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An Obstacle Control problem involving the $p$-Laplacian

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Abstract
In this paper, we consider the analogous of the obstacle problem in $H^1_0(\Omega)$, on the space $W^{1,p}_0(\Omega)$. We prove an existence and uniqueness of the result. In a second time, we define the optimal control problem associated. The results, here enclosed, generalize the one obtained by D.R. Adams, S. Lenhard in [1], [2] in the case $p = 2$.

Key words and phrases: Obstacle problem, Approximation, Optimal Pair.

1 Introduction
Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, whose boundary is $C^1$ piecewise. For $p > 1$ and for $\psi$ given in $W^{1,p}_0(\Omega)$, define

$$K(\psi) = \{v \in W^{1,p}_0(\Omega), v \geq \psi \text{ a.e. in } \Omega\}.$$ 

It is clear that $K(\psi)$ is a convex and weakly closed set in $L^p(\Omega)$. Let $p'$ be the conjugate of $p$, and $f \in L^{p'}(\Omega)$. We consider the following variational inequality called the obstacle problem:

$$\begin{cases}
    u \in K(\psi), \\
    \int_{\Omega} \sigma(u) \cdot \nabla (v - u) \, dx \geq \int_{\Omega} f(v - u) \, dx, \quad \forall v \in K(\psi),
\end{cases} \quad (1.1)$$

where $\sigma(u) = |\nabla u|^{p-2}\nabla u$. We shall say that $\psi$ is the obstacle and $f$ is the source term.

We begin to prove existence and uniqueness of a solution $u$ to (1.1), using variational formulation of the obstacle problem on the set $K(\psi)$. We shall then denote $u$ by: $u = T_f(\psi)$. Secondly, we characterize $T_f(\psi)$ as the lowest $f$—superharmonic function greater than $\psi$. 

1
2 Existence and uniqueness of the solution

**Proposition 1.** A function $u$ is a solution to the problem (1.1) if and only if $u$ satisfies the following:

\[
\begin{align*}
&u \in K(\psi), \\
&-\Delta_p u \geq f, \text{ a.e. in } \Omega, \\
&\int_\Omega \sigma(u) \cdot \nabla (\psi - u) \, dx = \int_\Omega f(\psi - u) \, dx.
\end{align*}
\] (2.1)

**Proof of Proposition 1.** Suppose that $u$ satisfies (1.1). Then taking $v = u + \varphi \in K(\psi)$ for $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, one gets that $-\Delta_p u \geq f$ in $\Omega$. Moreover, for $v = \psi$ and $v = 2\psi - u$, one gets that

\[
\int_\Omega \sigma(u) \cdot \nabla (\psi - u) \, dx = \int_\Omega f(\psi - u) \, dx,
\]

hence $u$ satisfies (2.1).

Conversely, let $u \in K(\psi)$ such that $-\Delta_p u \geq f$, let $v$ be in $K(\psi)$ and $\varphi_n \in \mathcal{D}(\Omega)$, $\varphi_n \geq 0$ such that $\varphi_n \to v - \psi$ in $W_0^{1,p}(\Omega)$. Then one gets

\[
\int_\Omega \sigma(u) \cdot \nabla (v - \psi) = \lim_{n \to \infty} \int_\Omega \sigma(u) \cdot \nabla \varphi_n
\]

\[
= \lim_{n \to \infty} \int_\Omega -\Delta_p u \varphi_n
\]

\[
\geq \lim_{n \to \infty} \int_\Omega f \varphi_n = \int_\Omega f(v - \psi), \forall v \in K(\psi).
\]

Using the last equality of (2.1), one gets that

\[
\int_\Omega \sigma(u) \cdot \nabla (v - u) \geq \int_\Omega f(v - u), \forall v \in K(\psi),
\]

hence $u$ satisfies (1.1). \hfill \Box

Let us prove now the existence and uniqueness of a solution to the obstacle problem (1.1).

