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# Sofic trace subshift of a cellular automaton<sup>\*</sup>

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**Abstract.** The trace subshift of a cellular automaton is the subshift of all possible columns that may appear in a space-time diagram. In this paper we study conditions for a sofic subshift to be the trace of a cellular automaton.

**Keywords:** discrete-time dynamical systems, cellular automata, symbolic dynamics, sofic systems, formal languages.

## 1 Introduction

Cellular automata are well-known formal models for complex systems. They are used in a huge variety of different scientific fields including mathematics, physics and computer science.

A *cellular automaton* (CA) consists in an infinite number of identical *cells* arranged on a regular lattice indexed by  $\mathbb{Z}$ . Each cell is a finite automaton which *state* takes value in a finite set  $A$ . All cells evolve synchronously according to their own state and those of their neighbors.

The study and classification of the evolutions of cellular automata is one of the standing open problems in the field [1,2,3,4,5,6]. Indeed, the simple definition of CA contrasts the wide variety of their evolutions. An interesting idea is to classify these behaviors according to some notion of complexity. Of course, the word “complexity” means different things to different researchers. For this reason, in literature one finds classifications according to topological entropy, measure theory, dimension theory, attractors, algorithmic complexity, etc.

In this paper we follow a formal languages approach. Each CA is associated with a language. The idea is that the more complex is the language, the more complex is the automaton.

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The associated language is defined as follows (see Section 2 for more precise definitions). Each CA can be seen as a discrete dynamical system  $(A^{\mathbb{Z}}, F)$ , where  $F$  is the global function. Let  $\beta = \beta_1, \beta_2, \dots, \beta_k$  a finite partition of  $A^{\mathbb{Z}}$ . Then an orbit of initial condition  $x$ ,  $\mathcal{O}_F(x) = (x, F(x), \dots, F^n(x), \dots)$  can be associated with the infinite word  $w$  such that  $\forall n \in \mathbb{N}, w_n = i$  if  $F^n(x) \in \beta_i$ . Then,  $w$  is the  $\beta$ -trace of  $F$  with initial condition  $x$ . The  $\beta$ -trace set  $\Sigma$  of  $F$  is the set of  $\beta$ -traces with all possible initial conditions. Remark that when  $A^{\mathbb{Z}}$  is endowed with the Cantor topology (see Section 2),  $\Sigma$  is a closed shift-stable set i.e. a subshift. The language  $\mathcal{L}(\Sigma)$  of factors occurring in configurations of  $\Sigma$  is the language associated with  $(A^{\mathbb{Z}}, F)$ .

In [7], Kůrka classified factor subshifts of CA according to their language complexity. He devised three classes: bounded periodic; regular but not bounded periodic; not regular. In this paper we address a somewhat complementary question, namely, given a subshift  $\Sigma$  of a certain language complexity we wonder if it can be the trace of a CA.

Motivations come both from classical symbolic dynamics but also from physics. Indeed, when observing natural phenomena due to physical constraints, one can keep trace only of a finite number of measurements. This set of measurements, usually, takes into account only a minor part of the parameters ruling the phenomenon under investigation. Hence, to some extent, what is observed is the “trace” of the phenomenon left on the “instruments” rather than the whole phenomenon in its globality.

It is a very important issue (Galilean principle) to find a formal model which can reproduce the observed trace. Restating those reasonings in our context: given a subshift  $\Sigma$ , one wonders which discrete dynamical system can produce it. In particular, one can ask if there exists a CA having  $\Sigma$  as a trace.

Giving a complete answer to this question seems very hard. In this paper we give some sufficient conditions for a regular language to be traceable. The proof is constructive. We believe that the construction of the CA is of some interest in its own.

Because of the lack of space some of the proofs are omitted. They can be found in the Appendices.

## 2 Definitions

Let  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . For  $i, j \in \mathbb{N}$  with  $i \leq j$ ,  $[i, j]$  denotes the set of integers between  $i$  and  $j$ . For any function  $F$  from  $A^{\mathbb{Z}}$  into itself,  $F^n$  denotes the  $n$ -fold composition of  $F$  with itself. A set  $S \subset A^{\mathbb{Z}}$  is  $F$ -stable if  $F(S) \subset S$ .

