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► **To cite this version:**

| Vincent Humilière. Hamiltonian pseudo-representations. 2007. hal-00136107v1

HAL Id: hal-00136107

<https://hal.science/hal-00136107v1>

Preprint submitted on 12 Mar 2007 (v1), last revised 20 Jul 2007 (v2)

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Hamiltonian pseudo-representations

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11/12/06

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Abstract

The question studied here is the behavior of the Poisson bracket under C^0 -perturbations. In this purpose, we introduce the notion of pseudo-representation and prove that for a nilpotent Lie algebra, it converges to a representation. This question remains open for general Lie algebras.

An unexpected consequence of this result is that for many non-closed symplectic manifolds (including cotangent bundles), the group of Hamiltonian diffeomorphisms (with no assumptions on supports) has no C^{-1} bi-invariant metric. Our methods also provide a new proof of Gromov-Eliashberg Theorem, it is to say that the group of symplectic diffeomorphisms is C^0 -closed in the group of all diffeomorphisms.

1 Statement of results

1.1 Poisson Brackets and C^0 -convergence

We consider a symplectic manifold (M, ω) and its set of compactly supported Hamiltonian functions $Ham_c(M)$. Endowed with the Poisson brackets $\{\cdot, \cdot\}$, it has the structure of a Lie algebra.

In the whole paper, we will denote X_H the symplectic gradient of an Hamiltonian function H , i.e., the only vector field satisfying $dH = \iota_{X_H}\omega$. Then, the Poisson brackets are given by $\{H, K\} = dH(X_K)$. Let us consider the following property.

Property (P) : For any sequence of linear maps $\rho_n : \mathfrak{g} \rightarrow \text{Ham}_c(M)$, such that for any $f \in \mathfrak{g}$, there exists a smooth Hamiltonian $\rho(f) \in \text{Ham}_c(M)$, such that

1. $\rho_n(f) \rightarrow \rho(f)$, for all $f \in \mathfrak{g}$,
2. $\{\rho_n(f), \rho_n(g)\} - \rho_n([f, g]) \rightarrow 0$, for all $f, g \in \mathfrak{g}$,

then for all $f, g \in \mathfrak{g}$, $\{\rho(f), \rho(g)\} = \rho([f, g])$.

Such a sequence ρ_n will be called *pseudo-representation* of \mathfrak{g} , and ρ will be referred to as the limit of the pseudo representation. We see that the limit of a pseudo-representation of a Lie algebra satisfying (P) is a representation of this Lie algebra.

It is proved in [1] that any abelian Lie algebra (nilpotent algebra of index 1) satisfies (P). The main result of this paper is that it holds for all nilpotent Lie algebras.

Theorem 1. Any nilpotent Lie algebra satisfies (P).

Remark 1. This result generalizes Gromov-Eliashberg's Theorem of C^0 closure of the symplectomorphisms group in the group of diffeomorphisms.

Indeed, a diffeomorphism of \mathbb{R}^{2n} is symplectic if and only if its coordinate functions $(f_i), (g_i)$ satisfy

$$\{f_i, g_j\} = \delta_{ij}, \quad \{f_i, f_j\} = \{g_i, g_j\} = 0.$$

Thus we can easily see that a sequence of symplectomorphisms gives a pseudo-representation of a 2-nilpotent Lie algebra. If the support of the coordinate functions were compact, we could immediately apply Theorem 1. In fact, for compactly supported symplectomorphisms, these functions are linear at infinity, and we have to adapt the proof to this case (See Appendix A for details).

Remark 2. Consider the following example, which is derived from Polterovich's example presented in Section 2.3. Let χ be a compactly supported smooth function on \mathbb{R} , and set the following Hamiltonians on \mathbb{R}^2 :

$$F_n(q, p) = \frac{\chi(p)}{\sqrt{n}} \cos(nq),$$

$$G_n(q, p) = \frac{\chi(p)}{\sqrt{n}} \sin(nq).$$

It is easy to see that F_n and G_n converge to 0, but that their Poisson bracket equals $\chi(p)\chi'(p) \neq 0$.

This example shows that when the Poisson brackets C^0 -converge, then its limit is not necessarily the brackets of the respective limits. But in that case, we can see that the Hamiltonians F_n and G_n do not generate a pseudo-representation.

Remark 3. The theorem holds if we replace the symplectic manifold with a general Poisson manifold. Indeed, Poisson manifolds are foliated by Poisson submanifolds that are symplectic, and we just have to apply theorem 1 to each leaf.

Question: What about general Lie algebras ? For instance, it is unclear whether $\mathfrak{su}(2)$ satisfies **(P)** or not. Propositions 12 and 13 give restrictions on the possible counter-examples.

1.2 Bi-invariant Metrics

Here we consider a subgroup \mathcal{G} of the group $\mathcal{H}(M)$ of Hamiltonian diffeomorphisms on M . If we denote ϕ_H^t the flow generated by X_H (when it exists), and $\phi_H = \phi_H^1$ the time-1 map, $\mathcal{H}(M)$ is the set of all diffeomorphisms ϕ for which it exists a path of Hamiltonian functions $H_t \in Ham$ such that $\phi = \phi_H$.

Definition 2. A bi-invariant metric on \mathcal{G} is a distance d on \mathcal{G} such that for any ϕ, ψ, χ in \mathcal{G} ,

$$d(\phi, \psi) = d(\phi\chi, \psi\chi) = d(\chi\phi, \chi\psi).$$

It will be said C^{-1} if its composition with the map $\Phi : H \mapsto \phi_H^1$ is a continuous map $\Phi^{-1}(\mathcal{G}) \times \Phi^{-1}(\mathcal{G}) \rightarrow \mathbb{R}$, where $\Phi^{-1}(\mathcal{G}) \subset Ham$ is endowed with the compact-open topology.