**Proposition 2.** There exists a solution to (1.1), which can be obtained as the minimizer of the following minimization problem

\[
\inf_{v \in K(\psi)} I(v), \tag{2.2}
\]
where $I$ is the following energy functional

$$I(v) = \frac{1}{p} \int_{\Omega} |\nabla v|^p - \int_{\Omega} f v.$$  

**Proof of Proposition 2.** Using classical arguments in the calculus of variations, since $K(\psi)$ is a weakly closed convex set in $W_0^{1,p}(\Omega)$, and the functional $I$ is convex and coercive on $W_0^{1,p}(\Omega)$, then one obtains that there exists a solution $u$ to (2.2). □

**Proposition 3.** The inequation (1.1) possesses a unique solution.

**Proof of Proposition 3.** Suppose that $u_1, u_2 \in W_0^{1,p}(\Omega)$ are two solutions of the variational inequality (1.1)

$$u_i \in K(\psi) : \int_{\Omega} \sigma(u_i) \cdot \nabla(v - u_i) \, dx \geq \int_{\Omega} f(v - u_i) \, dx, \ \forall \ v \in K(\psi), \ i = 1, 2$$

Taking $v = u_1$ for $i = 2$ and $v = u_2$ for $i = 1$ and adding, we have

$$\int_{\Omega} [\sigma(u_1) - \sigma(u_2)] \cdot \nabla(u_1 - u_2) \leq 0.$$  

Recall that we have

$$\int_{\Omega} [\sigma(u_1) - \sigma(u_2)] \cdot \nabla(u_1 - u_2) \geq 0,$$

which implies that

$$\int_{\Omega} [\sigma(u_1) - \sigma(u_2)] \cdot \nabla(u_1 - u_2) = 0,$$

and then, $u_1 = u_2$ a.e in $\Omega$. □

Thus, we get the existence and uniqueness of a solution to (1.1).

**Definition 1.** We shall say that $u$ is $f$–superharmonic in $\Omega$, if $u \in W_0^{1,p}(\Omega)$ is a weak solution to $-\Delta_p u \geq f$, in the sense of distributions.

**Proposition 4.** A function $u$ is a solution of (1.1), if and only if $u$ is the lowest $f$–superharmonic function, greater than $\psi$.  

3
Proof of Proposition 4. Let \( u \) be a solution of (1.1) and \( v \) be an \( f \)-superharmonic function, greater than \( \psi \). Let \( \xi = \max(u, v), \xi \in K(\psi) \). Recalling that \( v^- = \sup(0, -v) \), one has then \( (\xi - u) = -(v - u)^- \). From (1.1), one gets
\[
\int_{\Omega} \sigma(u) \cdot \nabla(\xi - u) \geq \int_{\Omega} f(\xi - u).
\]
On the other hand, since \( \xi - u \leq 0 \) and \( -\Delta_p v \geq f \), we have
\[
\int_{\Omega} \sigma(v) \cdot \nabla(\xi - u) \leq \int_{\Omega} f(\xi - u).
\]
We obtain, subtracting the above two inequalities:
\[
\int_{\Omega} [\sigma(v) - \sigma(u)] \cdot \nabla(\xi - u) \leq 0,
\]
which implies that
\[
-\int_{\Omega} [\sigma(v) - \sigma(u)] \cdot \nabla(v - u)^- \leq 0,
\]
and then \( (v - u)^- = 0 \), or equivalently \( u \leq v \) in \( \Omega \).

Recall that we define by \( T_f(\psi) \) the lowest \( f \)-superharmonic function, greater than \( \psi \).

Lemma 1. The mapping \( \psi \mapsto T_f(\psi) \) is increasing.

Proof of Lemma 1. Let \( u_1 = T_f(\psi_1) \) and \( u_2 = T_f(\psi_2) \), which are respectively solutions to the following variational inequalities
\[
\left\{\begin{array}{l}
-\Delta_p u_i \geq f \\
u_i \geq \psi_i, \ i = 1, 2
\end{array}\right.
\]
and let \( \psi_1 \leq \psi_2 \). It is clear that \( u_2 \geq \psi_1 \). Hence \( u_2 \) is \( f \)-superharmonic and using Proposition 4, one obtains \( u_1 \leq u_2 \).