*Languages.* Let  $A$  be a finite *alphabet* with at least two *letters*. A *word* is a finite sequence of letters  $w = w_0 \dots w_{|w|-1} \in A^*$ . Its *reverse*  $\bar{w}$  is  $w_{|w|-1} \dots w_0$  and its *rotation*  $\gamma(w)$  is  $w_1 \dots w_{|w|-1} w_0$ . A *factor* of a word  $w = w_0 \dots w_{|w|-1} \in A^*$  is a word  $w_{[i,j]} = w_i \dots w_j$ , for  $0 \leq i \leq j < |w|$ . We note  $w_{[i,j]} \sqsubset w$ . The empty word is denoted by  $\varepsilon$ . Given two languages  $C, D \subset A^*$ ,  $CD$

denotes their concatenation,  $C + D$  their union,  $C^* = \bigcup_{n \in \mathbb{N}} C^n$ , and  $C^\omega = \{z \in A^\mathbb{N} \mid \forall j \in \mathbb{N}, \exists k \geq j, z_{[0, k-1]} \in C^*\}$ . When no confusion is possible, given a word  $w$ , we also denote  $w$  the language  $\{w\}$ .

*Configurations.* A *configuration* is a biinfinite sequence of letters  $x \in A^\mathbb{Z}$ . The set  $A^\mathbb{Z}$  of configurations is the *phase space*. The definition of *factor* can be naturally extended to configurations: for  $x \in A^\mathbb{Z}$  and  $i \leq j$ ,  $x_{[i, j]} = x_i \dots x_j \sqsubset x$ . If  $u \in A^*$ , then  $u^\omega$  is the infinite word consisting in periodic repetitions of  $u$ , and  ${}^\omega u^\omega$  is the configuration consisting in periodic repetitions of  $u$ .

*Topology.* We endow the phase space with the *Cantor topology*. A base for open sets is given by cylinders. For  $j, k \in \mathbb{N}$  and a finite set  $W$  of words of length  $j$ , we will note  $[W]_k$  the *cylinder*  $\{w \in A^\mathbb{Z} \mid w_{[k, k+j-1]} \in W\}$ .  $[W]_k^C$  is the complement of the cylinder  $[W]_k$ .

*Cellular automata.* A (one-dimensional) *cellular automaton* is a parallel synchronous computation model consisting in cells distributed over a regular lattice  $\mathbb{Z}$ . Each cell has a state in the finite alphabet  $A$ , which evolves depending on the state of their neighbors according to a *local rule*  $f : A^d \rightarrow A$ , where  $m \in \mathbb{Z}$  and  $d \in \mathbb{N}^*$  are the *anchor* and the *diameter* of the CA, respectively. The *global function* of the CA is  $F : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$  such that  $F(x)_i = f(x_{[i-m, i-m+d]})$  for every  $x \in A^\mathbb{Z}$  and  $i \in \mathbb{Z}$ . The *space-time diagram* of initial configuration  $x \in A^\mathbb{Z}$  is the sequence of the configurations of the orbit  $(F^j(x))_{j \in \mathbb{N}}$ . Usually they are graphically represented by a two-dimensional diagram like in Figure 1.

The *shift map*  $\sigma : A^\mathbb{Z} \rightarrow A^\mathbb{Z}$  is a particular CA global function defined by  $\sigma(x)_i = x_{i+1}$  for every  $x \in A^\mathbb{Z}$  and  $i \in \mathbb{Z}$ . According to the Hedlund theorem [8], the global functions of CA are exactly the continuous self-maps of  $A^\mathbb{Z}$  commuting with the shift map.

Any local rule  $f$  of a CA can be extended naturally to an application on words  $f(w) = (f(w_{[i, i+d-1]}))_{0 \leq i \leq |w|-d}$ , for all  $w \in A^* A^d$ .

*Dynamical systems.* A *dynamical system* is a couple  $(X, F)$  where  $X$  is a set called the *phase space*, and  $F : X \rightarrow X$  is a continuous self-map.  $(Y, F)$  is a *subsystem* of  $(X, F)$  if  $Y$  is a closed  $F$ -stable subset of  $X$ . A set  $Y$  is *F-stable* if  $F(Y) \subset Y$ .

