcal{M}(E)$ there exists $h_M \in \mathcal{H}^\infty(U_E)$ (resp. $h_M \in L^\infty(U_E)$)

if $\mathring{U}_E \neq \emptyset$ (resp. $\mathring{U}_E = \emptyset$) such that

$$\widehat{Mf}(z) = h_M(z) \hat{f}(z), \quad \forall z \in U_E, \quad \forall f \in C_c(\mathbb{R}^k).$$

Here

$$\widehat{Mf}(z) = \int_{\mathbb{R}^k} (Mf)(x) e^{-i\langle z, x \rangle} dx, \quad \forall z \in U_E.$$

The function h_M is called the symbol of M .

In [5] some important properties of U_E has been proved but it was an open problem to characterize U_E . On the other hand, it is easy to prove (see [5]) that U_E is a not empty subset of the joint spectrum of the generators of the algebra generated by the translations. For these reasons, we introduced in [5] the following conjecture

(U) The set U_E coincides with the joint spectrum of the generators of the algebra generated by the translations.

Recall that the joint spectrum of m operators $A_j, j = 1, \dots, m$ in an algebra \mathcal{A} is the set of complex numbers $(\mu_1, \dots, \mu_m) \in \mathbb{C}^m$ such that the operator

$$\sum_{j=1}^m (A_j - \mu_j I) J_j$$

is non invertible in \mathcal{A} for every $(J_1, \dots, J_m) \in (\mathcal{A})^m$. Let $\sigma(A)$ denote the spectrum of A . It is well known that the joint spectrum is included in the set $\prod_{j=1}^m \sigma(A_j)$ but in general this inclusion is strict.

It is important to mention that if (U) holds, it will implies that the set U_E is the optimal set where a L^∞ symbol exists for every multiplier on E . Let δ be a weight on \mathbb{R} . More precisely, δ is a measurable positive function on \mathbb{R} satisfying:

$$0 < \sup_{x \in \mathbb{R}} \frac{\delta(x+y)}{\delta(x)} < +\infty, \quad \forall y \in \mathbb{R}.$$

The space $L_\delta^2(\mathbb{R})$ is the set of all measurable functions f on \mathbb{R} such that $f\delta \in L^2(\mathbb{R})$. Following [4], we know that if $E = L_\delta^2(\mathbb{R})$, then

$$U_E = \left\{ z \in \mathbb{C}; -\ln \rho(S_{-1}) \leq \operatorname{Im}(z) \leq \ln \rho(S_1) \right\},$$

where S_{-1} (resp. S_1) is the translation by -1 (resp. by 1) on $L_\delta^2(\mathbb{R})$. In this case the result in [4] says that $e^{-iU_E} = \sigma(S_1)$. If we consider the group $G = \mathbb{R}^k$, the set U_E has the form $\mathbb{R}^k + iK$, where K is a convex compact of \mathbb{R}^k but it is not known if K is the product of k intervals. The purpose of this paper is to prove the conjecture (U) in the case $G = \mathbb{R}^k$, provided E is a weighted $L_\omega^2(\mathbb{R}^k)$ space. For simplicity of the exposition we assume $k = 2$ but our arguments works with trivial modifications in the case $k > 2$.

Let ω_1 and ω_2 be two weights on \mathbb{R} . Define a weight ω on \mathbb{R}^2 by

$$\omega(x, y) = \omega_1(x)\omega_2(y), \quad \forall (x, y) \in \mathbb{R}^2.$$

Set

$$L_\omega^2(\mathbb{R}^2) = \left\{ f : \mathbb{R} \longrightarrow \mathbb{C}, \int_{\mathbb{R}^2} |f(x, y)|^2 \omega(x, y)^2 dx dy < +\infty \right\}$$

and consider the operator

$$S_{a,b} : L_\omega^2(\mathbb{R}^2) \longrightarrow L_\omega^2(\mathbb{R}^2)$$

defined by

$$(S_{a,b}f)(x, y) = f(x - a, y - b), \quad a.e.$$

Definition 2. A bounded operator M on $L_\omega^2(\mathbb{R}^2)$ is called multiplier if

$$MS_{a,b} = S_{a,b}M, \quad \forall (a, b) \in \mathbb{R}^2.$$

Let \mathcal{M}_ω be the set of multipliers on $L_\omega^2(\mathbb{R}^2)$. Taking into account the properties of ω , it is obvious that

$$M_\phi : L_\omega^2(\mathbb{R}^2) \ni f \longrightarrow f * \phi \in L_\omega^2(\mathbb{R}^2), \quad \forall \phi \in C_c(\mathbb{R}^2)$$

