



HAL
open science

EXISTENCE AND DECAY OF SOLUTIONS OF A NONLINEAR VISCOELASTIC PROBLEM WITH A MIXED NONHOMOGENEOUS CONDITION

Long Nguyen Thanh, Alain Pham Ngoc Dinh, Le Xuan Truong

► **To cite this version:**

Long Nguyen Thanh, Alain Pham Ngoc Dinh, Le Xuan Truong. EXISTENCE AND DECAY OF SOLUTIONS OF A NONLINEAR VISCOELASTIC PROBLEM WITH A MIXED NONHOMOGENEOUS CONDITION. 2007. hal-00136410v1

HAL Id: hal-00136410

<https://hal.science/hal-00136410v1>

Preprint submitted on 13 Mar 2007 (v1), last revised 27 May 2007 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Existence and decay of solutions of a nonlinear viscoelastic problem with a mixed nonhomogeneous condition

Nguyen Thanh Long⁽¹⁾, Alain Pham Ngoc Dinh⁽²⁾, Le Xuan Truong⁽³⁾

⁽¹⁾ Department of Mathematics and Computer Science, University of Natural Science, Vietnam National University HoChiMinh City, 227 Nguyen Van Cu Str., Dist.5, HoChiMinh City, Vietnam.

E-mail: longnt@hcmc.netnam.vn, longnt2@gmail.com

⁽²⁾ MAPMO, UMR 6628, bât. Mathématiques, University of Orléans, BP 6759, 45067 Orléans Cedex 2, France.

E-mail: alain.pham@univ-orleans.fr, alain.pham@math.cnrs.fr

⁽³⁾ Department of Mathematics, Faculty of General Science, University of Technical Education in HoChiMinh City, 01 Vo Van Ngan Str., Thu Duc Dist., HoChiMinh City, Vietnam.

E-mail: truong@math.net

Abstract. *We study the initial-boundary value problem for a nonlinear wave equation given by*

$$(1) \begin{cases} u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + |u_t|^{q-2}u_t = f(x, t, u), & 0 < x < 1, 0 < t < T, \\ u_x(0, t) = u(0, t), \quad u_x(1, t) + \eta u(1, t) = g(t), \\ u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x), \end{cases}$$

where $\eta \geq 0$, $q \geq 2$ are given constants and $\tilde{u}_0, \tilde{u}_1, g, k, f$ are given functions.

In this paper, we consider two main parts. In Part 1, under a certain local Lipschitzian condition on f with $(\tilde{u}_0, \tilde{u}_1) \in H^1 \times L^2$; $k, g \in H^1(0, T)$, $\eta \geq 0$; $q \geq 2$, a global existence and uniqueness theorem is proved. The proof is based on the paper [10] associated to a contraction mapping theorem and standard arguments of density. In Part 2, the asymptotic behavior of the solution u as $t \rightarrow +\infty$ is studied, under more restrictive conditions, namely $g \equiv 0$, $f(x, t, u) = -|u|^{p-2}u + F(x, t)$, $p \geq 2$, $F \in L^1(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; L^2)$, $\int_0^{+\infty} e^{\sigma t} \|F(t)\|^2 dt < +\infty$, with $\sigma > 0$, and $(\tilde{u}_0, \tilde{u}_1) \in H^1 \times L^2$, $k \in H^1(\mathbb{R}_+)$, and some others ($\|\cdot\|$ denotes the $L^2(0, 1)$ norm). It is proved that under these conditions, a unique solution $u(t)$ exists on \mathbb{R}_+ such that $\|u'(t)\| + \|u_x(t)\|$ decay exponentially to 0 as $t \rightarrow +\infty$.

Keywords: Faedo-Galerkin method; Global existence; contraction mapping theorem; nonlinear wave equation; Viscoelastic; Exponential decay.

AMS subject classification: 35L05, 35L15, 35L70, 37B25.

Address for correspondence: Alain Pham Ngoc Dinh.

1. INTRODUCTION

In this paper we will consider the following initial and boundary value problem:

$$(1.1) \quad u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + \lambda|u_t|^{q-2}u_t = f(x,t,u), \quad 0 < x < 1, 0 < t < T,$$

$$(1.2) \quad u_x(0,t) = \eta_0 u(0,t), \quad u_x(1,t) + \eta u(1,t) = g(t),$$

$$(1.3) \quad u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x),$$

where $\eta \geq 0$, $\eta_0 > 0$, $q \geq 2$; $\lambda > 0$ are given constants and $\tilde{u}_0, \tilde{u}_1, g, k, f$ are given functions satisfying conditions specified later.

In a recent paper [1], Berrimia and Messaoudi considered the problem

$$(1.4) \quad u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)ds = |u|^{p-2}u, \quad x \in \Omega, t > 0,$$

$$(1.5) \quad u = 0, \quad \text{on } \partial\Omega,$$

$$(1.6) \quad u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x), \quad x \in \Omega,$$

where $p > 2$ is a constant, k is a given positive function, and Ω is a bounded domain of \mathbb{R}^n ($n \geq 1$), with a smooth boundary $\partial\Omega$. This type of problems have been considered by many authors and several results concerning existence, nonexistence, and asymptotic behavior have been established. In this regard, Cavalcanti et al. [3] studied the following equation

$$(1.7) \quad u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)ds + |u|^{p-2}u + a(t)u_t = 0, \quad \text{in } \Omega \times (0, \infty),$$

for $a : \Omega \rightarrow \mathbb{R}_+$, a function, which may be null on a part of the domain Ω . Under the conditions that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and

$$(1.8) \quad -\zeta_1 k(t) \leq k'(t) \leq -\zeta_2 k(t), \quad t \geq 0,$$

the authors established an exponential rate of decay.

In [2] Bergounioux, Long and Dinh studied problem (1.1), (1.3) with $k \equiv 0$, $q = 2$, $f(x,t,u) = -Ku + F(x,t)$, and the mixed boundary conditions (1.2) standing for

$$(1.9) \quad u_x(0,t) = g(t) + hu(0,t) - \int_0^t H(t-s)u(0,s)ds,$$

$$(1.10) \quad u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0,$$

where $h \geq 0$, $K, \lambda, K_1, \lambda_1$ are given constants and g, H are given functions.

In [7], Long, Dinh and Diem obtained the unique existence, regularity and asymptotic expansion of the problem (1.1), (1.3), (1.9) and (1.10) in the case of $k \equiv 0$, $f(x,t,u) = -K|u|^{p-2}u + F(x,t)$, with $p \geq 2$, $q \geq 2$; K, λ are given constants.

In [8], Long, Ut and Truc gave the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of problem (1.1)- (1.3) when $k \equiv 0$, $q = 2$, $f(x,t,u) = -Ku + F(x,t)$ and $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^1$. In this case, the problem (1.1)-(1.3) is the mathematical model describing a shock problem involving a linear viscoelastic bar.

In [9], Long and Giai obtained the unique existence and asymptotic expansion for the solution of problem (1.1), (1.3) when $k \equiv 0$, $q = 2$, $f(x,t,u) = -Ku + F(x,t)$ and $(\tilde{u}_0, \tilde{u}_1) \in H^1 \times L^2$, and the mixed boundary conditions (1.2) standing for

$$(1.11) \quad u_x(0,t) = g(t) + K_1|u(0,t)|^{\alpha-2}u(0,t) + \lambda_1|u_t(0,t)|^{\beta-2}u_t(0,t) - \int_0^t H(t-s)u(0,s)ds,$$

$$(1.12) \quad u(1, t) = 0,$$

where $K, \lambda, K_1, \lambda_1, \alpha, \beta$ are given constants and g, H are given functions. In this case, the problem (1.1), (1.3), (1.11), (1.12) is the mathematical model describing a shock problem involving a nonlinear viscoelastic bar.

In [10], Long and Truong obtained the unique existence and asymptotic expansion for the solution of problem (1.1)-(1.3) when $f(x, t, u) = -K|u|^{p-2}u + F(x, t)$, $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^1$; $F, F_t \in L^2(Q_T)$, $k \in W^{2,1}(0, T)$, $g \in H^2(0, T)$; $K, \eta \geq 0, \eta_0 > 0$; $p, q \geq 2$.

In this paper, we consider two main parts. In Part 1, under a certain local Lipschitzian condition on f with $(\tilde{u}_0, \tilde{u}_1) \in H^1 \times L^2$; $k, g \in H^1(0, T)$, $\lambda > 0, \eta_0 > 0$; $\eta \geq 0$; $q \geq 2$, a global existence and uniqueness theorem is proved. The proof is based on the paper [10] associated to a contraction mapping theorem and standard arguments of density. In Part 2, the asymptotic behavior of the solution u as $t \rightarrow +\infty$ is studied, under more restrictive conditions, namely $f(x, t, u) = -|u|^{p-2}u + F(x, t)$, $p \geq 2$, $F \in L^1(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; L^2)$, $\int_0^{+\infty} e^{\sigma t} \|F(t)\|^2 dt < +\infty$, with $\sigma > 0$, and $(\tilde{u}_0, \tilde{u}_1) \in H^1 \times L^2$, $g \equiv 0$, $k \in H^1(\mathbb{R}_+)$, and some others ($\|\cdot\|$ denotes the $L^2(0, 1)$ norm). It is proved that under these conditions, a unique solution $u(t)$ exists on \mathbb{R}_+ such that $\|u'(t)\| + \|u_x(t)\|$ decay exponentially to 0 as $t \rightarrow +\infty$. The results obtained here relatively are in part generalizations of those in [1-3, 6-10].

2. PRELIMINARY RESULTS

Put $\Omega = (0, 1)$, $Q_T = \Omega \times (0, T)$, $T > 0$. We omit the definitions of usual function spaces: $C^m(\bar{\Omega})$, $L^p(\Omega)$, $W^{m,p}(\Omega)$.

We denote $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, \dots$

The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle \cdot, \cdot \rangle$ the scalar product in L^2 or pair of dual scalar product of continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X . We denote by $L^p(0, T; X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0, T) \rightarrow X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{0 < t < T} \|u(t)\|_X \quad \text{for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x, t)$, $\frac{\partial u}{\partial t}(x, t)$, $\frac{\partial^2 u}{\partial t^2}(x, t)$, $\frac{\partial u}{\partial x}(x, t)$, $\frac{\partial^2 u}{\partial x^2}(x, t)$, respectively.

Without loss of generality, we can suppose that $\eta_0 = \lambda = 1$. For every $\eta \geq 0$, we put

$$(2.1) \quad \begin{cases} a_\eta(u, v) = \int_0^1 u_x(x)v_x(x)dx + u(0)v(0) + \eta u(1)v(1), \text{ for } u, v \in H^1, \\ \|v\|_\eta = (a_\eta(v, v))^{1/2}. \end{cases}$$

On H^1 we shall use the following equivalent norm

$$(2.2) \quad \|v\|_1 = \left(v^2(0) + \int_0^1 |v_x(x)|^2 dx \right)^{1/2}.$$

Then we have the following lemmas.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$(2.3) \quad \|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2} \|v\|_1, \text{ for all } v \in H^1.$$

Lemma 2.2. *Let $\eta \geq 0$. Then, the symmetric bilinear form $a_\eta(\cdot, \cdot)$ defined by (2.1)₁ is continuous on $H^1 \times H^1$ and coercive on H^1 , i.e.,*

$$(i) \quad |a_\eta(u, v)| \leq C_\eta \|u\|_1 \|v\|_1, \text{ for all } u, v \in H^1,$$

$$(ii) \quad a_\eta(v, v) \geq \|v\|_1^2, \text{ for all } v \in H^1,$$

where $C_\eta = 1 + 2\eta$.

