# EXISTENCE AND DECAY OF SOLUTIONS OF A NONLINEAR VISCOELASTIC PROBLEM WITH A MIXED NONHOMOGENEOUS CONDITION

Long Nguyen Thanh, Alain Pham Ngoc Dinh, Le Xuan Truong

# EXISTENCE AND DECAY OF SOLUTIONS OF A NONLINEAR VISCOELASTIC PROBLEM WITH A MIXED NONHOMOGENEOUS CONDITION

NGUYEN THANH LONG, ALAIN PHAM NGOC DINH, LE XUAN TRUONG

Abstract. We study the initial-boundary value problem for a nonlinear wave equation given by

$$u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + |u_t|^{q-2}u_t = f(x,t,u),$$
$$u_x(0,t) = u(0,t), u_x(1,t) + \eta u(1,t) = g(t),$$
$$u(x,0) = u_0(x), u_t(x,0) = u_1(x),$$

where $\eta \geq 0$, $q \geq 2$ are given constants and $u_0, u_1, g, k, f$ are given functions. In this paper, we consider two main parts. In Part 1, under a certain local Lipschitzian condition on $f$ with $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$; $k, g \in H^1(0,T)$, $\eta \geq 0$; $q \geq 2$, a global existence and uniqueness theorem is proved. The proof is based on the paper [10] associated to a contraction mapping theorem and standard arguments of density. In Part 2, the asymptotic behavior of the solution $u$ as $t \to +\infty$ is studied, under more restrictive conditions, namely $g = 0$, $f(x,t,u) = -|u|^{p-2}u + F(x,t)$, $p \geq 2$, $F \in L^1(\mathbb{R}_+; L^2) \bigcap L^2(\mathbb{R}_+; L^2)$, $\int_0^{+\infty} e^{\sigma t}\|F(t)\|^2 dt < +\infty$, with $\sigma > 0$, and $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$, $k \in H^1(\mathbb{R}_+)$, and some others ($\|\cdot\|$ denotes the $L^2(0,1)$ norm). It is proved that under these conditions, a unique solution $u(t)$ exists on $\mathbb{R}_+$ such that $\|u'(t)\| + \|u_x(t)\|$ decay exponentially to 0 as $t \to +\infty$. Finally, we present some numerical results.

## 1. Introduction

In this paper we will consider the following initial and boundary value problem:

$$u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + |u_t|^{q-2}u_t = f(x,t,u), 0 < x < 1; 0 < t < T, \tag{1.1}$$

$$u_x(0,t) = u(0,t), u_x(1,t) + \eta u(1,t) = g(t), \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{1.3}$$

where $\eta \geq 0$, $q \geq 2$ are given constants and $u_0, u_1, g, k, f$ are given functions satisfying conditions specified later.

2000 *Mathematics Subject Classification.* 35L05, 35L15, 35L70, 37B25.
*Key words and phrases.* Faedo-Galerkin method; Global existence; contraction mapping theorem; nonlinear wave equation; Viscoelastic; Exponential decay.

In a recent paper [1], Berrimia and Messaoudi considered the problem

$$u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)ds = |u|^{p-2}u, x \in \Omega, t > 0, \tag{1.4}$$

$$u = 0, \quad \text{on} \quad \partial\Omega, \tag{1.5}$$

$$u(x,0) = \widetilde{u}_0(x), u_t(x,0) = \widetilde{u}_1(x), x \in \Omega, \tag{1.6}$$

where $p > 2$ is a constant, $k$ is a given positive function, and $\Omega$ is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. This type of problems have been considered by many authors and several results concerning existence, nonexistence, and asymptotic behavior have been established. In this regard, Cavalcanti et al. [3] studied the following equation

$$u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)ds + |u|^{p-2}u + a(t)u_t = 0, \quad \text{in} \quad \Omega \times (0,\infty), \tag{1.7}$$

for $a : \Omega \to \mathbb{R}_+$, a function, which may be null on a part of the domain $\Omega$. Under the conditions that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with $\omega$ satisfying some geometry restrictions and

$$-\zeta_1 k(t) = k^{/}(t) = -\zeta_2 k(t), t \geq 0, \tag{1.8}$$

the authors established an exponential rate of decay.

In [2] Bergounioux, Long and Dinh studied problem (1.1), (1.3) with $k = 0, q = 2, f(x,t,u) = -Ku + F(x,t)$, and the mixed boundary conditions (1.2) standing for

$$u_x(0,t) = g(t) + hu(0,t) - \int_0^t H(t-s)u(0,s)ds, \tag{1.9}$$

$$u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0, \tag{1.10}$$

where $h \geq 0$, $K, \lambda, K_1, \lambda_1$ are given constants and $g, H$ are given functions.

In [7], Long, Dinh and Diem obtained the unique existence, regularity and asymptotic expansion of the problem (1.1), (1.3), (1.9) and (1.10) in the case of $k = 0$, $f(x,t,u) = -K|u|^{p-2}u + F(x,t)$, with $p \geq 2$, $q \geq 2$; $K, \lambda$ are given constants.

In [8], Long, Ut and Truc gave the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of problem (1.1)- (1.3) when $k = 0$, $q = 2$, $f(x,t,u) = -Ku + F(x,t)$ and $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$. In this case, the problem (1.1)- (1.3) is the mathematical model describing a shock problem involving a linear viscoelastic bar.

In [9], Long and Giai obtained the unique existence and asymptotic expansion for the solution of problem (1.1), (1.3) when $k = 0$, $q = 2$, $f(x,t,u) = -Ku + F(x,t)$ and $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$, and the mixed boundary conditions (1.2) standing for

$$\begin{aligned} u_x(0,t) =& g(t) + K_1|u(0,t)|^{\alpha-2}u(0,t) + \lambda_1|u_t(0,t)|^{\beta-2}u_t(0,t) \\ &- \int_0^t H(t-s)u(0,s)ds, \end{aligned} \tag{1.11}$$

$$u(1,t) = 0, \tag{1.12}$$

where $K$, $\lambda$, $K_1$, $\lambda_1$, $\alpha$, $\beta$ are given constants and $g, H$ are given functions. In this case, the problem (1.1), (1.3), (1.11), (1.12) is the mathematical model describing a shock problem involving a nonlinear viscoelastic bar.

In [10], Long and Truong obtained the unique existence and asymptotic expansion for the solution of problem (1.1) -(1.3) when $f(x,t,u) = -K|u|^{p-2}u + F(x,t)$,

$(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$; $F, F_t \in L^2(Q_T)$, $k \in W^{2,1}(0,T)$, $g \in H^2(0,T)$; $K$, $\eta \geq 0$, $\eta_0 > 0$; $p, q \geq 2$.

In this paper, we consider two main parts. In Part 1, under a certain local Lipschitzian condition on $f$ with $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$; $k, g \in H^1(0,T)$,$\lambda > 0$, $\eta_0 > 0$; $\eta \geq 0$; $q \geq 2$, a global existence and uniqueness theorem is proved. The proof is based on the paper [10] associated to a contraction mapping theorem and standard arguments of density. In Part 2, the asymptotic behavior of the solution $u$ as $t \to \infty$ is studied, under more restrictive conditions, namely $f(x,t,u) = -|u|^{p-2}u + F(x,t)$, $p \geq 2$, $F \in L^1(\mathbb{R}_+; L^2) \bigcap L^2(\mathbb{R}_+; L^2)$, $\int_0^{+\infty} e^{\sigma t}\|F(t)\|^2 dt < +\infty$, with $\sigma > 0$, and $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$, $g = 0$, $k \in H^1(\mathbb{R}_+$, and some others ($\|\cdot\|$ denotes the $L^2(0,1)$ norm). It is proved that under these conditions, a unique solution $u(t)$ exists on $\mathbb{R}_+$ such that $\|u'(t)\| + \|u_x(t)\|$ decay exponentially to 0 as $t \to +\infty$. The results obtained here relatively are in part generalizations of those in [1-3, 6-10]. Finally, we present some numerical results.

## 2. Preliminary Results

Put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We omit the definitions of usual function spaces: $C^m(\overline{\Omega})$, $L^p(\Omega)$, $W^{m,p}(\Omega)$. We denote $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, \ldots$ The norm in $L^2$ is denoted by $\|\cdot\|$. We also denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2$ or pair of dual scalar product of continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space $X$ and by $X'$ the dual space of $X$. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \Big( \int_0^T \|u(t)\|_X^p dt \Big)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, and $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, and $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

Without loss of generality, we can suppose that $\eta_0 = \lambda = 1$. For every $\eta \geq 0$, we put

$$a_\eta(u,v) = \int_0^1 u_x(x)v_x(x)dx + u(0)v(0) + \eta u(1)v(1), \forall u, v \in H^1, \tag{2.1}$$

$$\|v\|_\eta = (a_\eta(v,v))^{1/2}. \tag{2.2}$$

On $H^1$ we shall use the following equivalent norm

$$\|v\|_1 = \left( v^2(0) + \int_0^1 |v_x(x)|^2 dx \right)^{1/2} \tag{2.3}$$

Then we have the following lemmas.

**Lemma 2.1.** *The imbedding $V \hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])} \leq \|v\|_V, \text{ for all } v \in V. \tag{2.4}$$

**Lemma 2.2.** *Let $\eta \geq 0$. Then, the symmetric bilinear form $a_\eta(\cdot,\cdot)$ defined by* (2.1) *is continuous on $H^1 \times H^1$ and coercive on $H^1$, i.e.,*
*(i)* $|a_\eta(u,v)| = C_\eta \|u\|_1 \|v\|_1, \quad$ *for all* $\quad u, v \in H^1$,
*(ii)* $a_\eta(v,v) = \|v\|_1^2, \quad$ *for all* $\quad v \in H^1$,
*where* $C_\eta = 1 + 2\eta$.

The proofs of these lemmas are straightforward, and we omit the details.
We also note that on $H^1$, $\|v\|_1, \|v\|_{H^1} = \left(\|v\|^2 + \|v^{/}\|^2\right)^{1/2}$, $\|v\|_\eta = \sqrt{(a_\eta(v,v))}$ are three equivalent norms.

$$\|v\|_1^2 \leq \|v\|_\eta^2 \leq C_\eta \|v\|_1^2, \quad \text{for all} \quad v \in H^1, \tag{2.5}$$

$$\frac{1}{3}\|v\|_{H^1}^2 \leq \|v\|_1^2 \leq 3\|v\|_{H^1}^2, \quad \text{for all} \quad v \in H^1, \tag{2.6}$$

## 3. The Existence and uniqueness theorem of the solution

In this section we study the global existence of solutions for problem (1.1)-(1.3). For this purpose, we consider, first, a related nonlinear problem. Then, we use the well-known Banach's fixed point theorem to prove the existence of solutions to the nonlinear problem (1.1)-(1.3).

We make the following assumptions:

(H1) $\eta \geq 0, q \geq 2$,
(H2) $k, g \in H^1(0,T)$,
(H3) $\widetilde{u}_0 \in H^1$ and $\widetilde{u}_1 \in L^2$,
(H4) $f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R})$ satisfies the conditions $D_2 f, D_3 f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R})$.

