Lie group structures on groups of smooth and holomorphic maps on non-compact manifolds

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Abstract. We study Lie group structures on groups of the form $C^\infty(M,K)$, where M is a non-compact smooth manifold and K is a, possibly infinite-dimensional, Lie group. First we prove that there is at most one Lie group structure with Lie algebra $C^\infty(M,\mathfrak{k})$ for which the evaluation map is smooth. We then prove the existence of such a structure if the universal cover of K is diffeomorphic to a locally convex space and if the image of the left logarithmic derivative in $\Omega^1(M,\mathfrak{k})$ is a smooth submanifold, the latter being the case in particular if M is one-dimensional. We also obtain analogs of these results for the group $\mathcal{O}(M,K)$ of holomorphic maps on a complex manifold with values in a complex Lie group. We show that there exists a natural Lie group structure on $\mathcal{O}(M,K)$ if K is Banach and M is a non-compact complex curve with finitely generated fundamental group.
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Introduction

If M is a finite-dimensional manifold (possibly with boundary) and K a Lie group (modeled on a locally convex space), then the group $C^\infty(M,K)$ of smooth maps with values in K has a natural group topology, called the *smooth compact open topology*. If, in addition, M is a complex manifold without boundary and K is a complex Lie group, then $C^\infty(M,K)$ contains the subgroup $\mathcal{O}(M,K)$ of holomorphic maps, on which the smooth compact open topology simply coincides with the compact open topology if M has no boundary. In this paper we discuss the question when the topological groups $C^\infty(M,K)$, resp. $\mathcal{O}(M,K)$, carry Lie group structures.

If $K=E$ is a locally convex space, then $C^\infty(M,E)$ also is a locally convex space, hence a Lie group. It is also well-known that if M is compact, then $C^\infty(M,K)$ carries a natural Lie group structure (cf. [Mi80], [Gl02a]; and [Wo06] for manifolds with boundary). In this case one obtains charts of $C^\infty(M,K)$ by composing with charts of K. This does no longer work for non-compact manifolds. A necessary condition for a topological group G to possess a compatible Lie group structure is that it is locally contractible, which implies in particular that the arc-component G_a of the identity is open. In some cases we shall prove that the topological group $C^\infty(M,K)$ is not a Lie group by showing that the latter condition fails.

In general we cannot expect the group $C^\infty(M,K)$ to carry a Lie group structure, but any "reasonable" Lie group structure on this group should have the property that for any smooth manifold N, a map $f\colon N\to C^\infty(M,K)$ is smooth if and only if the corresponding map

$$f^\wedge\colon N\times M\to K,\quad f^\wedge(n,m):=f(n)(m)$$

is smooth. We thus start our investigation in Section I with a characterization of smooth maps $N\times M\to K$ in terms of data associated to the group $G:=C^\infty(M,K)$. This leads to the main result of Section I, that for any regular Lie group K the group G carries at most one regular Lie group structure with Lie algebra $\mathfrak{g}=C^\infty(M,\mathfrak{k})$ for which all evaluation maps $\mathrm{ev}_m\colon C^\infty(M,K)\to K$ are smooth with $\mathbf{L}(\mathrm{ev}_m)=\mathrm{ev}_m$. We call such a Lie group structure *compatible with evaluations*.

From what we have said above, it easily follows that such Lie group structures exist if M is compact (Theorem I.3) or if the universal covering group $\widetilde{K}$ of K is diffeomorphic to a locally convex space (Theorem IV.2), which is the case in particular if K is regular abelian or finite-dimensional solvable. If $\widetilde{K}$ is diffeomorphic to a locally convex space E, the Lie group $C^\infty(M,\widetilde{K})$ is diffeomorphic to the locally convex space $C^\infty(M,E)$ and the underlying topology coincides with the smooth compact open topology. If K is not simply connected, the Lie topology might be finer than the smooth compact open topology, but both coincide on the arc-component of the identity, which need not be open in the smooth compact open topology. In Section IV we take a closer look at this subtle situation and show that if, f.i., K is finite-dimensional solvable, then both topologies coincide if and only if the group $H^1(M,\mathbb{Z})$ is finitely generated (Remark IV.13). We also derive analogous results for the group $\mathcal{O}(M,K)$ of holomorphic maps on a complex manifold M with values in a complex Lie group K.

Clearly, the condition that $\widetilde{K}$ is diffeomorphic to a vector space is quite restrictive. To find weaker sufficient conditions for the existence of Lie group structures, we study in Section II the left logarithmic derivative

$$\delta\colon C^\infty(M,K)\to \mathrm{MC}(M,\mathfrak{k}) := \{\alpha\in\Omega^1(M,\mathfrak{k})\colon d\alpha+\tfrac{1}{2}[\alpha,\alpha]=0\},\quad f\mapsto f^{-1}.df.$$

We show in Proposition II.1 that for any regular Lie group K with Lie algebra $\mathfrak{k}$, any connected manifold M and any $m_0\in M$, it maps the subgroup

$$C^\infty_*(M,K) := \{f\in C^\infty(M,K)\colon f(m_0)=\mathbf{1}\}$$

of based maps homeomorphically onto its image, which is characterized by the Fundamental Theorem (Theorem I.5). If $\mathrm{im}(\delta)$ carries a natural manifold structure, we thus obtain a manifold structure on the group $C^\infty_*(M,K)$ and hence on $C^\infty(M,K)\cong K\ltimes C^\infty_*(M,K)$ a regular Lie group structure compatible with evaluations (Theorem II.2). If M is 1-connected, this is the case if K is abelian, M is one-dimensional (cf. [KM97]) or for holomorphic maps on complex curves.

If M is not simply connected, the situation is more complicated. If K is abelian, we always have a regular Lie group structure on $C^\infty(M,K)$ since $\widetilde{K}$ is a vector space, but it is compatible with the smooth compact open topology only if $H^1(M,\mathbb{Z})$ is finitely generated (Theorem IV.8). If M is real 1-dimensional, then it either is compact or simply connected, but the situation becomes interesting if M is a complex curve which is not simply connected. It turns out that in this situation one can show that $\delta(\mathcal{O}(M,K))$ is a complex submanifold of the Fréchet space $\Omega^1_h(M,\mathfrak{k})$ of holomorphic $\mathfrak{k}$-valued 1-forms on M whenever K is a Banach–Lie group and $\pi_1(M)$ is finitely generated. The key tool in our argument is Glöckner's Implicit Function Theorem ([Gl03]) for smooth maps on locally convex spaces with values in Banach spaces, which is used to take care of the period conditions.

We collect some of our main results in the following two theorems:

Theorem 1. *Let K be a connected regular real Lie group and M a real finite-dimensional connected σ-compact manifold. Then the group $C^\infty(M,K)$ carries a Lie group structure compatible with evaluations if*

(1) *$\widetilde{K}$ is diffeomorphic to a locally convex space. If, in addition, $\pi_1(M)$ is finitely generated, the Lie group structure is compatible with the smooth compact open topology* (Theorem IV.2).
(2) $\dim M=1$ (Corollary II.3).
(3) *$M\cong\mathbb{R}^k\times C$, where C is compact* (Corollary II.8). ∎

For complex groups and holomorphic maps we have:

Theorem 2. *Let K be a regular complex Lie group and M a finite-dimensional connected σ-compact complex manifold without boundary. Then the group $\mathcal{O}(M,K)$ carries a Lie group structure with Lie algebra $\mathcal{O}(M,\mathfrak{k})$ compatible with evaluations if*

(1) *$\widetilde{K}$ is diffeomorphic to a locally convex space. If, in addition, $\pi_1(M)$ is finitely generated, the Lie group structure is compatible with the compact open topology* (Theorem IV.2).
(2) $\dim_{\mathbb{C}} M=1$, *$\pi_1(M)$ is finitely generated and K is a Banach–Lie group* (Theorem III.11). ∎

Actually (2) in the preceding theorem was the original source of motivation for this work. It provides in particular a Lie theoretic environment for Lie groups associated to Krichever-Novikov Lie algebras which form an interesting generalization of affine Kac–Moody algebras (cf. [Sch03]).

If M is a σ-compact finite-dimensional manifold and $M = \bigcup_n M_n$ is an exhaustion of M by compact submanifolds M_n with boundary, then the group $C^\infty(M,K)$ can be identified with the projective limit $\varprojlim C^\infty(M_n,K)$, where the connecting maps are given by restriction. Since each group $C^\infty(M_n,K)$ carries a natural Lie group structure, the topological group $C^\infty(M,K)$ is a projective limit of Lie groups. From this point of view, the present paper deals with Lie group structures on certain projective limits of infinite-dimensional Lie groups. For projective limits of finite-dimensional Lie groups, the corresponding problem has been solved completely in [HoNe06].

The paper is structured as follows. Section I contains generalities on smooth maps with values in Lie groups and the aforementioned uniqueness result on Lie group structures with smooth evaluation map (Corollary I.10). In Section II we exploit the method to obtain Lie group structures on $C^\infty(M,K)$ by submanifold structures on $\mathrm{im}(\delta)$ and in Section III we transfer this method to groups of holomorphic maps. Target groups K whose universal cover is diffeomorphic to a locally convex space are discussed in Section IV. We conclude with a short section on strange properties of the exponential map of groups of holomorphic maps on non-compact manifolds and an appendix with some technical tools necessary to deal with manifolds of smooth and holomorphic maps.

Preliminaries

Let X and Y be locally convex topological vector spaces, $U \subseteq X$ open and $f\colon U \to Y$ a map. Then the *derivative of f at x in the direction of h* is defined as

$$df(x)(h) := \lim_{t\to 0} \frac{1}{t}\big(f(x+th) - f(x)\big)$$

whenever the limit exists. The function f is called *differentiable at x* if $df(x)(h)$ exists for all $h \in X$. It is called *continuously differentiable or C^1* if it is continuous and differentiable at all points of U and

$$df\colon U \times X \to Y, \quad (x,h) \mapsto df(x)(h)$$

is a continuous map. It is called a C^n-map if it is C^1 and df is a C^{n-1}-map, and C^∞ (or *smooth*) if it is C^n for all $n \in \mathbb{N}$. This is the notion of differentiability used in [Mil84], [Ha82] and [Gl02b], where the latter reference deals with the modifications needed for incomplete spaces. If X and Y are complex, f is called *holomorphic* if it is smooth and its differentials $df(x)$ are complex linear. If Y is Mackey complete, it suffices that f is C^1.

Since we have a chain rule for C^1-maps between locally convex spaces, we can define smooth manifolds M as in the finite-dimensional case. A chart (φ, U) with respect to a given manifold structure on M is an open set $U \subset M$ together with a homeomorphism φ onto an open set of the model space. An atlas for the tangent bundle TM is obtained directly from an atlas of M, but we do not consider the cotangent bundle as a manifold because this requires to choose a topology on the dual spaces, for which there are many possibilities. Nonetheless, there is a natural concept of a smooth k-form on M. If E is a locally convex space, then an *E-valued k-form* ω on M is a function ω which associates to each $p \in M$ a k-linear alternating map $T_p(M)^k \to E$ such that in local coordinates the map $(p, v_1, \dots, v_k) \mapsto \omega(p)(v_1, \dots, v_k)$ is smooth. We write $\Omega^k(M,E)$ for the space of smooth k-forms on M with values in E. The differentials

$$d\colon \Omega^k(M,E) \to \Omega^{k+1}(M,E)$$

are defined by the same formula as in the finite-dimensional case (cf. [Beg87]).

If M is a smooth manifold modeled on the locally convex space E, a subset $N \subseteq M$ is called a *submanifold* of M if there exists a closed subspace $F \subseteq E$ and for each $n \in N$ there

exists an E-chart (φ, U) of M with $n \in U$ and $\varphi(U \cap N) = \varphi(U) \cap F$. The submanifold N is called a *split submanifold* if, in addition, there exists a subspace $G \subseteq E$ for which the addition map $F \times G \to E, (f, g) \mapsto f + g$ is a topological isomorphism.

M is a smooth manifold modeled on the locally convex space E with boundary ∂M in case the $m \in \partial M$ have smooth charts (φ, U) to open neighborhoods $\varphi(U)$ of the boundary of a half space of E. M is said to have corners in case corner points have smooth charts to open neighborhoods of the vertex of a quadrant in E. Boundaries and the set of corners may be empty, and M reduces in this case to an ordinary manifold. For a complex manifold M, the boundary is always supposed to be empty.

A *Lie group* G is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write $\mathbf{1} \in G$ for the identity element and $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on G. Then each $x \in T_{\mathbf{1}}(G)$ corresponds to a unique left invariant vector field x_l with $x_l(g) := d\lambda_g(\mathbf{1}).x, g \in G$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_{\mathbf{1}}(G)$ a continuous Lie bracket which is uniquely determined by $[x, y]_l = [x_l, y_l]$ for $x, y \in \mathfrak{g}$. The *Maurer–Cartan form* $\kappa_G \in \Omega^1(G, \mathfrak{g})$ is the unique left invariant 1-form on G with $\kappa_{G,\mathbf{1}} = \mathrm{id}_{\mathfrak{g}}$, i.e., $\kappa_G(x_l) = x$ for each $x \in \mathfrak{g}$. We write $q_G \colon \widetilde{G}_0 \to G_0$ for the universal covering map of the identity component G_0 of G and identify the discrete central subgroup $\ker q_G$ of $\widetilde{G}_0$ with $\pi_1(G) \cong \pi_1(G_0)$.

In the following we always write $I = [0, 1]$ for the unit interval in $\mathbb{R}$. A Lie group G is called *regular* if for each $\xi \in C^\infty(I, \mathfrak{g})$, the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = \gamma(t).\xi(t) = T(\lambda_{\gamma(t)})\xi(t)$$

has a solution $\gamma_\xi \in C^\infty(I, G)$, and the evolution map

$$\mathrm{evol}_G \colon C^\infty(I, \mathfrak{g}) \to G, \quad \xi \mapsto \gamma_\xi(1)$$

is smooth (cf. [Mil84]). We then also write

$$\mathrm{Evol}_G \colon C^\infty(I, \mathfrak{g}) \to C^\infty(I, G), \quad \xi \mapsto \gamma_\xi,$$

and recall that this is a smooth map if $C^\infty(I, G)$ carries its natural Lie group structure (cf. Theorem I.3 and Lemma A.5 below). For a locally convex space E, the regularity of the Lie group $(E, +)$ is equivalent to the Mackey completeness of E, i.e., to the existence of integrals of smooth curves $\gamma \colon I \to E$. We also recall that for each regular Lie group G, its Lie algebra $\mathfrak{g}$ is regular and that all Banach–Lie groups are regular ([GN07]). The evolution map evol_G is supposed to be holomorphic for a complex regular Lie group G.

Throughout this paper, K denotes a regular Lie group.

I. Generalities on groups of smooth maps and regular Lie groups

In this section we introduce the natural group topology on the group $G = C^\infty(M, K)$ of smooth maps from a manifold M with values in a Lie group K. We then describe some technical tools to deal with nonlinear maps between spaces of smooth maps and differential forms which leads to a characterization of maps $f \colon N \to G$ for which the corresponding map $f^\wedge \colon N \times M \to K$ is smooth (Proposition I.8). This in turn is used to show that G carries at most one regular Lie group structure compatible with evaluations (Corollary I.10).

Definition I.1. (Groups of differentiable maps as topological groups) (a) If X and Y are topological spaces, then the *compact open topology* on the space $C(X, Y)$ is defined as the topology generated by the sets of the form

$$W(K, U) := \{f \in C(X, Y) \colon f(K) \subseteq U\},$$

where K is a compact subset of X and U an open subset of Y. We write $C(X,Y)_c$ for the topological space obtained by endowing $C(X,Y)$ with the compact open topology.

(b) If K is a topological group and X is Hausdorff, then $C(X,K)$ is a group with respect to the pointwise product. Then the compact open topology on $C(X,K)$ coincides with the topology of uniform convergence on compact subsets of X, for which the sets $W(C,U)$, $C \subseteq X$ compact and $U \subseteq K$ a $\mathbf{1}$-neighborhood in K, form a basis of $\mathbf{1}$-neighborhoods. In particular, $C(X,K)_c$ is a topological group.

(c) In the following we topologize for two smooth manifolds M (possibly with boundary) and N, the space $C^\infty(M,N)$ by the embedding

$$C^\infty(M,N) \hookrightarrow \prod_{k=0}^{\infty} C(T^k(M), T^k(N))_c, \quad f \mapsto (T^k(f))_{k\in\mathbb{N}_0}, \tag{1.1}$$

where the spaces $C(T^k(M),T^k(N))_c$ carry the compact open topology. The so obtained topology on $C^\infty(M,N)$ is called the *smooth compact open topology*.

Now let K be a Lie group with Lie algebra $\mathfrak{k}$ and $r \in \mathbb{N}_0 \cup \{\infty\}$. The tangent map $T(m_K)$ of the multiplication map $m_K\colon K\times K \to K$ defines a Lie group structure on the tangent bundle TK (cf. [GN07]). Iterating this procedure, we obtain a Lie group structure on all higher tangent bundles T^nK. For each $n \in \mathbb{N}_0$, we thus obtain topological groups $C(T^nM, T^nK)_c$. We also observe that for two smooth maps $f_1, f_2\colon M \to K$, the functoriality of T yields

$$T(f_1 \cdot f_2) = T(m_K \circ (f_1 \times f_2)) = T(m_K) \circ (Tf_1 \times Tf_2) = Tf_1 \cdot Tf_2.$$

Therefore the inclusion map $C^\infty(M,K) \hookrightarrow \prod_{n=0}^\infty C(T^nM, T^nK)_c$ from (1.1) is a group homomorphism, so that the inverse image of the product topology from the right hand side is a group topology on $C^\infty(M,K)$, called the *smooth compact open topology*. It turns $C^\infty(M,K)$ into a topological group, even if M and K are infinite-dimensional.

(d) In the following we topologize the space $\Omega^1(M,E)$ of E-valued 1-forms on M as a closed subspace of $C^\infty(TM,E)$. ∎

For later reference, we first collect some information on the case where $K = E$ is a locally convex space or where M is compact.

Proposition I.2. *Let M be a finite-dimensional smooth manifold and E a locally convex space. Then the following assertions hold:*

(1) *$C^\infty(M,E)$ is a locally convex space, hence a Lie group and the evaluation map of $C^\infty(M,E)$ is smooth. If E is Mackey complete, then $C^\infty(M,E)$ is Mackey complete, hence a regular Lie group.*

(2) *If M and E are complex, then $\mathcal{O}(M,E) \hookrightarrow C^\infty(M,E)$ is a closed subspace, and the evaluation map $\mathrm{ev}\colon \mathcal{O}(M,E)\times M \to E$ is holomorphic. If E is Mackey complete, then $\mathcal{O}(M,E)$ is Mackey complete, hence a regular Lie group. If M has no boundary, then the subspace topology on $\mathcal{O}(M,E)$ coincides with the compact open topology.*

Proof. (1) All the spaces $C(T^kM, T^kE)_c$ are locally convex. Therefore the corresponding product topology is locally convex, and hence $C^\infty(M,E)$ is a locally convex space.

The continuity of the evaluation map follows from the continuity of the evaluation map for the compact open topology because the topology on $C^\infty(M,E)$ is finer. Next we observe that directional derivatives exist and lead to a map

$$d\,\mathrm{ev}\colon C^\infty(M,E)^2 \times T(M) \to E, \quad ((f,\xi), v_m) \mapsto \xi(m) + T_m(f)v$$

whose continuity follows from the first step, applied to the evaluation map of $C^\infty(TM,E)$. Hence ev is C^1, and iteration of this argument yields smoothness.

In view of Lemma A.3, we have $C^\infty(I, C^\infty(M,E)) \cong C^\infty(I\times M, E)$, and if E is Mackey complete, then we have an integration map

$$C^\infty(I\times M, E) \to C^\infty(M,E), \quad \xi \mapsto \int_0^1 \xi(t,\cdot)\,dt$$

which implies that each smooth curve with values in $C^\infty(M,E)$ has a Riemann integral, i.e., that $C^\infty(M,E)$ is Mackey complete, which, in view of $\operatorname{evol}(\xi) = \int_0^1 \xi(t)\,dt$, is equivalent to it being a regular Lie group.

