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► **To cite this version:**

Susumu Ariki, Nicolas Jacon. Dipper-James-Murphy's conjecture for Hecke algebras of type B. 2007.
hal-00136680v2

HAL Id: hal-00136680

<https://hal.science/hal-00136680v2>

Preprint submitted on 15 Mar 2007 (v2), last revised 11 Dec 2007 (v3)

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DIPPER-JAMES-MURPHY'S CONJECTURE FOR HECKE ALGEBRAS OF TYPE B_n

SUSUMU ARIKI AND NICOLAS JACON

ABSTRACT. We prove a conjecture by Dipper, James and Murphy that a bipartition is restricted if and only if it is Kleshchev. Hence the restricted bipartitions naturally label the crystal graphs of level two irreducible integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -modules and the simple modules of Hecke algebras of type B_n .

Dedicated to Toshiaki Shoji and Ken-ichi Shinoda on their 60th birthdays

1. INTRODUCTION

Let F be a field, q and Q invertible elements of F . The Hecke algebra of type B_n is the F -algebra defined by generators T_0, \dots, T_{n-1} and relations

$$\begin{aligned} (T_0 - Q)(T_0 + 1) &= 0, & (T_i - q)(T_i + 1) &= 0 \quad (1 \leq i < n) \\ (T_0 T_1)^2 &= (T_1 T_0)^2, & T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad (1 \leq i < n-1) \\ T_i T_j &= T_j T_i \quad (j \geq i+2). \end{aligned}$$

We denote it by $\mathcal{H}_n(Q, q)$, or \mathcal{H}_n for short. The representation theory of \mathcal{H}_n in the semisimple case was studied by Hoefsmit, which had applications in determining generic degrees and Lusztig's a -values. Motivated by the modular representation theory of $U_n(q)$ and $Sp_{2n}(q)$ in the non-defining characteristic case, Dipper, James and Murphy began the study of the modular case more than a decade ago. The first task was to obtain classification of simple modules. For this, they constructed Specht modules which are indexed by the set of bipartitions [7]. The work shows in particular that Hecke algebras of type B_n are cellular algebras in the sense of Graham and Lehrer.¹ Then they conjectured that the simple modules are labeled by (Q, e) -restricted bipartitions. Their philosophy to classify the simple \mathcal{H}_n -modules resembles the highest weight theory in Lie theory: let \mathfrak{g} be a semisimple Lie algebra. It has a commutative Lie subalgebra \mathfrak{h} , the Cartan subalgebra. One dimensional \mathfrak{h} -modules are called weights (by abuse of notion). When a \mathfrak{g} -module admits a simultaneous generalized eigenspace decomposition with respect to \mathfrak{h} , the decomposition is called the (generalized) weight decomposition. Let Λ be a weight. Suppose that a \mathfrak{g} -module M has the property that

Date: March, 2007.

2000 Mathematics Subject Classification. Primary 17B37; Secondary 20C08, 05E99.

¹This result has been recently generalized by Geck in [8].

- (i) Λ appears in the weight decomposition of M ,
- (ii) If N is a proper \mathfrak{g} -submodule of M then Λ does not appear in the weight decomposition of N .

Then the standard argument shows that M has a unique nonzero irreducible quotient. In fact, Verma modules enjoy the property and their irreducible quotients give a complete set of simple objects in the BGG category. Now we turn to the Hecke algebra \mathcal{H}_n . Define the Jucys-Murphy elements t_1, \dots, t_n by $t_1 = T_0$ and $t_{i+1} = q^{-1}T_i t_i T_i$, for $1 \leq i \leq n-1$. They generate a commutative subalgebra \mathcal{A}_n of the Hecke algebra \mathcal{H}_n , and \mathcal{A}_n plays the role of the Cartan subalgebra: one dimensional \mathcal{A}_n -modules are called weights and the generalized simultaneous eigenspace decomposition of an \mathcal{H}_n -module is called the weight decomposition. Any weight is uniquely determined by the values at t_1, \dots, t_n of the weight, and the sequence of these values in this order is called the residue sequence. Let \mathbf{t} be a standard bitableau of shape λ . Then, \mathbf{t} defines a weight whose values at t_i are given by $c_i q^{b_i - a_i}$ where a_i and b_i are the row number and the column number of the node of \mathbf{t} labelled by i respectively, $c_i = -Q$ if the node is in $\lambda^{(1)}$ and $c_i = 1$ if the node is in $\lambda^{(2)}$. By the theory of seminormal representations in the semisimple case and the modular reduction, a weight appears in some \mathcal{H}_n -module if and only if it is obtained from a bitableau this way.

Suppose that there is a weight obtained from a bitableau \mathbf{t} of shape λ such that it does not appear in S^μ when $\mu \triangleleft \lambda$. If such a bitableau exists, we say that λ is a (Q, e) -restricted bitableau. Recall from [7] that

$$[S^\lambda] = [D^\lambda] + \sum_{\mu \triangleleft \lambda} d_{\lambda\mu} [D^\mu],$$

where the summation is over μ such that $D^\mu \neq 0$, $d_{\lambda\mu}$ are decomposition numbers, and $\sum_{\mu \triangleleft \lambda} d_{\lambda\mu} [D^\mu]$ is represented by the radical of the bilinear form on S^λ . As D^μ is a surjective image of S^μ , it implies that the weight does not appear in the radical, while it appears in S^λ . Therefore, $D^\lambda \neq 0$ if λ is (Q, e) -restricted. Unlike the case of the BGG category, we may have $D^\lambda = 0$ and it is important to know when it occurs.