For each $T > 0$, we put

$$W(T) = v \in L^\infty(0,T;H^1) : v_t \in L^\infty(0,T;L^2) \bigcap L^q(Q_T). \tag{3.1}$$

Then $W(T)$ is a Banach space with respect to the norm (see[5]):

$$\|v\|_{W(T)} = \|v\|_{L^\infty(0,T;H^1)} + \|v_t\|_{L^\infty(0,T;L^2)} + \|v_t\|_{L^q(Q_T)}, v \in W(T). \tag{3.2}$$

For each $v \in W(T)$, we associate with the problem (1.1)-(1.3) the following variational problem.
Find $u \in W(T)$ which satisfies the variational problem

$$\begin{aligned} &< u^{//}(t), w > + a_\eta(u(t), w) - \int_0^t k(t-s) a_\eta(u(s), w) ds + < \psi_q(u^{/}(t)), w > \\ &= g_1(t) w(1) + < f(\cdot, t, v(\cdot, t)), w > \quad \text{for all} \quad w \in H^1, \end{aligned} \tag{3.3}$$

$$u(0) = \widetilde{u}_0, u_t(0) = \widetilde{u}_1, \tag{3.4}$$

where

$$\psi_q(z) = |z|^{q-2} z, g_1(t) = g(t) - \int_0^t k(t-s) g(s) ds. \tag{3.5}$$

Then, we have the following theorem

**Theorem 3.1.** *Let (H1)-(H4) hold. Then, for every $T > 0$ and $v \in W(T)$, problem* (3.3)- (3.5) *has a unique solution $u \in W(T)$ and such that*

$$u^{//}, u_{xx} \in L^{q^{/}}(0,T;(H^1)^{/}), \quad \text{where} \quad q^{/} = q/(q-1). \tag{3.6}$$

*Furthermore, we have*

$$\|u^{/}(t)\|^2 + \|u(t)\|_{\eta}^2 + 2\int_0^t \|u^{/}(s)\|_{L^q}^q ds \leq C_{1T}\exp(TC_{2T}), \forall t \in [0,T], \tag{3.7}$$

*where*

$$\begin{aligned} C_{1T} = C_{1T}(v, \widetilde{u}_0, \widetilde{u}_1, k, g) = 2\Bigg[&\|\widetilde{u}_1\|^2 + \|\widetilde{u}_0\|_{\eta}^2 + 2|g_1(0)\widetilde{u}_0(1)| \\ &+6\|g_1\|_{L^\infty(0,T)}^2 + 2\|g_1^{/}\|_{L^2(0,T)}^2 + \int_0^T \|f(\cdot, s, v(s))\|^2 ds\Bigg], \end{aligned} \tag{3.8}$$

$$C_{2T} = C_{2T}(k) = 2\left[3 + 2|k(0)| + 6\|k\|_{L^2(0,T)}^2 + T\|k^{/}\|_{L^2(0,T)}^2\right], \tag{3.9}$$

*and*

$$g_1(t) = g(t) - \int_0^t k(t-s)g(s)ds. \tag{3.10}$$

*Proof of theorem 3.1.* The proof consists of steps two steps
**a. The existence of solution.** We approximate $\widetilde{u}_0$, $\widetilde{u}_1$, $k$, $g$ by sequences $\{u_{0m}\} \subset C_0^\infty\left(\overline{\Omega}\right)$, $u_{1m} \subset C_0^\infty(\Omega)$, $k_m, g_m \subset C_0^\infty([0,T])$, respectively, such that

$$\begin{aligned} u_{0m} &\to \widetilde{u}_0 \quad \text{strongly in} \quad H^1, \\ u_{1m} &\to \widetilde{u}_1 \quad \text{strongly in} \quad L^2, \\ k_m &\to k \quad \text{strongly in} \quad H^1(0,T), \\ g_m &\to g \quad \text{strongly in} \quad H^1(0,T). \end{aligned} \tag{3.11}$$

Then we consider the following variational problem: Find $u_m \in W(T)$ which satisfies the variational problem

$$\begin{aligned} &< u_m^{//}(t), w > + a_\eta(u_m(t), w) - \int_0^t k_m(t-s)a_\eta(u_m(s), w)ds \\ &+ < \psi_q(u_m^{/}(t)), w >= g_{1m}(t)w(1) + < f(\cdot, t, v(\cdot, t)), w >, \forall w \in H^1, \end{aligned} \tag{3.12}$$

$$u(0) = u_{0m}, u^{/}(0) = u_{1m}, \tag{3.13}$$

and

$$u_m \in L^\infty(0,T;H^2), u_m^{/} \in L^\infty(0,T;H^1), u_m^{//} \in L^\infty(0,T;L^2), \tag{3.14}$$

where

$$g_{1m}(t) = g_m(t) - \int_0^t k_m(t-s)g_m(s)ds. \tag{3.15}$$

The existence of a sequence of solutions $u_m$ satisfying (3.12)-(3.15) is a direct result of the theorem 2.1 in [10]. We shall prove that $u_m$ is a Cauchy sequence in $W(T)$.
*(i) A priori estimates.*
We take $w = u_m^{/}(t)$ in (3.12), afterwards integrating with respect to the time variable from 0 to $t$, we get after some rearrangements

$$\begin{aligned}\sigma_m(t) =&\sigma_m(0) - 2g_{1m}(0)u_{0m}(1) + 2g_{1m}(t)u_m(1,t)\\ &- 2\int_0^t g_{1m}^{/}(r)u_m(1,r)dr - 2k_m(0)\int_0^t \|u_m(r)\|_\eta^2 dr\\ &+ 2\int_0^t k_m(t-s)a_\eta(u_m(s),u_m(t))ds\\ &- 2\int_0^t dr\int_0^r k_m^{/}(r-s)a_\eta(u_m(s),u_m(r))ds\\ &+ 2\int_0^t < f(\cdot,s,v(\cdot,s)),u_m^{/}(s) > ds,\end{aligned} \tag{3.16}$$

where

$$\sigma_m(t) = \|u_m^{/}(t)\|^2 + \|u_m(t)\|_\eta^2 + 2\int_0^t ||u_m^{/}(s)||_{L^q}^q ds. \tag{3.17}$$

Proving in the same manner as in [10], we have the following results:

$$\sigma_m(t) = C_{1T}(m) + C_{2T}(m)\int_0^t \sigma_m(s)ds, \forall t \in [0,T], \tag{3.18}$$

where

$$\begin{aligned}C_{1T}(m) =&2\Big[\|u_{1m}\|^2 + \|u_{0m}\|_\eta^2 + 2|g_{1m}(0)u_{0m}(1)| + 6\|g_{1m}\|_{L^\infty(0,T)}^2\\ &+2\|g_{1m}^{/}\|_{L^2(0,T)}^2 + \int_0^T \|f(\cdot,s,v(s))\|^2 ds\Big],\end{aligned} \tag{3.19}$$

$$C_{2T}(m) = 2\left[3 + 2|k_m(0)| + 6\|k_m\|_{L^2(0,T)}^2 + T\|k_m^{/}\|_{L^2(0,T)}^2\right]. \tag{3.20}$$

From the assumptions (H1)-(H4), afterwards using Gronwall's lemma, we deduce from (3.11), that

$$\sigma_m(t) \le \widetilde{C}_T, \quad \text{for all} \quad m \quad \text{and} \quad t \in [0,T], \tag{3.21}$$

where $C_T$ is a constant independent of $m$.

On the other hand, we deduce from (3.12), (3.21), that, for all $w \in H^1$, we have

$$\begin{aligned}| < u_m^{//}(t), w > | \le&\|u_m(t)\|_\eta\|w\|_\eta + \int_0^t |k_m(t-s)|\|u_m(s)\|_\eta\|w\|_\eta ds\\ &+ \|\psi_q(u_m^{/})\|_{L^{q'}(\Omega)}\|w\|_{L^q(\Omega)} + |g_{1m}(t)|\|w\|_\eta\\ &+ \|f(\cdot,t,v(\cdot,t))\|\|w\|\\ &\le C_T\sqrt{(3C_\eta)}\left[1 + \|\psi_q(u_m^{/})\|_{L^{q'}(\Omega)}\right]\|w\|_{H^1}.\end{aligned} \tag{3.22}$$

This implies that

$$\begin{aligned}\|u_m^{//}(t)\|_{(H^1)'} =& \sup_{0\ne w\in H^1}\frac{\left|< u_m^{//}(t), w >\right|}{\|w\|_{H^1}}\\ &\le C_T\sqrt{3C_\eta}\left[1 + \|\psi_q(u_m^{/})\|_{L^{q'}(\Omega)}\right].\end{aligned} \tag{3.23}$$

Hence

$$\begin{aligned}\|u_m^{//}\|^{q'}_{L^{q'}(0,T;(H^1)')} &= \int_0^T \|u_m^{//}(t)\|^{q'}_{(H^1)'}dt \\ &\le \left(C_T\sqrt{3C_\eta}\right)^{q'} 2^{q'-1}\int_0^T \left[1+\|u_m^{/}(t)\|^q_{L^q(\Omega)}\right]dt \\ &\le C_T,\end{aligned} \tag{3.24}$$

where $C_T$ always indicating a constant depending on $T$.
*(ii) The convergence of sequence* $\{u_m\}$
We shall prove that $u_m$ is a Cauchy sequence in $W(T)$. Let $\widehat{u} = u_m - u_\mu$. Then $\widehat{u}$ satisfies the variational problem

$$\begin{aligned}&< u^{//}(t), w > + a_\eta(\widehat{u}(t), w) - \int_0^t k_m(t-s)a_\eta(\widehat{u}(s), w)ds \\ &- \int_0^t \widehat{k}(t-s)a_\eta(u_\mu(s), w)ds + < \psi_q(u_m^{/}(t)) - \psi_q(u_\mu^{/}(t)), w > \\ &= g_1(t)w(1) \quad \text{for all} \quad w \in H^1,\end{aligned} \tag{3.25}$$

$$\widehat{u}(0) = \widehat{u}_0, \widehat{u}^{/}(0) = \widehat{u}1, \tag{3.26}$$

where

$$\begin{aligned}\widehat{u}_0 = u_{0m} - u_{0\mu}, \widehat{u}_1 = u_{1m} - u_{1\mu}, \\ \widehat{k} = k_m - k_\mu, \widehat{g} = g_m - g_\mu, \widehat{g}_1 = g_{1m} - g_{1\mu}, \\ \widehat{g}_1(t) = \widehat{g}(t) - \int_0^t k_m(t-s)\widehat{g}(s)ds - \int_0^t \widehat{k}(t-s)g_\mu(s)ds.\end{aligned} \tag{3.27}$$