(2) Since $\mathcal{O}(M,E)$ is a closed subspace of $C^\infty(M,E)$, $\mathcal{O}(M,E)\times M$ is a closed submanifold of $C^\infty(M,E)\times M$, and the first part of the proof shows that the evaluation map is smooth on this space. Clearly, it is separately holomorphic in both arguments, hence its differential is complex linear, and the assertion follows.

If E is Mackey complete, then $C^\infty(M,E)$ is Mackey complete by (1), and the closed subspace $\mathcal{O}(M,E)$ inherits this property.

If M has no boundary, then the Cauchy Formula entails that on the space $\mathcal{O}(M,E)$ uniform convergence on compact subsets implies in any local chart uniform convergence of all partial derivatives on compact subsets. Hence the inclusion map

$$\mathcal{O}(M,E) \hookrightarrow C^\infty(M,E)$$

is continuous and therefore a topological embedding.[1] ∎

Theorem I.3. *Let M be a smooth manifold and K be a Lie group with Lie algebra $\mathfrak{k}$. Then the following assertions hold:*

(1) *If M is compact (possibly with corners or boundary), then $C^\infty(M,K)$ carries a Lie group structure for which any $\mathfrak{k}$-chart (φ_K, U_K) of K yields a $C^\infty(M,\mathfrak{k})$-chart (φ, U) with*

$$U := \{f \in C^\infty(M,K)\colon f(M) \subseteq U_K\}, \quad \varphi(f) := \varphi_K \circ f,$$

and the evaluation map of $C^\infty(M,K)$ is smooth. The corresponding Lie algebra is $C^\infty(M,\mathfrak{k})$, and if K is regular, then $C^\infty(M,K)$ is regular.

(2) *If M is compact and complex (possibly with boundary) and K is a complex Lie group, then $\mathcal{O}(M,K)$, endowed with the smooth compact open topology, carries a Lie group structure for which any chart (φ_K, U_K) of K yields a chart (φ, U) with*

$$U := \{f \in \mathcal{O}(M,K)\colon f(M) \subseteq U_K\}, \quad \varphi(f) := \varphi_K \circ f,$$

and the evaluation map $\operatorname{ev}\colon \mathcal{O}(M,K)\times M \to K$ is holomorphic. The corresponding Lie algebra is $\mathcal{O}(M,\mathfrak{k})$.

Proof. (1) For the existence of the Lie group structure with the given charts we refer to [Gl02a] for the case without boundary which is also dealt with in [Mi80], and to [Wo05] for the case of manifolds with corners, including in particular manifolds with boundary.

The smoothness of the evaluation map follows on each domain U as above from the openness of $C^\infty(M,\varphi_K(U_K))$ in $C^\infty(M,\mathfrak{k})$ and the smoothness of the evaluation map of $C^\infty(M,\mathfrak{k})$, verified in (1).

Now we assume that K is regular and put $\mathfrak{g} := C^\infty(M,\mathfrak{k})$ and $G := C^\infty(M,K)$. Then we obtain for each $\xi \in C^\infty(I,\mathfrak{g}) \cong C^\infty(I\times M,\mathfrak{k})$ (Lemma A.3) a curve $\gamma\colon I \to G$ by $\gamma(t)(m) := \operatorname{Evol}_K(\xi^m)(t)$, defining a smooth map $I\times M \to K$ (Lemma A.6(3)), hence a smooth curve in G (Lemma A.2). Now $\delta(\gamma^m) = \xi^m$ implies that the evolution map of G is given by $\operatorname{evol}_G(\xi)(m) := \operatorname{evol}_K(\xi^m)$. Therefore the smoothness of evol_G follows from Lemma A.3 and the smoothness of the map $(\xi,m) \mapsto \operatorname{evol}_K(\xi^m)$ (Lemma A.6(3)).

(2) For the Lie group structure we refer to [Wo06]. The holomorphy of the evaluation map follows as in (1) from Proposition I.2(2). ∎

[1] Note that if M is a complex manifold with boundary, then we cannot expect that the topology $\mathcal{O}(M,\mathbb{C})$ inherits from $C^\infty(M,\mathbb{C})$ coincides with the compact open topology, as can be seen for the example $M=\{z\in\mathbb{C}:|z|<1\}$. In this case the space $\mathcal{O}(M,\mathbb{C})$ of holomorphic functions with smooth boundary values is not complete with respect to the compact open topology, but it is a closed subspace of $C^\infty(M,\mathbb{C})$ with respect to the smooth compact open topology.

Smooth maps with values in regular Lie groups

Definition I.4. Let M be a smooth manifold (with boundary) and K a Lie group with Lie algebra $\mathfrak{k}$ and Maurer–Cartan form $\kappa_K \in \Omega^1(K,\mathfrak{k})$. For an element $f \in C^\infty(M,K)$ we call $\delta(f) := f^*\kappa_K =: f^{-1}.df \in \Omega^1(M,\mathfrak{k})$ the *(left) logarithmic derivative of* f. This is a $\mathfrak{k}$-valued 1-form on M. We thus obtain a map

$$\delta\colon C^\infty(M,K) \to \Omega^1(M,\mathfrak{k})$$

satisfying the cocycle condition

$$\delta(f_1 f_2) = \mathrm{Ad}(f_2)^{-1}.\delta(f_1) + \delta(f_2). \tag{1.2}$$

(cf. [KM97, 38.1], [GN07]). From this it easily follows that if M is connected, then

$$\delta(f_1) = \delta(f_2) \quad\Longleftrightarrow\quad (\exists k \in K)\ f_2 = \lambda_k \circ f_1. \tag{1.3}$$

If K is abelian, then δ is a group homomorphism whose kernel consists of the locally constant maps $M \to K$.

We call $\alpha \in \Omega^1(M,\mathfrak{k})$ *integrable* if there exists a smooth function $f\colon M \to K$ with $\delta(f) = \alpha$. We say that α is *locally integrable* if each point $m \in M$ has an open neighborhood U such that $\alpha \mid_U$ is integrable. We note that for any smooth map $f\colon M \to K$ and any smooth curve $\gamma\colon [0,1] \to M$ we have

$$f(\gamma(1)) = f(\gamma(0))\,\mathrm{evol}_K(\gamma^*\delta(f)) = f(\gamma(0))\,\mathrm{evol}_K(\delta(f\circ\gamma)). \tag{1.4}$$

■

To describe necessary conditions for the integrability of an element $\alpha \in \Omega^1(M,\mathfrak{k})$, we define for a manifold M and a locally convex Lie algebra $\mathfrak{k}$, the bracket

$$[\cdot,\cdot]\colon \Omega^1(M,\mathfrak{k}) \times \Omega^1(M,\mathfrak{k}) \to \Omega^2(M,\mathfrak{k})$$

by

$$[\alpha,\beta]_p(v,w) := [\alpha_p(v),\beta_p(w)] - [\alpha_p(w),\beta_p(v)] \quad \text{for} \quad v,w \in T_p(M).$$

Note that $[\alpha,\beta] = [\beta,\alpha]$. For a locally convex Lie algebra $\mathfrak{k}$ and a smooth manifold M (with boundary), we write

$$\mathrm{MC}(M,\mathfrak{k}) := \big\{\alpha \in \Omega^1(M,\mathfrak{k})\colon d\alpha + \tfrac{1}{2}[\alpha,\alpha] = 0\big\}$$

for the set of solutions of the *Maurer–Cartan equation*.

The following theorem characterizes the image of δ for a regular Lie group K.

Theorem I.5. (Fundamental Theorem for Lie group-valued functions) *Let K be a regular Lie group and $\alpha \in \Omega^1(M,\mathfrak{k})$.*

(1) *α is locally integrable if and only if $\alpha \in \mathrm{MC}(M,\mathfrak{k})$.*
(2) *If M is 1-connected and α is locally integrable, then it is integrable.*
(3) *Suppose that M is connected, fix $m_0 \in M$ and let $\alpha \in \mathrm{MC}(M,\mathfrak{k})$. Using piecewise smooth representatives of homotopy classes, we obtain a well-defined group homomorphism*

$$\mathrm{per}_\alpha^{m_0}\colon \pi_1(M,m_0) \to K, \quad [\gamma] \mapsto \mathrm{evol}_K(\gamma^*\alpha),$$

and α is integrable if and only if this homomorphism is trivial.

Proof. (1) and (2) follow directly from [KM97, Th. 40.2] (see also [GN07]).

(3) (cf. [GN07]) Let $q_M\colon \widetilde{M} \to M$ denote a simply connected covering manifold of M and choose a base point $\widetilde{m}_0 \in \widetilde{M}$ with $q_M(\widetilde{m}_0) = m_0$. Then the $\mathfrak{k}$-valued 1-form $q_M^*\alpha$ on $\widetilde{M}$ also satisfies the Maurer–Cartan equation, so that (2) implies the existence of a unique smooth function $\widetilde{f}\colon \widetilde{M} \to K$ with $\delta(\widetilde{f}) = q_M^*\alpha$ and $\widetilde{f}(\widetilde{m}_0) = \mathbf{1}$.

We write

$$\sigma\colon \pi_1(M, m_0) \times \widetilde{M} \to \widetilde{M}, \quad (d, m) \mapsto d.m =: \sigma_d(m)$$

for the left action of the fundamental group $\pi_1(M, m_0)$ on $\widetilde{M}$. In view of (1.3) in Definition I.4, the relation $\delta(\widetilde{f} \circ \sigma_d) = \sigma_d^* q_M^* \alpha = q_M^* \alpha = \delta(\widetilde{f})$ for each $d \in \pi_1(M, m_0)$ implies the existence of a function

$$\chi\colon \pi_1(M, m_0) \to K \quad \text{with} \quad \widetilde{f} \circ \sigma_d = \chi(d) \cdot \widetilde{f} \quad \text{for } d \in \pi_1(M, m_0).$$

For $d_1, d_2 \in \pi_1(M, m_0)$ we then have

$$\widetilde{f} \circ \sigma_{d_1 d_2} = \widetilde{f} \circ \sigma_{d_1} \circ \sigma_{d_2} = (\chi(d_1) \cdot \widetilde{f}) \circ \sigma_{d_2} = \chi(d_1) \cdot (\widetilde{f} \circ \sigma_{d_2}) = \chi(d_1)\chi(d_2) \cdot \widetilde{f},$$

so that χ is a group homomorphism. We now pick a smooth lift $\widetilde{\gamma}\colon I \to \widetilde{M}$ with $q_M \circ \widetilde{\gamma} = \gamma$ and observe that

$$\delta(\widetilde{f} \circ \widetilde{\gamma}) = \widetilde{\gamma}^* q_M^* \alpha = \gamma^* \alpha,$$

which leads to $\chi([\gamma]) = \widetilde{f}([\gamma].\widetilde{m}_0) = \widetilde{f}(\widetilde{\gamma}(1)) = \operatorname{evol}_K(\gamma^*\alpha)$. This proves that $\operatorname{per}_\alpha^{m_0}$ is well-defined and a group homomorphism.

Clearly, $\operatorname{per}_\alpha^{m_0}$ vanishes if and only if the function $\widetilde{f}$ is invariant under the action of $\pi_1(M, m_0)$, which is equivalent to the existence of a smooth function $f\colon M \to K$ with $f \circ q_M = \widetilde{f}$, which in turn means that α is integrable. ∎

Remark I.6. Let M be a connected smooth manifold (with boundary), $m_0 \in M$, and $q_M\colon \widetilde{M} \to M$ a universal covering map. Further let K be a regular Lie group with Lie algebra $\mathfrak{k}$. Then we have an embedding

$$q_M^*\colon \operatorname{MC}(M, \mathfrak{k}) \to \operatorname{MC}(\widetilde{M}, \mathfrak{k})^{\pi_1(M, m_0)},$$

where the right hand side denotes the set of all solutions of the Maurer–Cartan equation on $\widetilde{M}$ which are invariant under the action of the fundamental group $\pi_1(M, m_0)$ by deck transformations.

(a) If $f \in C^\infty(\widetilde{M}, K)$ satisfies $\delta(f) = q_M^*\alpha$, then

$$f(d.x) = \operatorname{per}_\alpha^{m_0}(d) \cdot f(x) \quad \text{for all} \quad x \in \widetilde{M}.$$

If, conversely, $f \in C^\infty(\widetilde{M}, K)$ satisfies $f(d.x) = \chi(d) \cdot f(x)$ for $x \in \widetilde{M}$ and $d \in \pi_1(M, m_0)$, then $\delta(f)$ is $\pi_1(M, m_0)$-invariant, hence of the form $\delta(f) = q_M^*\alpha$, and $\chi = \operatorname{per}_\alpha^{m_0}$. This shows that

$$C^\infty(\widetilde{M}, K)^\sharp := \{f \in C^\infty(\widetilde{M}, K)\colon \delta(f) \in q_M^* \operatorname{MC}(M, \mathfrak{k})\}$$

is fibered over the set $\operatorname{Hom}(\pi_1(M, m_0), K)$ by

$$C^\infty(\widetilde{M}, K)^\sharp = \bigcup_{\chi \in \operatorname{Hom}(\pi_1(M, m_0), K)} C^\infty(\widetilde{M}, K)_\chi,$$

where

$$C^\infty(\widetilde{M}, K)_\chi := \{f \in C^\infty(\widetilde{M}, K)\colon (\forall x \in \widetilde{M})(\forall d \in \pi_1(M, m_0))\ f(d.x) = \chi(d) f(x)\}.$$

The set $C^\infty(\widetilde{M}, K)_\chi$ is invariant under multiplication with functions in $q_M^* C^\infty(M, K)$ from the right. Conversely, for $f, g \in C^\infty(\widetilde{M}, K)_\chi$, the function $g^{-1} \cdot f$ factors through a function F on

M with $g \cdot q_M^* F = f$. Therefore the fibers of $C^\infty(\widetilde{M}, K)^\sharp$ coincide with the orbits of the group $C^\infty(M,K)$, acting by right multiplication. On the set $\mathrm{MC}(M,\mathfrak{k})$, the corresponding action is given by

$$\alpha * f := \delta(f) + \mathrm{Ad}(f)^{-1}.\alpha$$

for $\alpha \in \mathrm{MC}(M,\mathfrak{k})$ and $f \in C^\infty(M,\mathfrak{k})$ (Definition I.4).

In general, not all the sets $C^\infty(\widetilde{M},K)_\chi$ are non-empty. Indeed, this condition can be interpreted as the smooth (which is equivalent to the topological [MW06]) triviality of the flat principal K-bundle $P_\chi := \widetilde{M} \times_\chi K := (\widetilde{M} \times K)/\pi_1(M)$, where $\pi_1(M)$ acts on $\widetilde{M} \times K$ by $d.(x,k) = (d.x, \chi(d)k)$. Not all such bundles are topologically trivial. If Σ is a compact Riemann surface of genus $g > 1$, then there are flat non-trivial $\mathrm{SL}_2(\mathbb{R})$-bundles over Σ (see p. 24 in [KT68], which uses results from [Mil58]). On the other hand, if K is a complex algebraic group, then [Gro68, Cor. 7.2] asserts that all rational characteristic classes of the bundles P_χ vanish.

(b) Note that for $f_i \in C^\infty(\widetilde{M},K)_{\chi_i}$, $i = 1,2$, and $\mathrm{im}(\chi_2) \subseteq Z(K)$, we have $\chi_1\chi_2 \in \mathrm{Hom}(\pi_1(M,m_0),K)$ with $f_1 f_2 \in C^\infty(\widetilde{M},K)_{\chi_1\chi_2}$. For $\delta(f_i) = q_M^*\alpha_i$ we then have $\delta(f_1 f_2) = q_M^*\beta$, where

$$\beta = \alpha_2 + \overline{\mathrm{Ad}(f_2)}^{-1}.\alpha_1$$

and $\overline{\mathrm{Ad}(f_2)}$ is the well-defined function $M \to \mathrm{Aut}(\mathfrak{k})$, defined by $\overline{\mathrm{Ad}(f_2)} \circ q_M = \mathrm{Ad}(f_2)$ (cf. Definition I.4). ∎

For later use in Section III, we record the following formula:

Lemma I.7. *Let $x \in \mathfrak{k}$ and $\beta \in \Omega^1(M,\mathbb{R})$ be a closed 1-form. Then $\alpha := \beta \cdot x \in \Omega^1(M,\mathfrak{k})$ satisfies*

$$\mathrm{per}_\alpha^{m_0} = \exp_K \circ (\mathrm{per}_\beta^{m_0} \cdot x). \tag{1.5}$$

Proof. That α satisfies the Maurer–Cartan equation follows from $d\alpha = d\beta \cdot x = 0 = [\alpha,\alpha]$. To calculate $\mathrm{per}_\alpha^{m_0}$, we first pick a smooth function $h \in C^\infty(\widetilde{M},\mathbb{R})$ with $dh = q_M^*\beta$. Then

$$f\colon \widetilde{M} \to K, \quad m \mapsto \exp_K(h(m)x)$$

is a smooth function with

$$\begin{aligned} T_m(f)v &= T_{h(m)x}(\exp_K)(dh(m)v \cdot x) = dh(m)v \cdot T_{h(m)x}(\exp_K)(x) \\ &= dh(m)v \cdot T_{\mathbf{1}}(\lambda_{\exp_K(h(m)x)})x = dh(m)v \cdot T_{\mathbf{1}}(\lambda_{f(m)})x, \end{aligned}$$

so that

$$\delta(f) = dh \cdot x = q_M^*(\beta \cdot x) = q_M^*\alpha.$$

Let $\widetilde{m}_0 \in q_M^{-1}(m_0)$ and assume w.l.o.g. that $h(\widetilde{m}_0) = 0$, so that $f(\widetilde{m}_0) = \mathbf{1}$. We then have

$$\mathrm{per}_\alpha^{m_0}(d) = f(d.\widetilde{m}_0) = \exp_K(h(d.\widetilde{m}_0)x) = \exp_K(\mathrm{per}_\beta^{m_0}(d)x),$$

which implies (1.5). ∎

Uniqueness of regular Lie group structures

The following theorem characterizes smooth maps $N \times M \to K$ in terms of smoothness of maps defined on N. We shall need it later to prove the uniqueness of a regular Lie group structure on $C^\infty(M,K)$ with smooth evaluation map.

Proposition I.8. *Let N be a locally convex manifold, M a connected finite-dimensional manifold and K a regular Lie group. Then a function $f: N \times M \to K$ is smooth if and only if*
(1) *there exists a point $m_0 \in M$ for which the map $f^{m_0}: N \to K, n \mapsto f(n, m_0)$ is smooth, and*
(2) *the functions $f_n: M \to K, m \mapsto f(n, m)$ are smooth and $F: N \to \Omega^1(M, \mathfrak{k}), n \mapsto \delta(f_n)$ is smooth.*

Proof. "$\Rightarrow$": If f is smooth, then all the maps f^m and f_n are smooth. To see that F is smooth, we recall that $\Omega^1(M, \mathfrak{k})$ is a closed subspace of $C^\infty(TM, \mathfrak{k})$ (Definition I.1(d)), so that it suffices to show that the map

$$\widetilde{F}: N \times TM \to \mathfrak{k}, \quad (n, v) \mapsto \delta(f_n)v = \kappa_K(T(f_n)v) = \kappa_K(T(f)(0, v))$$

is smooth (Lemma A.2). Since the Maurer–Cartan form κ_K of K is a smooth map $TK \to \mathfrak{k}$, the assertion follows from the smoothness of $T(f): T(N \times M) \cong TN \times TM \to TK$.

"$\Leftarrow$": **Step 1:** First we show that f^m is smooth for each $m \in M$. Pick a smooth path $\gamma: [0, 1] \to M$ with $\gamma(0) = m_0$ and $\gamma(1) = m$. Then

$$f^m(n) = f_n(m) = f_n(m_0)\,\mathrm{evol}_K(\delta(f_n \circ \gamma)) = f_n(m_0)\,\mathrm{evol}_K(\gamma^*\delta(f_n)) = f^{m_0}(n)\,\mathrm{evol}_K(\gamma^* F(n)).$$

Hence the smoothness of f^{m_0}, evol_K, F, and the continuity of the linear map $\gamma^*: \Omega^1(M, \mathfrak{k}) \to C^\infty([0, 1], \mathfrak{k})$ imply that f^m is smooth.