Proposition 5. The mapping \( \psi \mapsto T_f(\psi) \) is weak lower semicontinuous, in the sense that:

- If \( \psi_k \rightharpoonup \psi \) weakly in \( W_0^{1,p}(\Omega) \), then \( T_f(\psi) \leq \liminf_{k \to \infty} T_f(\psi_k) \).
- \( \int_{\Omega} |\nabla(T_f(\psi))|^p \leq \liminf_{k \to \infty} \int_{\Omega} |\nabla(T_f(\psi_k))|^p \).
Proof of Proposition 5. Let \((\psi_k)\) be a sequence in \(W^{1,p}_0(\Omega)\) which converges weakly in \(W^{1,p}_0(\Omega)\) to \(\psi\), and let \(\varphi_k = \min(\psi_k, \psi)\). Since \(T_f\) is increasing, one gets that \(T_f(\varphi_k) \leq T_f(\psi_k)\). We now prove that \(T_f(\varphi_k)\) converges strongly in \(W^{1,p}_0(\Omega)\) towards \(T_f(\psi)\). This will imply that

\[
T_f(\psi) = \lim_{k \to \infty} T_f(\varphi_k) \leq \liminf_{k \to \infty} T_f(\psi_k).
\]

We denote \(u_k = T_f(\varphi_k)\). It is clear that \(u_k\) is bounded in \(W^{1,p}_0(\Omega)\) since \(\varphi_k \leq \psi\). Hence for a subsequence, still denoted \(u_k\), there exists some \(u\) in \(W^{1,p}_0(\Omega)\) such that

\[
\nabla u_k \rightharpoonup \nabla u \text{ weakly in } L^p(\Omega), \ u_k \to u \text{ strongly in } L^p(\Omega).
\] (2.3)

On the other hand, using the fact that \(\varphi_k\) converges weakly to \(\psi\) in \(W^{1,p}_0(\Omega)\) (see Lemma 2 below), one gets the following assertion:

\[
u_k \geq \varphi_k \implies u \geq \psi.
\]

Let us prove now that \(u\) is a solution of the minimizing problem (2.2). For that aim, for \(v \in K(\psi)\), since \(v \geq \psi \geq \varphi_k\), we have

\[
\frac{1}{p} \int_{\Omega} |\nabla u|^p - \int_{\Omega} f u \leq \liminf_{k \to \infty} \frac{1}{p} \int_{\Omega} |\nabla u_k|^p - \int_{\Omega} f u_k \\
\leq \liminf_{k \to \infty} \inf_{w \geq \varphi_k} \left\{ \frac{1}{p} \int_{\Omega} |\nabla w|^p - \int_{\Omega} f w \right\} \\
\leq \frac{1}{p} \int_{\Omega} |\nabla v|^p - \int_{\Omega} f v.
\]

Then \(u\) realizes the infimum in (2.2). At the same time, since \(u \in K(\psi)\), one has the following convergence

\[
\frac{1}{p} \int_{\Omega} |\nabla u_k|^p - \int_{\Omega} f u_k \to \frac{1}{p} \int_{\Omega} |\nabla u|^p - \int_{\Omega} f u, \text{ when } k \to \infty,
\]

which implies that \(u_k\) converges strongly to \(u\) in \(W^{1,p}_0(\Omega)\). We can conclude that \(T_f(\varphi_k)\) converges strongly to \(T_f(\psi)\). \(\square\)

**Lemma 2.** Suppose that \(\psi_k\) converges weakly to some \(\psi\) in \(W^{1,p}_0(\Omega)\). Then, \(\varphi_k = \min(\psi_k, \psi)\) converges weakly to \(\psi\) in \(W^{1,p}_0(\Omega)\).
Proof of Lemma 2. We have

$$\psi_k \rightharpoonup \psi \quad \text{in} \quad L^p(\Omega).$$

Then

$$\varphi_k = \frac{\psi_k + \psi - |\psi_k - \psi|}{2} \rightharpoonup \psi \quad \text{in} \quad L^p(\Omega).$$

Let us prove now that $|\nabla \varphi_k|$ is bounded in $L^p(\Omega)$. For that aim, we write

$$\int_\Omega |\nabla \varphi_k|^p = \int_\Omega \left| \nabla \left( \frac{\psi_k + \psi - |\psi_k - \psi|}{2} \right) \right|^p \leq C_p \left( \int_\Omega |\nabla \psi_k|^p + \int_\Omega |\nabla \psi|^p \right).$$

Therefore the sequence $\varphi_k$ is bounded in $W^{1,p}_0(\Omega)$, so it converges weakly, up to a subsequence, to $\psi$ in $W^{1,p}_0(\Omega)$. □

Proposition 6. The mapping $T_f$ is an involution, i.e. $T_f^2 = T_f$.