We take $w = u^{/}(t)$ in (3.25), after integrating with respect to the time variable from 0 to $t$, we get after some rearrangements

$$\begin{aligned}Z(t) =& Z(0) - 2\widehat{g}_1(0)\widehat{u}_0(1) + 2\widehat{g}_1(t)\widehat{u}(1,t) - 2\int_0^t \widehat{g}1^{/}(r)\widehat{u}(1,r)dr \\ &- 2k_m(0)\int_0^t \|\widehat{u}(r)\|^2_\eta dr + 2\int_0^t k_m(t-s)a_\eta(\widehat{u}(s), \widehat{u}(t))ds \\ &- 2\int_0^t dr\int_0^r k_m^{/}(r-s)a_\eta(\widehat{u}(s), \widehat{u}(r))ds \\ &- 2\widehat{k}(0)\int_0^t a_\eta(u_\mu(s), \widehat{u}(s))ds + 2\int_0^t \widehat{k}(t-s)a_\eta(u_\mu(s), \widehat{u}(t))ds \\ &- 2\int_0^t dr\int_0^r \widehat{k}^{/}(r-s)a_\eta(u_\mu(s), \widehat{u}(r))ds,\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}Z(t) =& \|\widehat{u}^{/}(t)\|^2 + \|\widehat{u}(t)\|^2_\eta \\ &+ 2\int_0^t < \psi_q(u_m^{/}(s)) - \psi_q(u_\mu^{/}(s)), u_m^{/}(s) - u_\mu^{/}(s) > ds.\end{aligned} \tag{3.29}$$

Using the following inequality

$$\forall q \ge 2, \exists C_q > 0 : (|x|^{q-2}x - |y|^{q-2}y)(x-y) \ge C_q|x-y|^q, \forall x, y \in \mathbb{R}, \tag{3.30}$$

it follows from (3.29) that

$$Z(t) \geq \|\widehat{u}^{/}(t)\|^2 + \|\widehat{u}(t)\|_\eta^2 + 2C_q \int_0^t \|\widehat{u}^{/}(s)\|_{L^q}^q ds. \tag{3.31}$$

Using the inequality

$$2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2, \forall a, b \in \mathbb{R}, \forall \epsilon > 0, \tag{3.32}$$

and the following inequalities

$$|a_\eta(u,v)| \leq \|u\|_\eta \|v\|_\eta, \forall u, v \in H^1, \tag{3.33}$$

$$|\widehat{u}(1,t)| \leq \|\widehat{u}(t)\|_{C^0(\Omega)} \leq \sqrt{2}\|\widehat{u}(t)\|_1 \leq \sqrt{2}\|\widehat{u}(t)\|_\eta \leq \sqrt{2Z(t)}, \tag{3.34}$$

we shall estimate respectively the following terms on the right-hand side of (3.28) as follows

$$\begin{aligned} Z(0) - 2\widehat{g}_1(0)\widehat{u}_0(1) \leq & \|u_{1m} - u_{1\mu}\|^2 + \|u_{0m} - u_{0\mu}\|_\eta^2 \\ & + 2|g_{1m}(0) - g_{1\mu}(0)||u_{0m}(1) - u_{0\mu}(1)|, \end{aligned} \tag{3.35}$$

$$2\widehat{g}_1(t)\widehat{u}(1,t) \leq 8\|\widehat{g}_1\|_{L^\infty(0,T)}^2 + \frac{1}{4}Z(t), \quad \text{with} \quad \epsilon = \frac{1}{8}, \tag{3.36}$$

$$-2\int_0^t \widehat{g}_1^{/}(r)\widehat{u}(1,r)dr \leq 2\|\widehat{g}_1^{/}\|_{L^2(0,T)}^2 + \int_0^t Z(r)dr, \tag{3.37}$$

$$2\int_0^t k_m(t-s)a_\eta(\widehat{u}(s),\widehat{u}(t))ds \leq \frac{1}{8}Z(t) + 8\|k_m\|_{L^2(0,T)}^2 \int_0^t Z(s)ds, \tag{3.38}$$

$$-2k_m(0)\int_0^t \|\widehat{u}(r)\|_\eta^2 dr \leq 2|k_m(0)| \int_0^t Z(r)dr, \tag{3.39}$$

$$-2\int_0^t dr \int_0^r k_m^{/}(r-s)a_\eta(\widehat{u}(s),\widehat{u}(r))ds \leq \left(1 + T\|k_m^{/}\|_{L^2(0,T)}^2\right) \int_0^t Z(s)ds, \tag{3.40}$$

$$2\int_0^t \widehat{k}(t-s)a_\eta(u_\mu(s),\widehat{u}(t))ds \leq \frac{1}{8}Z(t) + 8\widetilde{C}_T\|\widehat{k}\|_{L^1(0,T)}^2, \tag{3.41}$$

$$-2\widehat{k}(0)\int_0^t a_\eta(u_\mu(s),\widehat{u}(s))ds \leq T\widetilde{C}_T|\widehat{k}(0)|^2 + \int_0^t Z(s)ds, \tag{3.42}$$

$$-2\int_0^t dr \int_0^r \widehat{k}^{/}(r-s)a_\eta(u_\mu(s),\widehat{u}(r))ds \leq T^2\widetilde{C}_T\|\widehat{k}^{/}\|_{L^2(0,T)}^2 + \int_0^t Z(s)ds. \tag{3.43}$$

Combining (3.28), (3.29), (3.31) and (3.35)-(3.43), we obtain

$$Z(t) \leq \rho_{1T}(m,\mu) + \rho_{2T}(m)\int_0^t Z(s)ds, \forall t \in [0,T], \tag{3.44}$$

where

$$\begin{aligned} \rho_{1T}(m,\mu) = 2\Big[&\|\widehat{u}_1\|^2 + \|\widehat{u}_0\|_\eta^2 + 2|\widehat{g}_1(0)\widehat{u}_0(1)| + 8\|\widehat{g}_1\|_{L^\infty(0,T)}^2 \\ &+2\|\widehat{g}_1^{/}\|_{L^2(0,T)}^2 + 8\widetilde{C}_T\|\widehat{k}\|_{L^1(0,T)}^2 + T\widetilde{C}_T|\widehat{k}(0)|^2 + T^2\widetilde{C}_T\|\widehat{k}^{/}\|_{L^2(0,T)}^2\Big], \\ \rho_{2T}(m) = 2\Big[&4 + 2|k_m(0)| + 8\|k_m\|_{L^2(0,T)}^2 + T\|k_m^{/}\|_{L^2(0,T)}^2\Big]. \end{aligned} \tag{3.45}$$

By Gronwall's lemma, we deduce from (3.31), (3.44), (3.45), that

$$\begin{aligned}\|\widehat{u}^{/}(t)\|^2 + \|\widehat{u}(t)\|_\eta^2 + 2C_q\int_0^t \|u^{/}(s)\|_{L^q}^q ds \le Z(t)\\ \le \rho_{1T}(m,\mu)exp(T\rho_{2T}(m)), \quad \text{for all} \quad t\in[0,T].\end{aligned} \tag{3.46}$$

By (3.11), (3.27) and (3.45), we obtain $\rho_{1T}(m,\mu)exp(T\rho_{2T}(m)) \to 0$ as $m,\mu \to +\infty$. Hence, it follows from (3.46) that $\{u_m\}$ is a Cauchy sequence in $W(T)$. Therefore there exists $u \in W(T)$ such that

$$u_m \to u \quad \text{strongly in} \quad W(T). \tag{3.47}$$

On the other hand, by (3.47) and the continuity of $\psi_q$, we obtain

$$\psi_q(u_m^{/}) \to \psi_q(u^{/}) \quad \text{a.e.} \quad (x,t)\in Q_T. \tag{3.48}$$

By means of (3.21), it follows that

$$\|\psi_q(u_m^{/})\|_{L^{q^{/}}(Q_T)}^{q^{/}} = \|u_m^{/}\|_{L^q(Q_T)}^q \le \frac{1}{2}\widetilde{C}_T, \tag{3.49}$$

for all $m$. By Lions's lemma [5, Lemma 1.3, p. 12], it follows from (3.48) and (3.49) that

$$\psi_q(u_m^{/}) \to \psi_q(u^{/}) \quad \text{in} \quad L^{q^{/}}(Q_T) \quad \text{weakly}. \tag{3.50}$$

Noticing $(3.11)_3$ and (3.47) we have

$$\begin{aligned}&\left|\int_0^T dt\int_0^t k_m(t-s)a_\eta(u_m(s),w(t))ds\right.\\ &\left.-\int_0^T dt\int_0^t k(t-s)a_\eta(u(s),w(t))ds\right|\\ &\le \left|\int_0^T dt\int_0^t k_m(t-s)a_\eta(u_m(s)-u(s),w(t))ds\right|\\ &+\left|\int_0^T dt\int_0^t [k_m(t-s)-k(t-s)]a_\eta(u(s),w(t))ds\right|\\ &\le 3C_\eta\|w\|_{L^1(0,T;H^1)}\left[\|k_m\|_{L^1(0,T)}\|u_m-u\|_{L^\infty(0,T;H^1)}\right.\\ &\left.+\|k_m-k\|_{L^1(0,T)}\|u\|_{L^\infty(0,T;H^1)}\right]\to 0\end{aligned} \tag{3.51}$$

for all $w\in L^1(0,T;H^1)$.

On the other hand, by $(3.11)_{3,4}$ and (3.15), we also obtain

$$g_{1m} \to g_1 \quad \text{strongly in} \quad H^1(0,T). \tag{3.52}$$

From (3.24), we deduce the existence of a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, such that

$$u_m^{//} \to u^{//} \quad \text{in} \quad L^{q^{/}}(0,T;(H^1)^{/}) \quad \text{weak}. \tag{3.53}$$

Passing to the limit in (3.12), (3.13) by (3.47) and (3.50)-(3.53) we have $u$ satisfying the equation

$$\begin{aligned}&\left\langle u^{//}(t),w\right\rangle + a_\eta(u(t),w) - \int_0^t k(t-s)a_\eta(u(s),w)ds + \left\langle \psi_q(u^{/}(t)),w\right\rangle\\ &= g_1(t)w(1) + < f(\cdot,t,v(\cdot,t)),w >, \forall w\in H^1, \quad \text{in} \quad L^{q^{/}}(0,T) \quad \text{weak},\end{aligned} \tag{3.54}$$

and

$$u(0) = \widetilde{u}_0, u^{/}(0) = \widetilde{u}_1. \tag{3.55}$$

On the other hand, we deduce from (3.54), that

$$u_{xx}(t) - \int_0^t k(t-s)u_{xx}(s)ds = \phi(t), \tag{3.56}$$

where

$$\phi(t) = u^{//}(t) + |u^{/}|^{q-2}u^{/} - f(\cdot, t, v(\cdot, t)) \in L^{q^{/}}(0, T; (H^1)^{/}). \tag{3.57}$$