Step 2: Now we show that f is smooth. To this end, let $m \in M$ and choose a chart (φ, U) of M for which $\varphi(U)$ is convex with $\varphi(m) = 0$. We have to show that the map

$$h: N \times U \to K, \quad (n, x) \mapsto f(n, \varphi^{-1}(x))$$

is smooth. For $\gamma_x(t) := tx$, $0 \le t \le 1$, we have

$$h(n, x) = h(n, \gamma_x(1)) = h(n, 0)\,\mathrm{evol}_K(\delta(f_n \circ \varphi^{-1} \circ \gamma_x)) = f^m(n)\,\mathrm{evol}_K(\gamma_x^*(\varphi^{-1})^* F(n)).$$

Since f^m and evol_K are smooth and

$$(\varphi^{-1})^*: \Omega^1(U, \mathfrak{k}) \to \Omega^1(\varphi(U), \mathfrak{k})$$

is a topological linear isomorphism, it suffices to show that the map

$$\Omega^1(\varphi(U), \mathfrak{k}) \times U \to C^\infty([0, 1], \mathfrak{k}), \quad (\alpha, x) \mapsto \gamma_x^*\alpha$$

is smooth. In view of Lemma A.2, this follows from the smoothness of the map

$$\Omega^1(\varphi(U), \mathfrak{k}) \times U \times [0, 1] \to \mathfrak{k}, \quad (\alpha, x, t) \mapsto (\gamma_x^*\alpha)_t = \alpha_{tx}(x),$$

which is a consequence of the smoothness of the evaluation map of $C^\infty(TM, \mathfrak{k})$ (Proposition I.2). ∎

The following theorem characterizes the "good" Lie group structures on $C^\infty(M, K)$ in various ways. Note that we do not assume that the Lie group structure is compatible with the topology on $C^\infty(M, K)$.

Proposition I.9. *Let M be a connected finite-dimensional smooth manifold and K a regular Lie group. Suppose that the group $G := C^\infty(M, K)$ carries a Lie group structure for which $\mathfrak{g} := C^\infty(M, \mathfrak{k})$ is the corresponding Lie algebra and all evaluation maps $\mathrm{ev}_m: G \to K$, $m \in M$, are smooth with*

$$\mathbf{L}(\mathrm{ev}_m) = \mathrm{ev}_m: \mathfrak{g} \to \mathfrak{k}.$$

Then the following assertions hold:
(1) *The evaluation map $\mathrm{ev}: G \times M \to K, (f, m) \mapsto f(m)$ is smooth.*

(2) *If, in addition, G is regular, then a map $f: N \to G$ is smooth if and only if the corresponding map $f^\wedge: N \times M \to K$ is smooth.*

Proof. (1) Let $N \subseteq M$ be a compact submanifold (with boundary). Then $C^\infty(N, K)$ carries the structure of a regular Lie group (Proposition I.3). Let $q_G: \widetilde{G}_0 \to G_0$ denote the universal covering of the identity component G_0 of G. Consider the continuous homomorphism of Lie algebras

$$\psi: \mathbf{L}(G) = C^\infty(M, \mathfrak{k}) \to C^\infty(N, \mathfrak{k}), \quad f \mapsto f|_N.$$

In view of the regularity of $C^\infty(N, K)$, there exists a unique morphism of Lie groups

$$\widetilde{\varphi}: \widetilde{G}_0 \to C^\infty(N, K) \quad \text{with} \quad \mathbf{L}(\widetilde{\varphi}) = \psi,$$

where $\widetilde{G}_0$ is the universal covering group of the identity component G_0 of G. Then, for each $n \in N$, the homomorphism $\mathrm{ev}_n \circ \widetilde{\varphi}: \widetilde{G}_0 \to K$ is smooth with differential $\mathbf{L}(\mathrm{ev}_n \circ \widetilde{\varphi}) = \mathrm{ev}_n$, so that $\mathrm{ev}_n \circ \widetilde{\varphi} = \mathrm{ev}_n \circ q_G$, where $q_G: \widetilde{G}_0 \to G$ is the universal covering map. We conclude that

$$\ker q_G \subseteq \ker \widetilde{\varphi},$$

and hence that $\widetilde{\varphi}$ factors through the restriction map $G_0 = C^\infty(M, K)_0 \to C^\infty(N, K)$. In particular, the restriction map $C^\infty(M, K) \to C^\infty(N, K)$ is a smooth homomorphism of Lie groups.

This implies in particular that for each relatively compact open subset $U \subseteq M$, the map $G \to \Omega^1(U, \mathfrak{k}), f \mapsto \delta(f|_U)$ is smooth (cf. Lemma A.5(1)), and since $\Omega^1(M, \mathfrak{k})$ embeds into $\prod_U \Omega^1(U, \mathfrak{k})$, it follows that δ is smooth.

(2) Now we assume that the evaluation map is smooth. If f is smooth, then $f^{m_0} = \mathrm{ev}_{m_0} \circ f$ is smooth and (1) entails that $\delta \circ f: N \to \Omega^1(M, \mathfrak{k})$ is smooth, so that Proposition I.8 implies that $f^\wedge$ is smooth.

If, conversely, $f^\wedge$ is smooth, we have to show that f is smooth. To this end, we may w.l.o.g. assume that N is 1-connected, since the assertion is local with respect to N. We define $\beta \in \Omega^1(N, \mathfrak{g})$ by

$$\beta v = \kappa_K(T(f^\wedge)(v, 0)),$$

which shows immediately that β defines a smooth map $TN \times M \to \mathfrak{k}$ which is linear on the tangent spaces of N, and with $C^\infty(TN \times M, \mathfrak{k}) \cong C^\infty(TN, \mathfrak{g})$ (Lemma A.3), we see that this is an element of $\Omega^1(N, \mathfrak{g})$.

We claim that β satisfies the Maurer–Cartan equation. In fact, for each $m \in M$, we have

$$\mathrm{ev}_m \circ \beta = \delta(\mathrm{ev}_m \circ f^\wedge),$$

which satisfies the Maurer–Cartan equation. Since the evaluation map $\mathrm{ev}_m: \mathfrak{g} \to \mathfrak{k}$ is a homomorphism of Lie algebras, and the corresponding maps $\mathrm{ev}_m: \Omega^1(M, \mathfrak{g}) \to \Omega^1(M, \mathfrak{k})$ separate the points, it follows that β satisfies the Maurer–Cartan equation.

Fix a point $n_0 \in N$. Since G is regular and N is 1-connected, the Fundamental Theorem (Theorem I.5) implies the existence of a unique smooth function $h: N \to G$ with $h(n_0) = f(n_0)$ and $\delta(h) = \beta$. For each $m \in M$ we then have $h(n_0)(m) = f(n_0)(m)$ and

$$\delta(\mathrm{ev}_m \circ h) = \mathrm{ev}_m \circ \delta(h) = \mathrm{ev}_m \circ \beta = \delta(\mathrm{ev}_m \circ f),$$

so that the uniqueness part of the Fundamental Theorem, applied to K-valued functions, yields $\mathrm{ev}_m \circ f = \mathrm{ev}_m \circ h$ for each m, which leads to $h = f$. This proves that f is smooth. ∎

In the following, we say that a Lie group structure on $C^\infty(M, K)$ is *compatible with evaluations* if it satisfies the assumptions of the preceding proposition. The following corollary contains one of the main results of this section. It asserts that there is at most one regular Lie group structure compatible with evaluations.

Corollary I.10. *Under the assumptions of the preceding theorem, there exists at most one regular Lie group structure on the group $G := C^\infty(M,K)$ compatible with evaluations.*

Proof. Let G_1 and G_2 be two regular Lie groups obtained from Lie group structures on $C^\infty(M,K)$ compatible with evaluations. In view of Proposition I.9, the smoothness of the evaluation maps $\mathrm{ev}_j\colon G_j \times M \to K$ implies that the identity maps $G_1 \to G_2$ and $G_2 \to G_1$ are smooth, hence that G_1 and G_2 are isomorphic Lie groups. ∎

In the preceding corollary, we do not have to assume that the Lie group structure on $C^\infty(M,K)$ is compatible with the smooth compact open topology, but even if we consider only regular Lie group structures compatible with this topology, the uniqueness of these structures does not directly follow from Lie theoretic considerations:

Remark I.11. If G is a regular Lie group and H any simply connected Lie group, then any continuous homomorphism $\psi\colon \mathbf{L}(H) \to \mathbf{L}(G)$ of Lie algebras integrates to a unique homomorphism $\varphi\colon H \to G$ with $\mathbf{L}(\varphi) = \psi$ (cf. [Mil84]). This implies in particular that two 1-connected regular Lie groups with isomorphic Lie algebras are isomorphic. In this sense regularity of a Lie group is crucial for uniqueness results.

On the other hand, we do not know if there exist topologically isomorphic regular Lie groups G_1 and G_2 which are not isomorphic as Lie groups. To prove that this is not the case, we would need a result on the automatic smoothness of continuous homomorphisms of Lie groups, but presently the optimal result in this direction requires at least Hölder continuity (cf. [Gl05]). ∎

II. Logarithmic derivatives and the Maurer–Cartan equation

In this section we describe a strategy to obtain a Lie group structure on the group $G := C^\infty(M,K)$ for a non-compact connected manifold M. It is based on the injectivity of the logarithmic derivative on the normal subgroup

$$G_* := \{f \in C^\infty(M,K)\colon f(m_0) = \mathbf{1}\},$$

where m_0 is a base point in M. We thus realize G_* as a subset of $\mathrm{MC}(M,\mathfrak{k}) \subseteq \Omega^1(M,\mathfrak{k})$. There are several situations, in which one can show that $\delta(G_*)$ is a manifold, and where transferring the manifold structure of $\mathrm{im}(\delta)$ to G_* leads to a Lie group structure on G_* and hence on $G \cong G_* \rtimes K$ because K (realized as the constant functions on M) acts smoothly by conjugation on $\Omega^1(M,\mathfrak{k}) \supseteq \delta(G_*)$ (Lemma A.5(2)).

One of the main results of this section is Theorem II.2, asserting that $\delta(G_*)$ is a Lie group whenever it is a submanifold of $\Omega^1(M,\mathfrak{k})$. This condition is satisfied in particular if M is one-dimensional (Corollary II.3). Then we establish an iterative procedure leading to regular Lie group structures on $C^\infty(\mathbb{R}^n \times M, K)$ for any compact smooth manifold M and any n (Corollary II.5).

Proposition II.1. *If K is a regular Lie group and M is a connected finite-dimensional smooth manifold, then the map*

$$\delta\colon C^\infty_*(M,K) \to \Omega^1(M,\mathfrak{k})$$

is a topological embedding. Let $\mathrm{Evol}_K := \delta^{-1}\colon \mathrm{im}(\delta) \to C^\infty_*(M,K)$ denote its inverse. Then δ is an isomorphism of topological groups if we endow $\mathrm{im}(\delta)$ with the group structure defined by

$$\alpha * \beta := \beta + \mathrm{Ad}(\mathrm{Evol}_K(\beta))^{-1}.\alpha \tag{2.1}$$

and

$$\alpha^{-1} = -\mathrm{Ad}(\mathrm{Evol}_K(\alpha)).\alpha, \tag{2.2}$$

Proof. First we show that δ is continuous. By definition of the topology on $C^\infty(M,K)$, the tangent map induces a continuous group homomorphism

$$T\colon C^\infty(M,K) \to C^\infty(TM,TK), \quad f \mapsto T(f).$$

Let $\kappa_K\colon TK \to \mathfrak{k}$ denote the (left) Maurer–Cartan form of K. Since $\delta(f) = f^*\kappa_K = \kappa_K \circ T(f)$, it follows that the composition

$$C^\infty(M,K) \to C^\infty(T(M),T(K)) \to C^\infty(T(M),\mathfrak{k}), \quad f \mapsto T(f) \mapsto \delta(f)$$

is continuous.

Next we show that δ is an embedding. Consider $\alpha = \delta(f)$ with $f \in C^\infty_*(M,K)$, i.e., $f(m_0) = \mathbf{1}$ holds for the base point $m_0 \in M$. To reconstruct f from α, we pick for $m \in M$ a piecewise smooth path $\gamma\colon [0,1] \to M$ with $\gamma(0) = m_0$ and $\gamma(1) = m$. Then $\delta(f\circ\gamma) = \gamma^*\delta(f) = \gamma^*\alpha$ implies $f(m) = \mathrm{evol}_K(\gamma^*\alpha)$.

We now choose an open neighborhood U of m and a chart (φ, U) of M such that $\varphi(U)$ is convex with $\varphi(m) = 0$. Then, for each $x \in U$, (1.4) in Definition I.4 yields

$$f(x) = f(m)\cdot \mathrm{evol}_K(\gamma_x^*\alpha), \tag{2.3}$$

where $\gamma_x(t) = \varphi^{-1}(t\varphi(x))$.

From Lemma A.6(1),(2), we immediately derive that the map

$$\Omega^1(M,\mathfrak{k}) \times U \to K, \quad (\alpha, x) \mapsto \mathrm{evol}_K(\gamma^*\alpha)\cdot \mathrm{evol}_K(\gamma_x^*\alpha) \tag{2.4}$$

is smooth, so that the corresponding map $\Omega^1(M,\mathfrak{k}) \to C^\infty(U,K)$ is in particular continuous (Lemma A.1). We conclude that the map

$$\delta(C^\infty_*(M,K)) \to C^\infty(U,K), \quad \delta(f) \mapsto f|_U$$

is continuous. We finally observe that for each open covering $M = \bigcup_{j\in J} U_j$, the restriction maps to U_i lead to a topological embedding $C^\infty(M,K) \hookrightarrow \prod_{j\in J} C^\infty(U_j,K)$, and this completes the proof. ∎

Theorem II.2. *Let M be a connected finite-dimensional smooth manifold (with boundary) and K a regular Lie group. Assume that the subset $\delta(C^\infty_*(M,K))$ is a smooth submanifold of $\Omega^1(M,\mathfrak{k})$ and endow $C^\infty_*(M,K)$ with the manifold structure for which $\delta\colon C^\infty_*(M,K) \to \mathrm{im}(\delta)$ is a diffeomorphism and*

$$C^\infty(M,K) \cong K \ltimes C^\infty_*(M,K)$$

with the product manifold structure. Then the following assertions hold:

(1) *For each locally convex manifold N, a map $f\colon N\times M \to K$ is smooth if and only if all the maps $f_n\colon M \to K, m \mapsto f(n,m)$ are smooth and the corresponding map*

$$f^\vee\colon N \to C^\infty(M,K), \quad n \mapsto f_n$$

is smooth.

(2) *K acts smoothly by conjugation on $C^\infty_*(M,K)$, and $C^\infty(M,K)$ carries a regular Lie group structure compatible with evaluations.*

Proof. (1) Let m_0 be the base point of M. According to Proposition I.8, $f\colon N\times M \to K$ is smooth if and only if f^{m_0} is smooth, all the maps f_n are smooth, and $\delta\circ f^\vee\colon N \to \Omega^1(M,\mathfrak{k})$ is smooth. In view of our definition of the manifold structure on $C^\infty_*(M,K)$, the latter condition is equivalent to the smoothness of the map $N \to C^\infty_*(M,K), n \mapsto f_n(m_0)^{-1}f_n = f^{m_0}(n)^{-1}f_n$. Since the evaluation in m_0 coincides with the projection

$$C^\infty(M,K) \cong K \ltimes C^\infty_*(M,K) \to K,$$

we see that f is smooth if and only if all the maps f_n are smooth and $f^\vee$ is smooth.

(2) For the evaluation map $f = \mathrm{ev}\colon G \times M \to K$, we have $\mathrm{ev}^\vee = \mathrm{id}_G$ and $\mathrm{ev}_g = g$ for each $g \in G$. Hence (1) implies that ev is smooth.

In view of Proposition II.1, δ is an isomorphism of topological groups if $\mathrm{im}(\delta)$ is endowed with the group structure (2.1). We now show that the operations (2.1) and (2.2) are smooth with respect to the submanifold structure on $\mathrm{im}(\delta)$.

The Lie group structure: It suffices to show that the map

$$\mathrm{im}(\delta) \times \Omega^1(M, \mathfrak{k}) \to \Omega^1(M, \mathfrak{k}), \quad (\alpha, \beta) \mapsto \mathrm{Ad}(\mathrm{Evol}_K(\alpha)).\beta$$

is smooth. For each open covering $(U_j)_{j\in J}$, we obtain an embedding $\Omega^1(M, \mathfrak{k}) \hookrightarrow \prod_{j\in J} \Omega^1(U_j, \mathfrak{k})$, so that it suffices to prove for each $m \in M$ the existence of an open neighborhood U of m, for which the map

$$\mathrm{im}(\delta) \times \Omega^1(U, \mathfrak{k}) \to \Omega^1(U, \mathfrak{k}), \quad (\alpha, \beta) \mapsto \mathrm{Ad}(\mathrm{Evol}_K(\alpha)).\beta$$

is smooth. Choosing U so small that it lies in a chart domain, we have $\Omega^1(U, \mathfrak{k}) \cong C^\infty(U, \mathfrak{k})^d$ for $d = \dim M$, so that it suffices to show that

$$\mathrm{im}(\delta) \times C^\infty(U, \mathfrak{k}) \to C^\infty(U, \mathfrak{k}), \quad (\alpha, f) \mapsto \mathrm{Ad}(\mathrm{Evol}_K(\alpha)).f,$$

is smooth. Now it suffices to see that the map

$$\mathrm{im}(\delta) \times C^\infty(U, \mathfrak{k}) \times U \to \mathfrak{k}, \quad (\alpha, f, x) \mapsto \mathrm{Ad}(\mathrm{Evol}_K(\alpha)(x)).f(x)$$

is smooth. Since the action map $K \times C^\infty(M, \mathfrak{k}) \to C^\infty(M, \mathfrak{k})$ is smooth (Lemma A.5(1)), it suffices to recall from Proposition I.2 that the evaluation map of $C^\infty(U, \mathfrak{k})$ is smooth and to show that the map

$$\mathrm{im}(\delta) \times U \to K, \quad (\alpha, x) \mapsto \mathrm{Evol}_K(\alpha)(x) \tag{2.5}$$

is smooth.

Let $m \in M$. To obtain $\mathrm{Evol}_K(\alpha)$, we pick a piecewise smooth path $\gamma\colon [0,1] \to M$ with $\gamma(0) = m_0$ and $\gamma(1) = m$. Then $\delta(\mathrm{Evol}_K(\alpha) \circ \gamma) = \gamma^*\delta(\mathrm{Evol}_K(\alpha)) = \gamma^*\alpha$ implies

$$\mathrm{Evol}_K(\alpha)(m) = \mathrm{evol}_K(\gamma^*\alpha).$$

We now choose an open neighborhood U of m and a chart (φ, U) of M such that $\varphi(U)$ is convex. Then, for each $x \in U$, (1.4) in Definition I.4 entails

$$\mathrm{Evol}_K(\alpha)(x) = \mathrm{Evol}_K(\alpha)(m) \cdot \mathrm{evol}_K(\gamma_x^*\alpha),$$

where $\gamma_x(t) = \varphi^{-1}(t\varphi(x))$. From Lemma A.6(1),(2), we derive that the map

$$\Omega^1(M, \mathfrak{k}) \times U \to K, \quad (\alpha, x) \mapsto \mathrm{evol}_K(\gamma^*\alpha) \cdot \mathrm{evol}_K(\gamma_x^*\alpha)$$

is smooth, so that restriction to the submanifold $\mathrm{im}(\delta)$ implies the smoothness of (2.5). We conclude that multiplication and the inversion in $\mathrm{im}(\delta)$ is smooth and hence that it is a Lie group.

The Lie algebra: Next we verify regularity. To this end, we first determine the tangent space $T_0(\mathrm{im}(\delta))$ to see the Lie algebra of this group. Let $\eta\colon I \to \mathrm{im}(\delta)$ be a smooth curve with $\eta(0) = 0$ and $\beta := \eta'(0)$. Then

$$d\eta(t) + \frac{1}{2}[\eta(t), \eta(t)] = 0$$

for each $t \in I$ yields $d\eta'(0) = 0$, so that β is closed. We also have

$$\mathbf{1} = \mathrm{per}^{m_0}_{\eta(t)}(\gamma) = \mathrm{evol}_K(\gamma^*\eta(t))$$

for each smooth loop γ in m_0 and each $t \in I$. Taking the derivative in $t = 0$, we get with Lemma A.5(1):

$$0 = T_0(\mathrm{evol}_K)(\gamma^*\beta) = \int_0^1 \gamma^*\beta = \int_\gamma \beta.$$

Hence all periods of β vanish, so that β is exact. If, conversely, $\beta \in \Omega^1(M,\mathfrak{k})$ is an exact 1-form, then $\beta = df$ for some $f \in C^\infty_*(M,\mathfrak{k})$, and the curve $\alpha(t) := \delta(\exp_K(tf))$ in $\mathrm{im}(\delta)$ satisfies $\alpha'(0) = T_{\mathbf{1}}(\delta)f = df = \beta$. This shows that

$$T_0(\mathrm{im}(\delta)) = B^1_{\mathrm{dR}}(M,\mathfrak{k}) = dC^\infty_*(M,\mathfrak{k}) \cong C^\infty_*(M,\mathfrak{k}),$$

as a topological vector space (apply Proposition II.1 to the Lie group $(\mathfrak{k},+)$).

