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# An interpolation theorem in toric varieties

WEIMANN Martin

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## Abstract

In the spirit of a theorem of Wood [21], we give necessary and sufficient conditions for a family of germs of analytic hypersurfaces in a smooth projective toric variety  $X$  to be interpolated by an algebraic hypersurface with a fixed class in the Picard group of  $X$ .

## 1 Introduction

Let  $X$  be a compact algebraic variety over  $\mathbb{C}$ . We are interested in the following problem:

*Let  $V_1, \dots, V_N$  be a collection of germs of smooth analytic hypersurfaces at pairwise distinct smooth points  $p_1, \dots, p_N$  of  $X$ , and fix  $\alpha$  in the Picard group  $\text{Pic}(X)$  of  $X$ . When does there exist an algebraic hypersurface  $\tilde{V} \subset X$  with class  $\alpha$  containing all the germs  $V_i$  ?*

A natural way to answer this question is to study sums and products of values of rational functions at points of intersection of the germs  $V_i$  with a "moving" algebraic curve<sup>1</sup>.

Let us recall a theorem of Wood [21] treating the case of  $N$  germs in an affine chart  $\mathbb{C}^n$  of  $X = \mathbb{P}^n$ , transversal to the line  $l_0 = \{x_1 = \dots = x_{n-1} = 0\}$ . Any line  $l_a$  close to  $l_0$  has affine equations  $x_k = a_{k0} + a_{k1}x_n$ ,  $k = 1, \dots, n-1$ . The trace on  $V = V_1 \cup \dots \cup V_N$  of any function  $f$  holomorphic in an analytic neighborhood of  $V$  is the function

$$a \longmapsto \text{Tr}_V(f)(a) := \sum_{p \in V \cap l_a} f(p),$$

defined and holomorphic for  $a = ((a_{10}, a_{11}), \dots, (a_{n-1,0}, a_{n-1,1}))$  close enough to  $0 \in \mathbb{C}^{2n-2}$ .

**Theorem** (Wood, [21]) *There exists an algebraic hypersurface  $\tilde{V} \subset \mathbb{P}^n$  of degree  $N$  which contains  $V$  if and only if the function  $a \mapsto \text{Tr}_V(x_n)(a)$  is affine in the constant coefficients  $a_0 = (a_{10}, \dots, a_{n-1,0})$ .*

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<sup>1</sup>This idea goes back to Abel in his studies of abelian integrals [1].

We show here that Wood's theorem admits a natural generalization to the case of germs  $V_1, \dots, V_N$  in a smooth toric compactification  $X$  of  $\mathbb{C}^n$  endowed with an ample line bundle. While our proof is constructive, we do not obtain (contrarily to [21]) the explicit construction of the polynomial equation of the interpolating hypersurface in the affine chart  $\mathbb{C}^n$ . Thus, in that toric context, we need more informations to characterize the class of  $\tilde{V}$  in  $Pic(X)$ .

For any projective variety  $X$ , there exist very ample line bundles  $L_1, \dots, L_{n-1}$  and a global section  $s_0 \in \Gamma(X, L_1) \oplus \dots \oplus \Gamma(X, L_{n-1})$  whose zero locus is a smooth irreducible curve  $C$  which intersects transversely each germ  $V_i$  at  $p_i$ . A generic point  $a$  in the associated parameter space

$$X^* := \mathbb{P}(\Gamma(X, L_1)) \times \dots \times \mathbb{P}(\Gamma(X, L_{n-1}))$$

determines a closed curve  $C_a$  in  $X$ , which, for  $a$  close enough to the class  $a^0 \in X^*$  of  $s_0$ , is smooth and intersects each germ  $V_i$  transversely at a point  $p_i(a)$  whose coordinates vary holomorphically with  $a$  by the implicit functions theorem. For any function  $f$  holomorphic at  $p_1, \dots, p_N$ , we define the trace of  $f$  on  $V := V_1 \cup \dots \cup V_N$  relatively to  $(L_1, \dots, L_{n-1})$  as the function

$$a \mapsto Tr_V(f)(a) := \sum_{i=1}^N f(p_i(a)),$$

which is defined and holomorphic for  $a$  in an analytic neighborhood of  $a^0$ .