is a multiplier. Denote $I_\omega = [-\ln \rho(S_{-1,0}), \ln \rho(S_{1,0})]$ and

$$\Omega_\omega = \{z \in \mathbb{C}, \operatorname{Im} z \in I_\omega\}.$$

Similarly, set $J_\omega = [-\ln \rho(S_{0,-1}), \ln \rho(S_{0,1})]$ and define

$$V_\omega = \{z \in \mathbb{C}, \operatorname{Im} z \in J_\omega\}.$$

We will prove the following

Theorem 3. 1) We have $U_E = U_\omega \times V_\omega$.

2) The joint spectrum of $S_{1,0}$ and $S_{0,1}$ is the set

$$\begin{aligned} & \sigma(S_{1,0}) \times \sigma(S_{0,1}) = e^{-iU_\omega} \times e^{-iV_\omega} \\ & = \left\{ z \in \mathbb{C}; \frac{1}{\rho(S_{-1,0})} \leq |z| \leq \rho(S_{1,0}) \right\} \times \left\{ z \in \mathbb{C}; \frac{1}{\rho(S_{0,-1})} \leq |z| \leq \rho(S_{0,1}) \right\}. \end{aligned}$$

From Theorem 3 and the result in [5] the following representation can be obtained directly

Theorem 4. Let $M \in \mathcal{M}_\omega$. For $(a, b) \in I_\omega \times J_\omega$, the function

$$\mathbb{R}^2 \ni (x, y) \longrightarrow (Mf)(x, y)e^{xa+yb}$$

is in $L^2(\mathbb{R}^2)$. Moreover, there exists $h_{a,b} \in L^\infty(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} (Mf)(x, y)e^{-ix(t+ia)-iy(s+ib)} dx dy = h_{a,b}(s, t) \int_{\mathbb{R}^2} f(x, y)e^{-ix(t+ia)-iy(s+ib)} dx dy, \forall f \in C_c(\mathbb{R}^2)$$

and

$$\|h_{a,b}\|_\infty \leq C_\omega \|M\|, \forall (a, b) \in I_\omega \times J_\omega,$$

where C_ω is a constant depending only on ω .

If $\overset{\circ}{I}_\omega \neq \emptyset$ and $\overset{\circ}{J}_\omega \neq \emptyset$, then there exists a function $h \in \mathcal{H}^\infty(\overset{\circ}{\Omega}_\omega \times \overset{\circ}{V}_\omega)$ such that

$$\int_{\mathbb{R}^2} (Mf)(x, y)e^{-ix\mu-iy\nu} dx dy = h(\mu, \nu) \int_{\mathbb{R}^2} f(x, y)e^{-ix\mu-iy\nu} dx dy.$$

If $\overset{\circ}{I}_\omega = \{a\}$ and $\overset{\circ}{J}_\omega \neq \emptyset$, then there exists a function $h_a \in \mathcal{H}^\infty(\overset{\circ}{V}_\omega)$ such that

$$\int_{\mathbb{R}^2} (Mf)(x, y)e^{ax-iy\nu} dx dy = h_a(\nu) \int_{\mathbb{R}^2} f(x, y)e^{ax-iy\nu} dx dy, \forall \nu \in \overset{\circ}{V}_\omega.$$

If $\overset{\circ}{J}_\omega = \{b\}$ and $\overset{\circ}{I}_\omega \neq \emptyset$, then there exists a function $h_b \in \mathcal{H}^\infty(\overset{\circ}{\Omega}_\omega)$ such that

$$\int_{\mathbb{R}^2} (Mf)(x, y)e^{-ix\mu+by} dx dy = h_b(\mu) \int_{\mathbb{R}^2} f(x, y)e^{-ix\mu+by} dx dy, \forall \mu \in \overset{\circ}{\Omega}_\omega.$$

2. CONSTRUCTION OF A SEQUENCE (f_m)

An important step in our proof is the following

Proposition 1. For every fixed pair $(\mu, \nu) \in \Omega_\omega \times V_\omega$ and every $\phi \in C_c(\mathbb{R}^2)$, we have

$$\left| \int_{\mathbb{R}^2} \phi(x, y)e^{-i\mu x - i\nu y} dx dy \right| \leq \|M_\phi\|.$$