Next, recall that for each $m \in M$ and $\gamma\colon [0,1] \to M$ from m_0 to m we have

$$\mathrm{Evol}_K(\alpha)(m) = \mathrm{evol}_K(\gamma^*\alpha),$$

so that we get for any smooth curve η in $\mathrm{im}(\delta)$ with $\eta(0) = 0$ and $\eta'(0) = df$ with $f \in C^\infty_*(M,\mathfrak{k})$ the relation

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Evol}_K(\eta(t))(m) = \frac{d}{dt}\Big|_{t=0} \mathrm{evol}_K(\gamma^*\eta(t)) = T_0(\mathrm{evol}_K)\gamma^*\eta'(0) = \int_0^1 \gamma^*\eta'(0) = f(m),$$

i.e.,

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Evol}_K(\eta(t)) = f.$$

Now we can determine the Lie bracket on $T_0(\mathrm{im}(\delta))$. Let $\eta_j\colon I \to \mathrm{im}(\delta)$, $j = 1,2$, be smooth curves in $\mathbf{1}$ and $f_j \in C^\infty_*(M,\mathfrak{k})$ with $df_j = \eta'(0)$. Then

$$\frac{\partial^2}{\partial s \partial t}\,|_{s=t=0}\ \eta_1(s) * \eta_2(t) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\mathrm{Evol}_K(\eta_2(t)))^{-1}.\eta_1'(0) = -[f_2, df_1].$$

For the Lie bracket in $\mathbf{L}(\mathrm{im}(\delta)) = dC^\infty(M,\mathfrak{k})$, we thus obtain the formula

$$[df_1, df_2] = -[f_2, df_1] + [f_1, df_2] = d[f_1, f_2],$$

showing that

$$d\colon C^\infty_*(M,\mathfrak{k}) \to T_0(\mathrm{im}(\delta))$$

is an isomorphism of Lie algebras if $C^\infty_*(M,\mathfrak{k})$ is endowed with the pointwise Lie bracket.

Regularity: It remains to verify the regularity of G, i.e., the smoothness of the map

$$\mathrm{evol}_G\colon C^\infty(I,\mathfrak{g}) \to G \cong C^\infty(M,K).$$

First we make this map more explicit. Let $\xi \in C^\infty(I,\mathfrak{g}) \cong C^\infty(I \times M,\mathfrak{k})$ (Lemma A.2). To see that the curve $\gamma(t)(m) := \mathrm{Evol}_K(\xi^m)(t)$ in G is smooth, we observe that the map $I \times M \to K, (t,m) \mapsto \gamma(t)(m)$ is smooth (Lemma A.6(3)), so that (1) implies that $\gamma\colon I \to G$ is smooth. We also obtain from Lemma A.6(3) that $\delta(\gamma)_t = \xi_t$, so that $\delta(\gamma) = \xi$. Hence

$$\mathrm{evol}_G(\xi)(m) = \mathrm{evol}_K(\xi^m) = \mathrm{evol}^\vee_G(\xi, m).$$

In view of (1), the smoothness of $\mathrm{evol}^\vee_G$ (Lemma A.6(3)) now implies the smoothness of evol_G. ■

If M is one-dimensional, then each $\mathfrak{k}$-valued 2-form on M is trivial, so that $d\alpha = 0 = [\alpha,\alpha]$ for each $\alpha \in \Omega^1(M,\mathfrak{k})$. Therefore all 1-forms trivially solve the Maurer–Cartan equation. We thus obtain:

Corollary II.3. *If M is a one-dimensional 1-connected σ-compact real manifold (with boundary), then the group $C^\infty_*(M,K)$ carries a regular Lie group structure for which*

$$\delta\colon C^\infty_*(M,K) \to \Omega^1(M,\mathfrak{k}) \cong C^\infty(M,\mathfrak{k})$$

is a diffeomorphism and $C^\infty(M,K) \cong C^\infty_*(M,K) \rtimes K$ carries the structure of a regular Lie group compatible with evaluations and the smooth compact open topology. ∎

For the case $M = \mathbb{R}$, the preceding corollary can also be found in the book of Kriegl and Michor ([KM97, Th. 38.12]). Note that any 1-connected σ-compact 1-dimensional manifold with boundary is diffeomorphic to $\mathbb{R}$, $[0,1]$ or $[0,\infty[$.

Lemma II.4. *If $K \neq \exp\mathfrak{k}$, then the exponential image of $C^\infty(\mathbb{R},\mathfrak{k})$ is not an identity neighborhood in $C^\infty(\mathbb{R},K)$.*

Proof. The exponential function $C^\infty(\mathbb{R},\mathfrak{k})$ is simply given by $\exp(\xi) := \exp_K \circ \xi$, where $\exp_K$ is the exponential function of K.

Let $k \in K \setminus \exp\mathfrak{k}$ and consider a smooth curve $g\colon \mathbb{R} \to K$ with $g(t) = \mathbf{1}$ for $t < 0$ and $g(t) = k$ for $t > 1$. Then $g_n(t) := g(t-n)$ defines a sequence in $C^\infty(\mathbb{R},K)$, converging to $\mathbf{1}$. As $g_n(n+1) = k \notin \exp\mathfrak{k}$, none of the curves g_n is contained in the image of the exponential function. ∎

Iterative constructions

Corollary II.3 is much more powerful than it appears at first sight because it can be applied inductively to show that for each compact manifold M and $k \in \mathbb{N}$ the group $C^\infty(\mathbb{R}^k \times M, K)$ carries a regular Lie group structure compatible with evaluations. This result is the main goal of this subsection. First we need two lemmas. The more general key result is Theorem II.7.

Lemma II.5. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of Lie groups, $\varphi_{nm}\colon G_m \to G_n$ morphisms of Lie groups defining an inverse system, $G := \varprojlim G_n$ the corresponding topological projective limit group and $\varphi_n\colon G \to G_n$ the canonical maps. Assume that G carries a Lie group structure with the following properties:*

(1) *A map $f\colon M \to G$ of a smooth manifold M with values in G is smooth if and only if all the maps $f_n := \varphi_n \circ f$ are smooth.*
(2) *$\mathbf{L}(G) \cong \varprojlim \mathbf{L}(G_n)$ as topological Lie algebras, with respect to the projective system defined by the morphisms $\mathbf{L}(\varphi_{nm})\colon \mathbf{L}(G_n) \to \mathbf{L}(G_m)$.*

Then the map

$$\Psi\colon C^\infty(M,G) \cong \varprojlim C^\infty(M,G_n), \quad f \mapsto (f_n)_{n\in\mathbb{N}}$$

is an isomorphism of topological groups.

Proof. First we note that our assumptions imply that

$$TG \cong \mathbf{L}(G) \rtimes G \cong \Big(\varprojlim \mathbf{L}(G_n) \rtimes G_n\Big) \cong \varprojlim T(G_n)$$

as topological groups. Moreover, writing $|\,\mathbf{L}(G)|$ for the topological vector space underlying $\mathbf{L}(G)$, considered as an abelian Lie algebra, we have

$$\mathbf{L}(TG) \cong |\,\mathbf{L}(G)| \rtimes \mathbf{L}(G) \cong \Big(\varprojlim |\,\mathbf{L}(G_n)| \rtimes \mathbf{L}(G_n)\Big) \cong \varprojlim \mathbf{L}(TG_n),$$

so that the Lie group TG inherits all properties assumed for G. Hence we may iterate this argument to obtain

$$T^kG \cong \varprojlim T^kG_n$$

for each k and that (1) holds for the Lie group T^kG.

We thus have topological embeddings

$$C(T^kM, T^kG)_c \hookrightarrow \varprojlim C(T^kM, T^kG_n)_c,$$

which leads to a topological embedding

$$C^\infty(M,G) \hookrightarrow \prod_{k\in\mathbb{N}_0} C(T^kM, T^kG)_c \hookrightarrow \prod_{k\in\mathbb{N}_0} \varprojlim C(T^kM, T^kG_n)_c \cong \varprojlim \prod_{k\in\mathbb{N}_0} C(T^kM, T^kG_n)_c,$$

showing that Ψ is a topological isomorphism. ∎

Lemma II.6. *If N and M are compact manifolds (possibly with boundary), then the map*

$$\Phi\colon C^\infty(N, C^\infty(M,K)) \to C^\infty(N\times M, K), \quad f \mapsto f^\wedge$$

is an isomorphism of Lie groups.

Proof. The bijectivity of Φ follows from the smoothness of the evaluation map of $C^\infty(M,K)$ (Proposition I.3) and Proposition I.9. To see that Φ is an isomorphism of Lie groups, let (φ, U) be a $\mathfrak{k}$-chart of K with $\varphi(\mathbf{1}) = 0$. Then $C^\infty(M,U)$ is an open identity neighborhood, so that $C^\infty(N, C^\infty(M,U))$ is an open identity neighborhood, and so is $C^\infty(N\times M, U)$. That Φ restricts to a diffeomorphism

$$C^\infty(N, C^\infty(M,U)) \to C^\infty(N\times M, U)$$

now follows from Lemma A.3 which asserts that

$$C^\infty(N, C^\infty(M,\mathfrak{k})) \to C^\infty(N\times M, \mathfrak{k}), \quad f \mapsto f^\wedge$$

is an isomorphism of topological vector spaces, hence restricts to diffeomorphisms on open subsets. ∎

Theorem II.7. *Let K be a regular Lie group and N and M finite-dimensional smooth σ-compact manifolds. We assume that $G := C^\infty(M,K)$ carries a regular Lie group structure compatible with evaluations and the smooth compact open topology. If $C^\infty(N,G)$ also carries a regular Lie group structure compatible with evaluations and the smooth compact open topology, then $C^\infty(N\times M, K)$ carries a regular Lie group structure compatible with evaluations. Moreover, the canonical map*

$$\Phi\colon C^\infty(N\times M, K) \to C^\infty(N, G), \quad f \mapsto f^\vee$$

is an isomorphism of Lie groups.

Proof. In view of Proposition I.9, the map Φ is a bijective group homomorphism. First we show that it is an isomorphism of topological groups.

Let $M = \bigcup_n M_n$ be an exhaustion of M by compact submanifolds M_n with boundary satisfying $M_n \subseteq M_{n+1}^0$. Then our definition of the group topology implies that

$$G = C^\infty(M,K) \cong \varprojlim C^\infty(M_n, K)$$

as topological groups. Put $G_n := C^\infty(M_n, K)$ and recall from Proposition I.3 that it carries a regular Lie group structure compatible with evaluations. We also have the isomorphism of topological Lie algebras

$$\mathbf{L}(G) = C^\infty(M,\mathfrak{k}) \cong \varprojlim \mathbf{L}(G_n) \cong \varprojlim C^\infty(M_n, \mathfrak{k}),$$

and Proposition I.9 implies that we have for each smooth manifold X:

$$C^\infty(X,G) \cong C^\infty(X\times M, K) \cong \varprojlim C^\infty(X\times M_n, K) \cong \varprojlim C^\infty(X, G_n)$$

on the level of groups (without topology).

Now let $(N_k)_{k\in\mathbb{N}}$ be an exhaustion of N by compact submanifolds with boundary. Then Lemmas II.5 and II.6 lead to the following isomorphisms of topological groups:

$$\begin{aligned} C^\infty(N,G) &\cong \varprojlim C^\infty(N,G_n) = \varprojlim C^\infty(N, C^\infty(M_n,K)) \\ &\cong \varprojlim {}_k\varprojlim {}_n C^\infty(N_k, C^\infty(M_n,K)) \cong \varprojlim {}_k\varprojlim {}_n C^\infty(N_k\times M_n, K) \\ &\cong C^\infty(N\times M, K). \end{aligned}$$

The preceding isomorphism leads to a regular Lie group structure on the topological group $C^\infty(N\times M,K)$. To see that this is the unique regular Lie group structure compatible with evaluations, we first observe that all evaluation maps

$$\mathrm{ev}_{(n,m)} = \mathrm{ev}_m \circ \mathrm{ev}_n \colon C^\infty(N, C^\infty(M,K)) \to K$$

are smooth and then apply Proposition I.9 to see that the evaluation map on $C^\infty(N\times M,K)$ is smooth. ∎

Applying Theorem II.7 and Corollary II.3 inductively, we obtain:

Corollary II.8. *Let K be a regular Lie group, M a finite-dimensional compact manifold, $k\in\mathbb{N}_0$ and $N := \mathbb{R}^k\times M$. Then $C^\infty(N,K)$ carries a regular Lie group structure compatible with evaluations and the smooth compact open topology.* ∎

Corollary II.9. *For each finite-dimensional connected Lie group M and each regular Lie group K, the group $C^\infty(M,K)$ carries a regular Lie group structure for which the evaluation map is smooth.*

Proof. This follows from the fact that each connected Lie group M is diffeomorphic to $\mathbb{R}^n\times C$, where C is a maximal compact subgroup and $n = \dim M - \dim C$ ([Ho65]). ∎

III. The complex case

In this section we assume that M is a connected complex manifold without boundary of dimension d and that K is a regular complex Lie group.

If E is a complex locally convex space, we write $\Omega^1_h(M,E)$ for the space of holomorphic E-valued 1-forms on M. For a complex Lie algebra $\mathfrak{k}$, we write

$$\mathrm{MC}_h(M,\mathfrak{k}) := \mathrm{MC}(M,\mathfrak{k}) \cap \Omega^1_h(M,\mathfrak{k})$$

for the set of holomorphic solutions of the Maurer–Cartan equation and topologize this space as a subspace of $\mathcal{O}(TM,\mathfrak{k})$, endowed with the compact open topology.

In the complex setting, the Fundamental Theorem is easily deduced from the real version (Theorem I.5):

Theorem III.1. (Complex Fundamental Theorem) *Let M be a complex manifold and K be a regular complex Lie group.*

(1) *A smooth function $f\colon M\to K$ is holomorphic if and only if $\delta(f)$ is a holomorphic $\mathfrak{k}$-valued 1-form.*

(2) *An element $\alpha\in\Omega^1_h(M,\mathfrak{k})$ is locally integrable to a holomorphic function if and only if it satisfies the Maurer–Cartan equation.*

(3) *Suppose that M is connected, fix $m_0\in M$ and assume that $\alpha\in\mathrm{MC}_h(M,\mathfrak{k})$. Then α is integrable to a holomorphic function $M\to K$ if and only if the period homomorphism $\mathrm{per}_\alpha^{m_0}$ is trivial.*

Proof. (1) If f is holomorphic, then $T(f)\colon T(M)\to T(K)$ is holomorphic, and since κ_K is holomorphic, the same holds for $\delta(f) = \kappa_K\circ T(f)$.

If, conversely, $\delta(f)$ is a holomorphic 1-form, then each map $T_x(f)\colon T_x(M)\to T_{f(x)}(K)$ is complex linear, so that f is holomorphic.

(2), (3) In view of (1), this follows from Theorem I.5. ∎

Remark III.2. Remark I.6(a) carries over to the holomorphic setting as follows. For each homomorphism $\chi\colon \pi_1(M,m_0)\to K$ we obtain a holomorphic flat K-bundle $P_\chi := \widetilde{M}\times_\chi K$. If M is a Stein manifold M and K is Banach, then the Oka principle (cf. [Rae77, Th. 2.1]) asserts that this bundle has a holomorphic section if and only if it has a continuous section. This is the case if and only if the corresponding space $\mathcal{O}(\widetilde{M},K)_\chi$ is non-empty.

A typical example where $\mathcal{O}(\widetilde{M},K)_\chi=\emptyset$ can be obtained as follows: Consider the complex manifold $M=\mathrm{PGL}_n(\mathbb{C})=\mathrm{PSL}_n(\mathbb{C})$ with the universal covering $\widetilde{M}\cong \mathrm{SL}_n(\mathbb{C})$ and identify $\pi_1(M)$ with the cyclic group $C_n := \mathrm{SL}_n(\mathbb{C})\cap \mathbb{C}^\times\mathbf{1}$ of order n. Define $\chi\colon\pi_1(M)\to\mathbb{C}^\times$ by $z=\chi(z)^{-1}\mathbf{1}$. Then

$$P_\chi=\widetilde{M}\times_\chi\mathbb{C}^\times=\mathrm{SL}_n(\mathbb{C})\times_\chi\mathbb{C}^\times\cong\mathrm{SL}_n(\mathbb{C})\cdot\mathbb{C}^\times\mathbf{1}=\mathrm{GL}_n(\mathbb{C}),$$

and this $\mathbb{C}^\times$-bundle over M is non-trivial because the corresponding surjective homomorphism

$$\pi_1(\mathrm{GL}_n(\mathbb{C}))\cong\mathbb{Z}\to\pi_1(M)\cong C_n$$

does not split since $\mathbb{Z}$ is torsion free. ∎

Proposition III.3. *If M is a connected complex manifold without boundary and K a regular complex Lie group, then the map*

$$\delta\colon\mathcal{O}_*(M,K)\to\Omega^1_h(M,\mathfrak{k})\subseteq\mathcal{O}(T(M),\mathfrak{k})$$

is a topological embedding if $\mathcal{O}(M,K)$ carries the compact open topology.

Proof. To see that the inclusion $\mathcal{O}(M,K)\hookrightarrow C^\infty(M,K)$ is a topological embedding for any complex Lie group K, it suffices to prove that uniform convergence of holomorphic functions in $\mathcal{O}(M,K)$ implies uniform convergence of all tangent maps $T^n(f)\colon T^n(M)\to T^n(K)$ on compact subsets.

Suppose that $f_i\to f$ in $\mathcal{O}(M,K)$ and let $C\subseteq M$ be a compact subset for which $f(C)$ lies in U_K for some holomorphic $\mathfrak{k}$-chart (φ,U_K) of K. Then we may w.l.o.g. assume that $f_i(C)\subseteq U_K$ for each i. As we have seen in Proposition I.2(2), this implies that, on the interior C^0, $\varphi\circ f_i|_{C^0}\colon C^0\to\mathfrak{k}$ converges to $\varphi\circ f|_{C^0}$ in $C^\infty(C^0,\mathfrak{k})$, but this also implies that $f_i|_{C^0}\to f|_{C^0}$ in $C^\infty(C^0,K)$ (Lemma A.4). Since each compact subset of M can be covered with finitely many sets of the form C^0, it follows that $f_i\to f$ in $C^\infty(M,K)$.

Now the assertion of the corollary follows from Proposition II.1. ∎

Lemma III.4. *If K is a regular complex Lie group, then* $\mathrm{evol}_K\colon C^\infty([0,1],\mathfrak{k})\to K$ *is holomorphic.*

Proof. From Corollary II.3, we know that $C^\infty([0,1],\mathfrak{k})$ carries a natural Lie group structure, and for this Lie group structure, the map evol_K is a group homomorphism. In view of the regularity of K, this map is smooth. To verify its holomorphy, it therefore suffices to show that $T_{\mathbf{1}}(\mathrm{evol}_K)$ is complex linear which is an immediate consequence of Lemma A.5(1). ∎

We now adapt Proposition I.8, Proposition I.9 and Corollary I.10 to complex Lie groups to derive a complex version of Theorems II.2.

Proposition III.5. *Let N be a locally convex complex manifold, M a connected finite-dimensional complex manifold and K a regular complex Lie group. Then a function $f\colon N\times M\to K$ is holomorphic if and only if*

(1) *there exists a point $m_0\in M$ for which the map $f^{m_0}\colon N\to K, n\mapsto f(n,m_0)$ is holomorphic, and*
(2) *the functions $f_n\colon M\to K, m\mapsto f(n,m)$ are holomorphic and $F\colon N\to\Omega^1_h(M,\mathfrak{k}), n\mapsto\delta(f_n)$ is holomorphic.*

Proof. "⇒" is verified as in the real case (Proposition I.8).