*Morphisms.* A *morphism* of the dynamical system  $(X, F)$  into the dynamical system  $(Y, G)$  is a continuous map  $\phi : X \rightarrow Y$  such that  $\phi \circ F = G \circ \phi$ . A *conjugacy* (resp. a *factorization*) is a bijective (resp. surjective) morphism; in that case we say that  $(Y, G)$  is conjugate to (resp. a factor of)  $(X, F)$  (we expect the reader not to confuse between the use of the word “factor” in the dynamical system context and the language theory one).

*Subshifts.* The *onesided shift*, also noted  $\sigma$  is the self-map of  $A^{\mathbb{N}}$  such that  $\sigma(z)_i = z_{i+1}$ , for every  $z \in A^{\mathbb{N}}$  and  $i \in \mathbb{N}$ . A *onesided subshift* (or simply a *subshift*)  $\Sigma \subset A^{\mathbb{N}}$  is a  $\sigma$ -stable closed set of infinite words. The language of  $\Sigma$  is  $\mathcal{L}(\Sigma) = \{w \in A^* \mid \exists z \in \Sigma, w \sqsubset z\}$ , and characterizes  $\Sigma$ , since  $\Sigma = \{z \in A^{\mathbb{N}} \mid \forall w \sqsubset z, w \in \mathcal{L}(\Sigma)\}$ . The *alphabet* of the subshift  $\Sigma$  is the set  $\{a \in A \mid \exists z \in \Sigma, az \in \Sigma\}$  i.e. the set of letters that appear in infinite words belonging to  $\Sigma$ . A subshift  $\Sigma$  is *transitive* if for every words  $u, v \in \mathcal{L}(\Sigma)$ , there exists  $w \in A^*$  such that  $uwv \in \mathcal{L}(\Sigma)$ . A subshift can also be characterized by a language  $\mathcal{F} \subset A^*$  of *forbidden words*, i.e. such that  $\Sigma = \{z \in A^{\mathbb{N}} \mid \forall u \in \mathcal{F}, u \not\sqsubset z\}$ . A subshift is of *finite type* (SFT for short) if it has a finite language of forbidden words. It is a  $k$ -SFT (for  $k \in \mathbb{N}$ ) if it has a finite set of forbidden words of length  $k$ . A subshift  $\Sigma$  is *sofic* if  $\mathcal{L}(\Sigma)$  is a regular language. The following characterization of sofic subshifts will be very useful in the sequel. For more about subshifts, see for instance [9].

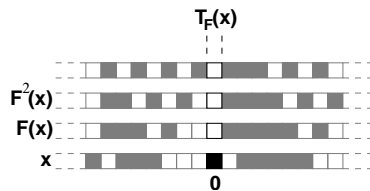
**Theorem 1 (Weiss [10]).** *A subshift is sofic if and only if it is a factor of a SFT.*

Hedlund's theorem can be extended as follows.

**Theorem 2 (Hedlund [8]).** *A function  $\phi$  is a morphism of a subshift  $(X, \sigma)$  on alphabet  $A$  into a subshift  $(Y, \sigma)$  on alphabet  $B$  if and only if there is a radius  $r \in \mathbb{N}$  and a local rule  $f : A^{r+1} \rightarrow B$  such that  $\forall j \in \mathbb{N}, \forall x \in X, \phi(x)_j = f(x_j \dots x_{j+r})$  (we say  $\phi$  is an  $r$ -block map).*

### 3 Traces

In this section we define the main notion introduced in the paper, namely, the trace of a CA and the traceability of a subshift. Moreover, we give a simple necessary condition for a subshift being traceable.



**Fig. 1.** The trace seen on the space-time diagram.

**Definition 1 (Trace).** *Given a CA  $F$ , the trace applications are defined for  $k \in \mathbb{Z}$  by  $T_F^k(x) = (F^j(x))_{j \in \mathbb{N}}$ . In other words,  $T_F^k(x)$  is the  $k^{\text{th}}$  column of the space-time diagram of initial configuration  $x$  (see Figure 1). We note  $T_F = T_F^0$ . We say that  $T_F(x)$  is the trace of  $F$  with initial condition  $x$ .*

The study of trace applications can be reduced to the study of  $T_F$ , because of shift-invariance of CA.