Proof of Proposition 6. Up to replacing $\psi$ by $u$ in the variational inequalities (1.1), and using proposition 4, one gets that $u = T_f(u)$. Then, we conclude that $T_f^2(\psi) = T_f(\psi)$. □

3 A method of penalization

Let $\mathcal{M}^+(\Omega)$ be the set of all nonnegative Radon measures on $\Omega$ and $W^{-1,p'}(\Omega)$ be the dual space of $W^{1,p}(\Omega)$ on $\Omega$ where $p'$ is the conjugate of $p$ $(1 < p < \infty)$. Suppose that $u$ solves (1.1). Using the fact that a nonnegative distribution on $\Omega$ is a nonnegative measure on $\Omega$ (cf. [7]), one gets the existence of $\mu \geq 0$, $\mu \in \mathcal{M}^+(\Omega)$, such that

$$\int_\Omega \sigma(u) \cdot \nabla \Phi \, dx - \int_\Omega f \Phi \, dx = \langle \mu, \Phi \rangle, \quad \forall \Phi \in \mathcal{D}(\Omega), \quad (3.1)$$

that we shall also write $-\Delta_p u = f + \mu, \quad \mu \geq 0$ in $\Omega$.

Let us introduce

$$\beta(x) = \begin{cases} 0, & x > 0, \\ x, & x \leq 0. \end{cases} \quad (3.2)$$

Clearly, $\beta$ is $C^1$ piecewise, $\beta(x) \leq 0$ and is nondecreasing. Let us consider, for some $\delta > 0$, the following semilinear elliptic equation:

$$\begin{cases} -\Delta_p u + \frac{1}{\delta} \beta(u - \psi) = f, & \text{in} \ \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \quad (3.3)$$
We have the following existence result:

**Theorem 1.** For any given \( \psi \in W^{1,p}_0(\Omega) \) and \( \delta > 0 \), (3.3) possesses a unique solution \( u^\delta \). Moreover,

1. \( u^\delta \rightharpoonup u \) strongly in \( W^{1,p}_0(\Omega) \), as \( \delta \to 0 \), with \( u := T_f(\psi) \).

2. There exists a unique \( \mu \in W^{-1,p'}(\Omega) \cap M^+(\Omega) \) such that:
   
   (i) \(-\frac{1}{\delta} \beta(u^\delta - \psi) \to \mu \) in \( W^{-1,p'}(\Omega) \cap M^+(\Omega) \).
   
   (ii) \( \langle \mu, T_f(\psi) - \psi \rangle = 0 \).

**Proof of Theorem 1.** (1) Let \( B \) be defined as \( B(r) = \int_0^r \beta(s)ds, \forall r \in \mathbb{R} \).

We introduce the following variational problem

\[
\inf_{v \in W^{1,p}_0(\Omega)} \left\{ \frac{1}{p} \int_\Omega |\nabla v|^p + \frac{1}{\delta} \int_\Omega B(v - \psi) - \int_\Omega f v \right\}.
\]  

(3.4)

The functional in (3.4) is coercive, strictly convex and continuous. As a consequence it possesses a unique solution \( u^\delta \in W^{1,p}_0(\Omega) \). Since \( B(0) = 0 \), one has

\[
\frac{1}{p} \int_\Omega |\nabla u^\delta|^p + \frac{1}{\delta} \int_\Omega B(u^\delta - \psi) - \int_\Omega f u^\delta \leq \frac{1}{p} \int_\Omega |\nabla \psi|^p - \int_\Omega f \psi,
\]

since \( B \geq 0 \), then \( u^\delta \) is bounded in \( W^{1,p}_0(\Omega) \). Extracting from \( u^\delta \) a subsequence, there exists \( u \) in \( W^{1,p}_0(\Omega) \), such that

\[
\nabla u^\delta \rightharpoonup \nabla u \text{ weakly in } L^p(\Omega), \quad u^\delta \to u \text{ strongly in } L^p(\Omega).
\]

Using \( \frac{1}{\delta} \int_\Omega B(u^\delta - \psi) \leq C \) and the continuity of \( B \) one has

\[
0 \leq \int_\Omega B(u - \psi) \leq \liminf_{\delta \to 0} \int_\Omega B(u^\delta - \psi) = 0,
\]

hence \( u \in K(\psi) \).