When $-Q$ is not a power of q , a bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ is (Q, e) -restricted if and only if both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e -restricted. Thus we know when a bipartition is (Q, e) -restricted. Further, [6, Thm 4.18] implies that $D^\lambda \neq 0$ if and only if λ is (Q, e) -restricted, that is, simple \mathcal{H}_n -modules are labelled by (Q, e) -restricted bipartitions. Now we suppose that $-Q$ is a power of q . More precisely, we suppose that

- (a) q is a primitive e^{th} root of unity with $e \geq 2$,
- (b) $-Q = q^m$, for some $0 \leq m < e$.

in the rest of the paper. We call (Q, e) -restricted bipartitions restricted bipartitions. They conjectured in this case that $D^\lambda \neq 0$ if and only if λ is restricted, and it has been known as the Dipper-James-Murphy conjecture for Hecke algebras of type B_n .

Later, connection with the theory of canonical bases in deformed Fock spaces in the sense of Hayashi and Misra-Miwa was discovered by Lascoux-Leclerc-Thibon [11] and its proof in the framework of cyclotomic Hecke algebras [1] allowed the first author and Mathas [3] [5] to label simple \mathcal{H}_n -modules by the crystal graph of the level two irreducible integrable $\mathfrak{g}(A_{e-1}^{(1)})$ -module $L_\nu(\Lambda_m + \Lambda_0)$. In the theory, the crystal graph is realized as a subcrystal of the crystal of bipartitions, and the nodes of the crystal graph are called Kleshchev bipartitions. Then, we have $D^\lambda \neq 0$ if and only if λ is Kleshchev. Hence, we obtained the classification of simple \mathcal{H}_n -modules, or more precisely description of the set $\{\lambda \mid D^\lambda \neq 0\}$, through a different approach and the Dipper-James-Murphy conjecture in the modern language is the statement that the Kleshchev bipartitions are precisely the restricted bipartitions.

The aim of this paper is to prove the Dipper-James-Murphy conjecture. Recall that Lascoux, Leclerc and Thibon considered Hecke algebras associated with the symmetric group and they showed that if λ is a e -restricted partition then we can find a_1, \dots, a_p and i_1, \dots, i_p such that we may write

$$f_{i_1}^{(a_1)} \dots f_{i_p}^{(a_p)} \emptyset = \lambda + \sum_{\nu \triangleright \lambda} c_{\nu, \lambda}(v) \nu$$

in the deformed Fock space, where $c_{\nu, \lambda}(v)$ are Laurent polynomials. This follows from the ladder decomposition of a partition. Then the LLT algorithm proves that Kleshchev partitions are precisely e -restricted partitions. The second author [10] proved the similar formula for FLOTW multipartitions in the Jimbo-Misra-Miwa-Okado higher level Fock space using certain a -values instead of the dominance order. Recalling that the Geck-Rouquier theory gives another method to label simple \mathcal{H}_n -modules by bipartitions, the result shows that the parametrizing set, which is called the canonical basic set, is precisely the set of the FLOTW bipartitions. Our strategy to prove the conjecture is to give the analogous formula for Kleshchev bipartitions. To establish the formula, a non recursive characterization of Kleshchev bipartitions given by the first author, Kreiman and Tsuchioka [4] plays a key role.

The paper is organized as follows. In the first section, we briefly recall the definition of Kleshchev bipartitions. We also recall the main result of [4]. In the second section, we use this result to give an analogue for bipartitions of the ladder decomposition. Finally, the last section gives a proof for the conjecture.

Acknowledgments : This paper was written when the second author visited RIMS in Kyoto. He would like to thank RIMS for the hospitality.

2. PRELIMINARIES

In this section, we recall the definition of Kleshchev bipartitions together with the main result of [4] which gives a non recursive characterization of

these bipartitions. We fix m as in the introduction. Namely, the parameter Q of the Hecke algebra is $Q = -q^m$ with $0 \leq m < e$.

2.1. First definitions. Recall that a partition λ is a sequence of weakly decreasing nonnegative integers $(\lambda_0, \lambda_1, \dots)$ such that $|\lambda| = \sum_{i \geq 0} \lambda_i$ is finite. A *bipartition* $\boldsymbol{\lambda}$ is an ordered pair of partitions $(\lambda^{(1)}, \lambda^{(2)})$. $|\boldsymbol{\lambda}| = |\lambda^{(1)}| + |\lambda^{(2)}|$ is called the *rank* of $\boldsymbol{\lambda}$. The (*Young*) *diagram* of $\boldsymbol{\lambda}$ is the set

$$\{(a, b, c) \mid 1 \leq c \leq 2, 1 \leq b \leq \lambda_a^{(c)}\}.$$

We often identify a bipartition with its diagram. The *nodes* of $\boldsymbol{\lambda}$ are the elements of the diagram. Let $\gamma = (a, b, c)$ be a node of $\boldsymbol{\lambda}$. Then the *residue* of γ is defined by

$$\text{res}(\gamma) \equiv \begin{cases} b - a + m \pmod{e} & \text{if } c = 1, \\ b - a \pmod{e} & \text{if } c = 2. \end{cases}$$

By assigning residues to the nodes of a bipartition, we always view a bipartition as a colored Young diagram with colors in $\mathbb{Z}/e\mathbb{Z}$.