Let us suppose now that  $X$  is a toric projective smooth compactification of  $U = \mathbb{C}^n$ , endowed with a linear action of an algebraic torus  $\mathbb{T}$  that preserves the coordinate hyperplanes  $x_i = 0$ ,  $i = 1, \dots, n$  (see [7]). Clearly, any germ  $V_i$  contained in the hypersurface at infinity  $X \setminus U$  is algebraic. We can thus suppose that  $V$  is contained in  $U$  and work with the affine coordinates  $x = (x_1, \dots, x_n)$ .

Since the Picard group of  $U = \mathbb{C}^n$  is trivial, the classes of the irreducible divisors  $G_1, \dots, G_s$  supported outside  $U$  form a basis for  $Pic(X)$ . Any globally generated line bundle  $L$  on  $X$  has thus a unique global section  $s_U \in \Gamma(X, L)$  such that  $div(s_U) \cap U = \emptyset$ . If  $s \in \Gamma(X, L)$ , the quotient  $\frac{s}{s_U}$  defines a rational function without poles on  $U \simeq \mathbb{C}^n$ , that is, a polynomial in  $x$ , which gives the local equation for the divisor  $H = div(s)$  in the affine chart  $U$ . Since  $L$  is globally generated, a generic section  $s \in \Gamma(X, L)$  does not vanish at  $0 \in U$  and the corresponding polynomial in  $x$  has a non-zero constant term.

In the context of very ample line bundles  $L_1, \dots, L_{n-1}$  on  $X$ , we can then use polynomial equations for  $C_a$  restricted to the affine chart  $U$ :

$$C_a \cap U = \{x = (x_1, \dots, x_n) \in U, a_{k0} = q_k(a'_k, x), k = 1, \dots, n-1\},$$

where  $a_k = (a_{k0}, a'_k)$  and  $q_k(a'_k, \cdot)$  are polynomials in  $x$  vanishing at  $0 \in U$ .

Since  $X$  is toric, we know from [9] that the Chow groups  $A_k(X)$  are isomorphic to the cohomology groups  $H^{2n-2k}(X, \mathbb{Z})$ , for any  $k = 0, \dots, n$ , and we can identify the Chow group  $A_0(X)$  of 0-cycles with  $\mathbb{Z} \simeq H^{2n}(X, \mathbb{Z})$ . We denote by  $[V]$  the class of any closed subvariety  $V$  of  $X$ ,  $c_1(L) \in H^2(X, \mathbb{Z})$  the first Chern class of any line bundle  $L$  on  $X$ , and we denote by  $\frown$  the usual cap product. Our first result is

**Theorem 1** *The set  $V := V_1 \cup \dots \cup V_N$  is contained in an algebraic hypersurface  $\tilde{V} \subset X$  such that*

$$[\tilde{V}] \frown \prod_{k=1}^{n-1} c_1(L_k) = N$$

*if and only if for all  $i = 1, \dots, n$  the functions  $a \mapsto Tr_V(x_i)(a)$  are affine in the constant coefficients  $a_0 = (a_{10}, \dots, a_{n-1,0})$ .*

Note that the left hand side in the formula of Theorem 1 is the intersection number, so that it must be at least  $N$  if the required algebraic hypersurface  $\tilde{V}$  exists. If the conditions of Theorem 1 are not satisfied,  $V$  can nevertheless be contained in a hypersurface  $\tilde{V}$  of  $X$  such that  $[\tilde{V}] \frown \prod_{k=1}^{n-1} c_1(L_k) > N$ . In this case, traces of affine coordinates are algebraic in  $a_0$  and no longer polynomials.

It is shown in [19] that in the projective case  $X = \mathbb{P}^n$ , Wood's theorem can be derived from the Abel-inverse theorem obtained in [13], using some rigidity properties of a particular system of PDE's. Using similar arguments, the following toric Abel-inverse theorem is proved in [18], Chapter 2, as a corollary of Theorem 1.

**Theorem** *Let  $\phi$  be a holomorphic form of maximal degree on  $V$ , not identically zero on any germs  $V_i$ , for  $i = 1, \dots, N$ . There exists an algebraic hypersurface  $\tilde{V} \subset X$  containing  $V$  such that  $[\tilde{V}] \frown \prod_{k=1}^{n-1} c_1(L_k) = N$  and a rational form  $\Psi$  on  $\tilde{V}$  such that  $\Psi|_V = \phi$ , if and only if the trace form  $Tr_V \phi(a) := \sum_{i=1}^N p_i^*(\phi)(a)$  is rational in  $a_0$ .*

Let us remark that it should be interesting to derive Theorem 1 from the previous theorem by choosing some form  $\phi$  related to the coordinate functions  $x_i$ .