It can be noticed that this notion of trace corresponds to the  $\beta$ -trace as defined in the introduction, with  $\beta = \{[a] \mid a \in A\}$  being the partition of  $A^{\mathbb{Z}}$  into cylinders of width 1.

**Definition 2 (Traceability).** *The trace subshift of a CA  $F$  is  $\tau(F) = T_F(A^{\mathbb{Z}})$ . It is a factor subshift of  $(A^{\mathbb{Z}}, F)$ , since  $T_F$  is continuous and commutes with  $\sigma$ . A subshift  $\Sigma$  is traceable if there exists a CA  $F$  for which  $\Sigma = \tau(F)$ .*

We begin with a condition for the traceability of a subshift. Proposition 2 proves that it is necessary.

**Definition 3 (T0 subshift).** *A subshift is T0 if it includes a 2-SFT with the same alphabet.*

**Proposition 1.** *A subshift is T0 if and only if there exists a map  $\phi : A \rightarrow A$  such that for every letter  $a \in A$ ,  $(\phi^j(a))_{j \in \mathbb{N}} \in \Sigma$ .*

*Proof.* If  $\forall a \in A, (\phi^j(a))_{j \in \mathbb{N}} \in \Gamma$ , then  $\{(\phi^j(a))_{j \in \mathbb{N}} \mid a \in A\}$  is a 2-SFT of alphabet  $A$  and set of forbidden words  $\bigcup_{a \in A} a(A \setminus \{\phi(a)\})$ . Conversely, consider a 2-SFT  $\Gamma$  of alphabet  $A$ . Define  $\phi(a) = b$  such that  $ab \in \mathcal{L}(\Gamma)$ . Since  $\Gamma$  is a 2-SFT, we have that  $\forall a \in A, (\phi^j(a))_{j \in \mathbb{N}} \in \Gamma$ .  $\square$

*Example 1.* Consider the following subshifts :

- $\Sigma = (1 + \varepsilon)(01)^\omega$ ; it is finite T0 (with  $\phi(0) = 1$  and  $\phi(1) = 0$ );
- $\Sigma = 0^\omega + 0^*1^\omega$ ; it is infinite T0 (with  $\phi(0) = \phi(1) = 1$ );
- $\Sigma = (01 + 1 + \varepsilon)(001)^\omega$ ; it is finite but not T0.

**Proposition 2.** *The trace subshift of a CA is T0.*

*Proof.* Consider a CA  $F$ . For any  $a \in A$ , define  $\phi(a)$  as  $F^{(\omega a^\omega)}_0$ . Then  $(\phi^j(a))_{j \in \mathbb{N}} = T_F^{(\omega a^\omega)} \in \tau(F)$ . By Proposition 1,  $\tau(F)$  is T0.  $\square$

**Theorem 3.** *Any 2-SFT is traceable.*

*Proof.* Consider a 2-SFT  $\Sigma$  and let  $A$  be its alphabet. For each  $a \in A$ , define  $\phi(a) = b$  such that  $ab \in \mathcal{L}(\Sigma)$ . Consider the CA of anchor 0, diameter 2 and local rule  $f$  defined by  $f(x_0, x_1) = x_1$  if  $x_0x_1 \in \mathcal{L}(\Sigma)$ , and  $\phi(x_0)$  otherwise. If  $x \in A^{\mathbb{Z}}$ , then the definition of the rule gives that every factor of length 2 of its trace is in  $\mathcal{L}(\Sigma)$ . Conversely, if  $z \in \Sigma$ , then we can see by induction that  $\Sigma$  is the trace of  $F$  with initial condition  $x$  as soon as  $x_{[0, +\infty)} = z$ .  $\square$

## 4 $k$ -traceability

In order to establish finer results we first need a weaker condition for traceability, namely  $k$ -traceability. A subshift of alphabet  $A$  is  $k$ -traceable if it is the set of columns of a CA on the alphabet  $A^k$ . The difference between a CA tracing a subshift of alphabet  $A$  and one tracing a subshift of alphabet  $A^k$  is that the rule of the latter can use the knowledge of the position of a letter of  $A$  in a word over  $A^k$ . This results in a much simpler construction.