The proofs of these lemmas are straightforward, and we omit the details.

We also note that on H^1 , $\|v\|_1$, $\|v\|_{H^1} = (\|v\|^2 + \|v'\|^2)^{1/2}$, $\|v\|_\eta = \sqrt{a_\eta(v, v)}$ are three equivalent norms.

$$(2.4) \quad \|v\|_1^2 \leq \|v\|_\eta^2 \leq C_\eta \|v\|_1^2, \text{ for all } v \in H^1,$$

$$(2.5) \quad \frac{1}{3} \|v\|_{H^1}^2 \leq \|v\|_1^2 \leq 3 \|v\|_{H^1}^2, \text{ for all } v \in H^1.$$

Then, we have the following theorem.

Theorem 2.3. *Let $T > 0$, $K, \lambda \geq 0$, $p, q \geq 2$; and $F, F_t \in L^2(Q_T)$ hold. Then, for every $(\tilde{u}_0, \tilde{u}_1, k, g) \in H^2 \times H^1 \times W^{2,1}(0, T) \times H^2(0, T)$, there exists a unique weak solution u of the problem*

$$(1) \quad \begin{cases} u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t = F(x, t), & 0 < x < 1, 0 < t < T, \\ u_x(0, t) = u(0, t), \quad u_x(1, t) + \eta u(1, t) = g(t), \\ u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x), \end{cases}$$

such that

$$(2.6) \quad u \in L^\infty(0, T; H^2), \quad u_t \in L^\infty(0, T; H^1), \quad u_{tt} \in L^\infty(0, T; L^2).$$

The proof of Theorem 2.3 can be found in [10, Theorem 2.1]. It can also be established by using the Faedo-Galerkin method as in [10].

3. THE EXISTENCE AND UNIQUENESS THEOREM OF THE SOLUTION

In this section we study the global existence of solutions for problem (1.1)-(1.3). For this purpose, we consider, first, a related nonlinear problem. Then, we use the well-known Schauder's fixed point theorem to prove the existence of solutions to the nonlinear problem (1.1)-(1.3).

We make the following assumptions:

$$(H_1) \quad \eta \geq 0, \quad q \geq 2,$$

$$(H_2) \quad k, g \in H^1(0, T),$$

$$(H_3) \quad \tilde{u}_0 \in H^1 \text{ and } \tilde{u}_1 \in L^2,$$

$$(H_4) \quad f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}) \text{ satisfies the conditions } D_2 f, D_3 f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}).$$

For each $T > 0$, we put

$$(3.1) \quad W(T) = \{v \in L^\infty(0, T; H^1) : v_t \in L^\infty(0, T; L^2) \cap L^q(Q_T)\}.$$

Then $W(T)$ is a Banach space with respect to the norm (see[5]):

$$(3.2) \quad \|v\|_{W(T)} = \|v\|_{L^\infty(0,T;H^1)} + \|v_t\|_{L^\infty(0,T;L^2)} + \|v_t\|_{L^q(Q_T)}, \quad v \in W(T).$$

For each $v \in W(T)$, we associate with the problem (1.1)-(1.3) the following variational problem.

Find $u \in W(T)$ which satisfies the variational problem

$$(3.3) \quad \langle u''(t), w \rangle + a_\eta(u(t), w) - \int_0^t k(t-s)a_\eta(u(s), w)ds + \langle \Psi_q(u'(t)), w \rangle \\ = g_1(t)w(1) + \langle f(\cdot, t, v(\cdot, t)), w \rangle \quad \text{for all } w \in H^1,$$

$$(3.4) \quad u(0) = \tilde{u}_0, \quad u_t(0) = \tilde{u}_1,$$

where

$$(3.5) \quad \Psi_q(z) = |z|^{q-2}z, \quad g_1(t) = g(t) - \int_0^t k(t-s)g(s)ds.$$

Then, we have the following

Theorem 3.1. *Let $T > 0$ and (H_1) , (H_4) hold. Then, for every $(\tilde{u}_0, \tilde{u}_1, k, g) \in H^2 \times H^1 \times W^{2,1}(0, T) \times H^2(0, T)$ and $v \in W(T)$ there exists a unique solution u of problem (3.3)- (3.5) such that*

$$(3.6) \quad u \in L^\infty(0, T; H^2), \quad u_t \in L^\infty(0, T; H^1), \quad u_{tt} \in L^\infty(0, T; L^2).$$

This Theorem 3.1 is a direct result of Theorem 2.3 with $K = 0$, $\lambda = 1$, $F(x, t) = f(x, t, v(x, t))$.

Remark 3.1. Noting that with the regularity obtain by (3.6), it follows that the problem (3.3)-(3.5) has a unique strong solution u satisfies

$$(3.7) \quad u \in C^0(0, T; H^1) \cap C^1(0, T; L^2) \cap L^\infty(0, T; H^2).$$

On the other hand, from (3.6) it is clear that $u, u_x, u_t, u_{xt}, u_{tt}, u_{xx} \in L^\infty(0, T; L^2) \subset L^2(Q_T)$. Also if the data $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^1$, then the solution u of problem (3.3)-(3.5) belongs to $H^2(Q_T)$. So the solution is almost classical which is rather natural since the initial data \tilde{u}_0 and \tilde{u}_1 do not belong necessarily to $C^2(\bar{\Omega})$ and $C^1(\bar{\Omega})$, respectively.

Theorem 3.2. *Let $(H_1) - (H_4)$ hold. Then, for every $T > 0$ and $v \in W(T)$, problem (3.3)- (3.5) has a unique solution $u \in W(T)$ and such that*

$$(3.8) \quad u'', \quad u_{xx} \in L^{q'}(0, T; (H^1)'), \quad \text{where } q' = q(q-1)^{-1}.$$

Furthermore, we have

$$(3.9) \quad \|u'(t)\|^2 + \|u(t)\|_\eta^2 + 2 \int_0^t \|u'(s)\|_{L^q}^q ds \leq C_{1T} \exp(TC_{2T}), \quad \forall t \in [0, T],$$

where

$$(3.10) \quad \begin{cases} C_{1T} = C_{1T}(v, \tilde{u}_0, \tilde{u}_1, k, g) = 2 \left[\|\tilde{u}_1\|^2 + \|\tilde{u}_0\|_\eta^2 + 2|g_1(0)\tilde{u}_0(1)| \right. \\ \quad \left. + 6\|g_1\|_{L^\infty(0,T)}^2 + 2\|g_1'\|_{L^2(0,T)}^2 + \int_0^T \|f(\cdot, s, v(s))\|^2 ds \right], \\ C_{2T} = C_{2T}(k) = 2 \left[3 + 2|k(0)| + 6\|k\|_{L^2(0,T)}^2 + T\|k'\|_{L^2(0,T)}^2 \right], \end{cases}$$

and

$$(3.11) \quad g_1(t) = g(t) - \int_0^t k(t-s)g(s)ds.$$

Proof.

a/ The existence of solution.

We approximate $\tilde{u}_0, \tilde{u}_1, k, g$ by sequences $\{u_{0m}\} \subset C_0^\infty(\bar{\Omega}), \{u_{1m}\} \subset C_0^\infty(\Omega), \{k_m\}, \{g_m\} \subset C_0^\infty([0, T])$, respectively, such that

$$(3.12) \quad \begin{cases} u_{0m} \rightarrow \tilde{u}_0 \text{ strongly in } H^1, \\ u_{1m} \rightarrow \tilde{u}_1 \text{ strongly in } L^2, \\ k_m \rightarrow k \text{ strongly in } H^1(0, T), \\ g_m \rightarrow g \text{ strongly in } H^1(0, T). \end{cases}$$

We then consider the following variational problem: Find $u_m \in W(T)$ which satisfies the variational problem

$$(3.13) \quad \begin{aligned} \langle u_m''(t), w \rangle + a_\eta(u_m(t), w) - \int_0^t k_m(t-s) a_\eta(u_m(s), w) ds + \langle \Psi_q(u_m'(t)), w \rangle \\ = g_{1m}(t)w(1) + \langle f(\cdot, t, v(\cdot, t)), w \rangle \text{ for all } w \in H^1, \end{aligned}$$

$$(3.14) \quad u(0) = u_{0m}, \quad u'(0) = u_{1m},$$

and

$$(3.15) \quad u_m \in L^\infty(0, T; H^2), \quad u_m' \in L^\infty(0, T; H^1), \quad u_m'' \in L^\infty(0, T; L^2),$$

where

$$(3.16) \quad g_{1m}(t) = g_m(t) - \int_0^t k_m(t-s)g_m(s)ds.$$

Theorem 3.1 guarantees the existence of a sequence of solutions $\{u_m\}$ satisfying (3.13)-(3.16). We shall prove that $\{u_m\}$ is a Cauchy sequence in $W(T)$.

(i) *A priori estimates*

We take $w = u_m'(t)$ in (3.13), afterwards integrating with respect to the time variable from 0 to t , we get after some rearrangements

$$(3.17) \quad \begin{aligned} \sigma_m(t) = \sigma_m(0) - 2g_{1m}(0)u_{0m}(1) + 2g_{1m}(t)u_m(1, t) - 2 \int_0^t g_{1m}'(r)u_m(1, r)dr \\ + 2 \int_0^t k_m(t-s) a_\eta(u_m(s), u_m(t)) ds - 2k_m(0) \int_0^t \|u_m(r)\|_\eta^2 dr \\ - 2 \int_0^t dr \int_0^r k_m'(r-s) a_\eta(u_m(s), u_m(r)) ds + 2 \int_0^t \langle f(\cdot, s, v(\cdot, s)), u_m'(s) \rangle ds, \end{aligned}$$

where

$$(3.18) \quad \sigma_m(t) = \|u_m'(t)\|^2 + \|u_m(t)\|_\eta^2 + 2 \int_0^t \|u_m'(s)\|_{L^q}^q ds.$$

Proving in the same manner as in [10], we have the following results:

$$(3.19) \quad \sigma_m(t) \leq C_{1T}(m) + C_{2T}(m) \int_0^t \sigma_m(s) ds, \quad \forall t \in [0, T],$$

where

$$(3.20) \quad \begin{cases} C_{1T}(m) = 2 \left[\|u_{1m}\|^2 + \|u_{0m}\|_\eta^2 + 2|g_{1m}(0)u_{0m}(1)| + 6\|g_{1m}\|_{L^\infty(0, T)}^2 \right. \\ \quad \left. + 2\|g_{1m}'\|_{L^2(0, T)}^2 + \int_0^T \|f(\cdot, s, v(s))\|^2 ds \right], \\ C_{2T}(m) = 2 \left[3 + 2|k_m(0)| + 6\|k_m\|_{L^2(0, T)}^2 + T\|k_m'\|_{L^2(0, T)}^2 \right]. \end{cases}$$

From the assumptions $(H_1) - (H_4)$, afterwards using Gronwall's lemma, we deduce from

(3.12), that

$$(3.21) \quad \sigma_m(t) \leq \tilde{C}_T, \text{ for all } m \text{ and } t \in [0, T],$$

where \tilde{C}_T is a constant independent of m .