Contrarily to the projective case handled in [21], Theorem 1 does not characterize the class of  $\tilde{V}$ . To do so, we introduce the norm on  $V$  relatively to  $(L_1, \dots, L_{n-1})$  of any function  $f$  holomorphic at  $p_1, \dots, p_N$ ,

$$a \longmapsto N_V(f)(a) := \prod_{i=1}^N f(p_i(a)),$$

which is defined and holomorphic for  $a \in X^*$  close to  $a^0$ . We then study the degree in  $a_0$  of norms of some rational functions on  $X$  whose polar

divisors generate  $Pic_{\mathbb{Q}}(X)$ . As in [19], let us fix very ample effective divisors  $E_1, \dots, E_s$  supported by  $X \setminus U$ , whose classes form a  $\mathbb{Q}$ -basis of  $Pic_{\mathbb{Q}}(X)$ . We can now characterize the class of the interpolating hypersurface.

**Theorem 2** *Suppose that conditions of Theorem 1 are satisfied. Then the equality  $[\tilde{V}] = \alpha \in Pic(X)$  holds if and only if there exist rational functions  $f_j \in H^0(X, \mathcal{O}_X(E_j))$  for  $j = 1, \dots, s$ , whose norms  $N_V(f_j)$  are polynomials in  $a_{10}$  of degree exactly*

$$deg_{a_{10}} N_V(f_j) = \alpha \cdot [E_j] \frown \prod_{k=2}^{n-1} c_1(L_k) \in \mathbb{Z}_{\geq 0}.$$

Note that Bernstein's theorem [4] allows to compute the degrees of intersection in Theorems 1 and 2 as mixed volume of the polytopes associated (up to translation) to the involved line bundles.

If  $X = \mathbb{P}^n$ , then  $Pic(X) \simeq \mathbb{Z}$  and Theorem 2 follows from Theorem 1: if  $Tr_V(x_n)$  is affine in  $a_0$ , then  $N_V(x_n)$  has degree  $N$  in  $a_0$ .

The proof of Theorem 1 uses a toric generalization of Abel-Jacobi's theorem [14] which gives combinatorial conditions for the vanishing of sums of Grothendieck residues associated to zero-dimensional complete intersections in toric varieties, those conditions being interpreted in terms of affine coordinates.

The difficulty to generalize Theorem 1 to other compactifications  $X$  of  $\mathbb{C}^n$ , as Grassmannians or flag varieties, is that there is no natural choice of affine coordinates, so *a priori* no grading for the algebra of regular functions over  $U = \mathbb{C}^n$  naturally associated to  $X$ . Such an interpolation result in Grassmannians would be important to generalize Theorem 1 to any projective variety  $X$  and to any union of germs of dimension  $k \leq n - 1$ , by using a grassmannian embedding of  $X$  associated to an adequate rank  $k$  ample bundle  $E$  on  $X$ . Nevertheless, we know now that there exist global intrinsic representations of residue currents, using some Chern connections acting on global sections of some vector bundle instead of usual differentials acting on holomorphic functions [2]. Then it has been recently shown [16] that such a global setting provides directly some generalizations of Abel-Jacobi's theorem obtained in [17]. We could hope that this approach should give an alternative proof for Theorem 1 (at least the direct part) which could admit generalizations to larger class of manifolds than toric varieties, for instance Grassmannians.

Finally, let us mention that we can hope for a generalization to the case of non-projective toric varieties, using blowing-up and essential families of globally generated line bundles, as presented in [20].

Section 2 is devoted to the proof of Theorem 1, and Section 3 to the proof of Theorem 2.

This article is part of my PhD thesis [18] “La trace en géométrie projective et torique”, which is available on the web page <http://tel.archives-ouvertes.fr/tel-00136109>.