*Notation.* If  $\Sigma$  is a subshift on an alphabet  $B \subset A^k$ , and  $q \in [0, k-1]$ , then the  $q^{\text{th}}$  projection is defined as

$$\pi_q : \begin{array}{c} B^{\mathbb{N}} \rightarrow A^{\mathbb{N}} \\ (z_j)_{j \in \mathbb{N}} \mapsto ((z_j)_q)_{j \in \mathbb{N}} \end{array} .$$

We also note  $\pi(\Sigma) = \bigcup_{0 \leq q < k} \pi_q(\Sigma)$ , which is a subshift on  $A$ .

In this section we will limit our study to onesided CA, i.e. with anchor 0.

**Definition 4 ( $k$ -traceability).** Given a CA  $F$  on the alphabet  $B \subset A^k$ , the  $k$ -trace subshift is defined by  $\overset{\circ}{\tau}(F) = \bigcup_{0 \leq q < k} \{(F^j(x)_0)_q\}_{j \in \mathbb{N}} \mid x \in B^{\mathbb{Z}}\} = \pi(\tau(F))$ .

A subshift is  $k$ -traceable if it is the  $k$ -trace of a onesided CA on the alphabet  $B \subset A^k$ .

Similarly to what done in the previous section we give a necessary condition for being  $k$ -traceable.

**Definition 5 (T1 subshift).** A subshift  $\Sigma$  on the alphabet  $A$  is T1 if there exists  $k \in \mathbb{N}^*$  and a 2-SFT  $\Gamma \subset (A^k)^{\mathbb{N}}$ , such that  $\pi_0(\Gamma) = \pi(\Gamma) = \Sigma$  (in particular,  $\Sigma$  is a factor of  $\Gamma$ ).

**Theorem 4.** A T1 subshift is  $k$ -traceable for some  $k \in \mathbb{N}^*$ .

*Proof.* By Theorem 3, the corresponding  $\Gamma \subset (A^k)^{\mathbb{N}}$  is the trace of a CA  $F$  on some alphabet  $B \subset A^k$ . Hence,  $\overset{\circ}{\tau}(F) = \pi(\tau(F)) = \pi(\Gamma) = \Sigma$ .  $\square$

*Example 2.* Consider the subshift  $\Sigma = (01 + 1 + \varepsilon)(001)^\omega$ . It is T1 (define the 2-SFT  $\Gamma = (uvw)^\omega$  on the alphabet  $B = \{u, v, w\} \subset A^3$ , where  $u = 001$ ,  $v = 010$  and  $w = 100$ ). It is thus 3-traceable, but not traceable since it is not T0.

**Theorem 5.** Any SFT is T1.

*Proof.* Let  $\Sigma$  be a  $k$ -SFT for some  $k \in \mathbb{N}^*$ .  $\Gamma = \{(z_{[j, j+k-1]})_{j \in \mathbb{N}} \mid z \in \Sigma\}$  is a 2-SFT, and  $\pi(\Gamma) = \bigcup_{0 \leq q < k} \{(z_{j+q})_{j \in \mathbb{N}} \mid z \in \Sigma\} = \bigcup_{0 \leq q < k} \sigma^q(\Sigma) = \Sigma = \pi_0(\Gamma)$ .  $\square$

This result allows us to prove the next proposition, which is a less restrictive condition for being T1.

**Proposition 3.** A subshift  $\Sigma$  is T1 if and only if it is a factor of a SFT  $\Gamma$  on alphabet  $A^k$  for some  $k \in \mathbb{N}^*$  such that  $\pi(\Gamma) \subset \Sigma$ .

Now we extend the results on  $k$ -traceability to sofic subshifts (with some additional properties).

**Definition 6 (T2 subshift).** A subshift is T2 if it is sofic and includes an infinite transitive subshift.

**Theorem 6.** Any T2 subshift is T1.

The proof of Theorem 6 is given using the following lemmata.