Hence, it follows from (3.56) and (3.57), that

$$\begin{aligned} \|u_{xx}(t)\|^{q^{/}}_{(H^1)^{/}} &\leq \left( \|\phi(t)\|_{(H^1)^{/}} + \int_0^t |k(t-s)| \|u_{xx}(s)\|_{(H^1)^{/}} ds \right)^{q^{/}} \\ &\leq 2^{q^{/}-1} \left[ \|\phi(t)\|^{q^{/}}_{(H^1)^{/}} + \left( \int_0^t |k(t-s)| \|u_{xx}(s)\|_{(H^1)^{/}} ds \right)^{q^{/}} \right] \\ &\leq 2^{q^{/}-1} \left[ \|\phi(t)\|^{q^{/}}_{(H^1)^{/}} + \|k\|^{q^{/}}_{L^q(0,T)} \left( \int_0^t \|u_{xx}(s)\|^{q^{/}}_{(H^1)^{/}} ds \right) \right]. \end{aligned} \tag{3.58}$$

This implies that

$$\begin{aligned} \int_0^r \|u_{xx}(t)\|^{q^{/}}_{(H^1)^{/}} dt \leq & 2^{q^{/}-1} \|\phi\|^{q^{/}}_{L^{q^{/}}(0,T;(H^1)^{/})} \\ & + 2^{q^{/}-1} \|k\|^{q^{/}}_{L^q(0,T)} \int_0^r dt \int_0^t \|u_{xx}(s)\|^{q^{/}}_{(H^1)^{/}} ds \end{aligned} \tag{3.59}$$

Using Gronwall's lemma, we obtain

$$\int_0^r \|u_{xx}(t)\|^{q^{/}}_{(H^1)^{/}} dt \leq 2^{q^{/}-1} \|\phi\|^{q^{/}}_{L^{q^{/}}(0,T;(H^1)^{/})} exp\left( 2^{q^{/}-1} \|k\|^{q^{/}}_{L^q(0,T)} r \right) \leq C_T, \tag{3.60}$$

where $C_T$ always indicating a constant depending on $T$
Thus

$$u_{xx} \in L^{q^{/}}(0, T; (H^1)^{/}). \tag{3.61}$$

On ther other hand, the estimate (3.7) hold by means of (3.11), (3.18), (3.19), (3.20), (3.47). The existence of the theorem is proved completely.

**b. Uniqueness of the solution.** First, we shall now require the following lemma.

**Lemma 3.2.** *Let $u$ be the weak solution of the following problem*

$$\begin{aligned} & u^{//} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds = \Phi, 0 < x < 1, 0 < t < T, \\ & u_x(0,t) = u(0,t), u_x(1,t) + \eta u(1,t) = 0, \\ & u(x,0) = \widetilde{u}_0(x), u^{/}(x,0) = \widetilde{u}_1(x), \\ & u \in L^{\infty}(0, T; H^1), u^{/}?L^{\infty}(0, T; L^2), \\ & k \in H^1(0, T), \Phi \in L^2(Q_T). \end{aligned} \tag{3.62}$$

*Then we have*

$$\begin{aligned}\frac{1}{2}\|u^{/}(t)\|^2+\frac{1}{2}\|u(t)\|_\eta^2&=\frac{1}{2}\|u_1\|^2+\frac{1}{2}\|u_0\|_\eta^2-k(0)\int_0^t\|u(r)\|_\eta^2dr\\&+\int_0^t k(t-s)a(u(s),u(t))ds-\int_0^t dr\int_0^r k^{/}(r-s)a(u(s),u(r))ds\\&+\int_0^t<\Phi(s),u^{/}(s)>ds,\quad a.e.\quad t\in[0,T].\end{aligned}\tag{3.63}$$

*Furthermore, if* $u_0=u_1=0$ *there is equality in* (3.63).

The idea of the proof is the same as in [4, Lemma 2.1, p. 79].

We now return to the proof of the uniqueness of a solution of the problem (3.3)-(3.5). Let $u_1$, $u_2$ be two weak solutions of problem (3.3)-(3.5), such that

$$u_i\in W(T),u_i^{//},u_{ixx}\in L^{q^{/}}(0,T;(H^1)^{/}),i=1,2.\tag{3.64}$$

Then $u=u_1-u_2$ is the weak solution of the following problem

$$\begin{aligned}&u^{//}-u_{xx}+\int_0^t k(t-s)u_{xx}(s)ds+\psi_q(u_1^{/})-\psi_q(u_2^{/})=0,\\&u_x(0,t)-u(0,t)=u_x(1,t)+\eta u(1,t)=0,\\&u(0)=u^{/}(0)=0,\\&u\in W(T),u^{//},u_{xx}\in L^{q^{/}}(0,T;(H^1)^{/}).\end{aligned}\tag{3.65}$$

By using Lemma 3.2 with $u_0=u_1=0$, $\Phi=-\psi_q(u_1^{/})+\psi_q(u_2^{/})$, we have

$$\begin{aligned}\sigma(t)=&2\int_0^t k(t-s)a(u(s),u(t))ds-2k(0)\int_0^t\|u(r)\|_\eta^2dr\\&-2\int_0^t dr\int_0^r k^{/}(r-s)a(u(s),u(r))ds,\end{aligned}\tag{3.66}$$

where

$$\sigma(t)=\|u^{/}(t)\|^2+\|u(t)\|_\eta^2+2\int_0^t\left\langle\psi_q(u_1^{/}(s))-\psi_q(u_2^{/}(s)),u^{/}(s)\right\rangle ds.\tag{3.67}$$

By using the same computations as in the above part we obtain from (3.66) that

$$\sigma(t)=2\left(1+2\|k\|_{L^2(0,T)}^2+2|k(0)|+\|k^{/}\|_{L^1(0,T)}^2\right)\int_0^t\sigma(r)dr.\tag{3.68}$$

By Gronwall's lemma, we deduce that $\sigma(t)=0$ and Theorem 3.1 is completely proved. □

**Theorem 3.3.** *Let* $T>0$ *and* $(H1)-(H4)$ *hold. Then there exists* $T_1\in(0,T)$ *such that problem* (1.1)- (1.3) *has a unique weak solution* $u\in W(T_1)$ *and such that*

$$u^{//},u_{xx}\in L^{q^{/}}(0,T_1;(H^1)^{/}).\tag{3.69}$$

*Proof.* For each $T_1>0$, we put

$$W_1(T_1)=\left\{v\in L^\infty(0,T_1;H^1):v_t\in L^\infty(0,T_1;L^2)\right\}.\tag{3.70}$$

Then $W_1(T_1)$ is a Banach space with respect to the norm (see[5]):

$$\|v\|_{W_1(T_1)}=\|v\|_{L^\infty(0,T_1;H^1)}+\|v_t\|_{L^\infty(0,T_1;L^2)},v\in W_1(T_1).\tag{3.71}$$

For $M > 0$ and $T_1 > 0$, we put

$$B(M, T_1) = \left\{ v \in W_1(T_1) : \|v\|_{W_1(T_1)} \leq M \right\}. \tag{3.72}$$

We also define the operator $F$ from $B(M, T_1)$ into $W(T_1)$ by $u = F(v)$, where $u$ is the unique solution of problem (3.3)- (3.5). We would like to show that $F$ is a contraction operator from $B(M, T_1)$ into itself. Applying the contraction mapping theorem, the operator $F$ has a fixed point in $B(M, T_1)$ that is also a weak solution of the problem (1.1)- (1.3).

First, by Theorem 3.1, we note that the unique solution of problem (3.3)- (3.5) satisfies (3.7), (3.8), (3.9). On the other hand, it follows from $(H3)$, that

$$\begin{aligned} \int_0^t \|f(\cdot, s, v(s))\|^2 ds &\leq 2 \int_0^t \|f(\cdot, s, v(s)) - f(\cdot, s, 0)\|^2 ds \\ &\quad + 2 \int_0^t \|f(\cdot, s, 0)\|^2 ds \\ &\leq 2T_1 K_1^2 M^2 + 2 \int_0^T \|f(\cdot, s, 0)\|^2 ds, \end{aligned} \tag{3.73}$$

where

$$\begin{aligned} K_1 &= K_1(M, T, f) \\ &= sup \left\{ |D_3 f(x, t, u)| : 0 \leq x \leq 1, 0 \leq t \leq T, |u| \leq \sqrt{2M} \right\}. \end{aligned} \tag{3.74}$$

It follows from (3.7)-(3.10) and (3.73) that

$$\begin{aligned} &\|u^{/}(t)\|^2 + \|u(t)\|_\eta^2 + 2 \int_0^t \|u^{/}(s)\|_{L^q}^q ds \\ &\leq \left( C_{1T} + 2T_1 K_1^2 M^2 \right) exp(T_1 C_{2T}), \forall t \in [0, T_1], \end{aligned} \tag{3.75}$$

where

$$\begin{aligned} C_{1T} &= C_{1T}\left(\widetilde{u}_0, \widetilde{u}_1, k, g\right) = 2 \Bigg[ \|\widetilde{u}_1\|^2 + \|\widetilde{u}_0\|_\eta^2 + 2 \left| g_1(0) \widetilde{u}_0(1) \right| \\ &\quad + 6\|g_1\|_{L^\infty(0,T)}^2 + 2\|g_1^{/}\|_{L^2(0,T)}^2 + 2 \int_0^T \|f(\cdot, s, 0)\|^2 ds \Bigg], \\ C_{2T} &= C_{2T}(k) = 2 \left[ 3 + 2|k(0)| + 6\|k\|_{L^2(0,T)}^2 + T\|k^{/}\|_{L^2(0,T)}^2 \right]. \end{aligned} \tag{3.76}$$

By choosing $M > 0$ large enough so that $C_{1_T} = \frac{1}{4} M^2$, then $T_1$ sufficiently small so that

$$\left( \frac{1}{4} M^2 + 2T_1 K_1^2 M^2 \right) exp(T_1 C_{2T}) \leq \frac{1}{2} M^2, \tag{3.77}$$

and

$$2\sqrt{2T_1} K_1 exp \left[ T_1 \left( 2 + 2|k(0)| + 2\|k\|_{L^2(0,T)}^2 + \|k^{/}\|_{L^1(0,T)}^2 \right) \right] < 1. \tag{3.78}$$

From (3.75), (3.77) we have $\|u\|_{W_1(T_1)} \leq M$, hence $u \in B(M, T_1)$. This shows that $F$ maps $B(M, T_1)$ into itself.