There are several well known examples of C^{-1} bi-invariant metrics, as, for example, Hofer's metric defined on $\mathcal{H}_c(M)$ (see [3] or [6]), Viterbo's metric defined on $\mathcal{H}_c(\mathbb{R}^{2n})$ (see [14]), and its analogous version defined by Schwarz in [11] for symplectically aspherical closed symplectic manifolds.

As far as we know, if we remove the assumption of compactness of the support, the question whether there exists such metrics is still open. Here we prove that the answer is negative for a large class of symplectic manifolds.

Let (N, ξ) be a contact manifold with contact form α (i.e., a smooth manifold N with a smooth hyperplane section ξ which is locally the kernel of a 1-form α whose differential $d\alpha$ is non-degenerate on ξ). Its *symplectization* is by definition the symplectic manifold $\mathcal{S}N = \mathbb{R} \times N$ endowed with the symplectic form $\omega = d(e^s\alpha)$, where s denotes the \mathbb{R} -coordinate in $\mathbb{R} \times N$. For any contact form α , one can define the *Reeb vector field* X_R by the identities $\iota_{X_R}d\alpha$ and $\alpha(X_R) = 1$. The trajectories of X_R are called *characteristics*. The question of the existence of a closed characteristic constitutes the famous

Weinstein's conjecture. It has now been proved for large classes of contact manifolds (see e.g. [2, 4, 5, 10, 9, 13, 12]).

Let us now state our result that will be proved in section 2.3

Theorem 3. *If M is the symplectization of a contact manifold that admits a closed characteristic, then there is no C^{-1} bi-invariant metric on $\mathcal{H}(M)$.*

Corollary 4. *If N is a smooth manifold with cotangent bundle T^*N , then there is no C^{-1} bi-invariant metric on $\mathcal{H}(T^*N)$.*

Remark. At least in the case of manifolds of finite volume, there probably exists non closed manifolds with such distances. Indeed, it follows from our previous work [7] that Viterbo's metric extends to Hamiltonians functions smooth out of a "small" compact set. Replacing Viterbo's metric with Schwarz's metric, we can reasonably expect to have: If M^{2n} is a closed symplectically aspherical manifold and K is a closed submanifold of dimension $\leq n - 2$, then Schwarz's metric on $\mathcal{H}(M)$ extends to $\mathcal{H}(M - K)$.

2 Proofs

2.1 Some identities for Hamiltonian representations

Lemma 5. *Let \mathfrak{g} be a p -nilpotent Lie algebra, \mathfrak{h} an ideal of \mathfrak{g} , and r be a linear map from \mathfrak{g} to $\text{Ham}_c(M)$ such that for all $h \in \mathfrak{h}$, $g \in \mathfrak{g}$, $\{r(h), r(g)\} = r([h, g])$. Let $h \in \mathfrak{h}$, $g \in \mathfrak{g}$, $H = r(h)$ and $G = r(g)$. Then, for all $\tau \in \mathbb{R}$,*

$$H \circ \phi_G^\tau = \sum_{j=0}^{p-1} r(\text{ad}(g)^j h) \frac{\tau^j}{j!},$$

where $\text{ad}(g) : \mathfrak{g} \rightarrow \mathfrak{g}$, $h \mapsto [h, g]$.

Furthermore, if $h \in \mathfrak{h}$, $g_1, g_2, g_3 \in \mathfrak{g}$ and $G_i = r(g_i)$, then for all s, t, τ ,

$$H \circ \phi_{G_1}^s \circ \phi_{G_2}^t \circ \phi_{G_3}^\tau = \sum_{i+j+k \leq p-1} r(\text{ad}(g_3)^k \text{ad}(g_2)^j \text{ad}(g_1)^i h) \frac{s^i}{i!} \frac{t^j}{j!} \frac{\tau^k}{k!}.$$

Proof: This lemma is based on the following simple observation:

$$\begin{aligned}
H \circ \phi_G^\tau &= H + \int_0^\tau \frac{d}{d\sigma} (H \circ \phi_G^\sigma) d\sigma \\
&= H + \int_0^\tau dH \cdot X_G \circ \phi_G^\sigma d\sigma \\
&= H + \int_0^\tau \{H, G\} \circ \phi_G^\sigma d\sigma \\
&= r(h) + \int_0^\tau r([h, g]) \circ \phi_G^\sigma d\sigma,
\end{aligned}$$

for $h \in \mathfrak{h}$ and $g \in \mathfrak{g}$. Then, a simple induction argument shows that for all integer N ,

$$H \circ \phi_G^\tau = \sum_{j=1}^{N-1} r(ad(g)^j h) \frac{\tau^j}{j!} + R_N(\tau),$$

where

$$R_N(\tau) = \int_0^\tau \int_0^{\sigma_1} \cdots \int_0^{\sigma_{N-1}} r(ad(g)^N h) \circ \phi_G^{\sigma_N} d\sigma_N \cdots d\sigma_1.$$

But for $N \geq p$, $ad(g)^N h = 0$ and hence $R_N(\tau) = 0$ which proves our claim.

The second assertion of Lemma 5 follows from the first one. Let us denote $R_N^{h,g}(\tau)$ instead of $R_N(\tau)$. Then we have

$$H \circ \phi_G^\tau = \sum_{i=1}^{p-1} r(ad(g_1)^i h) \frac{s^i}{i!},$$

and by a simple computation, the first part of our lemma applied to each couple $ad(g_1)^i h$ and g_2 gives

$$H \circ \phi_{G_1}^s \circ \phi_{G_2}^t = \sum_{i,j=0}^{p-1} r(ad(g_2)^j ad(g_1)^i h) \frac{s^i t^j}{i! j!},$$

and similarly,

$$H \circ \phi_{G_1}^s \circ \phi_{G_2}^t \circ \phi_{G_3}^\tau = \sum_{i,j,k=0}^{p-1} r(ad(g_3)^k ad(g_2)^j ad(g_1)^i h) \frac{s^i t^j \tau^k}{i! j! k!}.$$

It achieves the proof of our lemma, because the terms of the last sum are zero for $i + j + k \geq p$. \square

Lemma 5 can somehow be extended to the case of pseudo-representations, by replacing equalities with limits.