"$\Leftarrow$": The proof follows the line of the real case. In addition, we use that evol_K is holomorphic (Lemma III.4) and that the pull-back maps γ^* are complex linear. One point that requires some extra care is the verification of the holomorphy of the map

$$\Omega^1_h(\varphi(U),\mathfrak{k}) \times U \to C^\infty(I,\mathfrak{k}), \quad \gamma_x^*\alpha = \alpha \circ T(\gamma_x).$$

From the proof of Proposition I.8 we know that it is smooth and it is complex linear in α, so that its holomorphy follows from its holomorphy in x. ∎

Proposition III.6. *Let M be a connected finite-dimensional complex manifold and K a regular complex Lie group. For a complex Lie group structure on the group $G := \mathcal{O}(M,K)$ for which $\mathfrak{g} := \mathcal{O}(M,\mathfrak{k})$ is the corresponding Lie algebra and all evaluation maps $\mathrm{ev}_m\colon G \to K$, $m \in M$, are holomorphic with*

$$\mathbf{L}(\mathrm{ev}_m) = \mathrm{ev}_m\colon \mathfrak{g} \to \mathfrak{k}.$$

Then the following assertions hold:

(1) *The evaluation map $\mathrm{ev}\colon G \times M \to K, (f,m) \mapsto f(m)$ is holomorphic.*

(2) *If, in addition, G is regular, then a map $f\colon N \to G$ is holomorphic if and only if the corresponding map $f^\wedge\colon N \times M \to G$ is holomorphic.*

Proof. (1) From Proposition I.9 it follows that δ is smooth. It also satisfies the cocycle identity

$$\delta(f_1 f_2) = \mathrm{Ad}(f_2)^{-1}.\delta(f_1) + \delta(f_2).$$

Since the maps $\mathrm{Ad}(f)$ are complex linear on $\Omega^1_h(M,\mathfrak{k})$, it therefore suffices to observe that $T_{\mathbf{1}}(\delta)(f) = df$ is complex linear in f, to conclude that δ is holomorphic.

(2) If f is holomorphic, then $f^{m_0} = \mathrm{ev}_{m_0} \circ f$ is holomorphic and (1) entails that $\delta \circ f\colon N \to \Omega^1_h(M,\mathfrak{k})$ is holomorphic, so that Proposition III.5 implies that $f^\wedge$ is holomorphic.

If, conversely, $f^\wedge$ is holomorphic, we first use Proposition I.9 to see that f is smooth. That its differential is complex linear follows from the holomorphy of $f^\wedge$. ∎

In the following, we say that a Lie group structure on $\mathcal{O}(M,K)$ is *compatible with evaluations* if it satisfies the assumptions of the preceding proposition. As in the real case, we obtain:

Corollary III.7. *Under the assumptions of the preceding theorem, there exists at most one regular complex Lie group structure on the group $\mathcal{O}(M,K)$ which is compatible with evaluations.* ∎

Theorem III.8. *Let M be a connected complex manifold and K a complex regular Lie group. Assume that the subset $\delta(\mathcal{O}_*(M,K))$ is a complex submanifold of $\Omega^1_h(M,\mathfrak{k})$ and endow $\mathcal{O}_*(M,K)$ with the manifold structure for which $\delta\colon \mathcal{O}_*(M,K) \to \mathrm{im}(\delta)$ is biholomorphic and*

$$\mathcal{O}(M,K) \cong K \ltimes \mathcal{O}_*(M,K)$$

with the product manifold structure. Then the following assertions hold:

(1) *For each locally convex complex manifold N, a map $f\colon N \times M \to K$ is holomorphic if and only if all the maps $f_n\colon M \to K, m \mapsto f(n,m)$ are holomorphic and the corresponding map*

$$f^\vee\colon N \to \mathcal{O}(M,K), \quad n \mapsto f_n$$

is holomorphic.

(2) *K acts holomorphically by conjugation on $\mathcal{O}_*(M,K)$, and $\mathcal{O}(M,K)$ carries a regular complex Lie group structure compatible with evaluations.*

Proof. (1) is proved as Theorem II.2(1). Here we use Proposition III.5 instead of Proposition I.8.

(2) For the evaluation map $f = \mathrm{ev}\colon G \times M \to K$, we have $\mathrm{ev}^\vee = \mathrm{id}_G$ and $\mathrm{ev}_g = g$ for each $g \in G$. Hence (1) implies that ev is holomorphic.

The complex Lie group structure: In view of Propositions II.1 and III.3, δ is an isomorphism of topological groups if $\mathrm{im}(\delta)$ is endowed with the group structure (2.1). To see that the group operations are holomorphic, we have to show that the map

$$\mathrm{im}(\delta) \times \Omega^1_h(M,\mathfrak{k}) \to \Omega^1_h(M,\mathfrak{k}), \quad (\alpha,\beta) \mapsto \mathrm{Ad}(\mathrm{Evol}_K(\alpha)).\beta$$

is holomorphic. Its smoothness has already been verified in the proof of Theorem II.2. Since it is complex linear in β, it is holomorphic in the second argument.

We claim that it is also holomorphic in the first argument α. Since the evaluation maps

$$\mathrm{ev}_m \colon \Omega^1_h(M,\mathfrak{k}) \to \mathrm{Hom}(T_m(M),\mathfrak{k})$$

are complex linear, and the adjoint action of K on $\mathfrak{k}$ is holomorphic, it suffices to show that for each element $m \in M$, the map

$$\mathrm{im}(\delta) \to K, \quad \alpha \mapsto \mathrm{Evol}_K(\alpha)(m)$$

is holomorphic.

For any piecewise smooth path $\gamma \colon [0,1] \to M$ with $\gamma(0) = m_0$ and $\gamma(1) = m$ we have

$$\mathrm{Evol}_K(\alpha)(m) = \mathrm{evol}_K(\gamma^*\alpha),$$

so that the holomorphy of evol_K (Lemma III.4), combined with the complex linearity of the map

$$\Omega^1_h(M,\mathfrak{k}) \to C^\infty([0,1],\mathfrak{k}), \quad \alpha \mapsto \gamma^*\alpha$$

implies that $\mathrm{im}(\delta)$ is a complex Lie group. As in Theorem II.2(2), we see that

$$d \colon \mathcal{O}_*(M,\mathfrak{k}) \to T_0(\mathrm{im}(\delta))$$

is an isomorphism of Lie algebras if $\mathcal{O}_*(M,\mathfrak{k})$ is endowed with the pointwise Lie bracket.

Regularity: It remains to verify the regularity of G, i.e., the holomorphy of the map evol_G. As in the real case, we see that

$$\mathrm{evol}_G(\xi)(m) = \mathrm{evol}_K(\xi^m) = \mathrm{evol}^\vee_G(\xi,m),$$

which is smooth by (Lemma A.6(3)). Since evol_K is holomorphic (Lemma III.4) and ξ^m is complex linear in ξ and holomorphic in m, it follows that $\mathrm{evol}^\vee_G \colon C^\infty(I,\mathfrak{g}) \times M \to K$ is holomorphic, which implies that evol_G is holomorphic. ∎

As in the real case, there is a natural situation where $\mathrm{im}(\delta)$ is a submanifold, namely if M is one-dimensional and simply connected. In this case, each $\mathfrak{k}$-valued holomorphic 2-form on M vanishes, so that all holomorphic 1-forms satisfy the Maurer–Cartan equation, and if M is simply connected, Theorem III.1 implies that $\mathrm{im}(\delta) = \Omega^1_{\mathrm{dR,h}}(M,\mathfrak{k})$, which is in particular a submanifold. If M has no boundary, then the Riemann Mapping Theorem implies that it is isomorphic to $\mathbb{C}$, the unit disc $\Delta := \{z \in \mathbb{C} \colon |z| < 1\}$, or the Riemann sphere $\widehat{\mathbb{C}} \cong \mathbb{S}^2$.

Corollary III.9. *For each regular complex Lie group K and each 1-connected complex curve M without boundary, the group $\mathcal{O}_*(M,K)$ carries a regular complex Lie group structure for which*

$$\delta \colon \mathcal{O}_*(M,K) \to \Omega^1_h(M,\mathfrak{k})$$

is biholomorphic and $\mathcal{O}(M,K) \cong K \ltimes \mathcal{O}_*(M,K)$ carries a regular complex Lie group structure compatible with evaluations and the compact open topology. ∎

Remark III.10. Let $M := \widehat{\mathbb{C}}$ be the Riemann sphere. That the space $\Omega^1_h(\widehat{\mathbb{C}}, \mathbb{C})$ is trivial (which is well-known) can be seen as follows. Each holomorphic $\mathbb{C}$-valued 1-form α is closed, hence exact because $\widehat{\mathbb{C}}$ is simply connected. Now there exists a holomorphic function $f\colon \widehat{\mathbb{C}} \to \mathbb{C}$ with $df = \alpha$, and since f is constant, we get $\alpha = 0$. From this and the Hahn–Banach Theorem, we directly get $\Omega^1(\widehat{\mathbb{C}}, \mathfrak{k}) = \{0\}$ for any locally convex space $\mathfrak{k}$.

In view of Corollary III.9 and the fact that δ is injective on $\mathcal{O}_*(\widehat{\mathbb{C}}, K)$, we now derive

$$\mathcal{O}_*(\widehat{\mathbb{C}}, K) = \{\mathbf{1}\}$$

for any complex Lie group K, and this is independent of whether K has non-constant holomorphic functions $K \to \mathbb{C}$ or not. In particular, we see that there is no non-constant holomorphic function from $\widehat{\mathbb{C}}$ to any Lie group of the form E/Γ_E, where Γ_E is a discrete subgroup of the complex locally convex space E. ∎

We extract the following version of the Regular Value Theorem from Glöckner's Implicit Function Theorem ([Gl03]):

Theorem III.11. *Let M be a locally convex manifold, N a Banach manifold, $F\colon M \to N$ a smooth map and $n_0 \in N$. Assume that for each $m \in M$ with $F(m) = n_0$ there exists a continuous linear splitting of the tangent map*

$$T_m(F)\colon T_m(M) \to T_{n_0}(N).$$

Then $F^{-1}(n_0)$ is a split submanifold of M.

If, in addition, M and N are complex manifolds and F is holomorphic, then $F^{-1}(n_0)$ is a complex split submanifold of M.

Proof. Since the property of being a submanifold is local, it suffices to show that each $m_0 \in F^{-1}(n_0)$ has an open neighborhood U for which $U \cap F^{-1}(n_0)$ is a submanifold of U. In particular, we may assume that M is an open subset of a locally convex space $X \cong T_{m_0}(M)$. In view of the continuity of F, we may choose U in such a way that $F(U)$ is contained in a chart domain in N, so that we may further assume that $N \cong T_{n_0}(N)$ is a Banach space.

Fix a continuous linear splitting $\sigma\colon N \to X$ of the tangent map $T_{m_0}(F)$. Then $Y := \ker T_{m_0}(F)$ is a closed subspace of X and the map

$$Y \times N \to X, \quad (y, v) \mapsto y + \sigma(v)$$

is a linear topological isomorphism. We may therefore assume that $X = Y \times N$, write m_0 accordingly as (y_0, e_0), and that $T_{m_0}(F)\colon X = Y \times N \to N$ is the linear projection onto N.

Now Theorem 2.3 in [Gl03] implies the existence of an open neighborhood U of m_0 in $Y \times N$ and a diffeomorphism $\theta\colon U \to \theta(U)$ onto some open neighborhood of (y_0, n_0) in $Y \times N$ with

$$\theta(a, b) = (a, F(a, b)) \quad \text{for} \quad (a, b) \in U.$$

This implies that

$$F^{-1}(n_0) \cap U = \theta^{-1}(Y \times \{n_0\})$$

is a smooth submanifold of U. The remaining assertions are immediate from loc. cit. ∎

The following theorem is the second main result of this section.

Theorem III.12. *Let M be a non-compact connected complex curve without boundary. Assume further that $\pi_1(M)$ is finitely generated and that K is a complex Banach–Lie group. Then the group $\mathcal{O}_*(M, K)$ carries a regular complex Lie group structure for which*

$$\delta\colon \mathcal{O}_*(M, K) \to \Omega^1_h(M, \mathfrak{k})$$

is biholomorphic onto a complex submanifold, and $\mathcal{O}(M, K) \cong K \ltimes \mathcal{O}_*(M, K)$ carries a regular complex Lie group structure compatible with evaluations.

Proof. In view of Theorem III.8, it suffices to show that $\mathrm{im}(\delta)$ is a complex submanifold.

First we recall that the fundamental group $\pi_1(M)$ is free, because this is true for all non-compact surfaces without boundary. Let

$$\gamma_1, \dots, \gamma_r \colon [0,1] \to M$$

be piecewise smooth loops in the base point m_0 such that $[\gamma_1], \dots, [\gamma_r]$ are free generators of $\pi_1(M, m_0)$. Then the map

$$\mathrm{Hom}(\pi_1(M), K) \to K^r, \quad \chi \mapsto (\chi([\gamma_1]), \dots, \chi([\gamma_r])) \tag{3.1}$$

is a bijection.

Since the Maurer–Cartan equation is trivially satisfied for holomorphic 1-forms on a complex curve (cf. Corollary III.9), we have $\mathrm{im}(\delta) = P^{-1}(\mathbf{1})$ for the map

$$P \colon \Omega^1_h(M, \mathfrak{k}) \to K^r, \quad \alpha \mapsto (\mathrm{per}^{m_0}_\alpha([\gamma_1]), \dots, \mathrm{per}^{m_0}_\alpha([\gamma_r]))$$

(Theorem III.1). Since

$$\mathrm{per}^{m_0}_\alpha([\gamma]) = \mathrm{evol}_K(\gamma^*\alpha)$$

depends holomorphically on α for each piecewise smooth curve γ on M (Lemma III.4), P is a holomorphic map from the complex Fréchet space $\Omega^1_h(M, \mathfrak{k})$ to the complex Banach manifold K^r.

To see that $P^{-1}(\mathbf{1})$ is a submanifold, we have to verify the assumptions of Theorem III.11. The Behnke–Stein Theorem ([Fo77, Satz 28.6]) implies that each group homomorphism $\pi_1(M) \to \mathbb{C}$ can be realized by integration against a holomorphic 1-form. Hence there exist holomorphic 1-forms $\beta_1, \dots, \beta_r \in \Omega^1_h(M, \mathbb{C})$ with

$$\int_{\gamma_i} \beta_j = \delta_{ij}. \tag{3.2}$$

We define a linear map

$$\sigma \colon \mathfrak{k}^r \to \Omega^1_h(M, \mathfrak{k}), \quad (x_1, \dots, x_r) \mapsto \sum_{j=1}^r \beta_j \cdot x_j$$

whose continuity follows from Lemma A.5(3).

To verify for $\alpha \in P^{-1}(\mathbf{1})$ that the map $T_\alpha(P)$ has a continuous linear section, we consider the map

$$\sigma_\alpha \colon \mathfrak{k}^r \to \Omega^1_h(M, \mathfrak{k}), \quad x \mapsto \alpha + \mathrm{Ad}(f)^{-1}.\sigma(x) = \delta(f) + \mathrm{Ad}(f)^{-1}.\sigma(x),$$

where $f \in \mathcal{O}_*(M, K)$ is the unique function with $\delta(f) = \alpha$ (Theorem III.1). As f is fixed, σ_α is a continuous affine map, hence in particular holomorphic. From Remark I.6(a) we further know that

$$\mathrm{per}^{m_0}_{\sigma_\alpha(x)} = \mathrm{per}^{m_0}_{\sigma(x)},$$

so that $P \circ \sigma_\alpha = P \circ \sigma$.

In view of $\mathrm{per}^{m_0}_\beta([\gamma]) = \mathrm{evol}_K(\gamma^*\beta)$, the differential of the map $\beta \mapsto \mathrm{per}^{m_0}_\beta$ in 0 is given by

$$T_{\mathbf{1}}(\mathrm{evol}_K)(\gamma^*\beta) = \int_0^1 \gamma^*\beta = \int_\gamma \beta$$

(Lemma A.5(1)). Therefore

$$T_0(P)(\beta) = \Big(\int_{\gamma_1} \beta, \dots, \int_{\gamma_r} \beta \Big),$$

considered as an element of $\mathfrak{k}^r$. From

$$\int_{\gamma_i} \sigma(x_1, \dots, x_r) = \sum_{j=1}^r \int_{\gamma_i} \beta_j \cdot x_j = x_i,$$

we derive $T_0(P) \circ \sigma = \mathrm{id}_{\mathfrak{k}^r}$. We further have

$$T_\alpha(P) \circ \mathrm{Ad}(f)^{-1} \circ \sigma = T_\alpha(P) \circ T_0(\sigma_\alpha) = T_0(P \circ \sigma_\alpha) = T_0(P \circ \sigma) = T_0(P) \circ \sigma = \mathrm{id}_{\mathfrak{k}^r}.$$

Hence $\mathrm{Ad}(f)^{-1} \circ \sigma$ is a continuous linear section of $T_\alpha(P)$. Since $\alpha \in P^{-1}(\mathbf{1})$ was arbitrary, Theorem III.11 implies that $P^{-1}(\alpha)$ is a complex submanifold of $\Omega^1_h(M, \mathfrak{k})$. ∎

Corollary III.13. *Let Σ be a compact complex curve, $F \subseteq \Sigma$ a finite set and $M := \Sigma \setminus F$. Then, for each Banach–Lie group K, the group $\mathcal{O}(M,K)$ carries a regular complex Lie group structure compatible with the compact open topology and with evaluations.*

In particular, for each Banach–Lie group K, the topological group $\mathcal{O}(\mathbb{C}^\times, K)$ carries a compatible Lie group structure.

Proof. To apply Theorem III.12, it suffices to verify that $\pi_1(M)$ is finitely generated, but this follows from the fact that M is homotopic to a compact surface with $|F|$ boundary circles. ■

Example III.14. Let $M = \mathbb{C}^\times$ and $K = \mathrm{GL}_n(\mathbb{C})$. Then we associate to each holomorphic function $\xi\colon \mathbb{C}^\times \to \mathfrak{gl}_n(\mathbb{C})$ the 1-form

$$\alpha = \xi(z)dz.$$

Now $\delta(f) = \alpha$ is equivalent to the requirement that f is a solution of the linear differential equation

$$f'(z) = f(z)\xi(z).$$

If, for example, $\xi(z) = z^{-1}A$ for a matrix A, then the differential equation reads

$$f'(z) = z^{-1}Af(z), \tag{3.3}$$

and the corresponding 1-form on the group $(\mathbb{C}^\times, \cdot)$ is invariant. A solution of (3.3) exists if and only if

$$f(z) = e^{\log z \cdot A}$$

is well-defined. The corresponding period homomorphism is

$$P(\alpha)\colon \mathbb{Z} \to \mathrm{GL}_n(\mathbb{C}), \quad n \mapsto e^{2\pi i n A}.$$

Therefore $\alpha \in P^{-1}(\mathbf{1})$ is equivalent to $e^{2\pi i A} = \mathbf{1}$, which is equivalent to the diagonalizability of A and $\mathrm{Spec}(A) \subseteq \mathbb{Z}$.

Note that on the subspace $M_n(\mathbb{C})\frac{dz}{z}$ the matrix $\mathbf{1} \in \mathrm{GL}_n(\mathbb{C})$ is not a regular value of P because $\mathbf{1}$ is a not a regular value of the exponential function. Despite this fact, we have seen in the proof of Theorem III.12 that $\mathbf{1}$ is a regular value of the holomorphic function

$$P\colon \Omega^1(\mathbb{C}^\times, M_n(\mathbb{C})) \to \mathrm{GL}_n(\mathbb{C}), \quad \alpha \mapsto \mathrm{per}_\alpha\,. \qquad ■$$

Remark III.15. Throughout the present section we considered only complex manifolds without boundary to make sure that the compact open topology on $\mathcal{O}(M,K)$ is the right one. If M has non-empty boundary, all results remain true with respect to the finer smooth compact open topology. ■

IV. Maps with values in special Lie groups

In this section we discuss the group $C^\infty(M,K)$ under the assumption that the universal covering group $\widetilde{K}$ of K is diffeomorphic to a locally convex space. This includes in particular all regular connected abelian Lie groups ([MT99], [GN07]), all finite-dimensional solvable Lie groups and many interesting projective limits of Lie groups ([HoNe06]).