Proof. In order to prove the proposition we need to construct a special sequence $(f_m)_{m \in \mathbb{N}}$ of functions. First consider the group $\{S_{t,0}\}$, $t \in \mathbb{R}$ defined on $L^2_{\omega_1}(\mathbb{R})$. This group is clearly strongly continuous and we denote by A the generator of $S_{t,0} = e^{tA}$. We have the estimate

$$\|S_{t,0}\| \leq C e^{\alpha_1 |t|}, \quad \forall t \in \mathbb{R}, \quad (2.1)$$

where $C > 0$ and $\alpha_1 \geq 0$ are constants. This follows from the fact that ω_1 is equivalent to a special weight ω_0 constructed in [1]. The norm of the operator $S_{t,0}$ in $L^2_{\omega_0}(\mathbb{R})$ satisfies the estimate

$$\|S_{t,0}\|_{\omega_0} \leq e^{\alpha_1 |t|}, \quad \forall t \in \mathbb{R}, \quad (2.2)$$

where α_1 is a constant and this leads to (2.1). The details of the proof of (2.2) are given in [4] and [1].

Let $\mu \in \mathbb{C}$ be such that $e^\mu \in \sigma(S_{1,0})$. We denote by $\langle, \rangle_{\omega_i}$, $i = 1, 2$, the scalar product in $L^2_{\omega_i}(\mathbb{R})$, $i = 1, 2$, and by $\|\cdot\|_{\omega_i}$, $i = 1, 2$, the norm in these spaces. Next we denote by \langle, \rangle_ω and $\|\cdot\|_\omega$ the scalar product and the norm in the space $L^2_\omega(\mathbb{R}^2)$. First, our aim is to prove that there exists a sequence $(f_m)_{m \in \mathbb{N}}$ of functions of $L^2_{\omega_1}(\mathbb{R})$ such that

$$\lim_{m \rightarrow +\infty} \langle (e^{At} - e^{\mu t})f_m, f_m \rangle_{\omega_1} = 0, \quad \forall t \in \mathbb{R} \quad (2.3)$$

$$\|f_m\|_{\omega_1} = 1, \quad \forall m \in \mathbb{N}. \quad (2.4)$$

Since $e^{\sigma(A)} \subset \sigma(S_{1,0})$, we have to deal with two cases:

- (i) $\mu \in \sigma(A)$,
- (ii) $\mu \notin \sigma(A)$.

In the case (i) we have $\mu \in \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$, where $\sigma_p(A)$ is the point spectrum, $\sigma_c(A)$ is the continuous spectrum and $\sigma_r(A)$ is the residual spectrum of A . If we have

$$\mu \in \sigma_p(A) \cup \sigma_c(A),$$

it is easy to see that there exists a sequence $(f_m) \in L^2_{\omega_1}(\mathbb{R})$ such that

$$\|(A - \mu)f_m\|_{\omega_1} \xrightarrow{m \rightarrow +\infty} 0, \quad \|f_m\|_{\omega_1} = 1, \quad \forall m \in \mathbb{N}.$$

Then the equality

$$(e^{At} - e^{\mu t})f_m = \left(\int_0^t e^{\mu(t-s)} e^{As} ds \right) (A - \mu)f_m,$$

yields

$$\|(e^{At} - e^{\mu t})f_m\|_{\omega_1} \xrightarrow{m \rightarrow +\infty} 0, \quad \forall t \in \mathbb{R}$$

and we deduce (2.3). If $\mu \notin \sigma_p(A) \cup \sigma_c(A)$, we have $\mu \in \sigma_r(A)$ and

$$\overline{\text{Im}(A - \mu I)} \neq L_{\omega_1}^2(\mathbb{R}).$$

Therefore there exists $h \in D(A^*)$, $\|h\|_{\omega_1} = 1$, such that

$$\langle f, (A^* - \bar{\mu})h \rangle_{\omega_1} = 0, \forall f \in D(A).$$

This implies $(A^* - \bar{\mu})h = 0$ and we take $f = h$. Then

$$\begin{aligned} & \langle (e^{At} - e^{\mu t})f, f \rangle_{\omega_1} = \langle f, (e^{A^*t} - e^{\bar{\mu}t})f \rangle_{\omega_1} \\ & = \left\langle f, \left(\int_0^t e^{\bar{\mu}(t-s)} e^{A^*s} ds \right) (A^* - \bar{\mu})f \right\rangle_{\omega_1} = 0. \end{aligned}$$

In this case we set

$$f_m = f, \forall m \in \mathbb{N}$$

and we get again (2.3).