Lemma 6. Let ρ_n be a pseudo-representation of a p -nilpotent Lie algebra \mathfrak{g} . Let $f, g \in \mathfrak{g}$. Then for n large enough, $\rho_n(f) \circ \phi_{\rho_n(g)}^\tau$ is C^0 -close to $\sum_{j=0}^{p-1} \rho_n(\text{ad}(g)^j f) \frac{\tau^j}{j!}$.

Similarly, if $f, g_1, g_2, g_3 \in \mathfrak{g}$, then for n large enough,

$$\rho_n(f) \circ \phi_{\rho_n(g_1)}^s \circ \phi_{\rho_n(g_2)}^t \circ \phi_{\rho_n(g_3)}^\tau$$

is C^0 -close to

$$\sum_{i+j+k \leq p-1} \rho_n(\text{ad}(g_3)^k \text{ad}(g_2)^j \text{ad}(g_1)^i f) \frac{s^i t^j \tau^k}{i! j! k!}.$$

Proof: We start the proof with the same computation as in the proof of Lemma 5.

$$\begin{aligned} \rho_n(f) \circ \phi_{\rho_n(g)}^\tau &= \rho_n(f) + \int_0^\tau \{\rho_n(f), \rho_n(g)\} \circ \phi_{\rho_n(g)}^\sigma d\sigma \\ &= \rho_n(f) + \int_0^\tau \rho_n([f, g]) \circ \phi_{\rho_n(g)}^\sigma d\sigma + \Lambda_{\tau, n}(f, g) \end{aligned}$$

with

$$\Lambda_{\tau, n}(f, g) = \int_0^\tau \{\rho_n(f), \rho_n(g)\} - \rho_n([f, g]) \circ \phi_{\rho_n(g)}^\sigma d\sigma$$

By assumption, $\Lambda_{\tau, n}(f, g)$ C^0 -converges to 0 when n goes to infinity. Then, by a simple induction, we get for all integer N :

$$\rho_n(f) \circ \phi_{\rho_n(g)}^\tau - \rho_n(f) = \sum_{j=1}^N \rho_n(\text{ad}(g)^j f) \frac{\tau^j}{j!} + R_{N, n}(\tau) + S_{N, n}(\tau),$$

where,

$$R_{N, n}(\tau) = \int_0^\tau \int_0^{\sigma_1} \cdots \int_0^{\sigma_{N-1}} \rho_n(\text{ad}(g)^N f) \circ \phi_{\rho_n(g)}^{\sigma_N} d\sigma_N \cdots d\sigma_1$$

$$\begin{aligned} S_{N, n}(\tau) &= \Lambda_{\tau, n}(\text{ad}(g)^{N-1} f, g) \\ &+ \sum_{j=0}^{N-2} \int_0^\tau \int_0^{\sigma_1} \cdots \int_0^{\sigma_j} \Lambda_{\sigma_j, n}(\text{ad}(g)^{N-2-j} f, g) d\sigma_j \cdots d\sigma_1 \end{aligned}$$

For $N \geq p$, we have $R_{N, n}(\tau) = 0$ and $S_{N, n}(\tau)$ C^0 -converges to 0 when n goes to infinity, which proves our claim.

The second assertion of Lemma 6 follows from the first with exactly the same reasoning as in the proof of the second assertion of Lemma 5. \square

Although it is useless for the proof of our theorem, we end this section with a proposition that extends Lemma 5 to normed Lie algebras not necessarily nilpotent.

Proposition 7. *Let \mathfrak{g} be a Lie algebra endowed with a norm $|\cdot|$, \mathfrak{h} an ideal of \mathfrak{g} , and r a continuous linear map from \mathfrak{g} to $\text{Ham}_c(M)$ such that for all $h \in \mathfrak{h}$, $g \in \mathfrak{g}$, $\{r(h), r(g)\} = r([h, g])$. Let $h \in \mathfrak{h}$, $g \in \mathfrak{g}$, $H = r(h)$ and $G = r(g)$. Then, for all $\tau \in \mathbb{R}$,*

$$H \circ \phi_G^\tau = \sum_{j=0}^{+\infty} r(\text{ad}(g)^j h) \frac{\tau^j}{j!},$$

where $\text{ad}(g) : \mathfrak{g} \rightarrow \mathfrak{g}, h \mapsto [h, g]$.

Proof: We start the proof as for Lemma 5, and get for all integer N ,

$$H \circ \phi_G^\tau = \sum_{j=1}^{N-1} r(\text{ad}(g)^j h) \frac{\tau^j}{j!} + R_N(\tau),$$

where

$$R_N(\tau) = \int_0^\tau \int_0^{\sigma_1} \cdots \int_0^{\sigma_{N-1}} r(\text{ad}(g)^N h) \circ \phi_G^{\sigma_N} d\sigma_N \cdots d\sigma_1.$$

The proof of Proposition 7 will be complete if we prove that R^N converges to 0 uniformly on M , when N goes to $+\infty$.

First,

$$\|R_N(\tau)\| \leq \int_0^\tau \int_0^{\sigma_1} \cdots \int_0^{\sigma_{N-1}} 2\|\text{ad}(G)^N H\| d\sigma_N \cdots d\sigma_1 = 2 \frac{\tau^N}{N!} \|\text{ad}(G)^N H\|.$$

Since r is continuous, there exists $C > 0$ such that, for all $g \in \mathfrak{g}$, $\|r(g)\| \leq C|g|$. Therefore, using the inequality $\|[f, g]\| \leq |f||g|$, we get the following estimate:

$$\|R_N(\tau)\| \leq 2C|h| \frac{(\tau|g|)^N}{N!}.$$

And $R_N(\tau)$ converges to 0 uniformly on M , for all τ . \square

Remark: Using the same kind of estimates, Proposition 7 can be extended to the case of several compositions and to the case of pseudo-representations, as in the nilpotent case.