Next, we verify that $F$ is a contraction. Let $u_1 = F(v_1)$, $u_2 = F(v_2)$, where $v_1, v_2 \in B(M, T_1)$. Put $U = u_1 - u_2$ and $V = v_1 - v_2$. Then $U$ is the weak solution

of the following problem

$$\begin{aligned}
&U^{//} - U_{xx} + \int_0^t k(t-s)U_{xx}(s)ds + \psi_q(u_1^{/}) - \psi_q(u_2^{/}) \\
&= f(x,t,v_1(t)) - f(x,t,v_2(t)), 0 < x < 1, 0 < t < T_1, \\
&U_x(0,t) - U(0,t) = U_x(1,t) + \eta U(1,t) = 0, \\
&U(0) = U^{/}(0) = 0, \\
&U \in W(T_1); U^{//}, U_{xx} \in L^{q^{/}}(0,T_1;(H^1)^{/}).
\end{aligned} \tag{3.79}$$

By using Lemma 3.2 with $\widetilde{u}_0 = \widetilde{u}_1 = 0$, $\Phi = -\psi_q(u_1^{/}) + \psi_q(u_2^{/}) + f(x,t,v_1(t)) - f(x,t,v_2(t))$, we have

$$\begin{aligned}
\delta(t) =& -2k(0)\int_0^t \|U(r)\|_\eta^2 dr + 2\int_0^t k(t-s)a(U(s),U(t))ds \\
&- 2\int_0^t dr \int_0^r k^{/}(r-s)a(U(s),U(r))ds \\
&+ 2\int_0^t < f(\cdot,s,v_1(s)) - f(\cdot,s,v_2(s)), U^{/}(s) > ds, \text{a.e.} \quad t \in [0,T1],
\end{aligned} \tag{3.80}$$

where

$$\begin{aligned}
\delta(t) &= \|U^{/}(t)\|^2 + \|U(t)\|_\eta^2 + 2\int_0^t \left\langle \psi_q(u_1^{/}) - \psi_q(u_2^{/}), U^{/}(s) \right\rangle ds \\
&\geq \|U^{/}(t)\|^2 + \|U(t)\|_\eta^2 + 2C_q \int_0^t \|U^{/}(s)\|_{L^q}^q ds.
\end{aligned} \tag{3.81}$$

By the assumption $(H4)$, we have

$$\begin{aligned}
&2\int_0^t \left\langle f(\cdot,s,v_1(s)) - f(\cdot,s,v_2(s)), U^{/}(s) \right\rangle ds \\
&\leq \int_0^t \|U^{/}(s)\|^2 ds + \int_0^t \|f(\cdot,s,v_1(s)) - f(\cdot,s,v_2(s))\|^2 ds \\
&\leq \int_0^t \|U^{/}(s)\|^2 ds + 2T_1 K_1^2 \|V\|_{W_1(T_1)}^2,
\end{aligned} \tag{3.82}$$

Therefore, we can prove in a similar manner as above that

$$\begin{aligned}
\delta(t) \leq& 2T_1 K_1^2 \|V\|_{W_1(T_1)}^2 \\
&+ 2\left(2 + 2|k(0)| + 2\|k\|_{L^2(0,T)}^2 + \|k^{/}\|_{L^1(0,T)}^2\right)\int_0^t \delta(s)ds.
\end{aligned} \tag{3.83}$$

By Gronwall's lemma, we obtain from (3.83) that

$$\delta(t) = 2\left(\rho_1(k,K_1,T,T_1)\|V\|_{W_1(T_1)}\right)^2, \tag{3.84}$$

where

$$\begin{aligned}
&\rho_1(k,K_1,T,T_1) = \\
&\sqrt{2T_1}K_1 exp\left[T_1\left(2 + 2|k(0)| + 2\|k\|_{L^2(0,T)}^2 + \|k^{/}\|_{L^1(0,T)}^2\right)\right].
\end{aligned} \tag{3.85}$$

It follows from (3.81), (3.84) and (3.85) that

$$\|U\|_{W_1(T_1)} \leq 2\rho_1(k,K_1,T,T_1)\|V\|_{W_1(T_1)}, \tag{3.86}$$

where

$$2\rho_1(k, K_1, T, T_1) < 1, \tag{3.87}$$

since (3.78) and (3.85).

Hence, (3.86) shows that $\digamma : B(M, T_1) \to B(M, T_1)$ is a contraction. Applying the contraction mapping theorem, the operator $\digamma$ has a fixed point in $B(M, T_1)$ that is also a weak solution of the problem (1.1)- (1.3).

The solution of the problem (1.1)- (1.3) is unique, that can be showed using the same arguments as in the proof of Theorem 3.1. The proof of Theorem 3.3 is completed. □

**Remark 3.4.** *In the case of* $\lambda = 0$, $f(x,t,u) = |u|^{p-2}u$, $p > 2$, $k \in W^{2,1}(\mathbb{R}_+)$, $k \geq 0$, $k(0) > 0$, $0 < \int_0^{+\infty} k(t)dt < 1$, $k^{/}(t) + \zeta k(t) \leq 0$ *for all* $t \geq 0$, *with* $\zeta > 0$, *and the boundary condition* $u(0,t) = u(1,t) = 0$ *standing for* (1.2), *S. Berrimia, S. A. Messaoudi [1] has obtained a global existence and uniqueness theorem.*

## 4. Decay of solution

In this part, we will consider the problem of global existence and asymptotic behavior for $t \to +\infty$. We assume that $g(t) = 0$, $f(x,t,u) = F(x,t) - |u|^{p-2}u$, $p \geq 2$ and consider the following problem

$$\begin{aligned} &u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + |u|^{p-2}u + |u_t|^{q-2}u_t \\ &= F(x,t), 0 < x < 1, t > 0, \\ &u_x(0,t) = u(0,t), u_x(1,t) + \eta u(1,t) = 0, \\ &u(x,0) = \widetilde{u}_0(x), u_t(x,0) = \widetilde{u}_1(x), \end{aligned} \tag{4.1}$$

We make the following assumptions:

- $(\widetilde{H}1)$ $\eta \geq 0, p, q \geq 2$,
- $(\widetilde{H}2)$ $k \in W^{2,1}(\mathbb{R}_+)$, $k \geq 0$, satisfying
  - (i) $k(0) > 0$, $0 < 1 - \int_0^{+\infty} k(t)dt = k_\infty < 1$,
  - (ii) there exists a positive constant $\zeta$ such that $k^{/}(t) + \zeta k(t) \leq 0$ for all $t \geq 0$,
- $(\widetilde{H}3)$ $\widetilde{u}_0 \in H^2$ and $\widetilde{u}_1 \in H^1$,
- $(\widetilde{H}4)$ $F \in L^1(0,\infty; L^2) \cap L^2(0,\infty; L^2)$, $F_t \in L^1(0,\infty; L^2)$,
- $(\widetilde{H}5)$ There exists a constant $\sigma > 0$ such that $\int_0^{\infty} e^{st}\|F(t)\|^2 dt < +\infty$.

Under assumptions $(\widetilde{H}1)$-$(\widetilde{H}4)$ and let $T > 0$, by theorem 2.3, the problem (4.1) has a unique weak solution $u(t)$ such that

$$u \in L^\infty(0,T; H^2), u_t \in L^\infty(0,T; H^1), u_{tt} \in L^\infty(0,T; L^2). \tag{4.2}$$

Then, we have the following

**Lemma 4.1.** *Suppose that* $(\widetilde{H}1) - (\widetilde{H}4)$ *hold. Then there is a unique solution* $u(t)$ *of problem* (4.1) *defined on* $\mathbb{R}_+$. *Moreover*

$$\|u^{/}(t)\| + \|u(t)\|_\eta \leq C \quad \text{for all} \quad t \geq 0, \tag{4.3}$$

*where* $C$ *is a positive constant depending only on* $\widetilde{u}_0$, $\widetilde{u}_1$, $F$, $k_\infty$ *and* $p$.

*Proof.* By multiplying the equation $(4.1)_1$ by $u_t$ and integrate over $(0,1)\times(0,t)$ we obtain

$$\begin{aligned} E(t) + 2\int_0^t \|u^{/}(s)\|_{L^q}^q ds + \int_0^t k(s)\|u(s)\|_\eta^2 ds \\ - \int_0^t dr \int_0^r k^{/}(r-s)\|u(s)-u(r)\|_\eta^2 ds \\ = E(0) + 2\int_0^t < F(s), u^{/}(s) > ds, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} E(t) =& \|u^{/}(t)\|^2 + \left(1 - \int_0^t k(s)ds\right)\|u(t)\|_\eta^2 + \frac{2}{p}\|u(t)\|_{L^p}^p \\ &+ \int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds. \end{aligned} \tag{4.5}$$

On the other hand, by $(\widetilde{H}4)$ and the Cauchy's inequality, we obtain

$$\begin{aligned} 2\int_0^t \left\langle F(s), u^{/}(s)\right\rangle ds &\le \int_0^t \|F(s)\|ds + \int_0^t \|F(s)\|\|u^{/}(s)\|^2 ds \\ &\le \int_0^{+\infty} \|F(s)\|ds + \int_0^t \|F(s)\|E(s)ds. \end{aligned} \tag{4.6}$$

By Gronwall's lemma, we obtain from (4.4) and (4.6) that

$$\begin{aligned} E(t) &\le \left(E(0) + \int_0^{+\infty}\|F(s)\|ds\right) exp\left(\int_0^t \|F(s)\|ds\right) \\ &\le \left(E(0) + \int_0^{+\infty}\|F(s)\|ds\right) exp\left(\int_0^{+\infty} \|F(s)\|ds\right) = C, \forall t \ge 0. \end{aligned} \tag{4.7}$$

By $(\widetilde{H}3, i)$, we have

$$E(t) \ge \|u^{/}(t)\|^2 + \left(1 - \int_0^t k(s)ds\right)\|u(t)\|_\eta^2 \ge \|u^{/}(t)\|^2 + k_\infty \|u(t)\|_\eta^2. \tag{4.8}$$

Then we obtain (4.3) from (4.7) and (4.8). This completes the proof of Lemma 4.1. □

In this section we state and prove decay result.

**Theorem 4.2.** *Suppose that $(\widetilde{H}1) - (\widetilde{H}5)$ hold. Then the solution $u(t)$ of problem (4.1) decays exponentially to zero as $t \to +\infty$ in the following sense: there exist the positive constants $N$ and $\gamma$ such that*

$$\|u^{/}(t)\| + \|u(t)\|_\eta \le Ne^{-\gamma t} \quad \text{for all} \quad t \ge 0. \tag{4.9}$$

*Proof.* We use the following functional

$$\Gamma(t) = \Gamma(\varepsilon_1, \varepsilon_2, t) = E(t) + \varepsilon_1 E_1(t) + \varepsilon_2 E_2(t), \tag{4.10}$$

where

$$E_1(t) = < u(t), u^{/}(t) >, \tag{4.11}$$

$$E_2(t) = -\int_0^t k(t-s)\left\langle u^{/}(t), u(t)-u(s)\right\rangle ds. \tag{4.12}$$

*Estimating* $\Gamma(t)$.
By (2.3), (2.4), we obtain from $(\widetilde{H}2, i)$ that

$$|E_1(t)| = \left| < u(t), u^/(t) > \right| \leq \frac{1}{2}\|u^/(t)\|^2 + \|u(t)\|_\eta^2, \tag{4.13}$$

$$\begin{aligned} |E_2(t)| &= \left| \int_0^t k(t-s) \left\langle u^/(t), u(t) - u(s) \right\rangle ds \right| \\ &\leq \frac{1}{2}\|u^/(t)\|^2 + \frac{1}{2}\left( \int_0^t k(t-s)\|u(t) - u(s)\| ds \right)^2 \\ &\leq \frac{1}{2}\|u^/(t)\|^2 + (1 - k_\infty) \int_0^t k(t-s)\|u(t) - u(s)\|_\eta^2 ds. \end{aligned} \tag{4.14}$$

Hence, it follows from (4.10)-(4.14) that for $\varepsilon_1$, $\varepsilon_2$ small enough, there exist two positive constants $\alpha_1, \alpha_2$, such that

$$\alpha_1 E(t) \leq \Gamma(t) \leq \alpha_2 E(t). \tag{4.15}$$

*Estimating* $\Gamma^/(t)$.
Now differentiating (4.4) with respect to $t$, we have

$$\begin{aligned} E^/(t) &= -2\|u^/(t)\|_{L^q}^q + \int_0^t k^/(t-s)\|u(s) - u(t)\|_\eta^2 ds \\ &\quad - k(t)\|u(t)\|_\eta^2 + 2\left\langle F(t), u^/(t) \right\rangle \\ &\leq -2\|u^/(t)\|_{L^q}^q + \int_0^t k^/(t-s)\|u(s) - u(t)\|_\eta^2 ds + 2\left\langle F(t), u^/(t) \right\rangle, \end{aligned} \tag{4.16}$$

since $k(t) \geq 0$.