## 2 Proof of Theorem 1

### 2.1 Direct implication

Let us suppose that  $V$  is contained in an algebraic hypersurface  $\tilde{V}$  whose equation in the affine chart  $U$  is given by a polynomial  $f \in \mathbb{C}[x_1, \dots, x_n]$ . Since the line bundles  $L_1, \dots, L_{n-1}$  are very ample, the hypothesis on the degree of intersection is equivalent to the fact that for  $a$  near  $a^0$ , the intersection  $\tilde{V} \cap C_a$  is contained in  $U$  and equal to  $V \cap C_a$ . In particular, the  $n$  polynomials  $f, a_{10} - q_1(a'_1, \cdot), \dots, a_{n-1,0} - q_{n-1}(a'_{n-1}, \cdot)$  of  $x$  define a complete intersection in  $\mathbb{C}^n$ . Now, it is well known (see [12], Chapter 5, Section 2) that the trace of  $x_i$  is equal, for  $a$  close to  $a^0$ , to the action of the Grothendieck residue defined by these polynomials on the holomorphic form  $x_i df \wedge dq_1 \cdots \wedge dq_{n-1} / (2i\pi)^n$ , that is,

$$Tr_V(x_i)(a) = \text{Res} \left[ \frac{x_i df \wedge dq_1 \cdots \wedge dq_{n-1}}{f, a_{10} - q_1, \dots, a_{n-1,0} - q_{n-1}} \right],$$

where we use classical notations (see [3]) for Grothendieck residues<sup>2</sup>. This action is given by the integral formula

$$Tr_V(x_i)(a) = \int_{|a_{i0} - q_i| = \epsilon_i, i=1, \dots, n-1, |f| = \epsilon_n} \frac{x_i df \wedge dq_1 \cdots \wedge dq_{n-1}}{f(a_{10} - q_1) \cdots (a_{n-1,0} - q_{n-1})},$$

so that differentiation of the trace with respect to  $a_{k0}$  gives the equality

$$\partial_{a_{k0}}^{(l)} Tr_V(x_i)(a) = \text{Res} \left[ \frac{(-1)^l l! x_1 \cdots x_i^2 \cdots x_n \frac{df \wedge dq_1 \cdots \wedge dq_{n-1}}{x_1 \cdots x_n}}{f, a_{10} - q_1, \dots, (a_{k0} - q_k)^{l+1}, \dots, a_{n-1,0} - q_{n-1}} \right].$$

If  $h, f_1, \dots, f_n$  are Laurent polynomials in  $t = (t_1, \dots, t_n)$  with Newton polytopes  $P, P_1, \dots, P_n$ , the toric Abel-Jacobi theorem [14] asserts that

$$\text{Res} \left[ \frac{h \frac{dt_1 \cdots \wedge dt_n}{t_1 \cdots t_n}}{f_1, \dots, f_n} \right] = 0$$

<sup>2</sup>From a more conceptual point of view, the Grothendieck residue action on the form  $x_i df \wedge dq_1 \cdots \wedge dq_{n-1} / (2i\pi)^n$  coincides with the action of the logarithmic residue

$$dd^c \log|f| \wedge \cdots \wedge dd^c \log|a_{n-1,0} - q_{n-1}|$$

on the function  $x_i$ . It is well known that this logarithmic residue, considered as an  $(n, n)$ -current, is equal to the sum of the point masses at the points of intersection so that its action on  $x_i$  produces the trace of  $x_i$ .

as soon as  $P$  is contained in the interior of the Minkowski sum  $P_1 + \dots + P_n$ . Since  $L_k$  is very ample, the support of the polynomial  $P_k$  is  $n$ -dimensional and it is not difficult to check that the Newton polytope of the Jacobian of the map  $(f, q_1, \dots, q_{n-1})$  translated *via* the vector  $(1, \dots, 2, \dots, 1)$  (corresponding to multiplication by  $x_1 \dots x_i^2 \dots x_n$ ) is strictly contained in the Minkowski sum of the Newton polytopes of polynomials  $f, a_{10} - q_1, \dots, a_{n-1,0} - q_{n-1}$  for  $l \geq 2$ . This shows the direct part of Theorem 1.