2.2 Proof of theorem 1

Let \mathfrak{g} be a nilpotent Lie algebra. Its nilpotency index is by definition the first integer p for which $\mathfrak{g}^{p+1} = 0$, where $\mathfrak{g}^1 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$.

Proof of Theorem 1: We know from [1], that the theorem is true for 1-nilpotent Lie algebras. Let us suppose by induction that it holds for all k -nilpotent Lie algebras with $k \leq p-1$, and consider \mathfrak{g} a p -nilpotent Lie algebra, and ρ_n a pseudo-representation of \mathfrak{g} with limit ρ . Let $f, g \in \mathfrak{g}$ and $h = [f, g]$. We want to prove $\{\rho(f), \rho(g)\} = \rho(h)$.

Let us denote $\tilde{\mathfrak{g}}$ the Lie subalgebra of \mathfrak{g} generated by f and g . Then, ρ_n restricted to $\tilde{\mathfrak{g}}$ is a pseudo-representation of $\tilde{\mathfrak{g}}$, and we will be able to apply Lemma 6. Moreover, the Lie algebra generated by f and h is nilpotent of index at most $p-1$, and hence $\{\rho(f), \rho(h)\} = \rho([f, h])$. Similarly $\{\rho(g), \rho(h)\} = \rho([g, h])$. Now, if we denote $\tilde{\mathfrak{h}}$ the ideal of $\tilde{\mathfrak{g}}$ generated by h , we see that for all $\tilde{h} \in \tilde{\mathfrak{h}}$, $\tilde{g} \in \tilde{\mathfrak{g}}$, we have

$$\{\rho(\tilde{g}), \rho(\tilde{h})\} = \rho([\tilde{g}, \tilde{h}]),$$

so that we will be able to apply Lemma 5 with ρ in the place of r .

Let us denote $G = \rho(g)$, $F = \rho(f)$ and $H = \rho(h)$. For $s \in [0, 1]$, consider the Hamiltonian flow $\psi^t = \phi_H^{-ts} \phi_F^t \phi_G^s \phi_F^{-t} \phi_G^{-s}$. It is generated by the following Hamiltonian

$$L^s(t, x) = (F - F \circ \phi_G^{-s} \circ \phi_F^{-t} - sH) \circ \phi_H^{ts}(x),$$

Considering similarly, $G_n = \rho_n(g)$, $F_n = \rho_n(f)$ and $H_n = \rho_n(h)$, we get flows ψ_n^t and Hamiltonians $L_n^s(t, x)$.

We first consider the derivative

$$\begin{aligned} \frac{d}{ds}(F_n - F_n \circ \phi_{G_n}^{-s} \circ \phi_{F_n}^{-t} - sH_n) &= \{F_n, G_n\} \circ \phi_{G_n}^{-s} \circ \phi_{F_n}^{-t} - H_n \\ &= H_n \circ \phi_{G_n}^{-s} \circ \phi_{F_n}^{-t} - H_n + (\{F_n, G_n\} - H_n) \circ \phi_{G_n}^{-s} \circ \phi_{F_n}^{-t}. \end{aligned}$$

Let us consider, for s, t and τ in $[0, 1]$, the Hamiltonian function $Q_n(s, t, \tau) = H_n \circ \phi_{G_n}^{-s} \circ \phi_{F_n}^{-t} \circ \phi_{H_n}^\tau$. According to Lemma 6, for n large enough, it is C^0 -close to $\sum_{i+j+k < p} \rho_n(ad(h)^k ad(f)^j ad(g)^i h) \frac{s^i}{i!} \frac{t^j}{j!} \frac{\tau^k}{k!}$, which converges with n to $\sum_{i+j+k < p} \rho(ad(h)^k ad(f)^j ad(g)^i h) \frac{s^i}{i!} \frac{t^j}{j!} \frac{\tau^k}{k!}$. Then, by Lemma 5 the above triple sum equals $Q(s, t, \tau) = H \circ \phi_G^{-s} \circ \phi_F^{-t} \circ \phi_H^\tau$. Hence $Q_n(s, t, \tau)$ converges to $Q(s, t, \tau)$.

Let us now define $P_n(s, t, \tau) = \int_0^s Q_n(\sigma, t, \tau) d\sigma$, $R_n(s, t) = P_n(s, t, st)$ and similarly $P(s, t, \tau)$ and $R(s, t)$. Then $R_n(s, t)$ converges to $R(s, t)$.

Notice that R_n has been constructed to satisfy:

$$L_n^s(t) = R_n(s, t) + \left(\int_0^s (H_n - \{F_n, G_n\}) \circ \phi_{G_n}^{-\sigma} \circ \phi_{F_n}^{-t} d\sigma \right) \circ \phi_{H_n}^{ts}(x).$$

Since ρ_n is a pseudo-representation, $H_n - \{F_n, G_n\}$ converges to 0, and hence $L_n^s(t) - R_n(s, t)$ converges to 0 too. If we denote for s fixed ξ_n^t the flow generated by $R_n(s, t)$ (and similarly ξ^t the flow generated by $R(s, t)$), $\xi_n^{-t} \circ \psi_n^t$ is then generated by

$$(R_n(s, t) - L_n^s(t, \cdot)) \circ \xi_n^t,$$

which C^0 -converges to 0. Therefore, $\gamma(\xi_n^{-t} \circ \psi_n^t)$ converges to 0.