We want to prove now that \( u \) solves (1.1). Let \( v \in K(\psi) \), since \( B(r) \geq \)]
\begin{align*}
0, \forall \ r \in \mathbb{R} \ 	ext{one gets:} \\
\frac{1}{p} \int_{\Omega} |\nabla u|^p - \int_{\Omega} f u \leq \liminf_{\delta \to 0} \left( \frac{1}{p} \int_{\Omega} |\nabla u_{\delta}|^p - \int_{\Omega} f u_{\delta} \right) \\
\leq \liminf_{\delta \to 0} \left( \frac{1}{p} \int_{\Omega} |\nabla u_{\delta}|^p + \frac{1}{\delta} \int_{\Omega} B(u_{\delta} - \psi) - \int_{\Omega} f u_{\delta} \right) \\
\leq \liminf_{\delta \to 0} \inf_{u \geq \psi} \left\{ \frac{1}{p} \int_{\Omega} |\nabla u|^p + \frac{1}{\delta} \int_{\Omega} B(u - \psi) - \int_{\Omega} f u \right\} \\
\leq \frac{1}{p} \int_{\Omega} |\nabla v|^p - \int_{\Omega} f v.
\end{align*}

Then, one concludes that \( \nabla u_{\delta} \to \nabla u \) strongly in \( L^p(\Omega) \) and since \( u \in K(\psi) \), then \( u \) solves (1.1).

(2) (i) let \( u_{\delta} \) be the solution of (3.3), since \( \nabla u_{\delta} \) is uniformly bounded in \( L^p(\Omega) \) by some constant \( C \), we get that \( -\Delta_{\mu} u_{\delta} - f \) is bounded in \( W^{-1,p'}(\Omega) \), so it converges weakly, up to a subsequence, in \( W^{-1,p'}(\Omega) \). Hence, \( -\frac{1}{\delta} \beta(u_{\delta} - \psi) \) converges too, up to a subsequence, in \( W^{-1,p'}(\Omega) \), and we have

\(-\frac{1}{\delta} \beta(u_{\delta} - \psi) \rightharpoonup \mu \) weakly in \( W^{-1,p'}(\Omega) \),

where \( \mu \) is a positive distribution, hence a positive measure. Then, by (1), we see that \( u \) and \( \mu \) are linked by the relation (3.1).

We now prove (ii): let \( u \) be the solution of (1.1). Taking \( \varphi = (\psi - u) \in W^{0,p}(\Omega) \) in the above inequalities, one gets

\(-\frac{1}{\delta} \int_{\Omega} \beta(u_{\delta} - \psi) \ (u - \psi) \ dx \leq ||\nabla u_{\delta}||_{p^{-1}} ||\nabla(\psi - u)||_p + ||f||_{p'} ||\psi - u||_p.\)

Since \( u \in K(\psi) \), passing to the limit we obtain:

\[ (\mu, \psi - u) = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (\psi - u) - \int_{\Omega} f(\psi - u) = 0, \text{ by } (2.1) \]

Then (ii) follows. \( \Box \)