**Lemma 1.** *A sofic transitive subshift on alphabet  $A$  is infinite if and only if for all  $n \geq 2$ , it includes a subshift  $B^\omega$  with  $B \subset A^k$ ,  $|B| \geq n$  and some  $k \in \mathbb{N}^*$ .*

**Lemma 2.** *If  $\Sigma$  is a factor subshift of a SFT  $\Gamma$  on alphabet  $B$  such that  $B^\omega \subset \Sigma$ , then  $\Sigma$  is T1.*

*Proof (of Theorem 6).* Let  $\Sigma$  a T2 subshift. From Theorem 1, it is a factor of a SFT  $\Gamma$ . Thanks to Lemma 1, there is an arbitrarily large set of words  $B$  on  $A$  such that  $B^\omega \subset \Sigma$ , so we can assume without loss of generality that  $\Gamma$  is a subshift on such an alphabet  $B \subset A^k$  for some  $k \in \mathbb{N}^*$ . From Lemma 2, we conclude that  $\Sigma$  is T1.  $\square$

## 5 From $k$ -trace to trace

In the previous section, we gave a sufficient condition for a particular subshift  $\Sigma$  to be  $k$ -traced by a CA  $G$  on an alphabet  $B \subset A^k$ . In this section, we show how to simulate  $G$  with another CA, on alphabet  $A$ , in such way that its trace is  $\Sigma$ . This can be done if we add a further condition to our subshift.

**Definition 7 (T3 subshift).** *A subshift  $\Sigma$  is T3 if there is a map  $\phi : A \rightarrow A$  such that for every letter  $a \in A$ ,  $(\phi^j(a))_{j \in \mathbb{N}} \in \Sigma$  (it is T0) and there is a word  $w \in A^* \setminus \phi(A)^*$  such that  $w^\omega \in \Sigma$ .*

*Example 3.* Consider the following subshifts.

- $\Sigma = (1 + \varepsilon)(01)^\omega$  is not T3.
- $\Sigma' = 0^\omega + 0^*1^\omega$  is T3 (with  $\phi(0) = \phi(1) = 1$  and  $w = 0$ ).

**Theorem 7.** *Any T3  $k$ -traceable subshift (for some  $k \in \mathbb{N}^*$ ) is traceable.*

This section presents a sketch of the proof of Theorem 7. Remark that it is well known that a CA on any alphabet can be simulated by a CA on any other alphabet (with at least two letters), provided that its diameter is wide enough. In particular, any CA on  $B \subset A^k$  can be simulated by a CA on  $A$ . Each cell can see its neighborhood as words of  $A^k$  and evolve accordingly. The problem is that all cells must have the same local rule, so they have to find from the neighborhood which *column* of the  $A^k$  simulation they are representing. This is usually done using a special *border* word to delimit the words of  $A^k$ .

In this section,  $\Sigma$  denotes a T3 subshift on alphabet which is  $k$ -traceable by a onesided CA  $G$ . Let  $\phi$  and  $w$  be as in Definition 7. Assume  $G$  has diameter 2 (the construction can easily be generalized) and local rule  $g : B^2 \rightarrow B$ .

In order to achieve the simulation, we first define border words to delimit  $A^k$  cells. We have two *execution modes*: a *simulation mode* will simulate properly the execution of the CA on alphabet  $B$ , and a *default mode* will be applied if the neighborhood contains invalid information. This adds some issues: default evolution must be in  $\Sigma$ ; border evolution must also evolve according to  $\Sigma$ ; and



we have to ensure that when a mode is applied to a cell, the same mode keeps being applied there in the following generations, since a change of mode would produce an invalid trace. These problems will be solved in the three following subsections.

### 5.1 Borders

In order to make our simulations, we need to delimit computation zones. This is obtained by using some special words called borders and defined as follows:

$$\mathcal{Y} = \left\{ a^{|w|} v \bar{v} a^{k+3|w|} \mid a \in \phi(A), v \in \mathcal{O}_\gamma(w) \right\} \subset A^l,$$

where  $\mathcal{O}_\gamma(w) = \{ \gamma^q(w) \mid 0 \leq q < |w| \}$ , and  $l = k + 6|w|$ .