Example 2.1. Put $e = 4$, $m = 2$ and $\boldsymbol{\lambda} = ((3, 2), (4, 2, 1))$. Then the colored Young diagram associated with $\boldsymbol{\lambda}$ is as follows.

$$\left(\begin{array}{|c|c|c|} \hline 2 & 3 & 0 \\ \hline 1 & 2 & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline 3 & 0 & & \\ \hline 2 & & & \\ \hline \end{array} \right)$$

If γ is a node with residue i , we say that γ is an i -node. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two bipartitions such that $\boldsymbol{\mu} = \boldsymbol{\lambda} \sqcup \{\gamma\}$. Then, we denote $\boldsymbol{\mu}/\boldsymbol{\lambda} = \gamma$ and if $\text{res}(\gamma) = i$, we say that γ is a *removable i -node* for $\boldsymbol{\mu}$. We also say that γ is an *addable i -node* for $\boldsymbol{\lambda}$ by abuse of notion.²

Let $i \in \mathbb{Z}/e\mathbb{Z}$. We choose a total order on the set of i -nodes of a bipartition. Let $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ be two i -nodes of a bipartition. We say that γ is *above* γ' if either $c = 1$ and $c' = 2$, or $c = c'$ and $a < a'$.³

Let \mathcal{F} be the vector space over \mathbb{Q} with basis given by the set of all bipartitions. We call it the (level two) *Fock space*. We may equip it with \hat{sl}_e -module structure in which the action of the Chevalley generators is given by

$$e_i \boldsymbol{\lambda} = \sum_{\boldsymbol{\nu}: \text{res}(\boldsymbol{\lambda}/\boldsymbol{\nu})=i} \boldsymbol{\nu}, \quad f_i \boldsymbol{\lambda} = \sum_{\boldsymbol{\nu}: \text{res}(\boldsymbol{\nu}/\boldsymbol{\lambda})=i} \boldsymbol{\nu}.$$

Using the total order on the set of i -nodes given above, we deform the \hat{sl}_e -module structure to $U_v(\hat{sl}_e)$ -module structure on the deformed Fock space $\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}(v)$. We refer to [2] for the details.

²An addable i -node of $\boldsymbol{\lambda}$ is not a node of $\boldsymbol{\lambda}$.

³We now know that there are more than one Specht module theory, and different Specht module theories prefer different total orders on the set of i -nodes of a bipartition. Our choice of the total order is the one preferred by Dipper-James-Murphy's Specht module theory.

2.2. Kleshchev bipartitions. We are now ready to define Kleshchev bipartitions. Recall that the crystal basis of the deformed Fock space is given by the crystal of bipartitions. Then Kleshchev bipartitions are those bipartitions which belongs to the same connected component as the empty bipartition in the crystal of bipartitions. Equivalently, Kleshchev bipartitions are those bipartitions which may be obtained from the empty bipartition by applying the Kashiwara operators successively. Rephrasing it in combinatorial terms, we have a recursive definition of Kleshchev bipartitions as follows.

Let λ be a bipartition and let γ be an i -node of λ , we say that γ is a *normal* i -node of λ if, whenever η is an addable i -node of λ below γ , there are more removable i -nodes between η and γ than addable i -nodes between η and γ . If γ is the highest normal i -node of λ , we say that γ is a *good* i -node of λ .

Definition 2.2. A bipartition λ is *Kleshchev* if either it is the empty bipartition or there exists $i \in \mathbb{Z}/e\mathbb{Z}$ and a good i -node γ of λ such that $\lambda \setminus \{\gamma\}$ is Kleshchev.

Note that the definition depends on m . Note also that if $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ is Kleshchev then both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e -restricted.

In [4], the first author, Kreiman and Tsuchioka have given a characterization of Kleshchev bipartitions.

Let λ be a partition. Then the set of *beta numbers of charge* c is by definition the set J_c of decreasing integers

$$j_0 > j_1 > \cdots > j_k > \cdots$$

defined by $j_k = \lambda_k + c - k$, for $k \geq 0$.

Definition 2.3. Let λ be an e -restricted partition, J_m the corresponding set of beta numbers of charge m . Define

$$U(J_m) = \{x \in J_m \mid x - e \notin J_m\}.$$

If λ is an e -core then we define $\text{up}_m(\lambda) = \lambda$. Otherwise let $p = \max(U(J_m))$ and define

$$V(J_m) = \{x > p \mid x \not\equiv p \pmod{e}, x - e \in J, x \notin J\}.$$

Let $q = \min V(J_m)$. Then we define

$$\text{up}(J_m) = (J_m \setminus \{p\}) \sqcup \{q\}$$

and we denote the corresponding partition by $\text{up}_m(\lambda)$.

Definition 2.4. Let λ be an e -restricted partition. Apply up_m successively until we reach an e -core. We denote the resulting e -core by $\text{roof}_m(\lambda)$.