The starting point is the result that $C^\infty(M,\widetilde{K})$ always carries a Lie group structure compatible with evaluations. Then we study the passage from K to $\widetilde{K}$, which is encoded in an exact sequence

$$\mathbf{1} \to \pi_1(K) \to C^\infty(M,\widetilde{K}) \to C^\infty(M,K) \to \mathrm{Hom}(\pi_1(M),\pi_1(K)) \to \mathbf{1}$$

which shows that the Lie group $C^\infty(M,\widetilde{K})/\pi_1(K)$ can be identified with a normal subgroup of $C^\infty(M,K)$, which eventually leads to a Lie group structure on the whole group. We further show that, if the group $\mathrm{Hom}(\pi_1(M),\pi_1(K))$, endowed with the topology of pointwise convergence, is discrete, then this Lie group structure is compatible with the smooth compact open topology. To shed some light on these subtleties, we briefly discuss the groups $\mathrm{Hom}(A,\Gamma)$ for an abelian group A and a discrete subgroup Γ of a locally convex space, and this discussion leads to necessary conditions for the Lie group structure on $C^\infty(M,K)$ to be compatible with the smooth compact open topology if K is abelian or finite-dimensional.

Proposition IV.1. *Let M be a finite-dimensional manifold, K a Lie group with Lie algebra $\mathfrak{k}$, $\varphi_K: K \to E$ a diffeomorphism onto a locally convex space and $G := C^\infty(M,K)$. Then*

$$\varphi_G: G \to C^\infty(M,E), \quad f \mapsto \varphi_K \circ f$$

is a homeomorphism which defines a manifold structure on G, and this turns G into a Lie group compatible with evaluations. If, in addition, K is regular, then G is also regular.

Proof. It follows directly from the functoriality of the topology on $C^\infty(M,N)$ in N that φ_G is a homeomorphism. We consider (φ_G, G) as a smooth atlas of the topological group G.

To see that the group operations of G are smooth, let $m_K: K \times K \to K$ denote the multiplication of K and η_K its inversion map. Then

$$\varphi_K \circ m_K \circ (\varphi_K^{-1} \times \varphi_K^{-1}): E \times E \to E$$

is smooth, so that Lemma A.4 implies that the induced map

$$(\varphi_K \circ m_K \circ (\varphi_K^{-1} \times \varphi_K^{-1}))_*: C^\infty(M, E\times E) \to C^\infty(M,E)$$

is smooth, and this means that the multiplication in G is smooth. With a similar argument, we see that the inversion is also smooth. Hence G is a Lie group.

To calculate the Lie algebra of this group, we observe that for $m \in M$, we have for the multiplication in the chart (φ_G, G)

$$\begin{aligned}(f *_G g)(m) &:= \varphi_G\Big(\varphi_G^{-1}(f)\varphi_G^{-1}(g)\Big)(m) = \varphi_K\big(\varphi_K^{-1}(f(m))\varphi_K^{-1}(g(m))\big)\\ &= f(m) *_K g(m) = f(m) + g(m) + b_{\mathfrak{k}}(f(m),g(m)) + \cdots,\end{aligned}$$

where the Lie bracket in $\mathfrak{k} \cong T_{\mathbf{1}}(K) \cong E$ satisfies

$$[x,y] = b_{\mathfrak{k}}(x,y) - b_{\mathfrak{k}}(y,x)$$

(cf. [GN07]). Hence, we accordingly have $\big(b_{\mathfrak{g}}(f,g)\big)(m) = b_{\mathfrak{k}}(f(m),g(m))$, and thus

$$[f,g](m) = b_{\mathfrak{g}}(f,g)(m) - b_{\mathfrak{g}}(g,f)(m) = b_{\mathfrak{k}}(f(m),g(m)) - b_{\mathfrak{k}}(g(m),f(m)) = [f(m),g(m)].$$

Therefore $\mathfrak{g} = C^\infty(M,\mathfrak{k})$, endowed with the pointwise defined Lie bracket, is the Lie algebra of G.

The compatibility of the Lie group structure with evaluations follows directly from Proposition I.2 and the definition of the manifold structure on G.

Now we assume that K is regular. With Lemma A.6(3), we obtain for each $\xi \in C^\infty(I,\mathfrak{g})$ a curve $\gamma: I \to C^\infty(M,K)$ by $\gamma(t)(m) := \mathrm{Evol}_K(\xi^m)(t)$, defining a smooth map $I \times M \to K$, hence a smooth curve in G (Lemma A.2) because

$$C^\infty(I,G) \cong C^\infty(I, C^\infty(M,E)) \cong C^\infty(I\times M, E) \cong C^\infty(I\times M, K).$$

Further, $\delta(\gamma^m) = \xi^m$ implies that the evolution map of G is given by $\mathrm{evol}_G(\xi)(m) := \mathrm{evol}_K(\xi^m)$. Now the smoothness of evol_G follows from Lemma A.6(3) and Lemma A.2. ∎

Theorem IV.2. *Let M be a finite-dimensional connected σ-compact manifold, K a connected Lie group with Lie algebra $\mathfrak{k}$ whose universal covering group $\widetilde{K}$ is diffeomorphic to a locally convex space. Then the following assertions hold:*

(1) *$G := C^\infty(M,K)$ carries the structure of a Lie group compatible with evaluations. If K is regular, then G is regular.*

(2) *On the identity component G_0, this Lie group structure is compatible with the smooth compact open topology. In particular, the Lie group structure on G is compatible with the*

smooth compact open topology if and only if G_0 is open with respect to the smooth compact open topology.

(3) $\pi_0(G) \cong \operatorname{Hom}(\pi_1(M), \pi_1(K))$ *with respect to the Lie group structure and the smooth compact open topology.*

(4) *If $\pi_1(M)$ is finitely generated or, more generally, $\operatorname{Hom}(\pi_1(M), \pi_1(K))$ is discrete with respect to the topology of pointwise convergence, then G_0 is also open in the smooth compact open topology, so that the Lie group structure on G is compatible with this topology.*

Proof. (1) First we apply Proposition IV.1 to the Lie group $\widetilde{G}_0 := C^\infty(M, \widetilde{K})$ to obtain a Lie group structure compatible with evaluations. Let $q_K\colon \widetilde{K} \to K$ denote the universal covering group. Since $\ker q_K$ is central in $\widetilde{K}$, the conjugation action

$$C_{\widetilde{K}}\colon \widetilde{K} \times \widetilde{K} \to \widetilde{K}, \quad (x, y) \mapsto xyx^{-1}$$

factors through a smooth action

$$\widetilde{C}_K\colon K \times \widetilde{K} \to \widetilde{K}, \quad (x, y) \mapsto xyx^{-1}.$$

We claim that the corresponding action of $G = C^\infty(M, K)$ on $\widetilde{G}_0 = C^\infty(M, \widetilde{K})$ by $(f.g)(m) := \widetilde{C}_K(f(m))(g(m))$, is an action by smooth automorphisms. To see this, first observe that if $q_M\colon \widetilde{M} \to M$ is the universal covering manifold of M, then $C^\infty(\widetilde{M}, \widetilde{K})$ also carries a smooth Lie group structure for which we may identify $C^\infty(M, \widetilde{K})$ with a closed submanifold (corresponding to a closed vector subspace under the chart in Proposition IV.1). Since each smooth map $f\colon M \to K$ can be lifted to a smooth map $\widetilde{f}\colon \widetilde{M} \to \widetilde{K}$, the corresponding automorphism of $\widetilde{G}_0$ coincides with the restriction of the automorphism $c_{\widetilde{f}}$ of $C^\infty(\widetilde{M}, \widetilde{K})$ to $C^\infty(M, \widetilde{K})$, hence is smooth.

Since $\ker q_K \cong \pi_1(K) \subseteq \widetilde{K}$ is a discrete central subgroup of $\widetilde{K}$ and therefore also of the Lie group $\widetilde{G}_0$, the quotient $G_0 := \widetilde{G}_0/\pi_1(K)$ carries a unique Lie group structure for which the quotient map $\widetilde{G}_0 \to G_0$ is a covering (cf. [GN07]). This quotient map corresponds to the homomorphism

$$q_K^M\colon C^\infty(M, \widetilde{K}) \to C^\infty(M, K), \quad f \mapsto q_K \circ f$$

which is equivariant with respect to the aforementioned action of the group $C^\infty(M, K)$ on $C^\infty(M, \widetilde{K})$ by conjugation. Hence $G_0 \cong \operatorname{im}(q_K^M)$ is a normal subgroup which carries a Lie group structure and the other group elements act by smooth automorphisms. This implies that the Lie group structure extends uniquely to all of G in such a way that G_0 is the open identity component of G (cf. [GN07]). We thus obtain a Lie group structure on G for which $q_K^M\colon \widetilde{G}_0 \to G_0$ is the universal covering map, which implies that its Lie algebra also is $\mathfrak{g}$. The evaluation map $\operatorname{ev}\colon G \times M \to K$ can be obtained by factorization of the evaluation map $\widetilde{\operatorname{ev}}\colon \widetilde{G}_0 \times M \to \widetilde{K}$ because

$$q_K \circ \widetilde{\operatorname{ev}} = \operatorname{ev} \circ (q_K^M \times \mathrm{id}_M),$$

which shows that it is smooth on $G_0 \times M$ and hence on all of $G \times M$ because it is multiplicative in the first argument.

If, in addition, K is regular, then Proposition IV.1 implies that $\widetilde{G}_0$ is regular, and this easily implies that G_0 and G are regular Lie groups (cf. [GN07]).

(2) The construction in Proposition IV.1 implies that on $\widetilde{G}_0 = C^\infty(M, \widetilde{K})$ the Lie group structure is compatible with the smooth compact open topology. Writing this group as a semidirect product $\widetilde{G}_0 \cong C^\infty_*(M, \widetilde{K}) \rtimes \widetilde{K}$, we see that $G_0 \cong C^\infty_*(M, \widetilde{K}) \rtimes K$, so that it suffices to see that the injective continuous homomorphism

$$q_K^M\colon C^\infty_*(M, \widetilde{K}) \to C^\infty_*(M, K)$$

of topological groups is a topological embedding with respect to the smooth compact open topology.

To verify this claim, let $m \in M$ and C be a relatively compact open neighborhood of m. To see that

$$(q_K^M)^{-1}\colon q_K^M(C_*^\infty(M,\widetilde{K})) \to C_*^\infty(M,\widetilde{K})$$

is continuous in the smooth compact open topology, we note that for each $m \in \mathbb{N}$, the map

$$T^m(q_K)\colon T^m\widetilde{K} \to T^m K$$

is the universal covering morphism of the Lie group T^mK. If we can show that for all these coverings, the corresponding map

$$(q_{T^mK}^M)^{-1}\colon q_{T^mK}^M(C_*(M,T^m\widetilde{K})) \to C_*(M,T^m\widetilde{K})$$

is continuous in the compact open topology, the corresponding assertion follows. Hence it suffices to show that

$$(q_K^M)^{-1}\colon q_K^M(C_*(M,\widetilde{K}))_c \to C_*(M,\widetilde{K})_c \tag{4.1}$$

is continuous with respect to the compact open topology.

Evaluation in the base point m_0 maps the subgroup $\pi_1(K) \cong \ker q_K$ to a discrete subgroup of $\widetilde{K}$. Hence it is discrete as a subgroup of $C(M,\widetilde{K})_c$, so that for any sufficiently small $\mathbf{1}$-neighborhood V in $C(M,\widetilde{K})$ we have $C_*(M,\widetilde{K}) \cap V\pi_1(K) \subseteq V$. Let C be a compact connected subset of M and U be an open $\mathbf{1}$-neighborhood in $\widetilde{K}$ with

$$W(C,U)\pi_1(K) \cap C_*(M,K) \subseteq W(C,U) \quad \text{and} \quad UU^{-1} \cap \pi_1(K) = \{\mathbf{1}\}.$$

We claim that for any $\widetilde{f} \in C_*(M,\widetilde{K})$ the relation $q_K^M(\widetilde{f}) \in W(C, q_K(U))$ implies $\widetilde{f} \in W(C,U)$, and this implies the continuity of (4.1). Any such $\widetilde{f}$ maps the connected set C into the open subset $U\pi_1(K) = q_K^{-1}(q_K(U))$, and our assumption on U implies that the open sets Uz, $z \in \pi_1(K)$, are pairwise disjoint. Hence there exists a $z \in \pi_1(K)$ with $\widetilde{f} \in W(C,Uz) = W(C,U)z$, so that $\widetilde{f} \in C_*(M,\widetilde{K})$ leads to $z = \mathbf{1}$, i.e., $\widetilde{f} \in W(C,U)$.

This completes the proof of (2). We also note that if K is regular, then the assertion follows directly from Proposition II.1, because both maps

$$\delta_1\colon C_*^\infty(M,\widetilde{K}) \to \Omega^1(M,\mathfrak{k}) \quad \text{and} \quad \delta_2\colon C_*^\infty(M,K) \to \Omega^1(M,\mathfrak{k})$$

are topological embeddings with $\delta_2 \circ q_K^M = \delta_1$.

(3) The range of q_K^M consists of all smooth maps $f\colon M \to K$ lifting to maps $\widetilde{f}\colon M \to \widetilde{K}$. If $m_0 \in M$ is a base point, this condition is equivalent to the condition that the homomorphism

$$\Gamma(f)\colon \pi_1(M,m_0) \to [\mathbb{S}^1,K] \cong \pi_1(K), \quad [\alpha] \mapsto [f \circ \alpha]$$

is trivial. In view of

$$[(f_1 \cdot f_2) \circ \alpha] = [(f_1 \circ \alpha) \cdot (f_2 \circ \alpha)] = [f_1 \circ \alpha] \cdot [f_2 \circ \alpha],$$

Γ is a group homomorphism, and we obtain an exact sequence

$$\mathbf{1} \to \pi_1(K) \hookrightarrow C^\infty(M,\widetilde{K}) \xrightarrow{q_K^M} C^\infty(M,K) \xrightarrow{\Gamma} \mathrm{Hom}(\pi_1(M),\pi_1(K)) \tag{4.2}$$

of groups.

We also note that Γ is continuous with respect to the compact open topology on $C^\infty(M,K)$ and the topology of pointwise convergence on the abelian group $\mathrm{Hom}(\pi_1(M),\pi_1(K))$, which turns it into a totally disconnected group because $\pi_1(K)$ is discrete. This implies that $\mathrm{im}(q_K^M)$ also coincides with the arc-component of $\mathbf{1}$ with respect to the smooth compact open topology.

To see that Γ is surjective, let $\gamma \in \mathrm{Hom}(\pi_1(M),\pi_1(K))$, and consider the corresponding $\widetilde{K}$-principal bundle $P_\gamma := \widetilde{M} \times_\gamma \widetilde{K} \to M$ over M. Since $\widetilde{K}$ is contractible, this bundle is topologically trivial, and hence also smoothly trivial ([MW06]), so that there exists a smooth function $\widetilde{f}\colon \widetilde{M} \to \widetilde{K}$ with $\widetilde{f}(d.x) = \gamma(d)\widetilde{f}(x)$ for $d \in \pi_1(M)$ and $x \in \widetilde{M}$. Then $\widetilde{f}$ induces a smooth function $f\colon M \to K$ with $\Gamma(f) = \gamma$. As $\ker\Gamma$ is the arc-component of the identity for both topologies on $C^\infty(M,K)$, its surjectivity implies (3).

(4) If $\pi_1(M)$ is finitely generated, the group $\mathrm{Hom}(\pi_1(M),\pi_1(K))$ is discrete with respect to the topology of pointwise convergence. This in turn implies that G_0 is open with respect to the smooth compact open topology. Hence (2) implies that the Lie group structure on G is compatible with this topology because this is the case on G_0. ∎

Theorem IV.3. *Let M be a finite-dimensional connected σ-compact complex manifold, K a connected complex Lie group with Lie algebra $\mathfrak{k}$ whose universal covering group $\widetilde{K}$ is diffeomorphic to a locally convex space. Then the following assertions hold:*

(1) *$G := \mathcal{O}(M,K)$ carries a Lie group structure compatible with evaluations. If K is regular, then G is regular.*

(2) *On the identity component G_0, this Lie group structure is compatible with the smooth compact open topology.*

(3) *If $\pi_1(M)$ is finitely generated or $\mathrm{Hom}(\pi_1(M),\pi_1(K))$ is discrete with respect to the topology of pointwise convergence, then G_0 is also open in the smooth compact open topology, so that the Lie group structure on G is compatible with this topology.*

(4) *If M is Stein and K is Banach, then $\pi_0(G) \cong \mathrm{Hom}(\pi_1(M),\pi_1(K))$ holds with respect to the Lie group structure and the smooth compact open topology.*

Proof. Using Lemma A.4(3), (1) is proved exactly as in the real case, where the evaluation map is holomorphic by Proposition I.2(2). We omit the details.

(4) Here the crucial step is the surjectivity of the map Γ, for which we need that any bundle $P_\chi = M \times_\chi \widetilde{K} \to K$ is holomorphically trivial. In the proof of Theorem IV.2, we have argued that it is topologically trivial. Since M is assumed to be Stein and K Banach, the Oka principle (cf. [Rae77, Th. 2.1]) asserts that this bundle has a holomorphic section. The remaining arguments are similar to the real case. ∎

Remark IV.4. (a) It is quite plausible that under the assumptions of Theorem IV.2, the abelian group $\pi_1(K)$ is torsion free because each torsion element would lead to a fixed point free action of a cyclic group on the locally convex space $\widetilde{K}$.

If K is finite-dimensional, such an action does not exist (cf. [Sm41]), and if K is regular and abelian, we know anyhow that $\pi_1(K)$ is a subgroup of the additive group of $\mathfrak{k}$, hence torsion free.

(b) Theorem IV.2 applies in particular to all finite-dimensional Lie groups K which are diffeomorphic to some vector space. These groups are isomorphic to semidirect products of the form

$$R \rtimes \widetilde{\mathrm{SL}}_2(\mathbb{R})^m,$$

where R is a simply connected solvable Lie group and $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, which is diffeomorphic to $\mathbb{R}^3$.

Interesting infinite-dimensional Lie groups diffeomorphic to locally convex spaces can also be found among the simply connected pro-solvable pro-Lie groups (see [HoNe06] for more details).

(c) Theorem IV.3 applies to all finite-dimensional complex Lie groups K which are biholomorphic to a vector space, which is equivalent to K being solvable. ∎

Maps with values in abelian Lie groups

We have seen in Theorem IV.2 that the smooth compact open topology on $C^\infty(M,K)$ is compatible with the Lie group topology if and only if the arc-component of the identity is open. To get a better understanding of what is going wrong if $\pi_1(M)$ is not finitely generated, we now take a closer look at the relevant facts on abelian groups.

Let K be a regular connected abelian Lie group, hence of the form $K \cong \mathfrak{k}/\Gamma_K$, where $\mathfrak{k}$ is a Mackey complete locally convex space and $\Gamma_K \subseteq \mathfrak{k}$ is a discrete subgroup isomorphic to $\pi_1(K)$ (cf. [MT99], [GN07]). We write $q_K\colon \mathfrak{k} \to K$ for the quotient map with kernel Γ_K. Then $\widetilde{K} \cong \mathfrak{k}$ is a locally convex space and Theorem IV.2 applies.