**Remark 1** *If  $R_k$  is the unique divisor in  $|L_k|$  supported outside  $U$ , the previous argument yields the implication*

$$h \in H^0(X, \mathcal{O}_X(dR_k)) \Rightarrow \deg_{a_{k0}} \text{Tr}_V(h) \leq d$$

*with equality if the zero set of  $h$  has a proper intersection with  $X \setminus U$  (which is generically the case since  $L_k$  is globally generated). See [18], Corollary 3.6 p 127. In particular, the trace of the coordinate function  $x_i$  is affine in  $a_{k0}$  if the vector  $e_i := (0, \dots, 1, \dots, 0)$  is a vertice of  $P_k$ , and does not depend on  $a_{k0}$  otherwise.*

## 2.2 Converse implication

Let us show that  $\text{Tr}_V(x_i)$  being affine in  $a_0$  implies that  $\text{Tr}_V(x_i^l)$  is polynomial of degree at most  $l$  in  $a_0$  for any  $l \geq 1$ . We need an auxiliary lemma generalizing to the toric case the ‘‘Wave-shock equation’’ used in [13] to show the Abel-inverse theorem. We give a weak version of this lemma, which will be sufficient for our purpose. See [18], Proposition 3.8 p 128, for a stronger version.

For  $a$  near  $a^0$ , we use affine coordinates  $(x_1^{(j)}(a), \dots, x_n^{(j)}(a))$  for the unique point of intersection  $p_j(a)$  of  $V_j$  with  $C_a$ . Since  $L_k$  is very ample, the monomial  $x_i$  occurs in the polynomial  $q_k$  with a generically non zero coefficient denoted by  $a_{ki}$ , for  $i = 1, \dots, n$ .

**Lemma 1** *For any  $i \in \{1, \dots, n\}$ , and any  $j \in \{1, \dots, N\}$ , the function  $a \mapsto x_i^{(j)}(a)$  (holomorphic at  $a^0$ ) satisfies the following P.D.E:*

$$\partial_{a_{ki}} x_i^{(j)}(a) = -x_i^{(j)} \partial_{a_{k0}} x_i^{(j)}(a)$$

*for any  $k = 1, \dots, n - 1$  and any  $a$  close to  $a^0$ .*

*Proof.* Let us fix  $i = 1$  for simplicity. Trivially, the equality  $a_{k0} = q_k(a'_k, x)$  holds for all  $k = 1, \dots, n - 1$  if and only if  $x \in C_a \cap U$ , and the complex number

$$x_1^{(j)}((q_1(a'_1, x), a'_1), \dots, (q_{n-1}(a'_{n-1}, x), a'_{n-1}))$$

thus represents the  $x_1$ -coordinate of the unique point of intersection of  $V_j$  with the curve  $C_a$  passing through  $x$ . If  $x = (x_1, \dots, x_n)$  belongs to  $V_j$ , this

complex number, seen as a function of  $a' = (a'_1, \dots, a'_{n-1})$  is thus constant, equal to  $x_1$ . Differentiating according to the  $x_1$ -coefficient  $a_{k1}$  of  $q_k$  gives

$$\begin{aligned} 0 &= \partial_{a_{k1}} x_1^{(j)}((q_1(a'_1, x), a'_1), \dots, (q_{n-1}(a'_{n-1}, x), a'_{n-1})) \\ &\quad + x_1^{(j)}((q_1(a'_1, x), a'_1), \dots, (q_{n-1}(a'_{n-1}, x), a'_{n-1})) \\ &\quad \times \partial_{a_{k0}} x_1^{(j)}((q_1(a'_1, x), a'_1), \dots, (q_{n-1}(a'_{n-1}, x), a'_{n-1})). \end{aligned}$$

We can replace  $x \in V_j$  with  $(x_1^{(j)}(a), \dots, x_n^{(j)}(a)) \in V_j$ , and the desired relation follows from the equality  $q_k(a'_k, (x_1^{(j)}(a), \dots, x_n^{(j)}(a))) = a_{k0}$ .  $\square$