On the other hand, ξ_n^t γ -converges to ξ^t (because $R_n(s, t)$ C^0 -converges to $R(s, t)$), and $\psi_n^t = \phi_{H_n}^{-ts} \phi_{F_n}^t \phi_{G_n}^s \phi_{F_n}^{-t} \phi_{G_n}^{-s}$ converges for γ to $\psi^t = \phi_H^{-ts} \phi_F^t \phi_G^s \phi_F^{-t} \phi_G^{-s}$ (because, H_n, K_n, G_n converge respectively to H, K, G).

Finally we get $\gamma(\xi^{-t} \psi^t) = 0$ for all s and t , which implies $\xi^{-t} \psi^t = id$ for all s and t . It follows that its Hamiltonian is 0 for all s and t . Since its Hamiltonian function is exactly $(R(s, t) - L^s(t, \cdot)) \circ \xi^t$, we get:

$$\int_0^s (H - \{F, G\}) \circ \phi_G^{-\sigma} \circ \phi_F^{-t} d\sigma = 0,$$

and taking derivative with respect to s , we obtain $H = \{F, G\}$, namely

$$\rho([f, g]) = \{\rho(f), \rho(g)\}$$

as wanted. \square

2.3 Proof of theorem 3

Let us consider the following Hamiltonian functions on \mathbb{R}^2 (this example is due to Polterovich) with symplectic form written in polar coordinates $rdr \wedge d\theta$.

$$F_n(r, \theta) = \frac{r}{\sqrt{n}} \cos(n\theta),$$

$$G_n(r, \theta) = \frac{r}{\sqrt{n}} \sin(n\theta).$$

We see that $\{F_n, G_n\} = 1$ and that F_n and G_n converge to 0. Now, consider \mathfrak{g} the 3-dimensional Heisenberg Lie algebra (i.e., the Lie algebra with basis $\{f, g, h\}$ such that $[f, g] = h$ and $[f, h] = [g, h] = 0$) and set $\rho_n(f) = F_n$, $\rho_n(g) = G_n$ and $\rho_n(h) = 1$. Then, ρ_n is a pseudo-representation of \mathfrak{g} in $Ham(\mathbb{R}^2)$. The limit ρ of ρ_n satisfies $\rho(f) = 0$, $\rho(g) = 0$, $\rho(h) = 1$. Since $\{\rho(f), \rho(g)\} \neq \rho(h)$, ρ is not a representation of \mathfrak{g} .

Since \mathfrak{g} is nilpotent (of index 2), this example shows that Theorem 1 is false in general if we replace $Ham_c(M)$ with $Ham(M)$ for a non-compact manifold M , and uniform convergence with the uniform convergence on compact sets (compact-open topology).

If we read carefully the proof of Theorem 1, we see that the whole proof can be repeated in this settings except the two following points where the compactness of supports are needed

- Each time we consider the flows of the Hamiltonians, they must be complete. This is automatic for compactly supported Hamiltonians, but false in general. With the notations of the proof, the flows needed for the induction step are those of $F_n, F, G_n, G, H_n, H, R_n(s, t)$ and $R(s, t)$.
- We use a C^{-1} bi-invariant metric. This exists on $\mathcal{H}_c(M)$, but we do not know whether it exists on $\mathcal{H}(M)$.

Let us introduce some notations. For $f, g \in \mathfrak{g}$ generating a Lie algebra \mathfrak{h} and r a linear map $\mathfrak{g} \rightarrow Ham(M)$. We denote $\mathcal{P}_r(f, g)$ the set of all polynomial functions with indeterminate t and coefficients in $r(\mathfrak{h})$. We see that $F_n, F, G_n, G, H_n, H, R_n(s, t)$ and $R(s, t)$ (for s fixed) are in either $\mathcal{P}_{\rho_n}(f, g)$ or $\mathcal{P}_\rho(f, g)$. We also see that if h, k are elements of \mathfrak{h} , then $\mathcal{P}_r(h, k) \subset \mathcal{P}_r(f, g)$. Hence $\mathcal{P}_{\rho_n}(f, g) \cup \mathcal{P}_\rho(f, g)$ contains all the Hamiltonians needed, at any step of the induction.

The following lemma follows from the above discussion.

Lemma 8. *Let M be a non-compact symplectic manifold, \mathfrak{g} a nilpotent Lie algebra, and ρ_n a pseudo-representation of \mathfrak{g} in $Ham(M)$, with limit ρ . Suppose there exists two elements f and g in \mathfrak{g} , such that:*

- all the elements of $\mathcal{P}_{\rho_n}(f, g)$ and $\mathcal{P}_\rho(f, g)$ have complete flows,
- $\{\rho(f), \rho(g)\} \neq \rho([f, g])$,

then, the group of Hamiltonian diffeomorphisms $\mathcal{H}(M)$ admits no C^{-1} bi-invariant metric. \square

Proof of Theorem 3: Let us first consider the symplectization of S^1 . We just adapt Polterovich's example by setting :

$$\rho_n(f)(s, \theta) = \frac{e^{s/2}}{\sqrt{n}} \cos(n\theta),$$

$$\rho_n(g)(s, \theta) = \frac{e^{s/2}}{\sqrt{n}} \sin(n\theta).$$

The symplectic form being defined on $\mathbb{R} \times \mathbb{S}^1$ by $d(e^s d\theta) = e^s ds \wedge d\theta$, we get $\{\rho_n(f), \rho_n(g)\} = 2$. Since $\rho(f) = \rho(g) = 0$ we have a pseudo-representation of the 3-dimensional Heisenberg Lie algebra, and its limit is not a representation. According to Lemma 8, we then have to prove that all elements of $\mathcal{P}_{\rho_n}(f, g)$ and $\mathcal{P}_\rho(f, g)$ have complete flows, for f, g generators of the 3-dimensional Heisenberg Lie algebra, and ρ_n, ρ as in the example.