4 Optimal Control for a Non-Positive Source Term

4.1 Optimal control for a non positive source term

Proposition 7. Let \( f, \psi \) and \( T_f(\psi) \) be as in (1.1). One has

\[ \frac{1}{p} \int_{\Omega} |\nabla T_f(\psi)|^p \leq \frac{1}{p} \int_{\Omega} |\nabla \psi|^p \ dx + \int_{\Omega} f[T_f(\psi) - \psi] \ dx. \]
Proof of Proposition 7. From (1.1) taking \( v = \psi \) and using Hölder’s inequality, we have
\[
\int_{\Omega} |\nabla T_f(\psi)|^p \leq \frac{p-1}{p} \|\nabla T_f(\psi)\|^p + \frac{1}{p} \|\nabla \psi\|^p + \int_{\Omega} f[T_f(\psi) - \psi] \, dx.
\]
\[
\square
\]
Note that since \( T_f(\psi) \geq \psi \), it follows that if \( f \leq 0 \), then
\[
\int_{\Omega} |\nabla T_f(\psi)|^p \, dx \leq \int_{\Omega} |\nabla \psi|^p \, dx. \tag{4.1}
\]
Let us now introduce the following problem, said “optimal control problem”:
\[
\inf_{\tilde{\psi} \in W^{1,p}_0(\Omega)} J_f(\tilde{\psi}), \tag{4.2}
\]
where
\[
J_f(\tilde{\psi}) = \frac{1}{p} \int_{\Omega} \left\{ |T_f(\tilde{\psi}) - z|^p + |\nabla \tilde{\psi}|^p \right\} \, dx, \tag{4.3}
\]
for some given \( z \in L^p(\Omega) \). \( z \) is said to be the initial profile, \( \psi \) is the control variable and \( T_f(\psi) \) is the state variable. The pair \( (\psi^*, T_f(\psi^*)) \) where \( \psi^* \) is a solution for (4.2) is called an optimal pair and \( \psi^* \) an optimal control.

In this section, we establish the existence and uniqueness of the optimal pair in the case where \( f \leq 0 \).

Theorem 2. If \( f \in L^p(\Omega), \ f \leq 0 \) on \( \Omega \), then there exists a unique optimal control \( \psi^* \in W^{1,p}_0(\Omega) \) for (4.2). Moreover, the corresponding state \( u^* \) coincides with \( \psi^* \), i.e. \( T_f(\psi^*) = \psi^* \).

Proof of Theorem 2. In a first time we prove that there exists a pair of solutions of the form \( (u^*, u^*) \), hence \( (u^* = T_f(u^*)) \). Let \( (\psi_k)_k \) be a minimizing sequence for (4.3), then \( T_f(\psi_k) \) is bounded in \( W^{1,p}(\Omega) \), therefore \( T_f(\psi_k) \) converges for a subsequence towards some \( u^* \in W^{1,p}_0(\Omega) \). Moreover, using the lower semicontinuity of \( T_f \) as in proposition 5, one gets
\[
T_f(u^*) \leq \liminf_{k \to \infty} T_f(T_f(\psi_k)) \leq \lim_{k \to \infty} T_f(\psi_k) = u^*,
\]
and by the definition of \( T_f \), \( T_f(u^*) \geq u^* \). Hence \( u^* = T_f(u^*) \).
We prove that \((u^*, u^*)\) is an optimal pair. Using proposition 5, by the lower semicontinuity in \(W^{1,p}_0(\Omega)\) of \(T_f\):

\[
J_f(u^*) = \frac{1}{p} \int_{\Omega} \{|u^* - z|^p + |\nabla u^*|^p\} \, dx \\
\leq \liminf_{k \to \infty} \frac{1}{p} \int_{\Omega} \{|T_f(\psi_k) - z|^p + |\nabla \psi_k|^p\} \, dx \\
= \inf_{\psi \in W^{1,p}_0(\Omega)} J_f(\psi).
\]

Secondly, we prove that every optimal pair is of the form \((u^*, u^*)\). Observe that if \((\psi^*, T_f(\psi^*))\) is a solution then \((T_f(\psi^*), T_f(\psi^*))\) is a solution. Indeed

\[
\int_{\Omega} \{|T_f(\psi^*) - z|^p + |\nabla T_f(\psi^*)|^p\} \, dx \leq \int_{\Omega} \{|T_f(\psi^*) - z|^p + |\nabla \psi^*|^p\} \, dx.
\]

So

\[
\int_{\Omega} |\nabla T_f(\psi^*)|^p \, dx = \int_{\Omega} |\nabla \psi^*|^p \, dx, \quad (4.4)
\]

by inequality (4.1), using the Hölder’s inequality, one obtains then

\[
0 \leq \int_{\Omega} f(\psi^* - T_f(\psi^*)) \, dx \\
\leq \int_{\Omega} \sigma(T_f(\psi^*)) \cdot |\nabla(\psi^* - T_f(\psi^*))| \, dx \\
\leq \int_{\Omega} |\nabla T_f(\psi^*)|^p |\nabla(\psi^*)|^p - \int_{\Omega} |\nabla T_f(\psi^*)|^p \\
\leq \left( \int_{\Omega} |\nabla T_f(\psi^*)|^p \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |\nabla T_f(\psi^*)|^p \right)^{\frac{1}{p}} - \int_{\Omega} |\nabla T_f(\psi^*)|^p = 0,
\]

which implies

\[
\int_{\Omega} |\nabla T_f(\psi^*)|^p |\nabla(\psi^*)|^p - \int_{\Omega} |\nabla T_f(\psi^*)|^p = 0.
\]