In particular, Lemma 1 implies that

$$(l+1)\partial_{a_{ki}} \text{Tr}(x_i^l) = -l\partial_{a_{k0}} \text{Tr}(x_i^{l+1})$$

for any  $i = 1, \dots, n$ , any  $k = 1, \dots, n-1$ , and all integers  $l \in \mathbb{N}$ , from which we easily deduce

$$\deg_{a_{k0}} \text{Tr}(x_i^l) \leq l.$$

More generally, let

$$(y_1, \dots, y_n)^t = C(x_1, \dots, x_n)^t, \quad C \in GL_n(\mathbb{C})$$

be any linear change of coordinates in  $U$ . Then, we have equality

$$a_{k1}x_1 + \dots + a_{kn}x_n = \alpha_{k1}y_1 + \dots + \alpha_{kn}y_n$$

where  $\alpha_k = (\alpha_{k1}, \dots, \alpha_{kn})^t = (C^t)^{-1}(a_{k1}, \dots, a_{kn})^t$ , so that

$$q_k(a'_k, x) = \alpha_{k1}y_1 + \dots + \alpha_{kn}y_n + Q_k(a''_k, x)$$

where  $a'_k = (a_{k1}, \dots, a_{kn}, a''_k)$  and the polynomial

$$Q_k(a''_k, x) := q_k(a'_k, x) - (a_{k1}x_1 + \dots + a_{kn}x_n)$$

does not depend on  $(a_{k1}, \dots, a_{kn})$ . The proof of the previous lemma can be obviously adapted when differentiating with respect to the new parameter  $\alpha_{ki}$  (linear combination of  $a_{k1}, \dots, a_{kn}$ , coding for the new variable  $y_i$ ) instead of  $a_{ki}$ , and we obtain equality

$$\partial_{\alpha_{ki}} y_i(p_j(a)) = -y_i(p_j(a)) \times \partial_{a_{k0}} y_i(p_j(a)) = -\frac{1}{2} \partial_{a_{k0}} [y_i(p_j(a))]^2$$

for  $k = 1, \dots, n-1$ ,  $i = 1, \dots, n$  and  $j = 1, \dots, N$ .

Now, if  $y = c_1x_1 + \dots + c_nx_n$  is any linear combination of the affine coordinates  $x_i$ , its trace  $\text{Tr } y = \sum_{i=1}^n c_i \text{Tr } x_i$  is affine in  $a_0$ , and the previous equality implies that

$$\deg_{a_{k0}} \text{Tr}(y^l) \leq l \quad (*)$$



for any  $l \in \mathbb{N}$ .

To any such holomorphic function  $y = y(x)$ , we can associate its characteristic polynomial

$$P_y(X, a) := \prod_{j=1}^N (X - y(p_j(a))),$$

whose coefficients are holomorphic functions near  $a^0$ . Using Newton's formulas relating coefficients of  $P_y$  to the trace of the powers of  $y$ , we deduce from (\*) that  $P_y$  is polynomial in  $a_0 = (a_{01}, \dots, a_{0, n-1})$ . For any  $a$  near  $a^0$ , the function

$$Q_{y, a'}(x) := P_y(y(x), (q_1(x, a'_1), a'_1), \dots, (q_{n-1}(x, a'_{n-1}), a'_{n-1}))$$

is thus a polynomial in  $(x_1, \dots, x_n)$ , which, by construction, vanishes on  $V$  independently of  $a'$  and  $y$ . Let us consider the algebraic set

$$W_{a'} := \bigcap_{y=c_1 x_1 + \dots + c_n x_n} \{x \in U, Q_{y, a'}(x) = 0\}.$$

Then  $V \subset W_{a'}$  and  $x \in W_{a'} \cap C_a$  if and only if

$$y \in \{y(p_1(a), \dots, p_N(a))\}$$

for any linear combination  $y$  of the affine coordinates  $x_i$ . This implies that  $x \in \{p_1(a), \dots, p_N(a)\}$  by duality so that  $W_{a'} \cap C_a = V \cap C_a$  for any  $a$  near  $a^0$ . Consider now

$$\tilde{V} := \bigcap_{a \text{ near } a^0} \bar{W}_{a'},$$

where  $\bar{W}_{a'}$  denotes the Zariski closure in  $X$  of the affine algebraic hypersurface  $W_{a'}$ . Then

$$\text{codim}_X \tilde{V} \cap (X \setminus U) \geq 2,$$

so that the intersection  $\tilde{V} \cap C_a$  is generically contained in  $U$ . For  $a$  near  $a^0$ , there is thus equality  $\tilde{V} \cap C_a = V \cap C_a$ , so that  $[\tilde{V}] \cap \prod_{k=1}^{n-1} c_1(L_k) = N$ .  $\square$