Definition 2.5. Let λ be an e -restricted partition, J_0 the corresponding set of beta numbers of charge 0. Define

$$U(J_0) = \{x \in J_0 \mid x - e \notin J_0\}.$$

If λ is an e -core then we define $\text{down}_0(\lambda) = \lambda$. Otherwise let $p' = \min(U(J_0))$ and define

$$W(J_0) = \{x > p' - e \mid x \in J_0, x + e \notin J_0\} \cup \{p'\}.$$

Let $q' = \min W(J_0)$. Then we define

$$\text{down}(J_0) = (J_0 \setminus \{q'\}) \sqcup \{p' - e\}$$

and we denote the corresponding partition by $\text{down}_0(\lambda)$.

Definition 2.6. Let λ be an e -restricted partition. Apply down_0 successively until we reach an e -core. We denote the resulting e -core by $\text{base}_0(\lambda)$.

Finally, let λ be an e -restricted partition, J_0^{\max} the set of beta numbers of charge 0 for $\text{base}_0(\lambda)$. Define $M_i(\lambda)$ to be the maximal element of the i^{th} runner on the abacus of J_0^{\max} , for $i \in \mathbb{Z}/e\mathbb{Z}$. We write $M_i(\lambda)$ in decreasing order

$$M_{i_1}(\lambda) > M_{i_2}(\lambda) > \cdots > M_{i_e}(\lambda).$$

Then $J_0^{\max} \cup \{M_{i_k}(\lambda) + e\}_{1 \leq k \leq m}$ is the set of beta numbers of charge m , for some partition. We denote the partition by $\tau_m(\text{base}_0(\lambda))$.

Now, the characterization of Kleshchev bipartitions is as follows.

Theorem 2.7 ([4]). *Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be a bipartition such that both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e -restricted. Then λ is Kleshchev if and only if*

$$\text{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\text{base}_0(\lambda^{(2)})).$$

2.3. The Dipper-James-Murphy conjecture. We recall the dominance order for bipartition. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ and $\mu = (\mu^{(1)}, \mu^{(2)})$ be bipartitions. In this paper, we write $\mu \leq \lambda$ if

$$\sum_{k=1}^j \lambda_k^{(1)} \geq \sum_{k=1}^j \mu_k^{(1)} \text{ and } |\lambda^{(1)}| + \sum_{k=1}^j \lambda_k^{(2)} \geq |\mu^{(1)}| + \sum_{k=1}^j \mu_k^{(2)},$$

for all $j \geq 0$.

Definition 2.8. Let λ be a bipartition of rank n . A *standard bitableau of shape λ* is a sequence of bipartitions

$$\emptyset = \lambda[0] \subseteq \lambda[1] \subseteq \cdots \subseteq \lambda[n] = \lambda$$

such that the rank of $\lambda[k]$ is k , for $0 \leq k \leq n$. The *residue sequence* of \mathbf{t} is the sequence

$$(\text{res}(\gamma[1]), \dots, \text{res}(\gamma[n])) \in (\mathbb{Z}/e\mathbb{Z})^n$$

where $\gamma[k] = \lambda[k]/\lambda[k-1]$, for $1 \leq k \leq n$.

A standard bitableau may be viewed as filling of the nodes of λ with numbers $1, \dots, n$: we write k in the node $\gamma[k]$, for $1 \leq k \leq n$.

Definition 2.9. A bipartition λ is $(-q^m, e)$ -restricted, or *restricted* for short, if there exists a standard bitableau \mathbf{t} of shape λ such that the residue sequence of any standard bitableau of shape $\nu \triangleleft \lambda$ does not coincide with the residue sequence of \mathbf{t} .

Conjecture 2.10 ([7, Conj. 8.13]). *A bipartition λ is Kleshchev if and only if it is restricted.*

3. PROPERTIES OF KLESHCHEV BIPARTITIONS

The aim of this section is to prove some combinatorial results concerning Kleshchev bipartitions.

3.1. Admissible sequence. Let $i \in \mathbb{Z}/e\mathbb{Z}$. We say that a sequence of removable i -nodes R_1, \dots, R_s of a bipartition λ is an *admissible sequence* if

- there is no removable or addable i -node below R_j , for $j = 1, \dots, s$.
- if there is a removable i -node R above R_1, \dots, R_s , there must exist an addable i -node below R .

Thus, if we read removable and addable i -nodes in the total order of nodes, the smallest $s + 1$ i -nodes are A, R, \dots, R , where A stands for addable and R for removable.

The following lemma shows the existence of an admissible sequence of nodes for a Kleshchev bipartition: choose i as in the lemma and read addable and removable i -nodes in the total order of nodes. Let γ be the smallest addable i -node. Then the removable i -nodes below γ form an admissible sequence.

Lemma 3.1. *Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be a nonempty Kleshchev bipartition. Then there exists $i \in \mathbb{Z}/e\mathbb{Z}$ and a removable i -node γ such that if η is an addable i -node of λ then η is above γ .*

Proof. Recall that both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e -restricted. There are two cases to consider.