The case (ii) is more difficult. First, we observe that if $\mu \notin \sigma(A)$, we have $e^\mu \notin e^{\sigma(A)}$, hence $e^\mu \in \sigma(e^A) \setminus e^{\sigma(A)}$. Let ω_0 be a special weight equivalent to ω_1 (see the beginning of the section). Without loss of generality we can consider that $\{S_{t,0}\}$ and A are operating on $L_{\omega_0}^2(\mathbb{R}^2)$. The spectrum of $S_{t,0}$ (resp. A) operating on $L_{\omega_1}^2(\mathbb{R}^2)$ is the same as the spectrum of $S_{t,0}$ (resp. A) operating on $L_{\omega_0}^2(\mathbb{R}^2)$. Then, taking into account (2.2), we are in situation to apply in $L_{\omega_0}^2(\mathbb{R}^2)$ the results for the spectrum of the semi-groups in a Hilbert space (see [2], [3]). Following these results, there exists a sequence of integers n_k converging to ∞ such that

$$\|(A - (\mu + 2\pi i n_k))^{-1}\|_{\omega_0} \geq \frac{2}{k}, \forall k \in \mathbb{N}.$$

Consequently, the equivalence of the norms corresponding to ω_1 and ω_0 shows that there exists a sequence (g_{n_k}) such that $\|g_{n_k}\|_{\omega_1} = 1$ and

$$\|(A - (\mu + 2\pi i n_k))^{-1}g_{n_k}\|_{\omega_1} \geq \frac{C}{k},$$

where $C > 0$ is a constant independent on k and we set

$$f_{n_k} = \frac{(A - (\mu + 2\pi i n_k)I)^{-1}g_{n_k}}{\|(A - (\mu + 2\pi i n_k)I)^{-1}g_{n_k}\|_{\omega_1}}.$$

This implies

$$\begin{aligned} & (e^{At} - e^{\mu+2\pi i n_k t})f_{n_k} = (e^{At} - e^{\mu t})f_{n_k} = \\ & \left(\int_0^t e^{(\mu+2\pi i n_k)(t-s)} e^{As} ds \right) (A - (\mu + 2\pi i n_k))f_{n_k} \end{aligned}$$

and we deduce

$$\|(e^{At} - e^{\mu t})f_{n_k}\| \xrightarrow[k \rightarrow +\infty]{} 0.$$

In this case we set $f_m = f_{n_m}$ and we arrange again the properties (2.3) and (2.4).

Let $\nu \in \mathbb{C}$ be such that $e^\nu \in \sigma(S_{0,1})$. Repeating the above argument, we construct a sequence (u_m) satisfying properties (2.3) and (2.4) replacing A by the generator B of the semi-group $\{S_{0,t}\}_{t \in \mathbb{R}}$ and μ by ν . Now, we write

$$\begin{aligned} & \left\langle \int_{\mathbb{R}^2} \phi(t, s) (e^{\mu t + \nu s} - e^{At + Bs}) f_m u_m dt ds, f_m u_m \right\rangle_\omega \\ &= \int_{\mathbb{R}} \left\langle u_m e^{\nu s} \int_{\mathbb{R}} \langle \phi(t, s) (e^{\mu t} - e^{At}) f_m, f_m \rangle_{\omega_1} dt, u_m \right\rangle_{\omega_2} ds \\ &+ \int_{\mathbb{R}} \left\langle \int_{\mathbb{R}} \langle \phi(t, s) (e^{\nu s} - e^{Bs}) u_m, u_m \rangle_{\omega_2} ds e^{At} f_m, f_m \right\rangle_{\omega_1} dt \\ &= \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

We have

$$\begin{aligned} & \left| \int_{\mathbb{R}} \langle \phi(t, s) (e^{\mu t} - e^{At}) f_m, f_m \rangle_{\omega_1} dt \right| \\ & \leq \|\phi\|_\infty \int_{\text{supp}(\phi(t, \cdot))} | \langle (e^{\mu t} - e^{At}) f_m, f_m \rangle_{\omega_1} | dt. \end{aligned}$$

The property

$$\langle (e^{\mu t} - e^{At}) f_m, f_m \rangle_{\omega_1} \xrightarrow[m \rightarrow +\infty]{} 0, \quad \forall t \in \mathbb{R}.$$

implies

$$\mathcal{I}_1 \xrightarrow[m \rightarrow +\infty]{} 0.$$

In the same way we get

$$\mathcal{I}_2 \xrightarrow[m \rightarrow +\infty]{} 0.$$

It is clear that

$$| \langle f_m u_m, f_m u_m \rangle_\omega | = 1, \quad \forall m \in \mathbb{N}.$$