### 3 Proof of Theorem 2

We can associate to any codimension 2 closed subvariety  $W \subset X$  its dual set  $W^* \subset X^*$  associated to the line bundles  $(L_1, \dots, L_{n-1})$ , defined by

$$W^* := \{a \in X^*, C_a \cap W \neq \emptyset\}.$$

From [10], this is an hypersurface in the product of projective spaces  $X^*$ , irreducible if  $W$  is, whose multidegree  $(d_1, \dots, d_{n-1})$  in  $X^*$  is given by the intersection numbers

$$d_i = [W] \cap \prod_{i=1, i \neq j}^{n-1} c_1(L_i), \quad j = 1, \dots, n-1.$$

We call the  $(L_1, \dots, L_{n-1})$ -resultant of  $W$ , noted  $\mathcal{R}_W$ , the multihomogeneous polynomial of multidegree  $(d_1, \dots, d_{n-1})$  vanishing on  $W^*$  (it is defined up to a non zero scalar, but this has no consequence here). By linearity, we generalize this situation to the case of cycles:

$$\mathcal{R}_{\sum c_i W_i} := \prod (\mathcal{R}_{W_i})^{c_i}.$$

Duality respects rational equivalence so that the degree of the resultant of a cycle  $W$  only depends of the class of  $W$  in the Chow group of  $X$  (see [18], Proposition 7 p 100).

A generic rational function  $f_j \in H^0(X, \mathcal{O}_X(E_j))$  defines a principal divisor  $H_j - E_j$ , where the zero divisor  $H_j$  intersects properly  $\tilde{V}$  and  $X \setminus U$ . In that case, the product formula [15] gives rise to the equality :

$$N_{\tilde{V}}(f_j) = \frac{\mathcal{R}_{\tilde{V} \cdot H_j}}{\mathcal{R}_{\tilde{V} \cdot E_j}}.$$

Since the constant coefficients  $a_0 = (a_{10}, \dots, a_{n-1,0})$  do not influence the asymptotic behavior of the curves  $C_a$  outside the affine chart  $U$ , the resultant  $\mathcal{R}_{\tilde{V} \cdot E_j}(a)$  does not depend on  $a_0$ . We thus obtain

$$\deg_{a_{10}} N(f_j) = \deg_{a_{10}} \mathcal{R}_{\tilde{V} \cdot H_j} \leq \deg_{a_1} \mathcal{R}_{\tilde{V} \cdot H_j} = \deg_{a_1} \mathcal{R}_{\tilde{V} \cdot E_j}.$$

Since we deal with homogeneous polynomials in  $a_1$ , strict inequality in the previous expression is equivalent to the equality

$$\mathcal{R}_{\tilde{V} \cdot E_j}((a_{10}, 0, \dots, 0), a_2, \dots, a_{n-1}) \equiv 0.$$

This happens if and only if all subvarieties  $C = \{s = 0\}$  given by sections  $s \in \Gamma(X, \bigoplus_{k=2}^{n-1} L_k)$  intersect the set  $\tilde{V} \cap H_j \cap (X \setminus U)$ . By a dimension argument, this would imply that  $\tilde{V}$  has an irreducible branch contained in  $X \setminus U$ , and this can not occur since  $\tilde{V} \cap C_a = V \cap C_a \subset U$  for  $a$  close to  $a^0$ . Thus we have proved the equality:

$$\deg_{a_{10}} N_{\tilde{V}}(f_j) = [\tilde{V}] \cap [E_j] \cap \prod_{k=2}^{n-1} c_1(L_k)$$

Since the classes  $[E_j]$ ,  $j = 1, \dots, s$  determine a basis for  $A_{n-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ , the non degenerated natural pairing between the Chow groups  $A_1(X)$  and  $A_{n-1}(X)$  shows that the hypothesis of Theorem 2 is equivalent to

$$[\tilde{V}] \cap \prod_{k=2}^{n-1} c_1(L_k) = \alpha \cap \prod_{k=2}^{n-1} c_1(L_k).$$

Since the Strong Lefschetz theorem remains valid for any very ample algebraic vector bundle on a smooth projective variety  $X$  (see Prop. 1.1 in [5]), the preceding equality is equivalent to  $[\tilde{V}] = \alpha$ .  $\square$

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