- Assume that $\lambda^{(2)}$ is not the empty partition. Let $\lambda_j^{(2)}$ be the last row. Define $\gamma = (j, \lambda_j^{(2)}, 2)$ and $i = \text{res}(\gamma)$. Since $\lambda^{(2)}$ is e -restricted, the residue of the addable node $(j + 1, 1, 2)$ is not i . Hence all the addable i -node of λ are above γ .
- Assume that $\lambda^{(2)}$ is the empty partition. Let $\lambda_j^{(1)}$ be the last row. Define $\gamma = (j, \lambda_j^{(1)}, 1)$ and $i = \text{res}(\gamma)$. Since $\lambda^{(1)}$ is e -restricted, the residue of the addable node $(j + 1, 1, 1)$ is not i . We only have to show that the residue of the addable node $(1, 1, 2)$ is not i . Suppose to the contrary that the residue is i . As λ is Kleshchev, we may delete good nodes successively to obtain the empty bipartition. Hence $(j, \lambda_j^{(1)}, 1)$ must be deleted at some point in the process. However, it never be a good node since we always have an addable i -node $(1, 1, 2)$ just below it and there is no removable i -node between them.

Hence the claim follows. \square

Lemma 3.2. *Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be Kleshchev and we choose an admissible sequence as in the previous lemma, which we denote by R_1, \dots, R_s . Let*

$\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$ be the bipartition obtained by deleting these nodes from $\boldsymbol{\lambda}$:

$$\boldsymbol{\lambda} = \boldsymbol{\mu} \sqcup \{R_1, \dots, R_s\}.$$

Then $\boldsymbol{\mu}$ is Kleshchev.

Proof. Note that $\mu^{(1)}$ and $\mu^{(2)}$ are both e -restricted. We prove that $\mu^{(1)}$ is e -restricted. The proof for $\mu^{(2)}$ is the same. Since $\lambda^{(1)}$ is e -restricted, it occurs only when there exists j such that $\lambda_j^{(1)} = \lambda_{j+1}^{(1)} + e - 1$, and $\mu_j^{(1)} = \lambda_j^{(1)}$, $\mu_{j+1}^{(1)} = \lambda_{j+1}^{(1)} - 1$ and $\text{res}(j+1, \lambda_{j+1}^{(1)}, 1) = i$. But then $\text{res}(j, \lambda_j^{(1)}, 1) = i$, which implies $\mu_j^{(1)} = \lambda_j^{(1)} - 1$ by definition of $\boldsymbol{\mu}$.

First we consider the case when $\lambda^{(2)}$ has no addable i -node. Note that if $\lambda^{(1)}$ has no addable i -node then the claim holds. It follows from

- (i) $\boldsymbol{\lambda}$ has no addable i -node and $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by removing all the removable i -nodes.
- (ii) All the removable i -nodes of $\boldsymbol{\lambda}$ are normal.

Define $t \geq 0$ by

$$t = \min\{k \mid \text{roof}_m(\lambda^{(1)}) = \text{up}_m^k(\lambda^{(1)})\}.$$

We prove the claim by induction on t . Note that if $\boldsymbol{\lambda}$ satisfies the assumptions then so does $(\text{up}_m(\lambda^{(1)}), \lambda^{(2)})$ by

$$\text{roof}_m(\text{up}_m(\lambda^{(1)})) = \text{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\text{base}_0(\lambda^{(2)})),$$

which implies that $(\text{up}_m(\lambda^{(1)}), \lambda^{(2)})$ is Kleshchev.

Suppose that $t = 0$. In particular, $\lambda^{(1)}$ is an e -core. We may assume that $\lambda^{(1)}$ has an addable i -node. Thus $\lambda^{(1)}$ has no removable i -node, which implies $\mu^{(1)} = \lambda^{(1)}$ and that all the removable i -nodes of $\lambda^{(2)}$ are normal. Thus, $\boldsymbol{\mu}$ is Kleshchev.

Suppose that $t > 0$ and that the claim holds for $(\text{up}_m(\lambda^{(1)}), \lambda^{(2)})$. We may also assume that $\lambda^{(1)}$ has an addable i -node. Let p be the minimal addable i -node of J_m . If there is no removable i -node above p then all the removable i -nodes of $\lambda^{(1)}$ are normal and by deleting them, we obtain that $\boldsymbol{\mu}$ is Kleshchev. Hence, we assume that there is a removable i -node above p . This implies that there is $r \in U(J_m)$ on the $(i+1)^{\text{th}}$ runner such that $r > p$. Let $x = \max U(J_m)$. Then $r \in U(J_m)$ implies that $x \geq r > p$. Thus, p is an addable i -node of $\text{up}_m(\lambda^{(1)})$ and $\mu^{(1)}$ for $\text{up}_m(\lambda^{(1)})$ is equal to $\text{up}_m(\mu^{(1)})$. Therefore, $\boldsymbol{\mu}$ for $(\text{up}_m(\lambda^{(1)}), \lambda^{(2)})$ is $(\text{up}_m(\mu^{(1)}), \mu^{(2)})$, and it is Kleshchev by the induction hypothesis. Hence,

$$\text{roof}_m(\mu^{(1)}) = \text{roof}_m(\text{up}_m(\mu^{(1)})) \subseteq \tau_m(\text{base}_0(\mu^{(2)}))$$

and $\boldsymbol{\mu}$ is Kleshchev as desired.