By multiplying the equation $(4.1)_1$ by $u$ and integrate over $(0, 1)$ we obtain

$$\begin{aligned} E_1^/(t) &= \|u^/(t)\|^2 - \|u(t)\|_\eta^2 - \|u(t)\|_{L^p}^p + \langle F(t), u(t) \rangle \\ &\quad + \int_0^t k(t-s)a(u(s), u(t))ds - \left\langle |u^/(t)|^{q-2}u^/(t), u(t) \right\rangle \\ &= \|u^/(t)\|^2 - \|u(t)\|_\eta^2 - \|u(t)\|_{L^p}^p + \langle F(t), u(t) \rangle + I_1(t) + I_2(t). \end{aligned} \tag{4.17}$$

We now estimate the last two terms in the right side of (4.17) as follows

*Estimating* $I_1(t)$.
Using the inequality

$$ab \leq \frac{\delta}{r}a^r + \frac{r-1}{r}\delta^{\frac{-r}{r-1}} b^{\frac{r}{r-1}}, \forall a, b \geq 0, \forall r > 1, \forall \delta > 0, \tag{4.18}$$

we have

$$\begin{aligned} I_1(t) &= \int_0^t k(t-s)a(u(s),u(t))ds \\ &= \int_0^t k(t-s)a\left(u(s)-u(t),u(t)\right)ds + \int_0^t k(t-s)\|u(t)\|_\eta^2 ds \\ &\leq \delta_1\|u(t)\|_\eta^2 + \frac{1}{4\delta_1}\left(\int_0^t k(s)ds\right)\left(\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds\right) \\ &+ \left(\int_0^t k(s)ds\right)\|u(t)\|_\eta^2 \\ &\leq \delta_1\|u(t)\|_\eta^2 + \frac{1-k_\infty}{4\delta_1}\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds \\ &+ (1-k_\infty)\|u(t)\|_\eta^2 \\ &\leq (\delta_1+1-k_\infty)\|u(t)\|_\eta^2 + \frac{1-k_\infty}{4\delta_1}\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds, \end{aligned} \tag{4.19}$$

for all $\delta_1 > 0$.
*Estimating* $I_2(t)$.
We again use inequality (4.18) we obtain from (4.3) that

$$\begin{aligned} I_2(t) &= -\left\langle |u^/(t)|^{q-2}u^/(t), u(t)\right\rangle \leq \|u^/(t)\|_{L^q}^{q-1}\|u(t)\|_{L^q} \\ &\leq \frac{\delta_1^q}{q}\|u(t)\|_{L^q}^q + \frac{q-1}{q}\delta_1^{\frac{-q}{q-1}}\|u^/(t)\|_{L^q}^q \\ &\leq 2\frac{\delta_1^q}{q}\left(\sqrt{2}C\right)^{q-2}\|u(t)\|_\eta^2 + \frac{q-1}{q}\delta_1^{\frac{-q}{q-1}}\|u^/(t)\|_{L^q}^q, \end{aligned} \tag{4.20}$$

for all $\delta_1 > 0$.
By combining (4.17), (4.19) and (4.20), we obtain

$$\begin{aligned} E_1^/(t) \leq &-\|u(t)\|_{L^p}^p + \|u^/(t)\|^2 - \left(k_\infty - \delta_1 - 2\frac{\delta_1^q}{q}\left(\sqrt{2}C\right)^{q-2}\right)\|u(t)\|_\eta^2 \\ &+ \frac{q-1}{q}\delta_1^{\frac{-q}{q-1}}\|u^/(t)\|_{L^q}^q + \frac{1-k_\infty}{4\delta_1}\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds \\ &+ \langle F(t),u(t)\rangle . \end{aligned} \tag{4.21}$$

Then, we can always choose the constant $\delta_1 > 0$ such that

$$\gamma_1 = k_\infty - \delta_1 - 2\frac{\delta_1^q}{q}\left(\sqrt{2}C\right)^{q-2} > 0. \tag{4.22}$$

This implies that

$$\begin{aligned} E_1^/(t) \leq &-\|u(t)\|_{L^p}^p + \|u^/(t)\|^2 - \gamma_1\|u(t)\|_\eta^2 + \gamma_2\|u^/(t)\|_{L^q}^q \\ &+ \gamma_3\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds + \langle F(t),u(t)\rangle , \end{aligned} \tag{4.23}$$

where

$$\gamma_2 = \frac{q-1}{q}\delta_1^{\frac{-q}{q-1}}, \gamma_3 = \frac{1-k_\infty}{4\delta_1}. \tag{4.24}$$

Direct calculations give

$$\begin{aligned}
E_2^{/}(t) =& -\left(\int_0^t k(s)ds\right)\|u^{/}(t)\|^2 - \int_0^t k^{/}(t-s)\left\langle u^{/}(t), u(t)-u(s)\right\rangle ds \\
&+ \int_0^t k(t-s)a(u(t), u(t)-u(s))ds \\
&- \int_0^t k(t-s)a\left(\int_0^t k(t-\tau)u(\tau)d\tau, u(t)-u(s)\right)ds \\
&+ \int_0^t k(t-s)\left\langle |u(t)|^{p-2}u(t), u(t)-u(s)\right\rangle ds \\
&+ \int_0^t k(t-s)\left\langle |u^{/}(t)|^{q-2}u^{/}(t), u(t)-u(s)\right\rangle ds \\
&- \int_0^t k(t-s)\left\langle F(t), u(t)-u(s)\right\rangle ds = \sum_{i=1}^{7} J_i(t).
\end{aligned} \tag{4.25}$$

Similarly to (4.17), we estimate respectively the following terms on the right-hand side of (4.25) as follows.

*Estimating* $J_1(t)$.

Since $k$ is continuous and $k(0) > 0$ then there exists $t_0 > 0$, such that

$$\int_0^t k(s)ds \geq \int_0^{t_0} k(s)ds = k_0 > 0 \quad \text{for all} \quad t \geq t_0. \tag{4.26}$$

Hence,

$$J_1(t) = -\left(\int_0^t k(s)ds\right)\|u^{/}(t)\|^2 \leq -k_0\|u^{/}(t)\|^2 \quad \text{for all} \quad t \geq t_0. \tag{4.27}$$

*Estimating* $J_2(t)$.

$$\begin{aligned}
J_2(t) &= -\int_0^t k^{/}(t-s)\left\langle u^{/}(t), u(t)-u(s)\right\rangle ds \\
&\leq \delta_2\|u^{/}(t)\|^2 + \frac{1}{4\delta_2}\left(\int_0^t |k^{/}(t-s)|ds\right)\left(\int_0^t |k^{/}(t-s)|\|u(s)-u(t)\|^2 ds\right) \\
&\leq \delta_2\|u^{/}(t)\|^2 + \frac{1}{2\delta_2}\left(\int_0^t |k^{/}(t-s)|ds\right)\left(\int_0^t |k^{/}(t-s)|\|u(s)-u(t)\|_\eta^2 ds\right) \\
&\leq \delta_2\|u^{/}(t)\|^2 - \frac{k(0)}{2\delta_2}\int_0^t k^{/}(t-s)\|u(s)-u(t)\|_\eta^2 ds.
\end{aligned} \tag{4.28}$$

*Estimating* $J_3(t)$.

$$\begin{aligned}
J_3(t) &= \int_0^t k(t-s)a\left(u(t), u(t)-u(s)\right)ds \\
&\leq \delta_2\|u(t)\|_\eta^2 + \frac{1}{4\delta_2}\left(\int_0^t k(s)ds\right)\left(\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds\right) \\
&\leq \delta_2\|u(t)\|_\eta^2 + \frac{1-k_\infty}{4\delta_2}\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds
\end{aligned} \tag{4.29}$$

*Estimating* $J_4(t)$.

$$\begin{aligned}
J_4(t) &= -\int_0^t k(t-s)a\left(\int_0^t k(t-\tau)u(\tau)d\tau, u(t)-u(s)\right)ds \\
&\leq \int_0^t k(t-\tau)\|u(\tau)\|_\eta d\tau \int_0^t k(t-s)\|u(s)-u(t)\|_\eta ds \\
&\leq \delta_2\left(\int_0^t k(t-\tau)\|u(\tau)\|_\eta d\tau\right)^2 \\
&+ \frac{1}{4\delta_2}\left(\int_0^t k(t-s)\|u(s)-u(t)\|_\eta ds\right)^2 \\
&\leq 2\delta_2\left(\int_0^t k(t-\tau)\|u(\tau)\|_\eta d\tau\right)^2 \\
&+ \left(2\delta_2+\frac{1}{4\delta_2}\right)\left(\int_0^t k(t-s)\|u(s)-u(t)\|_\eta ds\right)^2 \\
&\leq 2\delta_2(1-k_\infty)^2\|u(t)\|_\eta^2 \\
&+ \left(2\delta_2+\frac{1}{4\delta_2}\right)(1-k_\infty)\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds.
\end{aligned} \tag{4.30}$$

*Estimating* $J_5(t)$.

$$\begin{aligned}
&J_5(t) = \int_0^t k(t-s)\left\langle |u(t)|^{p-2}u(t), u(t)-u(s)\right\rangle ds \\
&\leq 2\left(\sqrt{2}C\right)^{p-2}\int_0^t k(t-s)\|u(t)\|_\eta\|u(t)-u(s)\|_\eta ds \\
&\leq 2\left(\sqrt{2}C\right)^{p-2}\left[\delta_2\|u(t)\|_\eta^2 + \frac{1}{4\delta_2}\left(\int_0^t k(t-s)\|u(t)-u(s)\|_\eta ds\right)^2\right] \\
&\leq 2\left(\sqrt{2}C\right)^{p-2}\left[\delta_2\|u(t)\|_\eta^2 + \frac{1}{4\delta_2}(1-k_\infty)\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds\right].
\end{aligned} \tag{4.31}$$