In the case of ρ , we have $\mathcal{P}_\rho(f, g) = \{\lambda t^\alpha \mid \lambda \in \mathbb{R}, \alpha \in \mathbb{N}\}$. The flows of all these elements are Id at any time. In the case of ρ_n ,

$$\mathcal{P}_{\rho_n}(f, g) = \{\lambda \rho_n(f) t^\alpha + \mu \rho_n(g) t^\beta + \nu t^\gamma \mid \lambda, \mu, \nu \in \mathbb{R}, \alpha, \beta, \gamma \in \mathbb{N}\}.$$

The Hamiltonian vector field of $\lambda \rho_n(f) t^\alpha + \mu \rho_n(g) t^\beta + \nu t^\gamma$ is

$$\begin{aligned} & (e^{-s/2} \sqrt{n} (\lambda \sin(n\theta) t^\alpha - \mu \cos(n\theta) t^\beta)) \frac{\partial}{\partial \theta} \\ & - \left(\frac{1}{2\sqrt{n}} e^{-s/2} (\lambda \cos(n\theta) t^\alpha + \mu \sin(n\theta) t^\beta) \right) \frac{\partial}{\partial s}, \end{aligned}$$

which is equivalent through the symplectomorphism

$$(\mathbb{R} \times \mathbb{S}^1, d(e^s d\theta)) \rightarrow (\mathbb{R}^2 - \{0\}, r dr \wedge d\theta), \quad (s, \theta) \mapsto (e^{-s/2}, \theta),$$

to the vector field

$$\begin{aligned} & (r \sqrt{n} (\lambda \sin(n\theta) t^\alpha - \mu \cos(n\theta) t^\beta)) \frac{\partial}{\partial \theta} \\ & + \left(\frac{1}{\sqrt{n}} (\lambda \cos(n\theta) t^\alpha + \mu \sin(n\theta) t^\beta) \right) \frac{\partial}{\partial r}. \end{aligned}$$

The norm of this vector field is bounded by a polynomial function in t that does not depend on (r, θ) . Therefore, it is a consequence of Gronwall's lemma that it is complete.

In the case $\dim N = 1$, one of the connected component of N has to be \mathbb{S}^1 (otherwise there is no closed characteristic). The above example can be applied to this connected component and extended by 0 on the other components.

Let us consider now the case $d = \dim(N) > 1$. Denote by γ a closed characteristic, parameterized by $\theta \in \mathbb{S}^1$. Since the Reeb vector field is transverse to the contact structure ξ , there exists a diffeomorphism that maps a neighborhood \mathcal{V}_0 of the zero section in the restricted bundle $\xi|_\gamma$, onto a neighborhood \mathcal{V}_1 of γ in the contact manifold N . Since $\xi|_\gamma$ is a symplectic bundle over \mathbb{S}^1 , it is trivial. We thus have a neighborhood U of 0 in \mathbb{R}^{2n} and a diffeomorphism $\psi : \mathbb{S}^1 \times U \rightarrow \mathcal{V}_1 \subset N$. The pull back of ξ by ψ is a

contact structure on $\mathbb{S}^1 \times U$ which is contactomorphic (via Moser's argument) to the standard contact structure $d\theta - pdq$ on $\mathbb{S}^1 \times U$. Therefore, the above diffeomorphism ψ can be chosen as a contactomorphism.

Then the symplectization $\mathcal{S}\gamma$ of the closed characteristic gives a symplectic embedding $\mathcal{S}\mathbb{S}^1 \hookrightarrow \mathcal{S}N$. This embedding admits $\mathcal{S}(\mathbb{S}^1 \times U)$ as a neighborhood. Moreover, if we denote s, θ and x the coordinates in $\mathcal{S}(\mathbb{S}^1 \times U)$, ψ has been constructed so that s and θ are conjugated variables and the direction of x is symplectically orthogonal to those of s and θ . That will allow the following computations.

Just like in the above example, we have a pseudo-representation of \mathfrak{g} if we consider

$$\begin{aligned}(\rho_n(f))(s, \theta, x) &= \frac{\chi(x)e^{s/2}}{\sqrt{n}} \cos(n\theta), \\(\rho_n(g))(s, \theta, x) &= \frac{\chi(x)e^{s/2}}{\sqrt{n}} \sin(n\theta),\end{aligned}\tag{1}$$

and $(\rho_n(h))(s, \theta, x) = 2\chi(x)^2$. Indeed, we have again $\{\rho_n(f), \rho_n(g)\} = \rho_n(h)$, but its limit ρ satisfies $\{\rho(f), \rho(g)\} = 0 \neq 1 = \rho(h)$ and is not a representation. The fact that the elements of $\mathcal{P}_{\rho_n}(f, g)$ and $\mathcal{P}_\rho(f, g)$ have complete flows follows from the case $d = 1$. \square

Proof of Corollary 4 Let M be a smooth manifold, and choose a Riemannian metric on it. Then, consider the symplectization $\mathcal{S}ST^*M$ of the sphere cotangent bundle ST^*M . The cotangent bundle can be seen as the compactification of $\mathcal{S}ST^*M$, the set at infinity being the zero section of T^*M (or $\{-\infty\} \times ST^*M$ if we see $\mathcal{S}ST^*M$ as $\mathbb{R} \times ST^*M$).

The Reeb flow of ST^*M projects itself to the geodesic flow on M , and the closed characteristics are exactly the trajectories that project themselves to closed geodesics. Since any closed manifold carries a closed geodesic (see [8]), we can consider Example (1). It clearly extends to the compactification (the Hamiltonian functions involved and all their derivatives converges to 0 when s goes to $-\infty$), and we can achieve the proof as for Theorem 3. \square

A A proof of Gromov-Eliashberg theorem.

In this section, we show how our methods allow to recover Gromov-Eliashberg Theorem.