Next we consider the case when $\lambda^{(1)} = \mu^{(1)}$. Define $s \geq 0$ by

$$s = \min\{k \mid \text{base}_0(\lambda^{(2)}) = \text{down}_0^k(\lambda^{(2)})\}.$$

We prove the claim by induction on s . Note that if λ is Kleshchev then so is $(\lambda^{(1)}, \text{down}_0(\lambda^{(2)}))$ by

$$\text{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\text{base}_0(\lambda^{(2)})) = \tau_m(\text{base}_0(\text{down}_0(\lambda^{(2)}))).$$

If $s = 0$ then $\lambda^{(2)}$ is an e -core and it has a removable i -node by definition. Hence, $\lambda^{(2)}$ has no addable i -node and we are reduced to the previous case. Thus we suppose that $s > 0$ and that the claim holds for $(\lambda^{(1)}, \text{down}(\lambda^{(2)}))$. We may also assume that $\lambda^{(2)}$ has an addable i -node. Let

$$r = \min\{x \in J_0 \mid x - 1 \notin J_0\} - 1.$$

r is on the i^{th} runner by definition. Then there exists $N \geq 1$ such that

$$r, r + e, \dots, r + (N - 1)e \notin J_0 \quad \text{and} \quad r + Ne \in J_0.$$

Let $y = \min U(J_0)$. Then $y \leq r + Ne$. Suppose that y is not on the i^{th} runner and the $(i + 1)^{\text{th}}$ runner. If a node which is not on the both runners moves to $y - e$ by the down operation, $\mu^{(2)}$ for $\text{down}_0(\lambda^{(2)})$ is equal to $\text{down}_0(\mu^{(2)})$. Thus, by the induction hypothesis, $(\lambda^{(1)}, \text{down}_0(\mu^{(2)}))$ is Kleshchev. Hence,

$$\text{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\text{base}_0(\text{down}_0(\mu^{(2)}))) = \tau_m(\text{base}_0(\mu^{(2)}))$$

implies that μ is Kleshchev. If a node in one of the two runners moves to $y - e$, then $y < r + Ne$ and there exists $0 \leq k \leq N - 1$ such that $r + ke + 1 \in J_0$, $r + (k + 1)e + 1 \notin J_0$ and $r + ke + 1$ moves to $y - e$. Then, to obtain $\text{down}_0(\mu^{(2)})$, $r + ke$ moves to $y - e$. If $k < N - 1$ then $\mu^{(2)}$ for $\text{down}_0(\lambda^{(2)})$ is equal to $\text{down}_0(\mu^{(2)})$ again. Suppose that $k = N - 1$. As $r + (N - 1)e + 1$ moves, $r + Ne + 1 \notin J_0$ and we have that $r + Ne$ is an addable i -node. Let $J = J_0$ and K the set of beta numbers of charge 0 of $\mu^{(2)}$. We claim that

$$\text{base}(J_{\leq r+Ne}) = \text{base}(K_{\leq r+Ne}).$$

Let $y < y_1 < \dots < y_k < r + Ne$ be the nodes in J which are greater than y and smaller than $r + Ne$. We show that

$$\text{base}(J_{\leq y_j}) = s_i \text{base}(K_{\leq y_j}) \supseteq \text{base}(K_{\leq y_j}),$$

for $1 \leq j \leq k$, where s_i means switch of the i^{th} and $(i + 1)^{\text{th}}$ runners. Let $y_0 = y$. $J_{\leq y}$ and $K_{\leq y}$ are s_i -cores in the sense of [4], and direct computation shows the formula for $j = 0$. Now we use $\text{base}(J_{\leq y_{j+1}}) = \text{base}(\{y_{j+1}\} \cup \text{base}(J_{\leq y_j}))$ and $\text{base}(K_{\leq y_{j+1}}) = \text{base}(\{y_{j+1}\} \cup \text{base}(K_{\leq y_j}))$ ⁴ and continue the similar computation and comparison of $\text{base}(J_{\leq y_j})$ and $\text{base}(K_{\leq y_j})$. At the end of the inductive step, we get

$$\text{base}(J_{< r+Ne}) = s_i \text{base}(K_{< r+Ne}) \supseteq \text{base}(K_{< r+Ne}).$$

Now, one more direct computation shows

$$\text{base}(\{r + Ne\} \cup \text{base}(J_{< r+Ne})) = \text{base}(\{r + Ne\} \cup \text{base}(K_{< r+Ne})),$$

⁴See [4, Prop 7.8].

and we have the claim. Therefore, $\text{base}_0(\lambda^{(2)}) = \text{base}_0(\mu^{(2)})$ and we have

$$\text{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\text{base}_0(\lambda^{(2)})) = \tau_m(\text{base}_0(\mu^{(2)})).$$