*Estimating* $J_6(t)$. We again use inequality (4.18) with $r = q$, $\delta = \delta_2$, we obtain from (4.3) that

$$\begin{aligned}
&\left\langle |u^/(t)|^{q-2}u^/(t), u(t)-u(s)\right\rangle \leq \|u^/(t)\|_{L^q}^{q-1}\|u(t)-u(s)\|_{L^q} \\
&\leq \frac{\delta_2^q}{q}\|u(t)-u(s)\|_{L^q}^q + \frac{q-1}{q}\delta_2^{\frac{-q}{q-1}}\|u^/(t)\|_{L^q}^q \\
&\leq 2\frac{\delta_2^q}{q}\left(2\sqrt{2}C\right)^{q-2}\|u(t)-u(s)\|_\eta^2 + \frac{q-1}{q}\delta_2^{\frac{-q}{q-1}}\|u^/(t)\|_{L^q}^q.
\end{aligned} \tag{4.32}$$

It follows from (4.32) that

$$\begin{aligned} J_6(t) &= \int_0^t k(t-s)\left\langle |u^{/}(t)|^{q-2}u^{/}(t), u(t)-u(s)\right\rangle ds \\ &\leq 2\frac{\delta_2^q}{q}\left(2\sqrt{2}C\right)^{q-2}\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds \\ &+ \frac{q-1}{q}\delta_2^{\frac{-q}{q-1}}\|u^{/}(t)\|_{L^q}^q\int_0^t k(t-s)ds \\ &\leq 2\frac{\delta_2^q}{q}\left(2\sqrt{2}C\right)^{q-2}\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds \\ &+ \frac{q-1}{q}\delta_2^{\frac{-q}{q-1}}(1-k_\infty)\|u^{/}(t)\|_{L^q}^q \end{aligned} \tag{4.33}$$

*Estimating* $J_7(t)$

$$\begin{aligned} J_7(t) &= -\int_0^t k(t-s)\left\langle F(t), u(t)-u(s)\right\rangle ds \\ &\leq \int_0^t k(t-s)\|F(t)\|\|u(t)-u(s)\|ds \\ &\leq \frac{1}{4\delta_2}\|F(t)\|^2 + \delta_2\left(\int_0^t k(t-s)ds\right)\left(\int_0^t k(t-s)\|u(t)-u(s)\|^2 ds\right) \\ &\leq \frac{1}{4\delta_2}\|F(t)\|^2 + 2\delta_2(1-k_\infty)\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds. \end{aligned} \tag{4.34}$$

By combining (4.25), (4.27)-(4.31), (4.33) and (4.34), we obtain

$$\begin{aligned} E_2^{/}(t) &= -\left(k_0-\delta_2\right)\|u^{/}(t)\|^2 + \delta_2\widehat{\gamma}_1\|u(t)\|_\eta^2 + \widehat{\gamma}_2\|u^{/}(t)\|_{L^q}^q \\ &+ \widehat{\gamma}_3\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds \\ &- \widehat{\gamma}_4\int_0^t k^{/}(t-s)\|u(s)-u(t)\|_\eta^2 ds + \frac{1}{4\delta_2}\|F(t)\|^2, \end{aligned} \tag{4.35}$$

where

$$\begin{aligned} \widehat{\gamma}_1 &= 1 + 2(1-k_\infty)^2 + 2\left(\sqrt{2}C\right)^{p-2}, \\ \widehat{\gamma}_2 &= \frac{q}{q-1}\delta_2^{\frac{-q}{q-1}}(1-k_\infty), \\ \widehat{\gamma}_3 &= 2\frac{\delta_2^q}{q}\left(2\sqrt{2}C\right)^{q-2} + (1-k_\infty)\left[\frac{1}{2\delta_2}\left(\sqrt{2}C\right)^{p-2} + \left(4\delta_2 + \frac{1}{2\delta_2}\right)\right], \\ \widehat{\gamma}_4 &= \frac{k(0)}{2\delta_2}. \end{aligned} \tag{4.36}$$

Combining of (4.10), (4.16), (4.23) and (4.35), we obtain

$$\begin{aligned}
&\Gamma^{/}(t)+\varepsilon_1\|u(t)\|_{L^p}^p+\left((k_0-\delta_2)\varepsilon_2-\varepsilon_1\right)\|u^{/}(t)\|^2\\
&\quad+\left(\varepsilon_1\gamma_1-\varepsilon_2\delta_2\widehat{\gamma}_1\right)\|u(t)\|_\eta^2+\left(2-\varepsilon_1\gamma_2-\varepsilon_2\widehat{\gamma}_2\right)\|u^{/}(t)\|_{L^q}^q\\
&\quad-\left(\varepsilon_1\gamma_3+\varepsilon_2\widehat{\gamma}_3\right)\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds\\
&\quad-\left(1-\varepsilon_2\widehat{\gamma}_4\right)\int_0^t k^{/}(t-s)\|u(s)-u(t)\|_\eta^2 ds\\
&\quad\leq\left\langle F(t),2u^{/}(t)+\varepsilon_1u(t)\right\rangle+\frac{\varepsilon_2}{4\delta_2}\|F(t)\|^2.
\end{aligned}\tag{4.37}$$

Whence $\delta_1$ is fixed, choosing

$$\delta_2=\frac{1}{2}\frac{k(0)\gamma_1}{\gamma_1+\widehat{\gamma}_1},\varepsilon_2=\frac{2}{k_0}\varepsilon_1,\quad\text{where}\quad\varepsilon_1>0\quad\text{is arbitrary,}\tag{4.38}$$

we deduce from (4.37) and (4.38) that

$$\begin{aligned}
&\Gamma^{/}(t)+\varepsilon_1\|u(t)\|_{L^p}^p+\frac{\varepsilon_1\widehat{\gamma}_1}{\gamma_1+\widehat{\gamma}_1}\|u^{/}(t)\|^2+\frac{\varepsilon_1\gamma_1^2}{\gamma_1+\widehat{\gamma}_1}\|u(t)\|_\eta^2\\
&\quad+\left(2-\varepsilon_1\left(1+\frac{2}{k(0)}\widehat{\gamma}_2\right)\right)\|u^{/}(t)\|_{L^q}^q\\
&\quad-\varepsilon_1\left(\gamma_3+\frac{2}{k(0)}\widehat{\gamma}_3\right)\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds\\
&\quad-\left(1-\frac{2}{k(0)}\varepsilon_1\widehat{\gamma}_4\right)\int_0^t k^{/}(t-s)\|u(s)-u(t)\|_\eta^2 ds\\
&\quad\leq\left\langle F(t),2u^{/}(t)+\varepsilon_1u(t)\right\rangle+\frac{\varepsilon_1}{k_0^2}\left(1+\frac{\widehat{\gamma}_1}{\gamma_1}\right)\|F(t)\|^2.
\end{aligned}\tag{4.39}$$

Next, we choose $\varepsilon_1>0$, with

$$\varepsilon_1<min\left\{\frac{\zeta}{\gamma_3+\frac{2}{k_0}\widehat{\gamma}_3+\frac{2}{k_0}\widehat{\gamma}_4\zeta},\frac{2}{1+\frac{2}{k_0}\widehat{\gamma}_2}\right\}$$

and (4.15) is satisfied, then by the assumption $(\widetilde{H}2,ii)$, we deduce that

$$\begin{aligned}
&\Gamma^{/}(t)+\varepsilon_1\|u(t)\|_{L^p}^p+\frac{\varepsilon_1\widehat{\gamma}_1}{\gamma_1+\widehat{\gamma}_1}\|u^{/}(t)\|^2+\frac{\varepsilon_1\gamma_1^2}{\gamma_1+\widehat{\gamma}_1}\|u(t)\|_\eta^2+k_1\|u^{/}(t)\|_{L^q}^q\\
&\quad+k_2\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds\\
&\quad\leq\left\langle F(t),2u^{/}(t)+\varepsilon_1u(t)\right\rangle+k_3\|F(t)\|^2,
\end{aligned}\tag{4.40}$$

where

$$\begin{aligned}
k_1&=2-\varepsilon_1\left(1+\frac{2}{k_0}\widehat{\gamma}_2\right)>0,\\
k_2&=\zeta\left(1-\frac{2}{k_0}\varepsilon_1\widehat{\gamma}_4\right)-\varepsilon_1\left(\gamma_3+\frac{2}{k_0}\widehat{\gamma}_3\right)>0,\\
k_3&=\frac{\varepsilon_1}{k_0^2}\left(1+\frac{\widehat{\gamma}_1}{\gamma_1}\right).
\end{aligned}\tag{4.41}$$

By combining (4.5), (4.15) and (4.40), we can always choose the constant $\widetilde{\gamma} > 0$ is independent of $t$ such that

$$\Gamma^{/}(t) + 2\widetilde{\gamma}\Gamma(t) \leq \Big\langle F(t), 2u^{/}(t) + \varepsilon_1 u(t)\Big\rangle + k_3\|F(t)\|^2, \tag{4.42}$$

for all $t \geq t_0$.
On ther other hand,

$$\Big\langle F(t), 2u^{/}(t) + \varepsilon_1 u(t)\Big\rangle + k_3\|F(t)\|^2 \leq \widetilde{N}\|F(t)\|^2 + \widetilde{\gamma}\Gamma(t), \tag{4.43}$$

for some constant $\widetilde{N} > 0$. Therefore

$$\Gamma^{/}(t) + \widetilde{\gamma}\Gamma(t) \leq \widetilde{N}\|F(t)\|^2 \quad \text{for all} \quad t \geq t_0. \tag{4.44}$$

Putting $\gamma = \frac{1}{2}min\{\sigma, \widetilde{\gamma}\}$. A simple integration of (4.44) over $(t_0, t)$ gives

$$\Gamma(t) \leq \left[\Gamma(t_0)e^{\sigma t_0} + \widetilde{N}\int_{t_0}^{+\infty} e^{\sigma s}\|F(s)\|^2 ds\right] e^{-2\gamma t} = N_1 e^{-2\gamma t}, \tag{4.45}$$

for all $t \geq t_0$.
By the boundedness of $\Gamma(t)$ on $[0, t_0]$, we deduce from (4.45) that

$$\Gamma(t) = \|\Gamma\|_{L^\infty(0,t_0)} e^{-2\gamma(t-t_0)} + N_1 e^{-2\gamma t} = N_2 e^{-2\gamma t}, \tag{4.46}$$

for all $t \geq 0$.
By (4.15), it follows from (4.46) that

$$E(t) \leq \frac{1}{\alpha_1}\Gamma(t) \leq \frac{1}{\alpha_1} N_2 e^{-2\gamma t}, \quad \text{for all} \quad t \geq 0. \tag{4.47}$$