On the other hand, we deduce from (3.13), (3.21), that, for all $w \in H^1$, we have

$$(3.22) \quad \begin{aligned} |\langle u_m''(t), w \rangle| &\leq \|u_m(t)\|_\eta \|w\|_\eta + \int_0^t |k_m(t-s)| \|u_m(s)\|_\eta \|w\|_\eta ds \\ &\quad + \|\Psi_q(u_m')\|_{L^{q'}(\Omega)} \|w\|_{L^q(\Omega)} + |g_{1m}(t)| \|w\|_\eta + \|f(\cdot, t, v(\cdot, t))\| \|w\| \\ &\leq \left[\sqrt{3C_\eta} (1 + \|k_m\|_{L^1(0,T)}) \|u_m\|_{L^\infty(0,T;H^1)} \right. \\ &\quad \left. + C \|\Psi_q(u_m')\|_{L^{q'}(\Omega)} + |g_{1m}(t)| + C \|f(\cdot, t, v(\cdot, t))\| \right] \|w\|_\eta \\ &\leq C_T \sqrt{3C_\eta} \left[1 + \|\Psi_q(u_m')\|_{L^{q'}(\Omega)} \right] \|w\|_{H^1}. \end{aligned}$$

This implies that

$$(3.23) \quad \|u_m''(t)\|_{(H^1)'} = \sup_{0 \neq w \in H^1} \frac{|\langle u_m''(t), w \rangle|}{\|w\|_{H^1}} \leq C_T \sqrt{3C_\eta} \left[1 + \|\Psi_q(u_m'(t))\|_{L^{q'}(\Omega)} \right].$$

Hence

$$(3.24) \quad \begin{aligned} \|u_m''\|_{L^{q'}(0,T;(H^1)')}^{q'} &= \int_0^T \|u_m''(t)\|_{(H^1)'}^{q'} dt \\ &\leq (C_T \sqrt{3C_\eta})^{q'} \int_0^T \left[1 + \|\Psi_q(u_m'(t))\|_{L^{q'}(\Omega)} \right]^{q'} dt \\ &\leq (C_T \sqrt{3C_\eta})^{q'} 2^{q'-1} \int_0^T \left[1 + \|u_m'(t)\|_{L^q(\Omega)}^q \right] dt \leq C_T, \end{aligned}$$

where C_T always indicating a constant depending on T .

(ii) *The convergence of sequence $\{u_m\}$.*

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W(T)$. Let $\hat{u} = u_m - u_\mu$. Then \hat{u} satisfies the variational problem

$$(3.25) \quad \begin{aligned} \langle \hat{u}''(t), w \rangle + a_\eta(\hat{u}(t), w) - \int_0^t k_m(t-s) a_\eta(\hat{u}(s), w) ds - \int_0^t \hat{k}(t-s) a_\eta(u_\mu(s), w) ds \\ + \langle \Psi_q(u_m'(t)) - \Psi_q(u_\mu'(t)), w \rangle = \hat{g}_1(t) w(1) \quad \text{for all } w \in H^1, \end{aligned}$$

$$(3.26) \quad \hat{u}(0) = \hat{u}_0, \quad \hat{u}'(0) = \hat{u}_1,$$

where

$$(3.27) \quad \begin{cases} \hat{u}_0 = u_{0m} - u_{0\mu}, \quad \hat{u}_1 = u_{1m} - u_{1\mu}, \\ \hat{k} = k_m - k_\mu, \quad \hat{g} = g_m - g_\mu, \quad \hat{g}_1 = g_{1m} - g_{1\mu}, \\ \hat{g}_1(t) = \hat{g}(t) - \int_0^t k_m(t-s) \hat{g}(s) ds - \int_0^t \hat{k}(t-s) g_\mu(s) ds. \end{cases}$$

We take $w = \hat{u}'(t)$ in (3.25), after integrating with respect to the time variable from 0 to t , we get after some rearrangements

$$\begin{aligned}
(3.28) \quad Z(t) &= Z(0) - 2\widehat{g}_1(0)\widehat{u}_0(1) + 2\widehat{g}_1(t)\widehat{u}(1,t) - 2\int_0^t \widehat{g}'_1(r)\widehat{u}(1,r)dr \\
&\quad - 2k_m(0)\int_0^t \|\widehat{u}(r)\|_\eta^2 dr + 2\int_0^t k_m(t-s)a_\eta(\widehat{u}(s), \widehat{u}(t)) ds \\
&\quad - 2\int_0^t dr \int_0^r k'_m(r-s)a_\eta(\widehat{u}(s), \widehat{u}(r)) ds \\
&\quad - 2\widehat{k}(0)\int_0^t a_\eta(u_\mu(s), \widehat{u}(s)) ds + 2\int_0^t \widehat{k}(t-s)a_\eta(u_\mu(s), \widehat{u}(t)) ds \\
&\quad - 2\int_0^t dr \int_0^r \widehat{k}'(r-s)a_\eta(u_\mu(s), \widehat{u}(r)) ds,
\end{aligned}$$

where

$$(3.29) \quad Z(t) = \|\widehat{u}'(t)\|^2 + \|\widehat{u}(t)\|_\eta^2 + 2\int_0^t \langle \Psi_q(u'_m(s)) - \Psi_q(u'_\mu(s)), u'_m(s) - u'_\mu(s) \rangle ds.$$

Using the following inequality

$$(3.30) \quad \forall q \geq 2, \exists C_q > 0 : (|x|^{q-2}x - |y|^{q-2}y)(x - y) \geq C_q|x - y|^q \quad \forall x, y \in \mathbb{R},$$

it follows from (3.29) that

$$(3.31) \quad Z(t) \geq \|\widehat{u}'(t)\|^2 + \|\widehat{u}(t)\|_\eta^2 + 2C_q \int_0^t \|\widehat{u}'(s)\|_{L^q}^q ds.$$

Using the inequality

$$(3.32) \quad 2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \quad \forall a, b \in \mathbb{R}, \forall \varepsilon > 0,$$

and the following inequalities

$$(3.33) \quad |a_\eta(u, v)| \leq \|u\|_\eta \|v\|_\eta, \quad \forall u, v \in H^1,$$

$$(3.34) \quad |\widehat{u}(1, t)| \leq \|\widehat{u}(t)\|_{C^0(\overline{\Omega})} \leq \sqrt{2} \|\widehat{u}(t)\|_1 \leq \sqrt{2} \|\widehat{u}(t)\|_\eta \leq \sqrt{2Z(t)},$$

we shall estimate respectively the following terms on the right-hand side of (3.28) as follows

$$\begin{aligned}
(3.35) \quad Z(0) - 2\widehat{g}_1(0)\widehat{u}_0(1) &\leq \|\widehat{u}_1\|^2 + \|\widehat{u}_0\|_\eta^2 + 2|\widehat{g}_1(0)|\|\widehat{u}_0(1)| \\
&= \|u_{1m} - u_{1\mu}\|^2 + \|u_{0m} - u_{0\mu}\|_\eta^2 \\
&\quad + 2|g_{1m}(0) - g_{1\mu}(0)|\|u_{0m}(1) - u_{0\mu}(1)|,
\end{aligned}$$

$$\begin{aligned}
(3.36) \quad 2\widehat{g}_1(t)\widehat{u}(1, t) &\leq \frac{1}{\varepsilon} \widehat{g}_1^2(t) + \varepsilon \widehat{u}^2(1, t) \leq \frac{1}{\varepsilon} \|\widehat{g}_1\|_{L^\infty(0, T)}^2 + 2\varepsilon Z(t) \\
&= 8\|\widehat{g}_1\|_{L^\infty(0, T)}^2 + \frac{1}{4} Z(t), \quad \text{with } \varepsilon = \frac{1}{8},
\end{aligned}$$

$$(3.37) \quad -2\int_0^t \widehat{g}'_1(r)\widehat{u}(1, r)dr \leq 2\int_0^t |\widehat{g}'_1(r)| \sqrt{2Z(r)} dr \leq 2\|\widehat{g}'_1\|_{L^2(0, T)}^2 + \int_0^t Z(r)dr,$$

$$\begin{aligned}
(3.38) \quad 2\int_0^t k_m(t-s)a_\eta(\widehat{u}(s), \widehat{u}(t)) ds &\leq 2\int_0^t |k_m(t-s)| \|\widehat{u}(s)\|_\eta \|\widehat{u}(t)\|_\eta ds \\
&\leq 2\|\widehat{u}(t)\|_\eta \int_0^t |k_m(t-s)| \|\widehat{u}(s)\|_\eta ds \\
&\leq \frac{1}{8} Z(t) + 8\|k_m\|_{L^2(0, T)}^2 \int_0^t Z(s) ds,
\end{aligned}$$

$$(3.39) \quad -2k_m(0)\int_0^t \|\widehat{u}(r)\|_\eta^2 dr \leq 2|k_m(0)| \int_0^t Z(r)dr,$$

$$(3.40) \quad -2\int_0^t dr \int_0^r k'_m(r-s)a_\eta(\widehat{u}(s), \widehat{u}(r)) ds \leq (1 + T\|k'_m\|_{L^2(0, T)}^2) \int_0^t Z(s) ds,$$

$$(3.41) \quad 2 \int_0^t \widehat{k}(t-s) a_\eta(u_\mu(s), \widehat{u}(t)) ds \leq \frac{1}{8} Z(t) + 8 \widetilde{C}_T \|\widehat{k}\|_{L^1(0,T)}^2,$$

$$(3.42) \quad -2 \widehat{k}(0) \int_0^t a_\eta(u_\mu(s), \widehat{u}(s)) ds \leq T \widetilde{C}_T |\widehat{k}(0)|^2 + \int_0^t Z(s) ds,$$

$$(3.43) \quad -2 \int_0^t dr \int_0^r \widehat{k}'(r-s) a_\eta(u_\mu(s), \widehat{u}(r)) ds \leq T^2 \widetilde{C}_T \|\widehat{k}'\|_{L^2(0,T)}^2 + \int_0^t Z(s) ds.$$