Suppose that y is on one of the both runners. As $y \leq r + Ne$, we have either $y = r + Ne$ or $y = r + ke + 1$, for some $0 \leq k \leq N - 1$. If the latter occurs, $\text{down}_0(\lambda^{(2)})$ is obtained by moving a node outside the two runners to $y - e$ or moving y to $y - e$. Thus, $\mu^{(2)}$ for $\text{down}_0(\lambda^{(2)})$ is equal to $\text{down}_0(\mu^{(2)})$, which is obtained from $\mu^{(2)}$ by moving the same node outside the two runners to $y - 1 - e$ or moving $y - 1$ to $y - 1 - e$, respectively. If $y = r + Ne$ and $r + Ne + 1 \in J_0$ then the same is true, and if $y = r + Ne$ and $r + Ne + 1 \notin J_0$ then $\mu^{(2)} = \text{down}_0^N(\lambda^{(2)})$ and we have

$$\text{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\text{base}_0(\lambda^{(2)})) = \tau_m(\text{base}_0(\mu^{(2)}))$$

again. Hence, $\boldsymbol{\mu} = (\lambda^{(1)}, \mu^{(2)})$ is Kleshchev as desired. \square

3.2. Optimal sequence of a Kleshchev bipartition. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a Kleshchev bipartition. By the previous lemma, we may define by induction a sequence of Kleshchev bipartitions

$$\boldsymbol{\lambda} =: \boldsymbol{\lambda}[1], \boldsymbol{\lambda}[2], \dots, \boldsymbol{\lambda}[p], \boldsymbol{\lambda}[p+1] = \emptyset$$

and a sequence of residues

$$\underbrace{i_1, \dots, i_1}_{a_1 \text{ times}}, \dots, \underbrace{i_p, \dots, i_p}_{a_p \text{ times}}$$

such that $\boldsymbol{\lambda}[j+1] = \boldsymbol{\lambda}[j] \sqcup \{R_1^j, \dots, R_{a_j}^j\}$ and $R_1^j, \dots, R_{a_j}^j$ is an admissible sequence of i_j -nodes for $\boldsymbol{\lambda}[j]$, for $j = 1, \dots, p$. In this case, we say that $\underbrace{i_1, \dots, i_1}_{a_1 \text{ times}}, \dots, \underbrace{i_p, \dots, i_p}_{a_p \text{ times}}$ is an *optimal sequence* for $\boldsymbol{\lambda}$.

Example 3.3. Keeping example 2.1, it is easy to see that $((3.2), (4.2.1))$ is a Kleshchev bipartition and that its optimal sequence of residue is :

$$2, 2, 0, 3, 3, 2, 1, 1, 0, 0, 3, 2.$$

4. PROOF OF THE CONJECTURE

4.1. The result. We are now ready to prove the conjecture. As is explained in the introduction, it is enough to prove that Kleshchev bipartitions are restricted bipartitions.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ and $\boldsymbol{\nu} = (\nu^{(1)}, \nu^{(2)})$ be two bipartitions and let r_i ($i = 1, 2$) be their lengths. Thus $\lambda^{(i)} = (\lambda_1^{(i)}, \dots, \lambda_{r_i}^{(i)})$ and $\nu^{(i)} = (\nu_1^{(i)}, \dots, \nu_{r_i}^{(i)})$, for $i = 1, 2$. We introduce certain reverse lexicographic order on the set of bipartitions.

Definition 4.1. We write $\boldsymbol{\lambda} \prec \boldsymbol{\nu}$ if

- either there exists $j \in \{1, \dots, r_2\}$ such that $\lambda_k^{(2)} = \nu_k^{(2)}$, for $k > j$, and $\lambda_j^{(2)} < \nu_j^{(2)}$,

- or there exists $j \in \{1, \dots, r_1\}$ such that $\lambda^{(2)} = \nu^{(2)}$, $\lambda_k^{(1)} = \nu_k^{(1)}$, for $k > j$, and $\lambda_j^{(1)} < \nu_j^{(1)}$.

It is clear that if $\nu \triangleleft \lambda$ then $\lambda \prec \nu$. Recall that the deformed Fock space is given a specific $U_v(\widehat{sl}_e)$ -module structure which is suitable for the Dipper-James-Murphy's Specht module theory.

Proposition 4.2. *Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be Kleshchev and let*

$$\underbrace{i_1, \dots, i_1}_{a_1 \text{ times}}, \dots, \underbrace{i_p, \dots, i_p}_{a_p \text{ times}}$$

be an optimal sequence for λ . Then we have

$$f_{i_1}^{(a_1)} \dots f_{i_p}^{(a_p)} \emptyset = \lambda + \sum_{\nu \prec \lambda} c_{\nu, \lambda}(v) \nu,$$

for some Laurent polynomials $c_{\nu, \lambda}(v)$, in the deformed Fock space.

Proof. First note that the coefficient of λ is one because the sequence of admissible nodes is a sequence of normal nodes.

Now the proposition is proved by induction. Let $R_1^1, \dots, R_{a_1}^1$ be the admissible sequence of i -nodes for λ and let λ' be the Kleshchev bipartition such that

$$\lambda = \lambda' \sqcup \{R_1^1, \dots, R_{a_1}^1\}.$$

By induction, we have

$$f_{i_2}^{(a_2)} \dots f_{i_p}^{(a_p)} \emptyset = \lambda' + \sum_{\nu' \prec \lambda'} c_{\nu', \lambda'}(v) \nu'.$$