This completes the proof of Theorem 4.2. □

**Remark 4.3.** *The estimate* (4.9) *holds for any regular solution corresponding to* $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$. *This remains holds for solutions corresponding to* $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$ *by simple density argument.*

## 5. Numerical results

Consider the following problem:

$$u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + u_t^3 = u^2 + F(x,t), 0 < x < 1, 0 < t < T, \tag{5.1}$$

with boundary conditions

$$u_x(0,t) = u(0,t), u_x(1,t) + u(1,t) = 0, \tag{5.2}$$

and initial conditions

$$u(x,0) = \widetilde{u}_0(x), u_t(x,0) = \widetilde{u}_1(x), \tag{5.3}$$

where

$$\widetilde{u}_0(x) = -x^2 + x + 1, \widetilde{u}_1(x) = -\widetilde{u}_0(x), k(t) = \frac{1}{2}e^{-t}, \tag{5.4}$$

$$F(x,t) = (2-t)e^{-t} + U_{ex}(1 - U_{ex} - U_{ex}^2), \tag{5.5}$$

where

$$U_{ex}(x,t) = (-x^2 + x + 1)e^{-t}. \tag{5.6}$$

The exact solution of the problem (5.1)-(5.3) with $\widetilde{u}_0(x)$, $\widetilde{u}_1(x)$, $k(t)$ and $F(x,t)$ defined in (5.4) and (5.5) respectively, is the function $U_{ex}$ given in (5.6). To solve

problem (5.1)-(5.3) numerically, we consider the differential system for the unknowns $u_j(t) = u(x_j, t)$, $v_j(t) = \frac{du_j}{dt}(t)$, with $x_j = jh$, $h = \frac{1}{N}$, $j = 0, 1, ..., N$:

$$\begin{aligned}
&\frac{du_j}{dt}(t) = v_j(t), j = 0, 1, ..., N,\\
&\frac{dv_0}{dt}(t) = \frac{1}{h^2}\left[-(1+h)u_0(t) + u_1(t)\right]\\
&-\frac{1}{h^2}\int_0^t k(t-s)\left[-(1+h)u_0(s) + u_1(s)\right]ds - v_0^3(t) + u_0^2(t) + F(x_0, t),\\
&\frac{dv_j}{dt}(t) = \frac{1}{h^2}\left[u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)\right]\\
&-\frac{1}{h^2}\int_0^t k(t-s)\left[u_{j-1}(s) - 2u_j(s) + u_{j+1}(s)\right]ds\\
&-v_j^3(t) + u_j^2(t) + F(x_j, t), j = 1, 2, ..., N-1,\\
&\frac{dv_N}{dt}(t) = \frac{1}{h^2}\left[u_{N-1}(t) - (1+h)u_N(t)\right]\\
&-\frac{1}{h^2}\int_0^t k(t-s)\left[u_{N-1}(s) - (1+h)u_N(s)\right]ds - v_N^3(t) + u_N^2(t) + F(x_N, t),\\
&u_j(0) = \widetilde{u}_0(x_j), v_j(0) = \widetilde{u}_1(x_j), j = 0, 1, ..., N.
\end{aligned} \tag{5.7}$$

To solve the nonlinear differential system (5.7), we use the following linear recursive scheme generated by the nonlinear term

$$\begin{aligned}
&\frac{du_j^{(n)}}{dt}(t) = v_j^{(n)}(t), j = 0, 1, ..., N,\\
&\frac{dv_0^{(n)}}{dt}(t) = \frac{1}{h^2}\left[-(1+h)u_0^{(n)}(t) + u_1^{(n)}(t)\right]\\
&-\frac{\Delta t}{h^2}\sum_{i=1}^{N_1-1} k(t-i\Delta t)\left[-(1+h)u_0^{(n)}(i\Delta t) + u_1^{(n)}(i\Delta t)\right]\\
&-\left(v_0^{(n-1)}(t)\right)^3 + \left(u_0^{(n-1)}(t)\right)^3 + F(x_0, t),\\
&\frac{dv_j^{(n)}}{dt}(t) = \frac{1}{h^2}\left[u_{j-1}^{(n)}(t) - 2u_j^{(n)}(t) + u_{j+1}^{(n)}(t)\right]\\
&-\frac{\Delta t}{h^2}\sum_{i=1}^{N_1-1} k(t-i\Delta t)\left[u_{j-1}^{(n)}(i\Delta t) - 2u_j^{(n)}(i\Delta t) + u_{j+1}^{(n)}(i\Delta t)\right]\\
&-\left(v_j^{(n-1)}(t)\right)^3 + \left(u_j^{(n-1)}(t)\right)^2 + F(x_j, t), j = 1, 2, ..., N-1,\\
&\frac{dv_N^{(n)}}{dt}(t) = \frac{1}{h^2}\left[u_{N-1}^{(n)}(t) - (1+h)u_N^{(n)}(t)\right]\\
&-\frac{\Delta t}{h^2}\sum_{i=1}^{N_1-1} k(t-i\Delta t)\left[u_{N-1}^{(n)}(i\Delta t) - (1+h)u_N^{(n)}(i\Delta t)\right]\\
&-\left(v_N^{(n-1)}(t)\right)^3 + \left(u_N^{(n-1)}(t)\right)^2 + F(x_N, t),\\
&u_j^{(n)}(0) = \widetilde{u}_0(x_j), v_j^{(n)}(0) = \widetilde{u}_1(x_j), j = 0, 1, ..., N,
\end{aligned} \tag{5.8}$$

and where $u_j^{(n)}(i\Delta t)$, $i = 1, ..., N_1 - 1$, $j = 0, 1, ..., N$, of the system (5.8) being calculated at the time $t = N_1\Delta t$.

The latter system is solved by a spectral method and since the matrix of this system is very ill-conditioned so we have to regularize it by adding to the diagonal terms a small parameter in order to have a good accuracy of the convergence.

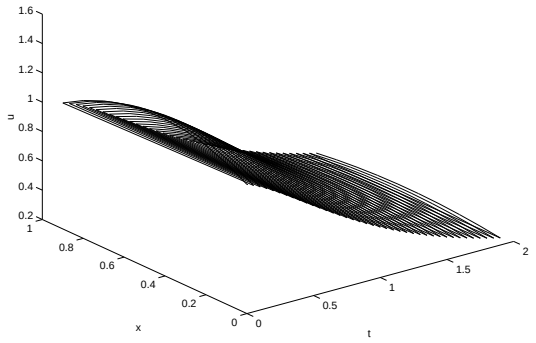


Figure 1

In fig.1 we have drawn the approximated solution of the problem (5.1)-(5.5) while fig.2 represents his corresponding exact solution (5.6).

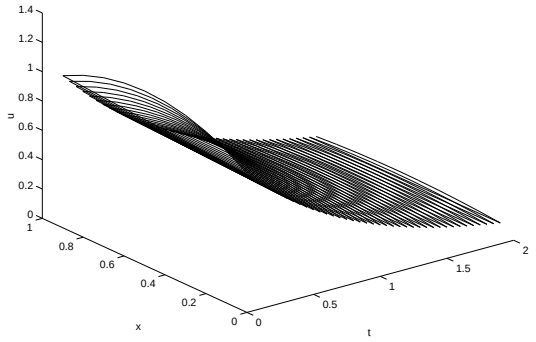


Figure 2

The fig.3 corresponds to the surface $(x, t) \mapsto u(x, t)$ approximated solution in the case where $F(x, t) = 0$. So in both cases we notice the very good decay of these surfaces to zero from $T = 2$.

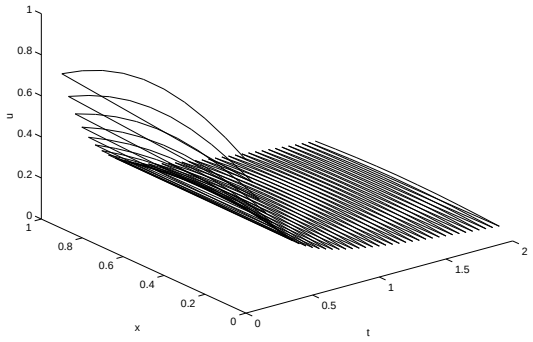


Figure 3

## References


[1] S. Berrimia, S. A. Messaoudi, *Existence and decay of solutions of a viscoelastic equation with a nonlinear source*, Nonlinear Anal. 64 (10) (2006) 2314-2331.

[2] M. Bergounioux, N. T. Long, A. P. N. Dinh, *Mathematical model for a shock problem involving a linear viscoelastic bar*, Nonlinear Anal. **43** (2001), 547-561.

[3] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.A. Soriano, *Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping*, Electron. J. Differential Equations 44 (2002) 1–14.

[4] J.L. Lions, W.A. Strauss, *Some nonlinear evolution equations*, Bull. Soc. Math., France, 93 (1965) 43-96.

[5] J. L. Lions, *Quelques méthodes de résolution des problè mes aux limites nonlinéaires*, Dunod; Gauthier- Villars, Paris. 1969.

[6] N.T. Long, A.P.N. Dinh, T.N. Diem, *Linear recursive schemes and asymptotic expansion associated with the Kirchhoff- Carrier operator*, J. Math. Anal. Appl. 267 (2002) 116-134.

[7] N. T. Long, A. P. N. Dinh, T. N. Diem, *On a shock problem involving a nonlinear viscoelastic bar*, J. Boundary Value Problems, Hindawi Publishing Corporation, **2005** (3) (2005), 337-358.

[8] N. T. Long, L. V. Ut, N. T. T. Truc, *On a shock problem involving a linear viscoelastic bar*, Nonlinear Analysis, Theory, Methods & Applications, Series A: Theory and Methods, **63** (2) (2005), 198-224.

[9] N.T. Long, V.G. Giai, *A wave equation associated with mixed nonhomogeneous conditions: Global existence and asymptotic expansion of solutions*, Nonlinear Analysis, Theory, Methods and Applications, Series A: Theory and Methods, 66 (7) (2007), 1526-1546.

[10] N. T. Long, L. X. Truong, *Existence and asymptotic expansion for a viscoelastic problem with a mixed nonhomogeneous condition*, Nonlinear Analysis, Theory, Methods & Applications, Series A: Theory and Methods, (accepted for publication).



Nguyen Thanh Long

Department of Mathematics, Hochiminh City National University, 227 Nguyen Van Cu, Q5, HoChiMinh City, Vietnam

*E-mail address*: longnt@hcmc.netnam.vn, longnt2@gmail.com

Alain Pham Ngoc Dinh

MAPMO, UMR 6628, Mathematics Dept., University of Orleans, BP 6759, 45067 Orleans Cedex 2, France.

*E-mail address*: alain.pham@univ-orleans.fr, alain.pham@math.cnrs.fr

Le Xuan Truong
Department of Mathematics, Faculty of General Science, University of Technical Education in HoChiMinh City, 01 Vo Van Ngan Str., Thu Duc Dist., HoChiMinh City, Vietnam
*E-mail address*: lxuantruong@gmail.com, truong@math.net