Combining (3.28), (3.29), (3.31) and (3.35)-(3.43), we obtain

$$(3.44) \quad Z(t) \leq \rho_{1T}(m, \mu) + \rho_{2T}(m) \int_0^t Z(s) ds, \quad \forall t \in [0, T],$$

where

$$(3.45) \quad \begin{cases} \rho_{1T}(m, \mu) = 2 \left[\|\widehat{u}_1\|^2 + \|\widehat{u}_0\|_\eta^2 + 2|\widehat{g}_1(0)\widehat{u}_0(1)| + 8\|\widehat{g}_1\|_{L^\infty(0,T)}^2 + 2\|\widehat{g}'_1\|_{L^2(0,T)}^2 \right. \\ \quad \left. + 8\widetilde{C}_T \|\widehat{k}\|_{L^1(0,T)}^2 + T \widetilde{C}_T |\widehat{k}(0)|^2 + T^2 \widetilde{C}_T \|\widehat{k}'\|_{L^2(0,T)}^2 \right], \\ \rho_{2T}(m) = 2 \left[4 + 2|k_m(0)| + 8\|k_m\|_{L^2(0,T)}^2 + T\|k'_m\|_{L^2(0,T)}^2 \right]. \end{cases}$$

By Gronwall's lemma, we deduce from (3.31), (3.44), (3.45), that

$$(3.46) \quad \|\widehat{u}'(t)\|^2 + \|\widehat{u}(t)\|_\eta^2 + 2C_q \int_0^t \|\widehat{u}'(s)\|_{L^q}^q ds \leq Z(t) \\ \leq \rho_{1T}(m, \mu) \exp(T\rho_{2T}(m)), \quad \text{for all } t \in [0, T].$$

By (3.12), (3.27) and (3.45), we obtain $\rho_{1T}(m, \mu) \exp(T\rho_{2T}(m)) \rightarrow 0$ as $m, \mu \rightarrow +\infty$. Hence, it follows from (3.46) that $\{u_m\}$ is a Cauchy sequence in $W(T)$. Therefore there exists $u \in W(T)$ such that

$$(3.47) \quad u_m \rightarrow u \text{ strongly in } W(T).$$

On the other hand, by (3.47) and the continuity of Ψ_q , we obtain

$$(3.48) \quad \Psi_q(u'_m) \rightarrow \Psi_q(u') \text{ a.e. } (x, t) \in Q_T.$$

By means of (3.21), it follows that

$$(3.49) \quad \|\Psi_q(u'_m)\|_{L^{q'}(Q_T)}^{q'} = \|u'_m\|_{L^q(Q_T)}^q \leq \frac{1}{2} \widetilde{C}_T,$$

for all m . By Lions's lemma [5, Lemma 1.3, p. 12], it follows from (3.48) and (3.49) that

$$(3.50) \quad \Psi_q(u'_m) \rightarrow \Psi_q(u') \text{ in } L^{q'}(Q_T) \text{ weakly.}$$

Noticing (3.12)₃ and (3.47) we have

$$(3.51) \quad \left| \int_0^T dt \int_0^t k_m(t-s) a_\eta(u_m(s), w(t)) ds - \int_0^T dt \int_0^t k(t-s) a_\eta(u(s), w(t)) ds \right| \\ \leq \left| \int_0^T dt \int_0^t k_m(t-s) a_\eta(u_m(s) - u(s), w(t)) ds \right| \\ + \left| \int_0^T dt \int_0^t [k_m(t-s) - k(t-s)] a_\eta(u(s), w(t)) ds \right| \\ \leq 3C_\eta \|w\|_{L^1(0,T;H^1)} \left[\|k_m\|_{L^1(0,T)} \|u_m - u\|_{L^\infty(0,T;H^1)} + \|k_m - k\|_{L^1(0,T)} \|u\|_{L^\infty(0,T;H^1)} \right] \rightarrow 0$$

for all $w \in L^1(0, T; H^1)$.

On the other hand, by (3.12)_{3,4} and (3.16), we also obtain

$$(3.52) \quad g_{1m} \rightarrow g_1 \text{ strongly in } H^1(0, T).$$

From (3.24), we deduce the existence of a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, such that

$$(3.53) \quad u_m'' \rightarrow u'' \text{ in } L^{q'}(0, T; (H^1)') \text{ weak.}$$

Passing to the limit in (3.13), (3.14) by (3.47) and (3.50)-(3.53) we have u satisfying the equation

$$(3.54) \quad \begin{cases} \langle u''(t), w \rangle + a_\eta(u(t), w) - \int_0^t k(t-s)a_\eta(u(s), w)ds + \langle \Psi_q(u'(t)), w \rangle \\ = g_1(t)w(1) + \langle f(\cdot, t, v(\cdot, t)), w \rangle, \text{ for all } w \in H^1, \text{ in } L^{q'}(0, T) \text{ weak,} \end{cases}$$

and

$$(3.55) \quad u(0) = \tilde{u}_0, \quad u'(0) = \tilde{u}_1.$$

On the other hand, we deduce from (3.54), that

$$(3.56) \quad u_{xx}(t) - \int_0^t k(t-s)u_{xx}(s)ds = \phi(t),$$

where

$$(3.57) \quad \phi(t) = u''(t) + |u'|^{q-2}u' - f(\cdot, t, v(\cdot, t)) \in L^{q'}(0, T; (H^1)').$$

Hence, it follows from (3.55) and (3.56), that

$$(3.58) \quad \begin{aligned} \|u_{xx}(t)\|_{(H^1)'}^{q'} &\leq \left(\|\phi(t)\|_{(H^1)'} + \int_0^t |k(t-s)| \|u_{xx}(s)\|_{(H^1)'} ds \right)^{q'} \\ &\leq 2^{q'-1} \left[\|\phi(t)\|_{(H^1)'}^{q'} + \left(\int_0^t |k(t-s)| \|u_{xx}(s)\|_{(H^1)'} ds \right)^{q'} \right] \\ &\leq 2^{q'-1} \left[\|\phi(t)\|_{(H^1)'}^{q'} + \|k\|_{L^q(0,T)}^{q'} \left(\int_0^t \|u_{xx}(s)\|_{(H^1)'}^{q'} ds \right) \right]. \end{aligned}$$

This implies that

$$(3.59) \quad \int_0^r \|u_{xx}(t)\|_{(H^1)'}^{q'} dt \leq 2^{q'-1} \|\phi\|_{L^{q'}(0,T;(H^1)')}^{q'} + 2^{q'-1} \|k\|_{L^q(0,T)}^{q'} \int_0^r \left(\int_0^t \|u_{xx}(s)\|_{(H^1)'}^{q'} ds \right) dt.$$

Using Gronwall's lemma, we obtain

$$(3.60) \quad \int_0^r \|u_{xx}(t)\|_{(H^1)'}^{q'} dt \leq 2^{q'-1} \|\phi\|_{L^{q'}(0,T;(H^1)')}^{q'} \exp\left(r2^{q'-1} \|k\|_{L^q(0,T)}^{q'}\right) \leq C_T,$$

where C_T always indicating a constant depending on T .

Thus

$$(3.61) \quad u_{xx} \in L^{q'}(0, T; (H^1)').$$

On the other hand, the estimate (3.9) hold by means of (3.12), (3.19), (3.20), (3.47).

The existence of the theorem is proved completely.

(b) Uniqueness of the solution.

First, we shall now require the following lemma.

Lemma 3.3. *Let u be the weak solution of the following problem*

$$(3.62) \quad \left\{ \begin{array}{l} u'' - u_{xx} + \int_0^t k(t-s)u_{xx}(s) ds = \Phi, \quad 0 < x < 1, \quad 0 < t < T, \\ u_x(0,t) = u(0,t), \quad u_x(1,t) + \eta u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \quad u'(x,0) = \tilde{u}_1(x), \\ u \in L^\infty(0,T;H^1), \quad u' \in L^\infty(0,T;L^2), \\ k \in H^1(0,T), \quad \Phi \in L^2(Q_T). \end{array} \right.$$

Then we have

$$(3.63) \quad \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} \|u(t)\|_\eta^2 \geq \frac{1}{2} \|u_1\|^2 + \frac{1}{2} \|u_0\|_\eta^2 - k(0) \int_0^t \|u(r)\|_\eta^2 dr \\ + \int_0^t k(t-s)a(u(s),u(t))ds - \int_0^t dr \int_0^r k'(r-s)a(u(s),u(r))ds \\ + \int_0^t \langle \Phi(s), u'(s) \rangle ds, \quad \text{a.e. } t \in [0, T].$$

Furthermore, if $\tilde{u}_0 = \tilde{u}_1 = 0$ there is equality in (3.63).

Proof of Lemma 3.3. The idea of the proof is the same as in [4, Lemma 2.1, p. 79]. Fix $t_1, t_2, 0 < t_1 < t_2 < T$ and let $v(x, t)$ be the function defined as follows

$$(3.64) \quad v(x, t) = [\theta_m(t)u'(x, t) * \rho_k(t) * \rho_k(t)]\theta_m(t),$$

where

i) θ_m is a continuous, piecewise linear function on $[0, T]$ defined as follows:

$$\theta_m(t) = \begin{cases} 0, & \text{if, } t \in [0, T] \setminus [t_1 + 1/m, t_2 - 1/m], \\ 1, & \text{if, } t \in [t_1 + 2/m, t_2 - 2/m], \\ m(t - t_1 - 1/m), & \text{if, } t \in [t_1 + 1/m, t_1 + 2/m], \\ -m(t - t_2 + 1/m), & \text{if, } t \in [t_2 - 2/m, t_2 - 1/m]. \end{cases}$$

ii) $\{\rho_k\}$ is a regularizing sequence in $C_c^\infty(\mathbb{R})$, i.e.,

$$\rho_k \in C_c^\infty(\mathbb{R}), \quad \rho_k(t) = \rho_k(-t), \quad \int_{-\infty}^{+\infty} \rho_k(t)dt = 1, \quad \text{supp } \rho_k \subset [-1/k, 1/k].$$

iii) $(*)$ is the convolution product in the time variable, i.e.,

$$(u * \rho_k)(x, t) = \int_{-\infty}^{+\infty} u(x, t-s)\rho_k(s)ds.$$

We take the scalar product of the function $v(x, t)$ in (3.64) with Eq.(3.62)₁, then integrate with respect to the time variable from 0 to T , we have

$$(3.65) \quad X_{mk} + Y_{mk} = Z_{mk},$$

where

$$(3.66) \quad \left\{ \begin{array}{l} X_{mk} = \int_0^T dt \int_0^1 u''(x, t)v(x, t)dx, \\ Y_{mk} = - \int_0^T dt \int_0^1 u_{xx}(x, t)v(x, t)dx + \int_0^T dt \int_0^1 \left(\int_0^t k(t-s)u_{xx}(x, s) ds \right) v(x, t)dx, \\ Z_{mk} = \int_0^T dt \int_0^1 \Phi(x, t)v(x, t)dx. \end{array} \right.$$