Consequently, for every $\phi \in C_c(\mathbb{R}^2)$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \phi(t, s) e^{\mu t + \nu s} dt ds \right| \\ &= \left| \left\langle \int_{\mathbb{R}^2} \phi(t, s) e^{\mu t + \nu s} f_m u_m dt ds, f_m u_m \right\rangle_\omega \right| \\ &\leq \mathcal{I}_1 + \mathcal{I}_2 + \left| \left\langle \int_{\mathbb{R}^2} \phi(t, s) e^{At + Bs} f_m u_m dt ds, f_m u_m \right\rangle_\omega \right| \end{aligned}$$

and we conclude that

$$\left| \int_{\mathbb{R}^2} \phi(t, s) e^{\mu t + \nu s} dt ds \right| \leq \|M_\phi\|.$$

This completes the proof of Proposition 1. \square

3. PROOFS OF THE CONJECTURE (U)

Now we pass to the proof of the main theorem. For $E = L_\omega^2(\mathbb{R}^2)$ denote U_E by U_ω .

Proof of Theorem 3. Recall that

$$\sigma(S_{1,0}) = \left\{ z \in \mathbb{C}; \frac{1}{\rho(S_{-1,0})} \leq |z| \leq \rho(S_{1,0}) \right\} = e^{-i\Omega_\omega}$$

and

$$\sigma(S_{0,1}) = \left\{ z \in \mathbb{C}; \frac{1}{\rho(S_{0,-1})} \leq |z| \leq \rho(S_{0,1}) \right\} = e^{-iV_\omega}$$

(see [4]). Taking into account these results and applying directly Proposition 1, we get

$$\Omega_\omega \times V_\omega \subset U_\omega.$$

Let \mathcal{G} be the set of the continuous morphisms from \mathbb{R}^2 into \mathbb{C}^* . Set

$$\mathcal{U}_\omega = \left\{ \theta \in \mathcal{G}; \left| \int_{\mathbb{R}^2} \phi(x, y) \theta(x, y)^{-1} dx dy \right| \leq \|M_\phi\|, \forall \phi \in C_c(\mathbb{R}^2) \right\}.$$

There exists a trivial isomorphism between \mathcal{U}_ω and U_ω . Indeed,

$$\theta \in \mathcal{U}_\omega \Leftrightarrow \theta(x, y) = e^{i(z_1 x + z_2 y)}, \text{ for some } (z_1, z_2) \in U_\omega. \quad (3.1)$$

In [5] it is proved that if $\theta \in \mathcal{U}_\omega$, then there exists a character Γ_θ on the algebra of the translations such that

$$\Gamma_\theta(S_{x,y}) = \theta(x, y), \forall (x, y) \in \mathbb{R}^2. \quad (3.2)$$

From the theory of commutative Banach algebras with unit, it follows that joint spectrum of $S_{1,0}$ and $S_{0,1}$ is the set of pairs $(\chi(S_{1,0}), \chi(S_{0,1}))$, where χ is a character on the algebra of the translations ([6]). This definition is equivalent to the definition of the joint spectrum given in the Introduction. It is clear that the pair

$$(\theta(1, 0), \theta(0, 1)) = (\Gamma_\theta(S_{1,0}), \Gamma_\theta(S_{0,1}))$$

is in the joint spectrum of $S_{1,0}$ and $S_{0,1}$. Hence, the property (3.1) implies that

$$e^{-i\Omega_\omega} \times e^{-iV_\omega} = \sigma(S_{1,0}) \times \sigma(S_{0,1})$$

is a subset of the joint spectrum of $S_{1,0}$ and $S_{0,1}$. Since the joint spectrum of two operators is a subset of the product of their spectra, we conclude that the joint spectrum of $S_{1,0}$ and $S_{0,1}$ is

$$e^{-i\Omega_\omega} \times e^{-iV_\omega} = \sigma(S_{1,0}) \times \sigma(S_{0,1}).$$

This completes the proof of 2).

If $z = (z_1, z_2) \in U_\omega$, we get by (3.1) and (3.2)

$$(e^{-iz_1}, e^{-iz_2}) \in \sigma(S_{1,0}) \times \sigma(S_{1,0}) = e^{-i\Omega_\omega} \times e^{-iV_\omega}$$

and hence

$$U_\omega \subset \Omega_\omega \times V_\omega.$$

We conclude that

$$U_\omega = \Omega_\omega \times V_\omega$$

and the proof of 1) is completed.

□

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