Let $\nu \neq \lambda$ be a bipartition which appears in $f_{i_1}^{(a_1)} \dots f_{i_p}^{(a_p)} \emptyset$ with non zero coefficient. Hence, there exist removable i_1 -nodes $R_1^1, \dots, R_{a_1}^1$ and a bipartition ν' such that

$$\nu = \nu' \sqcup \{R_1^1, \dots, R_{a_1}^1\}$$

with $\nu' \preceq \lambda'$. We may assume $\nu' \prec \lambda'$ since $\nu' = \lambda'$ implies $\nu \preceq \lambda$ by the choice of the nodes $\{R_1^1, \dots, R_{a_1}^1\}$. Assume that we have $\lambda \prec \nu$. If $\nu^{(2)} \neq \lambda^{(2)}$ then there exists t such that

- (i) $\nu_j^{(2)} = \lambda_j^{(2)}$, for $j > t$,
- (ii) $\nu_t^{(2)} = \lambda_t^{(2)} - 1$ and we have an addable i_1 -node in $\nu_t^{(2)}$,
- (iii) all the nodes $\{R_1^1, \dots, R_{a_1}^1\}$ are above $(t, \lambda_t^{(2)}, 2)$.

The condition (ii) implies that the residue of the node $(t, \lambda_t^{(2)}, 2)$ is i_1 . On the other hand, (iii) implies that the node $(t, \lambda_t^{(2)}, 2)$ is not a removable node for λ (otherwise, it has to be removed to obtain λ'). This implies that $\lambda_t^{(2)} = \lambda_{t+1}^{(2)}$. Then since we have $\nu_t^{(2)} \geq \nu_{t+1}^{(2)}$, we conclude that $\lambda_{t+1}^{(2)} > \nu_{t+1}^{(2)}$ hence $\nu \prec \lambda$, which is a contradiction.

If $\nu^{(2)} = \lambda^{(2)}$ then there exists t such that we have $\nu_j^{(1)} = \lambda_j^{(1)}$, for $j > t$, $\nu_t^{(1)} = \lambda_t^{(1)} - 1$, and we have an addable i_1 -node in $\nu_t^{(2)}$. Then we argue as above. \square

Corollary 4.3. *The Dipper-James-Murphy conjecture is true.*

Proof. Observe that ν appears in $f_{i_1} \cdots f_{i_n} \emptyset$ if and only if there exists a standard bitableau of shape ν such that its residue sequence is (i_1, \dots, i_n) . Let λ be Kleshchev. Then Proposition 4.2 shows that there is a standard bitableau \mathbf{t} such that if the residue sequence of \mathbf{t} appears as the residue sequence of a standard bitableau of shape ν then $\nu \preceq \lambda$. Suppose that the residue sequence of \mathbf{t} is the residue sequence of a standard bitableau of shape $\nu \triangleleft \lambda$. As $\nu \triangleleft \lambda$ implies $\lambda \prec \nu$, we cannot have $\nu \preceq \lambda$, a contradiction. Hence λ is restricted. \square

4.2. Remarks. We conclude the paper with two remarks.

Remark 4.4. We give the second proof of the fact that restricted implies Kleshchev. Note that we may assume that the characteristic of F is zero, and we freely use results in [2].

Assume that λ is restricted. Then there exist i_1, \dots, i_n such that

- (i) λ appears in $f_{i_1} \cdots f_{i_n} \emptyset$,
- (ii) $\nu \triangleleft \lambda$ implies that ν does not appear in $f_{i_1} \cdots f_{i_n} \emptyset$.

Let $\{G(\nu) | \nu \text{ is Kleshchev}\}$ be the canonical basis in the Fock space. As $f_{i_1} \cdots f_{i_n} \emptyset$ is represented by a projective \mathcal{H}_n -module, and $G(\nu) = [P^\nu]$, the indecomposable projective \mathcal{H}_n -module indexed by ν , we may write

$$f_{i_1} \cdots f_{i_n} \emptyset = \sum c_\nu G(\nu)$$

where $c_\nu \geq 0$. In particular, $c_\nu > 0$ implies that ν appears in $f_{i_1} \cdots f_{i_n} \emptyset$.

Suppose that λ is not Kleshchev. As λ appears in $f_{i_1} \cdots f_{i_n} \emptyset$, λ appears in $G(\nu)$ for some Kleshchev bipartition ν with $c_\nu > 0$. Since

$$G(\nu) = \nu + \sum_{\lambda \succ \nu} d_{\lambda\nu} \lambda$$

we must have $\nu \triangleleft \lambda$. However, this implies that ν does not appear in $f_{i_1} \cdots f_{i_n} \emptyset$, which contradicts $c_\nu > 0$.

Remark 4.5. There are several ways to label the crystal $B(\Lambda_m + \Lambda_0)$ by a set of bipartitions. These bipartitions are named Uglov bipartitions. Recent results of Geck [8] and Geck and the second author [9] show that Uglov bipartitions naturally label simple \mathcal{H}_n -modules, and Rouquier's theory of the BGG category of the rational Cherednik algebra as quasihereditary covers of Hecke algebras naturally explains the existence of various Specht module theories which depends on $\log Q$.

We conjecture that Uglov bipartitions satisfy an analogue of Proposition 4.2 except that the dominance order is replaced by an appropriate a -value (in the sense of [9, Prop 2.1]). As is mentioned in the introduction, it is

known that this conjecture is true in the case where Uglov bipartitions are FLOTW bipartitions [10, Prop. 4.6].

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