By using the properties of the functions $\theta_m(t)$ and $\rho_k(t)$ we can show after some lengthy calculation:

$$(3.67) \quad \left\{ \begin{array}{l} \lim_{k \rightarrow +\infty} X_{mk} = - \int_0^T \theta_m(t) \theta'_m(t) \|u'(t)\|^2 dt, \\ \lim_{k \rightarrow +\infty} Y_{mk} = - \int_0^T \theta_m(t) \theta'_m(t) \|u(t)\|_\eta^2 dt + 2 \int_0^T \theta'_m(t) \theta_m(t) dt \int_0^t k(t-s) a(u(s), u(t)) ds \\ \quad + k(0) \int_0^T \theta_m^2(t) \|u(t)\|_\eta^2 dt + \int_0^T \theta_m^2(t) dt \int_0^t k'(t-s) a(u(s), u(t)) ds, \\ \lim_{k \rightarrow +\infty} Z_{mk} = \int_0^T \theta_m^2(t) \langle \Phi(t), u'(t) \rangle dt. \end{array} \right.$$

Letting $k \rightarrow \infty$, from (3.65)-(3.67) we obtain

$$(3.68) \quad \begin{aligned} & \frac{1}{2} \|u'(t_2)\|^2 + \frac{1}{2} \|u(t_2)\|_\eta^2 + \int_0^{t_1} k(t_1-s) a(u(s), u(t_1)) ds - \int_0^{t_2} k(t_2-s) a(u(s), u(t_2)) ds \\ & \quad + k(0) \int_{t_1}^{t_2} \|u(t)\|_\eta^2 dt + \int_{t_1}^{t_2} dt \int_0^t k'(t-s) a(u(s), u(t)) ds \\ & = \frac{1}{2} \|u'(t_1)\|^2 + \frac{1}{2} \|u(t_1)\|_\eta^2 + \int_{t_1}^{t_2} \langle \Phi(t), u'(t) \rangle dt, \text{ a.e., } t_1, t_2 \in (0, T), t_1 < t_2. \end{aligned}$$

From (3.68) we obtain (3.63) by taking $t_2 = t$ and passing to the limit as $t_1 \rightarrow 0_+$ and using the property of weak lower semicontinuity of two functionals $v \mapsto \|v\|^2$ and $v \mapsto \|v\|_\eta^2$. In the case of $\tilde{u}_0 = \tilde{u}_1 = 0$, we prolong u, Φ, k by 0 as $t < 0$ and we deduce equality (3.68) is true for almost $t_1 < t_2 < T$.

Taking $t_1 < 0$ in (3.68), its right-hand side is 0, we take $t_1 \rightarrow 0_-$ and we have equality (3.63) when $\tilde{u}_0 = \tilde{u}_1 = 0$.

The proof of Lemma 3.3 is completed. ■

Remark 3.2. Lemma 3.3 is a relative generalization of a lemma contained in Lions's book [5, Lemma 6.1, p.224].

We now return to the proof of the uniqueness of a solution of the problem (3.3)-(3.5).

Let u_1, u_2 be two weak solutions of problem (3.3)-(3.5), such that

$$(3.69) \quad u_i \in W(T), u_i'', u_{ixx} \in L^q(0, T; (H^1)'), i = 1, 2.$$

Then $u = u_1 - u_2$ is the weak solution of the following problem

$$(3.70) \quad \left\{ \begin{array}{l} u'' - u_{xx} + \int_0^t k(t-s) u_{xx}(s) ds + \Psi_q(u_1') - \Psi_q(u_2') = 0, 0 < x < 1, 0 < t < T, \\ u_x(0, t) - u(0, t) = u_x(1, t) + \eta u(1, t) = 0, \\ u(0) = u'(0) = 0, \\ u \in W(T), u'', u_{xx} \in L^q(0, T; (H^1)'). \end{array} \right.$$

By using Lemma 3.3 with $\tilde{u}_0 = \tilde{u}_1 = 0, \Phi = -\Psi_q(u_1') + \Psi_q(u_2')$, we have

$$(3.71) \quad \begin{aligned} \sigma(t) &= 2 \int_0^t k(t-s) a(u(s), u(t)) ds - 2k(0) \int_0^t \|u(r)\|_\eta^2 dr \\ & \quad - 2 \int_0^t dr \int_0^r k'(r-s) a(u(s), u(r)) ds, \end{aligned}$$

where

$$(3.72) \quad \sigma(t) = \|u'(t)\|^2 + \|u(t)\|_\eta^2 + 2 \int_0^t \langle \Psi_q(u_1'(s)) - \Psi_q(u_2'(s)), u'(s) \rangle ds.$$

By using the same computations as in the above part we obtain from (3.71) that

$$(3.73) \quad \sigma(t) \leq 2 \left(1 + 2\|k\|_{L^2(0,T)}^2 + 2|k(0)| + \|k'\|_{L^1(0,T)}^2 \right) \int_0^t \sigma(r) dr.$$

By Gronwall's lemma, we deduce that $\sigma(t) \equiv 0$ and Theorem 3.2 is completely proved. ■

Theorem 3.4. *Let $T > 0$ and $(H_1) - (H_4)$ hold. Then there exists $T_1 \in (0, T)$ such that problem (1.1)- (1.3) has a unique weak solution $u \in W(T_1)$ and such that*

$$(3.74) \quad u'', u_{xx} \in L^q(0, T_1; (H^1)').$$

Proof.

For each $T_1 > 0$, we put

$$(3.75) \quad W_1(T_1) = \{v \in L^\infty(0, T_1; H^1) : v_t \in L^\infty(0, T_1; L^2)\}.$$

Then $W_1(T_1)$ is a Banach space with respect to the norm (see[5]):

$$(3.76) \quad \|v\|_{W_1(T_1)} = \|v\|_{L^\infty(0, T_1; H^1)} + \|v_t\|_{L^\infty(0, T_1; L^2)}, \quad v \in W_1(T_1).$$

For $M > 0$ and $T_1 > 0$, we put.

$$(3.77) \quad B(M, T_1) = \{v \in W_1(T_1) : \|v\|_{W_1(T_1)} \leq M\}.$$

We also define the operator \mathcal{F} from $B(M, T_1)$ into $W(T_1)$ by $u = \mathcal{F}(v)$, where u is the unique solution of problem (3.3)- (3.5). We would like to show that \mathcal{F} is a contraction operator from $B(M, T_1)$ into itself. Applying the contraction mapping theorem, the operator \mathcal{F} has a fixed point in $B(M, T_1)$ that is also a weak solution of the problem (1.1)- (1.3).

First, by theorem 3.2, we note that the unique solution of problem (3.3)- (3.5) satisfies (3.9), (3.10). On the other hand, it follows from (H_3) , that

$$(3.78) \quad \begin{aligned} \int_0^t \|f(\cdot, s, v(s))\|^2 ds &\leq 2 \int_0^t \|f(\cdot, s, v(s)) - f(\cdot, s, 0)\|^2 ds + 2 \int_0^t \|f(\cdot, s, 0)\|^2 ds \\ &\leq 2T_1 K_1^2 M^2 + 2 \int_0^T \|f(\cdot, s, 0)\|^2 ds, \end{aligned}$$

where

$$(3.79) \quad K_1 = K_1(M, T, f) = \sup\{|D_3 f(x, t, u)| : 0 \leq x \leq 1, 0 \leq t \leq T, |u| \leq \sqrt{2}M\}.$$

It follows from (3.9)-(3.11) and (3.78) that

$$(3.80) \quad \|u'(t)\|^2 + \|u(t)\|_\eta^2 + 2 \int_0^t \|u'(s)\|_{L^q}^q ds \leq (C_{1T} + 2T_1 K_1^2 M^2) \exp(T_1 C_{2T}), \quad \forall t \in [0, T_1],$$

where

$$(3.81) \quad \begin{cases} C_{1T} = C_{1T}(\tilde{u}_0, \tilde{u}_1, k, g) = 2 \left[\|\tilde{u}_1\|^2 + \|\tilde{u}_0\|_\eta^2 + 2|g_1(0)\tilde{u}_0(1)| + 6\|g_1\|_{L^\infty(0,T)}^2 \right. \\ \quad \left. + 2\|g_1'\|_{L^2(0,T)}^2 + 2 \int_0^T \|f(\cdot, s, 0)\|^2 ds \right], \\ C_{2T} = C_{2T}(k) = 2 \left[3 + 2|k(0)| + 6\|k\|_{L^2(0,T)}^2 + T\|k'\|_{L^2(0,T)}^2 \right]. \end{cases}$$

By choosing $M > 0$ large enough so that $C_{1T} \leq \frac{1}{4}M^2$, then T_1 sufficiently small so that

$$(3.82) \quad \left(\frac{1}{4}M^2 + 2T_1 K_1^2 M^2 \right) \exp(T_1 C_{2T}) \leq \frac{1}{2}M^2,$$

and

$$(3.83) \quad 2\sqrt{2T_1} K_1 \exp \left[T_1 \left(2 + 2|k(0)| + 2\|k\|_{L^2(0,T)}^2 + \|k'\|_{L^1(0,T)}^2 \right) \right] < 1.$$

From (3.80), (3.82) we have $\|u\|_{W_1(T_1)} \leq M$, hence $u \in B(M, T_1)$. This shows that \mathcal{F} maps $B(M, T_1)$ into itself.

Next, we verify that \mathcal{F} is a contraction. Let $u_1 = \mathcal{F}(v_1)$, $u_2 = \mathcal{F}(v_2)$, where $v_1,$

$v_2 \in B(M, T_1)$. Put $U = u_1 - u_2$ and $V = v_1 - v_2$. Then U is the weak solution of the following problem

$$(3.84) \quad \begin{cases} U'' - U_{xx} + \int_0^t k(t-s)U_{xx}(s)ds + \Psi_q(u_1') - \Psi_q(u_2') \\ \quad = f(x, t, v_1(t)) - f(x, t, v_2(t)), 0 < x < 1, 0 < t < T_1, \\ U_x(0, t) - U(0, t) = U_x(1, t) + \eta U(1, t) = 0, \\ U(0) = U'(0) = 0, \\ U \in W(T_1); U'', U_{xx} \in L^q(0, T_1; (H^1)'). \end{cases}$$

By using Lemma 3.3 with $\tilde{u}_0 = \tilde{u}_1 = 0$, $\Phi = -\Psi_q(u_1') + \Psi_q(u_2') + f(x, t, v_1(t)) - f(x, t, v_2(t))$, we have

$$(3.85) \quad \begin{aligned} \delta(t) &= -2k(0) \int_0^t \|U(r)\|_{\eta}^2 dr + 2 \int_0^t k(t-s) a(U(s), U(t)) ds \\ &\quad - 2 \int_0^t dr \int_0^r k'(r-s) a(U(s), U(r)) ds \\ &\quad + 2 \int_0^t \langle f(\cdot, s, v_1(s)) - f(\cdot, s, v_2(s)), U'(s) \rangle ds, \text{ a.e. } t \in [0, T_1], \end{aligned}$$

where

$$(3.86) \quad \begin{aligned} \delta(t) &= \|U'(t)\|^2 + \|U(t)\|_{\eta}^2 + 2 \int_0^t \langle \Psi_q(u_1') - \Psi_q(u_2'), U'(s) \rangle ds \\ &\geq \|U'(t)\|^2 + \|U(t)\|_{\eta}^2 + 2C_q \int_0^t \|U'(s)\|_{L^q}^q ds. \end{aligned}$$