Since $\mathfrak{k}$ is abelian, $\alpha \in \Omega^1(M,\mathfrak{k})$ satisfies the Maurer–Cartan equation if and only if α is closed and for each $m_0 \in M$ we have

$$\mathrm{per}_\alpha^{m_0}\colon \pi_1(M,m_0) \to K, \quad [\gamma] \mapsto q_K\Big(\int_\gamma \alpha\Big).$$

Therefore a closed 1-form α is integrable if and only if all its periods are contained in Γ_K. Let

$$Z^1_{\mathrm{dR}}(M,\mathfrak{k},\Gamma_K) := \mathrm{im}(\delta) \subseteq Z^1_{\mathrm{dR}}(M,\mathfrak{k})$$

denote the subgroup of all 1-forms satisfying this condition (Theorem I.5). Proposition II.1 now implies that

$$\delta\colon C^\infty_*(M,K) \to Z^1_{\mathrm{dR}}(M,\mathfrak{k},\Gamma_K) \tag{4.3}$$

is an isomorphism of topological groups, inducing an isomorphism of the path components of the identity

$$C^\infty_*(M,K)_a \cong C^\infty_*(M,\mathfrak{k}) \to dC^\infty(M,\mathfrak{k})$$

because we have

$$\Gamma\colon C^\infty(M,K) \to \mathrm{Hom}(\pi_1(M),\Gamma_K), \quad \Gamma(f)([\gamma]) = \int_\gamma \delta(f).$$

We conclude that the path component of the identity in $C^\infty(M,K)$ is open if and only if

$$H^1_{\mathrm{dR}}(M,\mathfrak{k},\Gamma_K) := Z^1_{\mathrm{dR}}(M,\mathfrak{k},\Gamma_K)/dC^\infty(M,\mathfrak{k})$$

is a discrete subgroup of the topological vector space $H^1_{\mathrm{dR}}(M,\mathfrak{k}) = Z^1_{\mathrm{dR}}(M,\mathfrak{k})/dC^\infty(M,\mathfrak{k})$. In view of the de Rham Theorem (cf. [KM97]) and the Hurewicz Homomorphism, we have isomorphisms of groups

$$H^1_{\mathrm{dR}}(M,\mathfrak{k}) \cong H^1_{\mathrm{sing}}(M,\mathfrak{k}) \cong \mathrm{Hom}(H_1(M),\mathfrak{k}) \cong \mathrm{Hom}(\pi_1(M),\mathfrak{k})$$

because $H_1(M) \cong \pi_1(M)/[\pi_1(M),\pi_1(M)]$ (cf. [Bre93]). By restriction, we thus obtain the isomorphism of groups

$$H^1_{\mathrm{dR}}(M,\mathfrak{k},\Gamma_K) \cong \mathrm{Hom}(\pi_1(M),\Gamma_K), \tag{4.4}$$

and, in view of Theorem IV.2, it is interesting to see when this actually is an isomorphism of topological groups, resp., when the subgroup $H^1_{\mathrm{dR}}(M,\mathfrak{k},\Gamma_K)$ is discrete.

Lemma IV.5. *If M is connected and σ-compact and $\mathfrak{k}$ is a Fréchet space, then the natural map*

$$\Phi\colon H^1_{\mathrm{dR}}(M,\mathfrak{k}) \to \mathrm{Hom}(H_1(M),\mathfrak{k})$$

is an isomorphism of Fréchet spaces, where the space $\mathrm{Hom}(H_1(M),\mathfrak{k})$ *carries the topology of pointwise convergence.*

Proof. First we observe that it is a continuous bijection of Fréchet spaces, where the topology of pointwise convergence on $\mathrm{Hom}(H_1(M),\mathfrak{k})$ defines a Fréchet space structure because the group $H_1(M)$ is countably generated (cf. [Ne04, Prop. IV.9]). Now the Open Mapping Theorem ([Ru73]) implies that Φ is an isomorphism of Fréchet spaces. ∎

In view of the preceding lemma, we are left with the question when $\mathrm{Hom}(A,\Gamma_K)$ is discrete in $\mathrm{Hom}(A,\mathfrak{k})$ for an abelian group A. The following theorem provides some information on the structure of Γ_K.

Theorem IV.6. (Sidney) *Countable discrete subgroups of locally convex spaces are free.*

Proof. If Γ is a discrete subgroup of the locally convex space $\mathfrak{k}$, then there exists a continuous seminorm p on $\mathfrak{k}$ such that $\inf\{p(\gamma)\colon 0 \neq \gamma \in \Gamma\} > 0$. If $\mathfrak{k}_p$ is the completion of the normed space $\mathfrak{k}/p^{-1}(0)$, it follows that Γ embeds as a subgroup of $\mathfrak{k}_p$ whose intersection with a sufficiently small ball is trivial, and therefore Γ is realized as a discrete subgroup of some Banach space. This implies that every discrete subgroup of a locally convex space is isomorphic to a discrete subgroup of some Banach space. Now the assertion follows from Sidney's Theorem that countable discrete subgroups of Banach spaces are free ([Si77, p.983]). ∎

Lemma IV.7. *Let A be a countable abelian group. Then the following are equivalent:*
(1) $\mathrm{Hom}(A,\Gamma)$ *is discrete in* $\mathrm{Hom}(A,E)$ *for each discrete subgroup* Γ *of a locally convex space* E.
(2) $\mathrm{Hom}(A,\Gamma)$ *is discrete in* $\mathrm{Hom}(A,E)$ *for one non-zero discrete subgroup* Γ *of some locally convex space* E.
(3) $\mathrm{Hom}(A,\mathbb{Z})$ *is discrete in* $\mathrm{Hom}(A,\mathbb{R})$.
(4) $\mathrm{Hom}(A,\mathbb{Z})$ *is finitely generated.*

Proof. Let $A_1 \subseteq A$ denote the intersection of all kernels of homomorphisms $A \to \mathbb{Z}$. Then we have an embedding

$$A/A_1 \hookrightarrow \mathbb{Z}^{\mathrm{Hom}(A,\mathbb{Z})}, \quad a + A_1 \mapsto (\chi \mapsto \chi(a))$$

and the group A/A_1 is countable. As all countable subgroups of groups of the form $\mathbb{Z}^I$ are free ([Fu70, Th. 19.2]), we have

$$A \cong A_1 \oplus A_2,$$

where $\mathrm{Hom}(A_1,\mathbb{Z}) = \mathbf{0}$ and $A_2 \cong \mathbb{Z}^{(J)}$ is free (cf. [Fu70, Cor. 19.3] for this result due to K. Stein). In particular, we have $\mathrm{Hom}(A,\mathbb{Z}) \cong \mathrm{Hom}(A_2,\mathbb{Z}) \cong \mathbb{Z}^J$, and this group is finitely generated if and only if J is finite.

If $\chi\colon A \to \Gamma$ is a homomorphism, then $\chi(A)$ is a countable discrete subgroup of the locally convex space $\mathfrak{k}$, hence free by Sidney's Theorem IV.6. Therefore the homomorphisms $\chi(A) \to \mathbb{Z}$ separate points, which implies that $A_1 \subseteq \ker\chi$. We conclude that

$$\mathrm{Hom}(A,\Gamma) \cong \mathrm{Hom}(A_2,\Gamma) \cong \mathrm{Hom}(\mathbb{Z}^{(J)},\Gamma) \cong \Gamma^J.$$

The topology of pointwise convergence on $\mathrm{Hom}(A,\Gamma)$ corresponds to the product topology on Γ^J. Hence this group is discrete if and only if either $\Gamma = \{0\}$ or J is finite. In particular, $\mathrm{Hom}(A,\mathbb{Z})$ is discrete if and only if J is finite. ∎

From the proof of Lemma IV.7 we obtain in particular that $\mathrm{Hom}(A,\mathbb{Z}) \cong \mathbb{Z}^J$, where J is a countable set. If J is finite, then $\mathbb{Z}^J \cong \mathbb{Z}^{|J|}$ is discrete, and otherwise it is isomorphic to $\mathbb{Z}^{\mathbb{N}}$, which is not discrete.

Theorem IV.8. *Let M be a connected σ-compact manifold, $\mathfrak{k}$ a Fréchet space, $\Gamma_K \subseteq \mathfrak{k}$ a non-trivial discrete subgroup, and $K := \mathfrak{k}/\Gamma_K$. Then the following are equivalent:*
(1) *The Lie group structure on $C^\infty(M,K)$ from* Theorem IV.2 *is compatible with the smooth compact open topology.*
(2) *The arc-component $C^\infty(M,K)_a = q_K^M(C^\infty(M,\mathfrak{k}))$ is open with respect to the smooth compact open topology.*
(3) *$H^1_{\mathrm{dR}}(M,\mathfrak{k},\Gamma_K)$ is a discrete subgroup of $H^1_{\mathrm{dR}}(M,\mathfrak{k})$.*
(4) *$H^1(M,\mathbb{Z})$ is finitely generated.*

Proof. The equivalence of (1) and (2) follows from Theorem IV.2(2).

The equivalence of (1) and (3) follows from the discussion preceding Lemma IV.5. To prove the equivalence between (3) and (4), we recall from (4.4) and Lemma IV.5 the isomorphism

$$H^1_{\mathrm{dR}}(M,\mathfrak{k},\Gamma_K) \cong \mathrm{Hom}(H_1(M),\Gamma_K)$$

of topological groups. As $H_1(M)$ is a countable abelian group (cf. [Ne04, Prop. IV.8]), the equivalence of (3) and (4) follows from $\Gamma_K \neq \{0\}$, Lemma IV.7, and $\mathrm{Hom}(H_1(M),\mathbb{Z}) \cong H^1(M,\mathbb{Z})$. ∎

The complex case

Now we assume that M is a complex manifold and $K \cong \mathfrak{k}/\Gamma_K$ is a regular abelian complex Lie group, i.e., $\mathfrak{k}$ is a complex Mackey complete space.

Theorem IV.9. *Let M be a connected σ-compact complex manifold without boundary, $\mathfrak{k}$ a complex Fréchet space, $\Gamma_K \subseteq \mathfrak{k}$ a non-zero discrete subgroup and $K := \mathfrak{k}/\Gamma_K$. Then*

$$\delta\colon \mathcal{O}_*(M,K) \to Z^1_{\mathrm{dR,h}}(M,\mathfrak{k},\Gamma_K)$$

is an isomorphism of topological groups, and the following are equivalent:

(1) *The Lie group structure on $\mathcal{O}(M,K)$ is compatible with the compact open topology.*

(2) *The arc-component $\mathcal{O}_*(M,K)_a = q_K^M(\mathcal{O}(M,\mathfrak{k}))$ is open with respect to the compact open topology.*

(3) *$H^1_{\mathrm{dR,h}}(M,\mathfrak{k},\Gamma_K) = \{[\alpha]\colon \alpha \in \mathrm{MC}_h(M,\mathfrak{k}), (\forall \gamma \in C^\infty(\mathbb{S}^1,M)) \int_\gamma \alpha \in \Gamma_K\}$ is a discrete subgroup of $H^1_{\mathrm{dR,h}}(M,\mathfrak{k})$. If, in addition, M is a Stein manifold, then* (1)-(3) *are equivalent to:*

(4) *$H^1(M,\mathbb{Z})$ is finitely generated.*

Proof. The equivalence of (1)–(3) is shown precisely as in the real case. Suppose, in addition, that M is a Stein manifold. Then the Oka principle ([Gr58], Satz I, p. 268) implies that each continuous map $M \to \mathbb{C}^\times$ is homotopic to a holomorphic map. As the homotopy classes $[M,\mathbb{C}^\times]$ are classified by

$$\mathrm{Hom}(\pi_1(M),\pi_1(\mathbb{C}^\times)) \cong \mathrm{Hom}(\pi_1(M),\mathbb{Z}) \cong H^1(M,\mathbb{Z}),$$

each homomorphism $\pi_1(M) \to \mathbb{Z}$ arises as $\pi_1(f)$ for a holomorphic map $f\colon M \to \mathbb{C}^\times$. Hence each class in $H^1(M,\mathbb{Z})$ is represented by a holomorphic 1-form. In view of Lemma IV.5, this implies that

$$H^1_{\mathrm{dR,h}}(M,\mathbb{C},\mathbb{Z}) = H^1_{\mathrm{dR}}(M,\mathbb{C},\mathbb{Z}) \cong \mathrm{Hom}(H_1(M),\mathbb{Z})$$

as topological groups, and the discreteness of this group is equivalent to (4) (Proposition IV.8). ■

Remark IV.10. The assumption that M is Stein is crucial in (4) above, because, in general, not every homomorphism $H_1(M) \to \mathbb{C}$ is represented by integration of a holomorphic 1-form. A typical example is given by a complex torus $M = \mathbb{C}/D$, where $\mathbb{Z}^2 \cong D \subseteq \mathbb{C}$ is a discrete subgroup. Since holomorphic functions on M are constant, each holomorphic 1-form on M is a constant multiple of dz. Therefore $H^1_{\mathrm{dR,h}}(M,\mathbb{C}) = \mathbb{C}[dz]$ is one-dimensional, and the homomorphism $\pi_1(M) \cong D \to \mathbb{C}$ corresponding to $\lambda \cdot dz$ is given by $d \mapsto \lambda d$. The group homomorphism $D \to \mathbb{C}, d \mapsto \overline{d}$ is not represented by integration of a holomorphic 1-form. ■

Remark IV.11. If M is a non-compact Riemann surface, then M is Stein and $\pi_1(M)$ is a free group, so that $H_1(M)$ is a free abelian group. Therefore Theorem IV.9 implies that for any abelian complex Fréchet–Lie group of the form $K = \mathfrak{k}/\Gamma_K$, the group $\mathcal{O}(M,K)$ is a Lie group with respect to the compact open topology if and only if the free abelian group $H_1(M)$ has finite rank. ■

Remark IV.12. (a) If M is a compact Kähler manifold (and $\mathfrak{k}$ still abelian), then $\Omega^1_h(M,\mathfrak{k}) \cong \mathrm{MC}_h(M,\mathfrak{k})$ because each holomorphic 1-form is automatically closed (cf. [We80]).

(b) For any compact complex manifold we have $\mathcal{O}_*(M,\mathfrak{k}) = \{0\}$ because all holomorphic functions are constant. Therefore the Lie group structure on $\mathcal{O}_*(M,K)$ is discrete. ■

Remark IV.13. (a) Let K be a connected finite-dimensional Lie group whose universal covering group $\widetilde{K}$ is diffeomorphic to a vector space, which is equivalent to $\mathfrak{k}/\operatorname{rad}(\mathfrak{k}) \cong \mathfrak{sl}_2(\mathbb{R})^m$ for some $m \in \mathbb{N}_0$ and this in turn is equivalent to the maximal compact subgroup $T \subseteq K$ being a torus (cf. [HoNe06]). Let $d := \dim T$. Since T is a maximal compact subgroup, $\pi_1(K) \cong \pi_1(T) \cong \mathbb{Z}^d$ is a free abelian group (cf. [Ho65]). Therefore

$$\operatorname{Hom}(\pi_1(M), \pi_1(K)) \cong \operatorname{Hom}(H_1(M), \pi_1(T)) \cong H^1(M, \mathbb{Z})^d.$$

If K is not simply connected, then $d > 0$, so that this group is discrete if and only if $H^1(M, \mathbb{Z})$ is finitely generated (Lemma IV.7).

If this is the case, then Theorem IV.2 implies that the Lie group structure on $C^\infty(M, K)$ is compatible with the smooth compact open topology. If $H^1(M, \mathbb{Z})$ is not finitely generated, then

$$C^\infty_*(M, \widetilde{T}) = C^\infty_*(M, \widetilde{K}) \cap C^\infty_*(M, T)$$

is not an open subgroup of $C^\infty_*(M, T) \subseteq C^\infty_*(M, K)$, and therefore $C^\infty_*(M, \widetilde{K})$ is not an open subgroup of $C^\infty_*(M, K)$. Now the topological decomposition

$$C^\infty(M, K) \cong C^\infty_*(M, K) \rtimes K$$

implies that the arc-component of the identity in $C^\infty(M, K)$ is not open, hence that the Lie group structure is not compatible with the smooth compact open topology.

(b) If K is a finite-dimensional complex Lie group, the Levi decomposition implies that the condition that $\widetilde{K}$ is diffeomorphic to a vector space is equivalent to K being solvable.

Suppose that $H^1(M, \mathbb{Z})$ is not finitely generated and let $T \subseteq K$ be a real maximal torus. Then the inclusion $T \hookrightarrow K$ extends to a holomorphic Lie group morphism $T_{\mathbb{C}} \to K$. Note that for $T \cong \mathbb{T}^n$ the universal complexification is $T_{\mathbb{C}} \cong (\mathbb{C}^\times)^n$. If the map $T_{\mathbb{C}} \hookrightarrow K$ is an embedding, then we also have an embedding

$$T_{\mathbb{C}} \hookrightarrow \mathcal{O}(M, T_{\mathbb{C}}) \hookrightarrow \mathcal{O}(M, K).$$

If M is a Stein manifold, then $\mathcal{O}(M, T_{\mathbb{C}})$ is not a Lie group because $\mathcal{O}(M, \widetilde{T}_{\mathbb{C}}) \cong \mathcal{O}(M, T_{\mathbb{C}})_a$ is not open (Theorem IV.9). As in (a) above, this implies that $\mathcal{O}(M, \widetilde{K})$ is not open in $\mathcal{O}(M, K)$, and therefore that $\mathcal{O}(M, K)$ is not a Lie group with respect to the compact open topology. ∎

V. Some strange properties of the exponential map

In this subsection, we collect some interesting properties of the exponential function of the groups $\mathcal{O}(M, K)$ on a finite-dimensional complex manifold M to a complex Banach–Lie group K, which is simply given by $\exp(\xi) := \exp_K \circ \xi$, where $\exp_K$ is the exponential function of K.

Proposition V.1. *Let M be a complex manifold which has non-constant holomorphic functions, and K be a complex connected Banach-Lie group. If $K \neq \exp_K \mathfrak{k}$, then the image of the exponential function of $\mathcal{O}(M, K)$ is not an identity neighborhood.*

Proof. **Step 1:** First we claim the existence of a holomorphic function $\ell: M \to \mathbb{C}$ with real part unbounded from above. Suppose that such a function does not exist. Replacing f by if, $-f$ and $-if$, we conclude that for each holomorphic function $f: M \to \mathbb{C}$ the functions $\operatorname{Re} f$ and $\operatorname{Im} f$ are bounded, and hence that f is bounded. If f is non-constant, then $f(M)$ is an open subset of $\mathbb{C}$, hence has a boundary point $z_0 \notin f(M)$. But then the function $(f - z_0)^{-1}$ is

unbounded, a contradiction. We therefore find a holomorphic function $\ell: M \to \mathbb{C}$ and a sequence $x_n \in M$ with $\operatorname{Re} \ell(x_n) \to \infty$.

Step 2: Let

$$K_1 := \{f(1): f \in \mathcal{O}(\mathbb{C}, K), f(0) = \mathbf{1}\}.$$

Then K_1 is a subgroup of K, because it is the homomorphic image of the subgroup $\mathcal{O}_*(\mathbb{C}, K)$ under the evaluation map in 1. If $k = \exp_K x$ for some $x \in \mathfrak{k}$, then the map $f(z) := \exp_K(zx)$ satisfies $f(0) = \mathbf{1}$ and $f(1) = k$. Hence $K_1 \supseteq \exp_K \mathfrak{k}$, and since the connected Banach–Lie group K is generated by $\exp_K \mathfrak{k}$, we obtain $K = K_1$.

Step 3: Let $k \in K \setminus \exp_K \mathfrak{k}$. In view of the preceding paragraph, there exists a holomorphic map $f: \mathbb{C} \to K$ with $f(1) = k$ and $f(0) = \mathbf{1}$. We define $h_n(x) := f(e^{\ell(x)-\ell(x_n)})$. Then $h_n(x_n) = f(1) \notin \exp_K \mathfrak{k}$, so that h_n is not contained in the image of the exponential function of $\mathcal{O}(M, K)$. On the other hand $h_n \to \mathbf{1}$ uniformly on compact subsets of M, hence in $\mathcal{O}(M, K)$. ∎

Corollary V.2. *Let M be a complex manifold with non-constant holomorphic functions and $K_1 \leq K$ a Banach–Lie subgroup whose exponential function is not surjective. Then there exist 0-neighborhoods in $\mathcal{O}(M, \mathfrak{k})$ whose image under the exponential function is not an identity neighborhood in $\mathcal{O}(M, K)$.*

Proof. Let $U_{\mathfrak{k}} \subseteq \mathfrak{k}$ be an open 0-neighborhood which is relatively compact and for which $\exp_K|_{2U_{\mathfrak{k}}}$ is a diffeomorphism onto its open image, satisfying

$$\exp_K(U_{\mathfrak{k}}) \cap K_1 = \exp_K(U_{\mathfrak{k}} \cap \mathfrak{k}_1). \tag{5.1}$$

Pick $m_0 \in M$ and a compact neighborhood C of m_0. Then we consider the identity neighborhood $W(C, \exp_K(U_{\mathfrak{k}}))$ of $\mathcal{O}(M, K)$. Let $\mathfrak{k}_1 \leq \mathfrak{k}$ be the Lie algebra of K_1 and observe that $W(C, U_{\mathfrak{k}})$ is a 0-neighborhood in $\mathcal{O}(M, \mathfrak{k})$.