Let us recall that by convexity, one has the following inequality

\[
\frac{1}{p} \int_{\Omega} |\nabla \psi^*|^p + \frac{p-1}{p} \int_{\Omega} |\nabla T_f(\psi^*)|^p - \int_{\Omega} |\nabla T_f(\psi^*)|^p \cdot |\nabla \psi^*| \geq 0.
\]

Then the equality holds and by the strict convexity, one gets \(\nabla(\psi^*) = \nabla(T_f(\psi^*))\) a.e., hence \(\psi^* = T_f(\psi^*)\). Finally, we deduce from the two previous steps that the pair is unique. Suppose that \((u_1, u_1)\) and \((u_2, u_2)\) are
two solutions, and consider \((\frac{u_1 + u_2}{2}, T_f(\frac{u_1 + u_2}{2}))\). We prove that it is also a solution. Indeed:

\[
\int_{\Omega} \left| \frac{u_1 + u_2}{2} - z \right|^p + \left| \nabla T_f(\frac{u_1 + u_2}{2}) \right|^p \, dx
\]

\[
\leq \int_{\Omega} \left| \frac{u_1 + u_2}{2} - z \right|^p + \left| \nabla \left( \frac{u_1 + u_2}{2} \right) \right|^p \, dx
\]

\[
\leq \frac{1}{2} (J_f(u_1) + J_f(u_2)) = \inf_{\psi \in W^{1,p}_0(\Omega)} J_f(\psi),
\]

which implies that \(u_1 = u_2\). Thus, the uniqueness of the optimal pair for \(f \leq 0\) holds.

\[\square\]

### 4.2 Optimal control for a nonnegative source term

We are interested here to the case \(f \geq 0\) on \(\Omega\). In what follows we will denote by \(Gf\) the unique function in \(W^{1,p}_0(\Omega)\) which verifies

\[
\begin{cases}
-\Delta_p(Gf) = f, & \text{in } \Omega \text{ a.e.} \\
Gf = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where \(f \in L^{p'}(\Omega)\) and \(Gf \in W^{1,p}_0(\Omega)\).

**Theorem 3.** Suppose that \(f \in L^{p'}(\Omega)\) is a nonnegative function. Suppose that \(z \in L^p(\Omega)\), satisfying \(z \leq Gf\) a.e on \(\Omega\). Then the minimizing problem (4.2) has a unique optimal pair \((0, Gf)\).

**Lemma 3.** Let \(T_f(\psi)\) be a solution to (1.1) and \(Gf\) defined as above. Then \(T_f(\psi)\) is greater than \(Gf\).

**Proof of Lemma 3.** We have that \(-\Delta_p(Gf) = f\), and \(T_f(\psi)\) realizes \(-\Delta_p(T_f(\psi)) \geq f\). Then, by the Comparison Theorem for \(-\Delta_p\) we get that \(Gf \leq T_f(\psi)\). \(\square\)

**Proof of Theorem 3.** In a first time we prove that \((0, Gf)\) is an optimal pair. Indeed, for all \(\psi \in W^{1,p}_0(\Omega)\)

\[
J_f(\psi) = \frac{1}{p} \int_{\Omega} \{|Gf - z + T_f(\psi) - Gf|^p + |\nabla \psi|^p\}
\]

\[
\geq \frac{1}{p} \int_{\Omega} \{|Gf - z|^p + p|Gf - z|^{p-2}(Gf - z)(T_f(\psi) - Gf)\}
\]

\[
\geq \frac{1}{p} \int_{\Omega} \{|Gf - z|^p\}
\]

\[= J_f(0).
\]

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The equality with \((\psi^*, T_f(\psi^*))\) implies that we have equality in each step, so we get \(\|\nabla \psi^*\|_p = 0\), then \(\psi^* = 0\) a.e. in \(\Omega\). Thus, \((0, Gf)\) is the unique optimal control pair.  

References