By the assumption (H_4) , we have

$$(3.87) \quad \begin{aligned} 2 \int_0^t \langle f(\cdot, s, v_1(s)) - f(\cdot, s, v_2(s)), U'(s) \rangle ds &\leq \int_0^t \|U'(s)\|^2 ds \\ &\quad + \int_0^t \|f(\cdot, s, v_1(s)) - f(\cdot, s, v_2(s))\|^2 ds \\ &\leq \int_0^t \|U'(s)\|^2 ds + 2T_1 K_1^2 \|V\|_{W_1(T_1)}^2, \end{aligned}$$

Therefore, we can prove in a similar manner as above that

$$(3.88) \quad \delta(t) \leq 2T_1 K_1^2 \|V\|_{W_1(T_1)}^2 + 2 \left(2 + 2|k(0)| + 2\|k\|_{L^2(0, T)}^2 + \|k'\|_{L^1(0, T)} \right) \int_0^t \delta(s) ds.$$

By Gronwall's lemma, we obtain from (3.88) that

$$(3.89) \quad \delta(t) \leq 2 \left(\rho_1(k, K_1, T, T_1) \|V\|_{W_1(T_1)} \right)^2,$$

where

$$(3.90) \quad \rho_1(k, K_1, T, T_1) = \sqrt{2T_1} K_1 \exp \left[T_1 \left(2 + 2|k(0)| + 2\|k\|_{L^2(0, T)}^2 + \|k'\|_{L^1(0, T)} \right) \right].$$

It follows from (3.86), (3.89) and (3.90) that

$$(3.91) \quad \|U\|_{W_1(T_1)} \leq 2\rho_1(k, K_1, T, T_1) \|V\|_{W_1(T_1)},$$

where

$$(3.92) \quad 2\rho_1(k, K_1, T, T_1) < 1,$$

since (3.83) and (3.90).

Hence, (3.91) shows that $\mathcal{F} : B(M, T_1) \rightarrow B(M, T_1)$ is a contraction. Applying the contraction mapping theorem, the operator \mathcal{F} has a fixed point in $B(M, T_1)$ that is also a weak

solution of the problem (1.1)- (1.3).

The solution of the problem (1.1)- (1.3) is unique, that can be showed using the same arguments as in the proof of Theorem 3.2. The proof of Theorem 3.4 is completed. ■

Remark 3.3. In the case of $\lambda = 0, f(x, t, u) = |u|^{p-2}u, p > 2, k \in W^{2,1}(\mathbb{R}_+), k \geq 0, k(0) > 0, 0 < \int_0^{+\infty} k(t)dt < 1, k'(t) + \zeta k(t) \leq 0$ for all $t \geq 0$, with $\zeta > 0$, and the boundary condition $u(0, t) = u(1, t) = 0$ standing for (1.2), S. Berrimia, S. A. Messaoudi [1] has obtained a global existence and uniqueness theorem.

4. DECAY OF SOLUTIONS

In this part, we will consider the problem of global existence and asymptotic behavior for $t \rightarrow +\infty$. We assume that $g(t) \equiv 0, f(x, t, u) = F(x, t) - |u|^{p-2}u, p \geq 2$ and consider the following problem

$$(4.1) \quad \begin{cases} u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + |u|^{p-2}u + |u_t|^{q-2}u_t = F(x, t), & 0 < x < 1, t > 0, \\ u_x(0, t) = u(0, t), \quad u_x(1, t) + \eta u(1, t) = 0, \\ u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x), \end{cases}$$

We make the following assumptions:

$$(H'_1) \quad \eta \geq 0, p, q \geq 2,$$

$$(H'_2) \quad k \in W^{2,1}(\mathbb{R}_+), k \geq 0, \text{ satisfying}$$

$$(i) \quad k(0) > 0, 0 < 1 - \int_0^{+\infty} k(t)dt \equiv k_\infty < 1,$$

$$(ii) \quad \text{there exists a positive constant } \zeta \text{ such that } k'(t) + \zeta k(t) \leq 0 \text{ for all } t \geq 0,$$

$$(H'_3) \quad \tilde{u}_0 \in H^2 \text{ and } \tilde{u}_1 \in H^1,$$

$$(H'_4) \quad F \in L^1(0, \infty; L^2) \cap L^2(0, \infty; L^2), F_t \in L^1(0, \infty; L^2),$$

$$(H'_5) \quad \text{There exists a constant } \sigma > 0 \text{ such that } \int_0^{+\infty} e^{\sigma t} \|F(t)\|^2 dt < +\infty.$$

Under assumptions $(H'_1) - (H'_4)$ and let $T > 0$, by theorem 2.3, the problem (4.1) has a unique weak solution $u(t)$ such that

$$(4.2) \quad u \in L^\infty(0, T; H^2), u_t \in L^\infty(0, T; H^1), u_{tt} \in L^\infty(0, T; L^2).$$

Then, we have the following

Lemma 4.1. *Suppose that $(H'_1) - (H'_4)$ hold. Then there is a unique solution $u(t)$ of problem (4.1) defined on \mathbb{R}_+ .*

Moreover

$$(4.3) \quad \|u'(t)\| + \|u(t)\|_\eta \leq C \text{ for all } t \geq 0,$$

where C is a positive constant depending only on $\tilde{u}_0, \tilde{u}_1, F, k_\infty$ and p .

Proof. By multiplying the equation (4.1)₁ by u_t and integrate over $(0, 1) \times (0, t)$ we obtain

$$(4.4) \quad \begin{aligned} E(t) + 2 \int_0^t \|u'(s)\|_{L^q}^q ds + \int_0^t k(s) \|u(s)\|_\eta^2 ds \\ - \int_0^t dr \int_0^r k'(r-s) \|u(s) - u(r)\|_\eta^2 ds = E(0) + 2 \int_0^t \langle F(s), u'(s) \rangle ds, \end{aligned}$$

where

$$(4.5) \quad E(t) = \|u'(t)\|^2 + \left(1 - \int_0^t k(s)ds\right) \|u(t)\|_\eta^2 + \frac{2}{p} \|u(t)\|_{L^p}^p + \int_0^t k(t-s) \|u(s) - u(t)\|_\eta^2 ds.$$

On the other hand, by (H_4') and the Cauchy's inequality, we obtain

$$(4.6) \quad 2 \int_0^t \langle F(s), u'(s) \rangle ds \leq \int_0^t \|F(s)\| ds + \int_0^t \|F(s)\| \|u'(s)\|^2 ds \leq \int_0^{+\infty} \|F(s)\| ds + \int_0^t \|F(s)\| E(s) ds.$$

By Gronwall's lemma, we obtain from (4.4) and (4.6) that

$$(4.7) \quad E(t) \leq \left(E(0) + \int_0^{+\infty} \|F(s)\| ds\right) \exp\left(\int_0^t \|F(s)\| ds\right) \leq \left(E(0) + \int_0^{+\infty} \|F(s)\| ds\right) \exp\left(\int_0^{+\infty} \|F(s)\| ds\right) = C, \text{ for all } t \geq 0.$$

By (A_3, i) , we have

$$(4.8) \quad E(t) \geq \|u'(t)\|^2 + \left(1 - \int_0^t k(s)ds\right) \|u(t)\|_\eta^2 \geq \|u'(t)\|^2 + k_\infty \|u(t)\|_\eta^2.$$

Then we obtain (4.3) from (4.7) and (4.8). This completes the proof of Lemma 4.1. ■

In this section we state and prove decay result.

Theorem 4.2. *Suppose that (H_1') – (H_5') hold. Then the solution $u(t)$ of problem (4.1) decays exponentially to zero as $t \rightarrow +\infty$ in the following sense: there exist the positive constants N and γ such that*

$$(4.9) \quad \|u'(t)\| + \|u(t)\|_\eta \leq Ne^{-\gamma t} \text{ for all } t \geq 0.$$

Proof. We use the following functional

$$(4.10) \quad \Gamma(t) = \Gamma(\varepsilon_1, \varepsilon_2, t) = E(t) + \varepsilon_1 E_1(t) + \varepsilon_2 E_2(t),$$

where

$$(4.11) \quad E_1(t) = \langle u(t), u'(t) \rangle,$$

$$(4.12) \quad E_2(t) = - \int_0^t k(t-s) \langle u'(t), u(t) - u(s) \rangle ds.$$

Estimating $\Gamma(t)$.

By (2.3), (2.4), we obtain from (H_2', i) that

$$(4.13) \quad |E_1(t)| = |\langle u(t), u'(t) \rangle| \leq \frac{1}{2} \|u'(t)\|^2 + \|u(t)\|_\eta^2,$$

$$(4.14) \quad |E_2(t)| = \left| \int_0^t k(t-s) \langle u'(t), u(t) - u(s) \rangle ds \right| \leq \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} \left(\int_0^t k(t-s) \|u(t) - u(s)\| ds \right)^2 \leq \frac{1}{2} \|u'(t)\|^2 + (1 - k_\infty) \int_0^t k(t-s) \|u(t) - u(s)\|_\eta^2 ds.$$

Hence, it follows from (4.10)-(4.14) that for $\varepsilon_1, \varepsilon_2$ small enough, there exist two positive constants α_1, α_2 , such that

$$(4.15) \quad \alpha_1 E(t) \leq \Gamma(t) \leq \alpha_2 E(t).$$

Estimating $\Gamma'(t)$.

Now differentiating (4.4) with respect to t , we have

$$(4.16) \quad \begin{aligned} E'(t) &= -2\|u'(t)\|_{L^q}^q + \int_0^t k'(t-s)\|u(s) - u(t)\|_\eta^2 ds - k(t)\|u(t)\|_\eta^2 + 2\langle F(t), u'(t) \rangle \\ &\leq -2\|u'(t)\|_{L^q}^q + \int_0^t k'(t-s)\|u(s) - u(t)\|_\eta^2 ds + 2\langle F(t), u'(t) \rangle, \end{aligned}$$

since $k(t) \geq 0$.

By multiplying the equation (4.1)₁ by u and integrate over $(0, 1)$ we obtain

$$(4.17) \quad \begin{aligned} E'_1(t) &= \|u'(t)\|^2 - \|u(t)\|_\eta^2 - \|u(t)\|_{L^p}^p + \langle F(t), u(t) \rangle \\ &\quad + \int_0^t k(t-s)a(u(s), u(t))ds - \langle |u'(t)|^{q-2}u'(t), u(t) \rangle \\ &= \|u'(t)\|^2 - \|u(t)\|_\eta^2 - \|u(t)\|_{L^p}^p + \langle F(t), u(t) \rangle + I_1(t) + I_2(t). \end{aligned}$$

We now estimate the last two terms in the right side of (4.17) as follows

Estimating $I_1(t)$.