In view Proposition V.1, each identity neighborhood in $\mathcal{O}(M, K_1)$ contains a holomorphic function $h: M \to K_1$, not contained in the image of the exponential function of $\mathcal{O}(M, K_1)$. Suppose that $h = \exp \xi = \exp_K \circ \xi$ holds for some holomorphic function $\xi: M \to \mathfrak{k}$, contained in $W(C, U_{\mathfrak{k}})$. Then the injectivity of $\exp_K|_{U_{\mathfrak{k}}}$ and (5.1) imply that $\xi(C) \subseteq U_{\mathfrak{k}} \cap \mathfrak{k}_1$. Since f is holomorphic, we obtain $\xi(M) \subseteq \mathfrak{k}_1$, contradicting the construction of h. Therefore $\exp(W(C, U_{\mathfrak{k}}))$ is not an identity neighborhood in $\mathcal{O}(M, K)$. ∎

The preceding corollary implies in particular that the exponential function of $\mathcal{O}(M, \mathrm{GL}_n(\mathbb{C}))$ is not locally surjective for any Stein manifold M. The following lemma shows that it is locally injective.

Lemma V.3. *If M is a connected complex manifold, then the exponential function*

$$\exp: \mathcal{O}(M, \mathfrak{k}) \to \mathcal{O}(M, K)$$

is locally injective.

Proof. Let $C \subseteq M$ be a non-empty compact subset, $U_{\mathfrak{k}} \subseteq \mathfrak{k}$ be an open 0-neighborhood on which the exponential function $\exp_K: \mathfrak{k} \to K$ is injective, and define

$$U := W(C, U_{\mathfrak{k}}) = \{\xi \in \mathcal{O}(M, \mathfrak{k}): (\forall x \in C)\ \xi(x) \in U_{\mathfrak{k}}\}.$$

Then U is an open 0-neighborhood in $\mathcal{O}(M, \mathfrak{k})$. If $\xi, \eta \in U$ satisfy $\exp \xi = \exp \eta$, then the injectivity of $\exp_K$ on $U_{\mathfrak{k}}$ implies that $\xi|_C = \eta|_C$, and since M is connected, we obtain $\xi = \eta$ by analytic continuation. ∎

Appendix. Technical tools

Lemma A.1. *Let M, N and L be locally convex manifolds, $f \in C^\infty(M \times N, L)$ and put $f_x(y) := f(x,y)$. Then the map $f^\vee\colon M \to C^\infty(N,L), x \mapsto f_x$ is continuous.*

Proof. **Step 1:** (cf. [Ne01, Lemma III.2]) For Hausdorff spaces M, N and L and $f \in C(M \times N, L)$, the map $f^\vee\colon M \to C(N,L)_c$, is continuous: Suppose that $f_x \in W(K,U)$ for some compact subset $K \subseteq N$ and some open subset $U \subseteq L$, i.e., $\{x\} \times K \subseteq f^{-1}(U)$. Since $f^{-1}(U)$ is an open subset of $M \times N$ and $\{x\} \times K \subseteq M \times N$ is compact, there exists an open neighborhood $O \subseteq M$ of x such that $O \times K \subseteq f^{-1}(U)$. This means that $x \in O \subseteq \{p \in M\colon f_p \in W(K,U)\}$, which proves the assertion.

Step 2: $f^\vee$ is continuous. For each $k \in \mathbb{N}$ we have a natural product decomposition $T^k(M \times N) \cong T^k(M) \times T^k(N)$, so that Step 1 implies the continuity of the maps

$$M \to C(T^k(N), T^k(L))_c, \quad x \mapsto T^k(f_x)$$

for each $k \in \mathbb{N}$. In view of the definition of the topology on $C^\infty(N,L)$, this proves that $f^\vee$ is continuous. ∎

Lemma A.2. *Let N and M be smooth manifolds.*

(1) *If E is a locally convex space and $f \in C^\infty(N \times M, E)$, then $f^\vee\colon N \to C^\infty(M,E)$ is smooth.*

(2) *If M is compact (possibly with boundary) and K is a Lie group, then for each smooth map $f \in C^\infty(N \times M, K)$, the map $f^\vee\colon N \to C^\infty(M,K)$ is smooth with respect to the natural Lie group structure on $C^\infty(M,K)$.*

Proof. (1) We may w.l.o.g. assume that N is an open convex subset of a locally convex space X and identify $T(N)$ with $N \times X$. First we show that $f^\vee$ is C^1 with tangent map

$$\Psi\colon T(N) \to C^\infty(M, T(E)), \quad \Psi_m(v)(n) = T_{(m,n)}(f)v$$

whose continuity follows from Lemma A.1 and the smoothness of the tangent map

$$T(f)\colon T(N) \times T(M) \to T(E).$$

Fix $(x,h) \in T(N)$. For a sufficiently small $\varepsilon > 0$ the map

$$]-\varepsilon, \varepsilon[\times [0,1] \to C^\infty(M,E), \quad (t,u) \mapsto \Psi(x + uth, h)$$

is continuous by Lemma A.1. Therefore

$$]-\varepsilon, \varepsilon[\to C^\infty(M,E), \quad t \mapsto \int_0^1 \Psi(x + uth, h)\, du$$

is continuous, and so

$$\lim_{t \to 0} \frac{1}{t}\big(f^\vee(x+th) - f^\vee(x)\big) = \lim_{t \to 0} \int_0^1 \Psi(x + uth, h)\, du = \int_0^1 \Psi(x,h)\, du = \Psi(x,h).$$

Thus $T(f^\vee)(x,h) = \Psi(x,h)$, and the continuity of Ψ implies that $f^\vee$ is C^1.

Applying this argument to the tangent map $T(f^\vee)$, we see that $T(f^\vee)$ is also C^1, so that $f^\vee$ is C^2. Proceeding inductively, it follows that $f^\vee$ is smooth.

(2) To see that $f^\vee$ is smooth in a neighborhood of some $n_0 \in N$, it suffices to prove the smoothness of the map $n \mapsto f^\vee(n_0)^{-1} f^\vee(n)$, so that we may assume that $f^\vee(n_0) = \mathbf{1}$.

Let (φ_K, U_K) be a $\mathfrak{k}$-chart of K and (φ, U) the corresponding $C^\infty(M, \mathfrak{k})$-chart of the group $C^\infty(M,K)$, given by

$$U := C^\infty(M, U_K) \quad \text{and} \quad \varphi(\xi) = \varphi_K \circ \xi$$

(cf. Theorem I.3). Then the continuity of $f^\vee$ implies that $f^\vee$ maps a neighborhood $U_N \subseteq N$ of n_0 into U, we may assume that $f(N \times M) \subseteq U$, and we have to show that $\varphi \circ f^\vee|_{U_N}$ is smooth. Since

$$(\varphi \circ f^\vee)(n)(m) = \varphi_K(f(n,m)) = (\varphi_K \circ f)^\vee(n)(m),$$

we have $\varphi \circ f^\vee = (\varphi_K \circ f)^\vee\colon N \to C^\infty(M, \mathfrak{k})$, and the smoothness of this map follows from (1). ∎

Lemma A.3. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, N be a locally convex smooth $\mathbb{K}$-manifold, M a finite-dimensional smooth $\mathbb{K}$-manifold (without boundary in case $\mathbb{K} = \mathbb{C}$) and E a topological $\mathbb{K}$-vector space. Then the following assertions hold:*

(1) *For $\mathbb{K} = \mathbb{R}$, a map $f: N \to C^\infty(M, E)$ is smooth if and only if the map*

$$f^\wedge: N \times M \to E, \quad f^\wedge(n)(m) := f(n, m)$$

is smooth. The map $\Psi: C^\infty(N, C^\infty(M, E)) \to C^\infty(N \times M, E), f \mapsto f^\wedge$ is an isomorphism of topological vector spaces.

(2) *For $\mathbb{K} = \mathbb{C}$, a map $f: N \to \mathcal{O}(M, E)$ is holomorphic if and only if $f^\wedge$ is holomorphic. The map $\Psi: \mathcal{O}(N, \mathcal{O}(M, E)) \to \mathcal{O}(N \times M, E), f \mapsto f^\wedge$ is an isomorphism of topological vector spaces.*

Proof. [Gl04, Prop. 12.2] directly implies (1). To verify (2), we first observe that the Cauchy Formula implies that on the closed subspace $\mathcal{O}(M, E)$, uniform convergence on compact subsets implies uniform convergence of all partial derivatives on compact subsets. Hence the inclusion map

$$\mathcal{O}(M, E) \hookrightarrow C^\infty(M, E)$$

is continuous and therefore a topological embedding. In this sense the compact open topology on $\mathcal{O}(M, E)$ coincides with the $\mathbb{C}$-smooth compact open topology, which is used in [Gl04]. Therefore (2) follows from [Gl04, Prop. 12.2] for $\mathbb{K} = \mathbb{C}$ or by observing that the map Ψ in (1) maps the closed subspace $\mathcal{O}(N, \mathcal{O}(M, E))$ of $C^\infty(N, C^\infty(M, E))$ homeomorphically onto $\mathcal{O}(N \times M, E)$. ∎

Lemma A.4. *Let E_1 and E_2 be locally convex spaces, $U_j \subseteq E_j$ open subsets, and $\varphi: U_1 \to U_2$ be a smooth map.*

(1) *The map*

$$\varphi_*: C^\infty(M, U_1) \to C^\infty(M, U_2), \quad f \mapsto \varphi \circ f$$

is continuous.

(2) *If, in addition, M is compact or $U_j = E_j$ for $j = 1, 2$, so that the subsets $C^\infty(M, U_j)$ are open in $C^\infty(M, E_j)$, then the map φ_* is smooth.*

(3) *If, in addition to the assumptions in (2), E_1 and E_2 are complex and φ is holomorphic, then φ_* is holomorphic.*

Proof. (1) The continuity of φ_* follows directly from the definition of the topology and the continuity of left compositions with respect to the compact open topology.

(2) Assume that M is compact or $U_j = E_j$ for $j = 1, 2$. In view of Lemma A.2, the smoothness of φ_* follows from the smoothness of the corresponding map

$$C^\infty(M, U_1) \times M \to U_2, \quad (f, m) \mapsto \varphi(f(m)) = \varphi \circ \mathrm{ev}(f, m),$$

where $\mathrm{ev}: C^\infty(M, U_1) \times M \to U_1$ is the smooth evaluation map (Proposition I.2).

(3) In view of (2), it remains to show that the differentials of φ_* are complex linear, which follows from

$$d(\varphi_*)(f)(\xi)(x) = d\varphi(f(x))\xi(x).$$ ∎

The following two lemmas collect some technical smoothness properties of regular Lie groups.

Lemma A.5. *For a connected finite-dimensional smooth manifold M (with boundary) and a regular Lie group K with Lie algebra $\mathfrak{k}$, the following assertions hold:*

(1) *The map $\mathrm{Evol}_K: C^\infty([0, 1], \mathfrak{k}) \to C^\infty_*([0, 1], K)$ is a diffeomorphism with*

$$T_0(\mathrm{Evol}_K)(\xi)(t) = \int_0^t \xi(s)\, ds \quad \text{and} \quad T_0(\mathrm{evol}_K)(\xi) = \int_0^1 \xi(s)\, ds.$$

Its inverse is

$$\delta: C^\infty_*([0, 1], K) \to C^\infty([0, 1], \mathfrak{k}) \quad \text{with} \quad T_{\mathbf{1}}(\delta)(\xi) = \xi'.$$

(2) *The action of* K *on* $\Omega^1(M,\mathfrak{k})$ *by* $\mathrm{Ad}(k).\alpha := \mathrm{Ad}(k)\circ\alpha$ *is smooth.*
(3) *The multiplication map* $\Omega^1(M,\mathbb{R})\times\mathfrak{k}\to\Omega^1(M,\mathfrak{k}), (\alpha,x)\mapsto\alpha\cdot x$ *is continuous.*

Proof. (1) First we show that Evol_K is smooth. For $\gamma\in C^\infty([0,1],K)$ and $\gamma_s(t):=\gamma(st)$ we have $\delta(\gamma_s)(t) = s\delta(\gamma)(st) = S(s,\delta(\gamma))(t)$, where

$$S\colon [0,1]\times C^\infty([0,1],\mathfrak{k})\to C^\infty([0,1],\mathfrak{k}),\quad S(s,\xi)(t) = s\xi(st)$$

is smooth by Proposition I.2 and Lemma A.2. For $\delta(\gamma)=\xi$ we have

$$\mathrm{Evol}_K(\xi)(s) = \gamma(s) = \gamma_s(1) = \mathrm{evol}_K(S(s,\xi)),$$

showing that the map $C^\infty([0,1],\mathfrak{k})\times[0,1]\to K, (\xi,s)\mapsto\mathrm{Evol}_K(\xi)(s)$ is smooth, and this implies that Evol_K is smooth (Lemma A.2).

To see that δ is smooth, we write $\delta(\gamma)(t) = \kappa_K(\gamma'(t))$. Since κ_K is smooth, the assertion follows from the smoothness of the homomorphism of Lie groups

$$C^\infty([0,1],K)\to C^\infty([0,1],TK),\quad \gamma\mapsto\gamma',$$

Lemma A.2 and the smoothness of the evaluation map of $C^\infty([0,1],TK)$ (Theorem I.3).

From $\delta\circ\mathrm{Evol}_K = \mathrm{id}_{C^\infty([0,1],\mathfrak{k})}$ and the smoothness of Evol_K, we derive that

$$T_{\mathbf{1}}(\delta)\circ T_0(\mathrm{Evol}_K) = \mathrm{id}_{C^\infty([0,1],\mathfrak{k})}.$$

Using the Chain Rule, we obtain directly $T_{\mathbf{1}}(\delta)(\xi)(t) = \xi'(t)$, and since $T_{\mathbf{1}}(\delta)$ is injective on $C^\infty_*([0,1],\mathfrak{k})$, the tangent space of $C^\infty_*([0,1],K)$ in the constant function $\mathbf{1}$, we get

$$T_0(\mathrm{Evol}_K)(\xi)(t) = \int_0^t \xi(s)\,ds.$$

Now $\mathrm{ev}_1\circ\mathrm{Evol}_K = \mathrm{evol}_K$ leads to the asserted formula for $T_0(\mathrm{evol}_K)$.

(2) Since $\Omega^1(M,\mathfrak{k})$ is a closed subspace of $C^\infty(TM,\mathfrak{k})$, it suffices to observe that the action of K on $C^\infty(TM,\mathfrak{k})$, given by $k.f := \mathrm{Ad}(k)\circ f$ is smooth. In view of Lemma A.2, it suffices to show that the map

$$K\times C^\infty(TM,\mathfrak{k})\times TM\to\mathfrak{k},\quad (k,f,m)\mapsto\mathrm{Ad}(k).f(m)$$

is smooth, which in turn follows from the smoothness of the adjoint action of K on $\mathfrak{k}$ and the smoothness of the evaluation map of $C^\infty(TM,\mathfrak{k})$ (Proposition I.2).

(3) With the same argument as in (2), it suffices to show that

$$C^\infty(TM,\mathbb{R})\times\mathfrak{k}\to C^\infty(TM,\mathfrak{k}),\quad (f,x)\mapsto f\cdot x$$

is smooth. This in turn follows from the smoothness of the map

$$C^\infty(TM,\mathbb{R})\times TM\times\mathfrak{k}\to\mathfrak{k},\quad (f,v,x)\mapsto f(v)\cdot x = \mathrm{ev}(f,v)\cdot x$$

(Proposition I.2, Lemma A.2). ∎

Lemma A.6. *Let* M *be a connected finite-dimensional smooth manifold (with boundary) and* K *a regular Lie group with Lie algebra* $\mathfrak{k}$*.*
(1) *If* $\gamma\colon[0,1]\to M$ *is a piecewise smooth curve, then the map*

$$\Omega^1(M,\mathfrak{k})\mapsto K,\quad \alpha\mapsto\mathrm{evol}_K(\gamma^*\alpha)$$

is smooth.

(2) *Let* (φ, U) *be a chart of* M *for which* $\varphi(U)$ *is a convex* 0*-neighborhood and* $\gamma_x(t) := \varphi^{-1}(t\varphi(x))$*. Then the map*

$$\Omega^1(M,\mathfrak{k}) \times U \to K, \quad (\alpha, x) \mapsto \mathrm{evol}_K(\gamma_x^*\alpha)$$

is smooth.

(3) *For* $\xi \in C^\infty(I \times M, \mathfrak{k})$ *put* $\xi^m(t) := \xi(t,m)$*. Then the map*

$$\gamma\colon I \times M \to K, \quad (t,m) \mapsto \mathrm{Evol}_K(\xi^m)(t)$$

is smooth with

$$\delta(\gamma)_t(m) := \gamma(t,m)^{-1}.\frac{d}{dt}\gamma(t,m) = \xi^m(t),$$

and the map

$$\mathrm{evol}_G^\wedge\colon C^\infty(I,\mathfrak{g}) \times M \to K, \quad (\xi, m) \mapsto \mathrm{evol}_K(\xi^m)$$

is also smooth.

Proof. (1) This follows from the smoothness of evol_K and the fact that for each smooth path $\eta\colon [a,b] \to M$ the map

$$\Omega^1(M,\mathfrak{k}) \to C^\infty([a,b],\mathfrak{k}), \quad \alpha \mapsto \eta^*\alpha = \alpha \circ T\eta$$

is continuous and linear, hence smooth.

(2) Since K is regular, we have to show that the map

$$\Omega^1(M,\mathfrak{k}) \times U \to C^\infty([0,1],\mathfrak{k}), \quad (\alpha, x) \mapsto \gamma_x^*\alpha$$

is smooth. In view of Lemma A.2, this follows from the smoothness of the map

$$\Omega^1(U,\mathfrak{k}) \times U \times [0,1] \to \mathfrak{k}, \quad (\alpha, x, t) \mapsto (\gamma_x^*\alpha)_t = \alpha_{\gamma_x(t)}\gamma_x'(t),$$

which is a consequence of the smoothness of the evaluation map of $C^\infty(TU,\mathfrak{k})$ (Proposition I.2) and of the map $U \times [0,1] \to TM, (x,t) \mapsto \gamma_x'(t)$.

(3) First we recall from Lemma A.3 that for $\mathfrak{g} := C^\infty(M,\mathfrak{k})$ we have

$$C^\infty(I,\mathfrak{g}) \cong C^\infty(I \times M, \mathfrak{k}) \cong C^\infty(M, C^\infty(I,\mathfrak{k}))$$

as topological vector spaces. In this sense, we consider each $\xi \in C^\infty(I,\mathfrak{g})$ as a smooth map $I \times M \to \mathfrak{k}$. In particular, $\xi^m \in C^\infty(I,\mathfrak{k})$, $\mathrm{evol}_K(\xi^m) \in K$, and the map $\xi^\vee\colon M \to C^\infty(I,\mathfrak{k}), m \mapsto \xi^m$ is smooth. Hence the smoothness of γ follows from

$$\gamma(t,m) = \mathrm{Evol}_K(\xi^m)(t) = \mathrm{ev} \circ (\mathrm{Evol}_K(\xi^\vee(m)), t) = \mathrm{ev} \circ ((\mathrm{Evol}_K \circ \xi^\vee) \times \mathrm{id}_I)(m,t)$$

because the evaluation map of $C^\infty(I,K)$ is smooth (Theorem I.3). The formula for $\delta(\gamma)$ follows immediately from the definition.

To see that $\mathrm{evol}_G^\wedge$ is smooth, we first recall that evol_K is smooth. Hence it suffices to observe that the map

$$C^\infty(I \times M, \mathfrak{k}) \times M \to C^\infty(I,\mathfrak{k}), \quad (\xi, m) \mapsto \xi^m$$

is smooth because it corresponds to the evaluation map of the space $C^\infty(M, C^\infty(I,\mathfrak{k}))$ (cf. Proposition I.2). ∎

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Karl-Hermann Neeb
Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstr. 7
64285 Darmstadt
Germany
neeb@mathematik.tu-darmstadt.de

Friedrich Wagemann
Laboratoire de Mathématiques
Jean Leray
Faculté des Sciences et Techniques
Université de Nantes
2, rue de la Houssinière
44322 Nantes cedex 3
France
wagemann@math.univ-nantes.fr