Theorem 9 (Gromov, Eliashberg). *The group of compactly supported symplectomorphisms $\text{Symp}_c(\mathbb{R}^{2n})$ is C^0 -closed in the group of all diffeomorphisms of \mathbb{R}^{2n} .*

Proof. Let ϕ_n be a sequence of diffeomorphisms that converges uniformly to a diffeomorphism ϕ . Denote $(f_i^n), (g_i^n)$ (resp. f_i, g_i) the coordinate functions of ϕ_n (resp. ϕ). These coordinate functions can be seen as Hamiltonian functions linear at infinity (i.e., that can be written $H + u$ with $H \in Ham_c$ and u linear map). Moreover, for a given sequence (f_i^n) or (g_i^n) , the linear part does not depend on n .

Since ϕ_n is symplectic, we have:

$$\{f_i^n, g_j^n\} = \delta_{ij}, \quad \{f_i^n, f_j^n\} = \{g_i^n, g_j^n\} = 0.$$

Thus the coordinate functions of ϕ_n give a pseudo-representation of the 2-nilpotent Lie algebra \mathfrak{g} generated by elements a_i, b_i, c , with the relations

$$[a_i, b_j] = \delta_{ij}, \quad [a_i, a_j] = [b_i, b_j] = 0, \quad \text{and} \quad [a_i, c] = [b_i, c] = 0.$$

Since ϕ is symplectic if and only if

$$\{f_i, g_j\} = \delta_{ij}, \quad \{f_i, f_j\} = \{g_i, g_j\} = 0$$

the proof will be achieved if we prove that the limit of this pseudo-representation is a representation. Consequently, we have to adapt the proof of Theorem 1 to the case of Hamiltonian functions linear at infinity, for 2-nilpotent Lie algebras. Gromov-Eliashberg Theorem then follows from the next two lemmas.

Lemma 10. *Let u, v be two linear forms on \mathbb{R}^{2n} and H_n, K_n be compactly supported Hamiltonians, such that*

$$H_n \rightarrow H, \quad K_n \rightarrow K, \quad \{H_n + u, K_n + v\} \rightarrow 0.$$

Then $\{H + u, K + v\} = 0$.

Lemma 11. *Let u, v, w be linear forms on \mathbb{R}^{2n} , and H_n, K_n, G_n , be compactly supported Hamiltonians such that*

$$\begin{aligned} H_n &\rightarrow H, \quad K_n \rightarrow K, \quad G_n \rightarrow G, \\ \{H_n + u, G_n + w\} &\rightarrow 0, \\ \{K_n + v, G_n + w\} &\rightarrow 0, \\ \{H_n + u, K_n + v\} - (G_n + w) &\rightarrow 0. \end{aligned}$$

Then $\{H + u, G + w\} = 0$, $\{K + v, G + w\} = 0$ and $\{H + u, K + v\} = G + w$.

Let us consider a C^{-1} biinvariant distance γ on $\mathcal{H}_c(\mathbb{R}^{2n})$ which is invariant under the action of linear at infinity Hamiltonians (such a condition is clearly satisfied by Hofer's distance). For a sequence of Hamiltonian functions that are linear at infinity with the same linear part, we can speak of its limit for γ by setting:

$$(\phi_{H_n+u}) \xrightarrow{\gamma} \phi_{H+u} \text{ if and only if } \gamma((\phi_{H+u})^{-1}\phi_{H_n+u}, Id) \rightarrow 0.$$

Moreover, if $(\phi_{H_n+u}) \xrightarrow{\gamma} \phi_{H+u}$ and $(\phi_{K_n+v}) \xrightarrow{\gamma} \phi_{K+v}$ then

$$(\phi_{H_n+u}\phi_{K_n+v}) \xrightarrow{\gamma} \phi_{H+u}\phi_{K+v}.$$

Indeed, we have

$$\begin{aligned} & \gamma((\phi_{H_n+u}\phi_{K_n+v})^{-1}(\phi_{H+u}\phi_{K+v}), Id) \\ &= \gamma(\phi_{K+v}^{-1}(\phi_{H+u}^{-1}\phi_{H_n+u})\phi_{K+v}(\phi_{K+v}^{-1}\phi_{K_n+v}), Id) \\ &\leq \gamma(\phi_{H+u}^{-1}\phi_{H_n+u}, Id) + \gamma(\phi_{K+v}^{-1}\phi_{K_n+v}, Id). \end{aligned}$$

Finally notice that if $\|H_n - H\|_{C^0} \rightarrow 0$, then $\phi_{H_n+u} \xrightarrow{\gamma} \phi_{H+u}$.

We are now ready for our proofs.

Proof of lemma 10. We just adapt the proof of Cardin and Viterbo [1] to the "linear at infinity" case.

First remark that the assumptions imply $\{u, v\} = 0$. Then, a simple computation shows that the flow

$$\psi_n^t = \phi_{H_n+u}^t \phi_{K_n+v}^s \phi_{H_n+u}^{-t} \phi_{K_n+v}^{-s}$$

is generated by the Hamiltonian function linear at infinity

$$\int_0^s \{H_n + u, K_n + v\}(\phi_{K_n+v}^\sigma \phi_{H_n+u}^t(x)) d\sigma,$$

which C^0 -converges to $0 = \{u, v\}$ by assumption. Therefore, ψ_n^t converges for any s and any t to Id . But on the another hand, according to the above remark, it converges to $\phi_{H+u}^t \phi_{K+v}^s \phi_{H+u}^{-t} \phi_{K+v}^{-s}$. Hence $\phi_{H+u}^t \phi_{K+v}^s \phi_{H+u}^{-t} \phi_{K+v}^{-s} = Id$ which proves $\{H + u, K + v\} = 0$. \square

Proof of lemma 11. First notice that the assumptions imply $\{u, v\} = w$, $\{u, w\} = 0$ and $\{v, w\} = 0$, and that the equalities $\{H + u, G + w\} = 0$, $\{K + v, G + w\} = 0$ follow from lemma 10. Here we consider the flow

$$\psi_n^t = \phi_{G_n+w}^{-ts} \phi_{H_n+u}^t \phi_{K_n+v}^s \phi_{H_n+u}^{-t} \phi_{K_n+v}^{-s}$$

which is generated by

$$\left(-s(G_n + w) + \int_0^s \{H_n + u, K_n + v\}(\phi_{K_n+v}^\sigma \phi_{H_n+u}^t) d\sigma\right) \circ \phi_{G_n+w}^{ts}.$$

This expression can be written

$$\left(\int_0^s (A_n + B_n) d\sigma\right) \circ \phi_{G_n+w}^{ts},$$

where $A_n = G_n - G_n(\phi_{K_n+v}^\sigma \phi_{H_n+u}^t)$ and $B_n = (\{H_n + u, K_n + v\} - (G_n + w))(\phi_{K_n+v}^\sigma \phi_{H_n+u}^t)$.