Using the inequality

$$(4.18) \quad ab \leq \frac{\delta^r}{r} a^r + \frac{r-1}{r} \delta^{-\frac{r}{r-1}} b^{\frac{r}{r-1}}, \quad \forall a, b \geq 0, \forall r > 1, \forall \delta > 0,$$

we have

$$(4.19) \quad \begin{aligned} I_1(t) &= \int_0^t k(t-s)a(u(s), u(t))ds \\ &= \int_0^t k(t-s)a(u(s) - u(t), u(t))ds + \int_0^t k(t-s)\|u(t)\|_\eta^2 ds \\ &\leq \delta_1 \|u(t)\|_\eta^2 + \frac{1}{4\delta_1} \left(\int_0^t k(s)ds \right) \left(\int_0^t k(t-s)\|u(s) - u(t)\|_\eta^2 ds \right) \\ &\quad + \left(\int_0^t k(s)ds \right) \|u(t)\|_\eta^2 \\ &\leq \delta_1 \|u(t)\|_\eta^2 + \frac{1-k_\infty}{4\delta_1} \int_0^t k(t-s)\|u(s) - u(t)\|_\eta^2 ds + (1 - k_\infty) \|u(t)\|_\eta^2 \\ &\leq (\delta_1 + 1 - k_\infty) \|u(t)\|_\eta^2 + \frac{1-k_\infty}{4\delta_1} \int_0^t k(t-s)\|u(s) - u(t)\|_\eta^2 ds, \end{aligned}$$

for all $\delta_1 > 0$.

Estimating $I_2(t)$.

We again use inequality (4.18) we obtain from (4.3) that

$$(4.20) \quad \begin{aligned} I_2(t) &= -\langle |u'(t)|^{q-2}u'(t), u(t) \rangle \leq \|u'(t)\|_{L^q}^{q-1} \|u(t)\|_{L^q} \\ &\leq \frac{\delta_1^q}{q} \|u(t)\|_{L^q}^q + \frac{q-1}{q} \delta_1^{-\frac{q}{q-1}} \|u'(t)\|_{L^q}^q \\ &\leq 2 \frac{\delta_1^q}{q} (\sqrt{2} C)^{q-2} \|u(t)\|_\eta^2 + \frac{q-1}{q} \delta_1^{-\frac{q}{q-1}} \|u'(t)\|_{L^q}^q, \end{aligned}$$

for all $\delta_1 > 0$.

By combining (4.17), (4.19) and (4.20), we obtain

$$(4.21) \quad \begin{aligned} E'_1(t) &\leq -\|u(t)\|_{L^p}^p + \|u'(t)\|^2 - \left(k_\infty - \delta_1 - 2 \frac{\delta_1^q}{q} (\sqrt{2} C)^{q-2} \right) \|u(t)\|_\eta^2 \\ &\quad + \frac{q-1}{q} \delta_1^{-\frac{q}{q-1}} \|u'(t)\|_{L^q}^q + \frac{1-k_\infty}{4\delta_1} \int_0^t k(t-s)\|u(s) - u(t)\|_\eta^2 ds + \langle F(t), u(t) \rangle. \end{aligned}$$

Then, we can always choose the constant $\delta_1 > 0$ such that

$$(4.22) \quad \gamma_1 = k_\infty - \delta_1 - 2 \frac{\delta_1^q}{q} (\sqrt{2} C)^{q-2} > 0.$$

This implies that

$$(4.23) \quad E_1'(t) \leq -\|u(t)\|_{L^p}^p + \|u'(t)\|^2 - \gamma_1 \|u(t)\|_\eta^2 + \gamma_2 \|u'(t)\|_{L^q}^q \\ + \gamma_3 \int_0^t k(t-s) \|u(s) - u(t)\|_\eta^2 ds + \langle F(t), u(t) \rangle,$$

where

$$(4.24) \quad \gamma_2 = \frac{q-1}{q} \delta_1^{\frac{-q}{q-1}}, \quad \gamma_3 = \frac{1-k_\infty}{4\delta_1}.$$

Direct calculations give

$$(4.25) \quad E_2'(t) = -\left(\int_0^t k(s) ds\right) \|u'(t)\|^2 - \int_0^t k'(t-s) \langle u'(t), u(t) - u(s) \rangle ds \\ + \int_0^t k(t-s) a(u(t), u(t) - u(s)) ds \\ - \int_0^t k(t-s) a\left(\int_0^t k(t-\tau) u(\tau) d\tau, u(t) - u(s)\right) ds \\ + \int_0^t k(t-s) \langle |u(t)|^{p-2} u(t), u(t) - u(s) \rangle ds \\ + \int_0^t k(t-s) \langle |u'(t)|^{q-2} u'(t), u(t) - u(s) \rangle ds \\ - \int_0^t k(t-s) \langle F(t), u(t) - u(s) \rangle ds \\ = \sum_{i=1}^7 J_i(t).$$

Similarly to (4.17), we estimate respectively the following terms on the right-hand side of (4.25) as follows.

Estimating $J_1(t)$. Since k is continuous and $k(0) > 0$ then there exists $t_0 > 0$, such that

$$(4.26) \quad \int_0^t k(s) ds \geq \int_0^{t_0} k(s) ds = k_0 > 0 \text{ for all } t \geq t_0.$$

Hence,

$$(4.27) \quad J_1(t) = -\left(\int_0^t k(s) ds\right) \|u'(t)\|^2 \leq -k_0 \|u'(t)\|^2 \text{ for all } t \geq t_0.$$

Estimating $J_2(t)$.

$$(4.28) \quad J_2(t) = -\int_0^t k'(t-s) \langle u'(t), u(t) - u(s) \rangle ds \\ \leq \delta_2 \|u'(t)\|^2 + \frac{1}{4\delta_2} \left(\int_0^t |k'(t-s)| ds\right) \left(\int_0^t |k'(t-s)| \|u(s) - u(t)\|^2 ds\right) \\ \leq \delta_2 \|u'(t)\|^2 + \frac{1}{2\delta_2} \left(\int_0^t |k'(t-s)| ds\right) \left(\int_0^t |k'(t-s)| \|u(s) - u(t)\|_\eta^2 ds\right) \\ \leq \delta_2 \|u'(t)\|^2 - \frac{k(0)}{2\delta_2} \int_0^t k'(t-s) \|u(s) - u(t)\|_\eta^2 ds.$$

Estimating $J_3(t)$.

$$(4.29) \quad J_3(t) = \int_0^t k(t-s) a(u(t), u(t) - u(s)) ds \\ \leq \delta_2 \|u(t)\|_\eta^2 + \frac{1}{4\delta_2} \left(\int_0^t k(s) ds\right) \left(\int_0^t k(t-s) \|u(s) - u(t)\|_\eta^2 ds\right) \\ \leq \delta_2 \|u(t)\|_\eta^2 + \frac{1-k_\infty}{4\delta_2} \int_0^t k(t-s) \|u(s) - u(t)\|_\eta^2 ds.$$

Estimating $J_4(t)$.

$$\begin{aligned}
(4.30) \quad J_4(t) &= -\int_0^t k(t-s)a \left(\int_0^t k(t-\tau)u(\tau)d\tau, u(t) - u(s) \right) ds \\
&\leq \int_0^t k(t-\tau) \|u(\tau)\|_\eta d\tau \int_0^t k(t-s) \|u(s) - u(t)\|_\eta ds \\
&\leq \delta_2 \left(\int_0^t k(t-\tau) \|u(\tau)\|_\eta d\tau \right)^2 + \frac{1}{4\delta_2} \left(\int_0^t k(t-s) \|u(s) - u(t)\|_\eta ds \right)^2 \\
&\leq 2\delta_2 \left(\int_0^t k(t-\tau) \|u(t)\|_\eta d\tau \right)^2 + \left(2\delta_2 + \frac{1}{4\delta_2} \right) \left(\int_0^t k(t-s) \|u(s) - u(t)\|_\eta ds \right)^2 \\
&\leq 2\delta_2(1 - k_\infty)^2 \|u(t)\|_\eta^2 + \left(2\delta_2 + \frac{1}{4\delta_2} \right) (1 - k_\infty) \int_0^t k(t-s) \|u(s) - u(t)\|_\eta^2 ds.
\end{aligned}$$

Estimating $J_5(t)$.

$$\begin{aligned}
(4.31) \quad J_5(t) &= \int_0^t k(t-s) \langle |u(t)|^{p-2} u(t), u(t) - u(s) \rangle ds \\
&\leq 2(\sqrt{2}C)^{p-2} \int_0^t k(t-s) \|u(t)\|_\eta \|u(t) - u(s)\|_\eta ds \\
&\leq 2(\sqrt{2}C)^{p-2} \left[\delta_2 \|u(t)\|_\eta^2 + \frac{1}{4\delta_2} \left(\int_0^t k(t-s) \|u(t) - u(s)\|_\eta ds \right)^2 \right] \\
&\leq 2(\sqrt{2}C)^{p-2} \left[\delta_2 \|u(t)\|_\eta^2 + \frac{1}{4\delta_2} (1 - k_\infty) \int_0^t k(t-s) \|u(t) - u(s)\|_\eta^2 ds \right].
\end{aligned}$$

Estimating $J_6(t)$. We again use inequality (4.18) with $r = q$, $\delta = \delta_2$, we obtain from (4.3) that

$$\begin{aligned}
(4.32) \quad \langle |u'(t)|^{q-2} u'(t), u(t) - u(s) \rangle &\leq \|u'(t)\|_{L^q}^{q-1} \|u(t) - u(s)\|_{L^q} \\
&\leq \frac{\delta_2^q}{q} \|u(t) - u(s)\|_{L^q}^q + \frac{q-1}{q} \delta_2^{-\frac{q-1}{q}} \|u'(t)\|_{L^q}^q \\
&\leq 2\frac{\delta_2^q}{q} (2\sqrt{2}C)^{q-2} \|u(t) - u(s)\|_\eta^2 + \frac{q-1}{q} \delta_2^{-\frac{q-1}{q}} \|u'(t)\|_{L^q}^q.
\end{aligned}$$

It follows from (4.32) that

$$\begin{aligned}
(4.33) \quad J_6(t) &= \int_0^t k(t-s) \langle |u'(t)|^{q-2} u'(t), u(t) - u(s) \rangle ds \\
&\leq 2\frac{\delta_2^q}{q} (2\sqrt{2}C)^{q-2} \int_0^t k(t-s) \|u(t) - u(s)\|_\eta^2 ds + \frac{q-1}{q} \delta_2^{-\frac{q-1}{q}} \|u'(t)\|_{L^q}^q \int_0^t k(t-s) ds \\
&\leq 2\frac{\delta_2^q}{q} (2\sqrt{2}C)^{q-2} \int_0^t k(t-s) \|u(t) - u(s)\|_\eta^2 ds + \frac{q-1}{q} \delta_2^{-\frac{q-1}{q}} (1 - k_\infty) \|u'(t)\|_{L^q}^q.
\end{aligned}$$

Estimating $J_7(t)$.

$$\begin{aligned}
(4.34) \quad J_7(t) &= -\int_0^t k(t-s) \langle F(t), u(t) - u(s) \rangle ds \leq \int_0^t k(t-s) \|F(t)\| \|u(t) - u(s)\| ds \\
&\leq \frac{1}{4\delta_2} \|F(t)\|^2 + \delta_2 \int_0^t k(t-s) ds \left(\int_0^t k(t-s) \|u(t) - u(s)\|^2 ds \right) \\
&\leq \frac{1}{4\delta_2} \|F(t)\|^2 + 2\delta_2(1 - k_\infty) \int_0^t k(t-s) \|u(t) - u(s)\|_\eta^2 ds.
\end{aligned}$$

By combining (4.25), (4.27)-(4.31), (4.33) and (4.34), we obtain

$$\begin{aligned}
(4.35) \quad E_2'(t) &\leq -(k_0 - \delta_2) \|u'(t)\|^2 + \delta_2 \hat{\gamma}_1 \|u(t)\|_\eta^2 + \hat{\gamma}_2 \|u'(t)\|_{L^q}^q \\
&\quad + \hat{\gamma}_3 \int_0^t k(t-s) \|u(t) - u(s)\|_\eta^2 ds \\
&\quad - \hat{\gamma}_4 \int_0^t k'(t-s) \|u(s) - u(t)\|_\eta^2 ds + \frac{1}{4\delta_2} \|F(t)\|^2,
\end{aligned}$$

where

$$(4.36) \quad \begin{cases} \hat{\gamma}_1 = 1 + 2(1 - k_\infty)^2 + 2(\sqrt{2}C)^{p-2}, & \hat{\gamma}_2 = \frac{q-1}{q}\delta_2^{\frac{-q}{q-1}}(1 - k_\infty), \\ \hat{\gamma}_3 = 2\frac{\delta_2^q}{q}(2\sqrt{2}C)^{q-2} + (1 - k_\infty)\left[\frac{1}{2\delta_2}(\sqrt{2}C)^{p-2} + \left(4\delta_2 + \frac{1}{2\delta_2}\right)\right], \\ \hat{\gamma}_4 = \frac{k(0)}{2\delta_2}. \end{cases}$$

Combining of (4.10), (4.16), (4.23) and (4.35), we obtain

$$(4.37) \quad \begin{aligned} & \Gamma'(t) + \varepsilon_1 \|u(t)\|_{L^p}^p + \left((k_0 - \delta_2)\varepsilon_2 - \varepsilon_1\right) \|u'(t)\|^2 \\ & + (\varepsilon_1\gamma_1 - \varepsilon_2\delta_2\hat{\gamma}_1) \|u(t)\|_\eta^2 + \left(2 - \varepsilon_1\gamma_2 - \varepsilon_2\hat{\gamma}_2\right) \|u'(t)\|_{L^q}^q \\ & - (\varepsilon_1\gamma_3 + \varepsilon_2\hat{\gamma}_3) \int_0^t k(t-s) \|u(t) - u(s)\|_\eta^2 ds \\ & - (1 - \varepsilon_2\hat{\gamma}_4) \int_0^t k'(t-s) \|u(s) - u(t)\|_\eta^2 ds \\ & \leq \langle F(t), 2u'(t) + \varepsilon_1 u(t) \rangle + \frac{\varepsilon_2}{4\delta_2} \|F(t)\|^2. \end{aligned}$$

Whence δ_1 is fixed, choosing

$$(4.38) \quad \delta_2 = \frac{1}{2} \frac{k_0\gamma_1}{\gamma_1 + \hat{\gamma}_1}, \quad \varepsilon_2 = \frac{2}{k_0} \varepsilon_1, \text{ where } \varepsilon_1 > 0 \text{ is arbitrary,}$$

we deduce from (4.37) and (4.38) that

$$(4.39) \quad \begin{aligned} & \Gamma'(t) + \varepsilon_1 \|u(t)\|_{L^p}^p + \frac{\varepsilon_1\hat{\gamma}_1}{\gamma_1 + \hat{\gamma}_1} \|u'(t)\|^2 + \frac{\varepsilon_1\gamma_1^2}{\gamma_1 + \hat{\gamma}_1} \|u(t)\|_\eta^2 \\ & + \left(2 - \varepsilon_1\left(1 + \frac{2}{k_0}\hat{\gamma}_2\right)\right) \|u'(t)\|_{L^q}^q \\ & - \varepsilon_1\left(\gamma_3 + \frac{2}{k_0}\hat{\gamma}_3\right) \int_0^t k(t-s) \|u(t) - u(s)\|_\eta^2 ds \\ & - \left(1 - \frac{2}{k_0}\varepsilon_1\hat{\gamma}_4\right) \int_0^t k'(t-s) \|u(s) - u(t)\|_\eta^2 ds \\ & \leq \langle F(t), 2u'(t) + \varepsilon_1 u(t) \rangle + \frac{\varepsilon_1}{k_0^2} \left(1 + \frac{\hat{\gamma}_1}{\gamma_1}\right) \|F(t)\|^2. \end{aligned}$$

Next, we choose $\varepsilon_1 > 0$, with $\varepsilon_1 < \min\left\{\frac{\zeta}{\gamma_3 + \frac{2}{k_0}\hat{\gamma}_3 + \frac{2}{k_0}\hat{\gamma}_4\zeta}, \frac{2}{1 + \frac{2}{k_0}\hat{\gamma}_2}\right\}$ and (4.15) is satisfied,

then by the assumption (H_2', ii) , we deduce that

$$(4.40) \quad \begin{aligned} & \Gamma'(t) + \varepsilon_1 \|u(t)\|_{L^p}^p + \frac{\varepsilon_1\hat{\gamma}_1}{\gamma_1 + \hat{\gamma}_1} \|u'(t)\|^2 + \frac{\varepsilon_1\gamma_1^2}{\gamma_1 + \hat{\gamma}_1} \|u(t)\|_\eta^2 + k_1 \|u'(t)\|_{L^q}^q \\ & + k_2 \int_0^t k(t-s) \|u(s) - u(t)\|_\eta^2 ds \\ & \leq \langle F(t), 2u'(t) + \varepsilon_1 u(t) \rangle + k_3 \|F(t)\|^2, \end{aligned}$$

where

$$(4.41) \quad \begin{cases} k_1 = 2 - \varepsilon_1\left(1 + \frac{2}{k_0}\hat{\gamma}_2\right) > 0, & k_2 = \zeta\left(1 - \frac{2}{k_0}\varepsilon_1\hat{\gamma}_4\right) - \varepsilon_1\left(\gamma_3 + \frac{2}{k_0}\hat{\gamma}_3\right) > 0, \\ k_3 = \frac{\varepsilon_1}{k_0^2}\left(1 + \frac{\hat{\gamma}_1}{\gamma_1}\right). \end{cases}$$

By combining (4.5), (4.15) and (4.40), we can always choose the constant $\tilde{\gamma} > 0$ is independent of t such that

$$(4.42) \quad \Gamma'(t) + 2\tilde{\gamma}\Gamma(t) \leq \langle F(t), 2u'(t) + \varepsilon_1 u(t) \rangle + k_3 \|F(t)\|^2 \text{ for all } t \geq t_0.$$

On the other hand,

$$(4.43) \quad \langle F(t), 2u'(t) + \varepsilon_1 u(t) \rangle + k_3 \|F(t)\|^2 \leq \tilde{N} \|F(t)\|^2 + \tilde{\gamma} \Gamma(t),$$

for some constant $\tilde{N} > 0$. Therefore

$$(4.44) \quad \Gamma'(t) + \tilde{\gamma} \Gamma(t) \leq \tilde{N} \|F(t)\|^2 \quad \text{for all } t \geq t_0.$$

Putting $\gamma = \frac{1}{2} \min\{\sigma, \tilde{\gamma}\}$. A simple integration of (4.44) over (t_0, t) gives

$$(4.45) \quad \Gamma(t) \leq \left[\Gamma(t_0) e^{\sigma t_0} + \tilde{N} \int_{t_0}^{+\infty} e^{\sigma s} \|F(s)\|^2 ds \right] e^{-2\gamma t} \equiv N_1 e^{-2\gamma t}, \quad \text{for all } t \geq t_0.$$

By the boundedness of $\Gamma(t)$ on $[0, t_0]$, we deduce from (4.45) that

$$(4.46) \quad \Gamma(t) \leq \|\Gamma\|_{L^\infty(0, t_0)} e^{-2\gamma(t-t_0)} + N_1 e^{-2\gamma t} = N_2 e^{-2\gamma t}, \quad \text{for all } t \geq 0.$$

By (4.15), it follows from (4.46) that

$$(4.47) \quad E(t) \leq \frac{1}{\alpha_1} \Gamma(t) \leq \frac{1}{\alpha_1} N_2 e^{-2\gamma t}, \quad \text{for all } t \geq 0.$$

This completes the proof of Theorem 4.2. ■

Remark 4.1. The estimate (4.9) holds for any regular solution corresponding to $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^1$. This remains holds for solutions corresponding to $(\tilde{u}_0, \tilde{u}_1) \in H^1 \times L^2$ by simple density argument.

REFERENCES

- [1] S. Berrimia, S. A. Messaoudi, *Existence and decay of solutions of a viscoelastic equation with a nonlinear source*, Nonlinear Anal. **64** (10) (2006) 2314-2331.
- [2] M. Bergounioux, N.T. Long, A.P.N. Dinh, *Mathematical model for a shock problem involving a linear viscoelastic bar*, Nonlinear Anal. **43** (2001) 547-561.
- [3] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.A. Soriano, *Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping*, Electron. J. Differential Equations **44** (2002) 1–14.
- [4] J.L. Lions, W.A. Strauss, *Some nonlinear evolution equations*, Bull. Soc. Math., France, **93** (1965) 43-96.
- [5] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites nonlinéaires*, Dunod; Gauthier- Villars, Paris. 1969.
- [6] N.T. Long, A.P.N. Dinh, T.N. Diem, *Linear recursive schemes and asymptotic expansion associated with the Kirchhoff- Carrier operator*, J. Math. Anal. Appl. **267** (2002) 116-134.
- [7] N.T. Long, A.P.N. Dinh, T.N. Diem, *On a shock problem involving a nonlinear viscoelastic bar*, J. Boundary Value Problems, Hindawi Publishing Corporation, **2005** (3) (2005) 337–358.
- [8] N.T. Long, L.V. Ut, N.T.T. Truc, *On a shock problem involving a linear viscoelastic bar*, Nonlinear Anal. **63** (2) (2005) 198–224.
- [9] N.T. Long, V.G. Giai, *A wave equation associated with mixed nonhomogeneous conditions: Global existence and asymptotic expansion of solutions*, Nonlinear Analysis, Theory, Methods & Applications, Series A: Theory and Methods, **66** (7) (2007), 1526-1546.
- [10] N.T. Long, L.X. Truong, *Existence and asymptotic expansion for a viscoelastic problem with a mixed nonhomogeneous condition*, Nonlinear Analysis, Theory, Methods & Applications, Series A: Theory and Methods, (accepted for publication).