By assumption, B_n C^0 -converges to 0 and A_n can be written:

$$\begin{aligned} A_n &= (G_n - G_n(\phi_{H_n+u}^t)) + (G_n - G_n(\phi_{K_n+v}^\sigma)) \circ \phi_{H_n+u}^t \\ &= \int_0^t \{G_n, H_n + u\} d\tau + \left(\int_0^\sigma \{G_n, K_n + v\} d\tau\right) \circ \phi_{H_n+u}^t \\ &= \int_0^t \{G_n + w, H_n + u\} d\tau + \left(\int_0^\sigma \{G_n + w, K_n + v\} d\tau\right) \circ \phi_{H_n+u}^t, \end{aligned}$$

which implies that A_n C^0 -converges to 0 too. It follows that the generating Hamiltonian of ψ_n^t C^0 -converges to 0, and hence that ψ_n^t γ -converges to Id . Since it also converges to $\psi^t := \phi_{G+w}^{-ts} \phi_{H+u}^t \phi_{K+v}^s \phi_{H+u}^{-t} \phi_{K+v}^{-s}$, we get $\psi^t = Id$ for any s and t . Thus, the generating Hamiltonian of ψ_t vanishes identically:

$$\left(-s(G + w) + \int_0^s \{H + u, K + v\}(\phi_{K+v}^\sigma \phi_{H+u}^t) d\sigma\right) \circ \phi_{G+w}^{ts} = 0.$$

But since $G + w$ commutes with $H + U$ and $K + v$, we get:

$$\int_0^s (\{H + u, K + v\} - (G + w))(\phi_{K+v}^\sigma \phi_{H+u}^t) d\sigma = 0.$$

Taking derivative with respect to s , we obtain $\{H + u, K + v\} - (G + w) = 0$. \square

B Few additional remarks using the theory of distributions.

The following results on Poisson brackets are obtained with the help of distributions. No assumptions are made on the Lie algebra generated by the Hamiltonian functions. They show in a certain way why it is difficult to find examples of pseudo-representations whose limit is not a representation.

Proposition 12. *If F_n C^2 -converges to F and G_n C^0 -converges to G . Then, $\{F_n, G_n\}$ converges to $\{F, G\}$ in the sense of distributions. As a consequence, if $\{F_n, G_n\}$ C^0 -converges to H , then $\{F, G\} = H$.*

Proof. For any smooth compactly supported function ϕ ,

$$\begin{aligned} \langle \{F_n, G_n\}, \phi \rangle &= \int \frac{\partial G_n}{\partial q} \frac{\partial F_n}{\partial p} \phi - \int \frac{\partial G_n}{\partial p} \frac{\partial F_n}{\partial q} \phi \\ &= - \int G_n \frac{\partial}{\partial q} \left(\frac{\partial F_n}{\partial p} \phi \right) + \int G_n \frac{\partial}{\partial p} \left(\frac{\partial F_n}{\partial q} \phi \right). \end{aligned}$$

By assumption, the integrands C^0 -converge and hence the integrals converge to $-\int G \frac{\partial}{\partial q} \left(\frac{\partial F}{\partial p} \phi \right) + \int G \frac{\partial}{\partial p} \left(\frac{\partial F}{\partial q} \phi \right)$ which equals $\langle \{F, G\}, \phi \rangle$. \square

Proposition 13. *If F_n C^0 -converges to F , G_n C^0 -converges to G and $\{F_p, G_q\}$ C^0 -converges to H when p and q go to infinity, then $\{F, G\} = H$.*

Proof. Take once again a compactly supported smooth function ϕ . Write

$$\langle \{F_p, G_q\} - \{F, G\}, \phi \rangle = \langle \{F_p - F, G_q\}, \phi \rangle + \langle \{F, G_q - G\}, \phi \rangle.$$

By Proposition 12, the first term converges to 0. Hence for all $\varepsilon > 0$, there exists an integer q_0 such that for any $q > q_0$, $|\langle \{F, G_q - G\}, \phi \rangle| \leq \varepsilon$.

Similarly, for each fixed q , there exists an integer p_0 such that for any $p > p_0$, $|\langle \{F_p - F, G_q\}, \phi \rangle| \leq \varepsilon$.

Therefore, for all ε and all integers p_1, q_1 , we can find $p > p_1, q > q_1$ such that $|\langle \{F_p, G_q\} - \{F, G\}, \phi \rangle| \leq 2\varepsilon$.

Thus we can construct two extractions χ, ψ such that $\langle \{F_{\chi(n)}, G_{\psi(n)}\} - \{F, G\}, \phi \rangle$ converges to 0. Since we have $\langle \{F_{\chi(n)}, G_{\psi(n)}\} - H, \phi \rangle \rightarrow 0$, it implies $\langle \{F, G\}, \phi \rangle = \langle H, \phi \rangle$, and this equality holds for any ϕ . \square

Acknowledgments. I warmly thank my supervisor Claude Viterbo for all his advices and for hours of fruitful discussion. I also thank Nicolas Roy for innumerable interesting conversations on multiple subjects.

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