Borders have the property that they cannot have a too wide overlap.

*Border evolution.* The border words of  $\mathcal{Y}$  must have an evolution in  $\Sigma$ . The following rule (of diameter 1) respects that condition:

$$\Delta_{\mathcal{Y}} : \begin{array}{c} \mathcal{Y} \rightarrow \mathcal{Y} \\ a^{|w|} v \bar{v} a^{k+3|w|} \mapsto \phi(a)^{|w|} \gamma(v) \overline{\gamma(v)} \phi(a)^{k+3|w|} \end{array} .$$

*Macrocells.* We will decompose our configurations into *macrocells*. A macrocell is the concatenation of a border word and a valid word of  $B$ . We can simulate the local rule  $g$  by a *macroevolution rule* (local rule on macrocells of  $B\mathcal{Y} \subset A^h$ , where  $h = k + l = 2k + 6|w|$ , and of diameter 2):

$$\Delta : \begin{array}{c} (B\mathcal{Y})^2 \rightarrow B\mathcal{Y} \\ (u, v) \mapsto g(u_{[0, k-1]}, v_{[0, k-1]}) \Delta_{\mathcal{Y}}(u_{[k, h-1]}) \end{array} .$$

### 5.2 Default mode

Our CA will work as follows. Valid zones (with macrocells), which evolve according to the macroevolution rule  $\Delta$  so that they remain valid zones. Invalid zones run a microdefault mode so that they remain invalid zones. Nevertheless, frontiers between the zones must not move. In the frontiers, a microdefault mode is applied in order for a macrocell to have the opportunity to evolve without taking into account its neighbors; that way, each cell will keep the same execution mode.

*Macrodefault mode.* To do so, we extend the macroevolution to a function on  $\Theta A^h$ , where  $\Theta = B\mathcal{Y}A^h \setminus \bigcup_{0 < i < h} A^i B\mathcal{Y}A^{h-i}$  because it does not take into account

overlapping macrocells. This is crucial in order to define a local rule. If the central macrocell has a neighbor macrocell in  $\Theta$ , we apply a simulation step of the CA. Otherwise, we evolve as a macrodefault mode (simulation from a monochromatic configuration):

$$\Delta : \begin{array}{c} \Theta A^h \rightarrow B\mathcal{Y} \\ u \mapsto \begin{cases} g(u_{[0, k-1]}, u_{[h, h+k-1]}) \Delta_{\mathcal{Y}}(u_{[k, h-1]}) & \text{if } u \in B\mathcal{Y}\Theta \\ g(u_{[0, k-1]}, u_{[0, k-1]}) \Delta_{\mathcal{Y}}(u_{[k, h-1]}) & \text{otherwise} \end{cases} \end{array} .$$

*Microdefault mode.* The function  $\phi$  (corresponding to the fact  $\Sigma$  is T0) allows to define a microdefault mode for a neighborhood that does not contain any macrocell. We are now able to transform the function  $\Delta$  into a local rule on  $A$ . Indeed, we can define, for anchor  $m = h - 1$  and diameter  $d = 3h - 1$ :

$$f : A^d \rightarrow A$$

$$w \mapsto \begin{cases} \Delta(u)_i & \text{if } w \in A^{m-i}uA^i, \text{ where } u \in \Theta A^h, i \in [0, h-1] \\ \phi(w_0) & \text{otherwise} \end{cases}$$

since such an integer  $i$ , and such a word  $u$  would be unique (from the construction of  $\Theta$ ). This local rule is such that  $f(A^m u A^m) = \Delta(u)$  for every  $u \in \Theta A^h$ , which is what we wanted: it can simulate in one step the behavior of our CA on  $B$ . Let  $F$  be the corresponding global rule.

The following lemma guarantees that no column changes its evolution mode.

**Lemma 3.** *The preimage of cylinder  $[B\Upsilon]$  is cylinder  $[\Theta]$ . Moreover, cylinder  $[\Theta]$  and its complementary  $[\Theta]^C$  are  $F$ -stable (in particular, we cannot create a border).*

A configuration which is a valid encoding of some  $y \in B^{\mathbb{Z}}$  (simulation mode), then its trace is some projection of the trace of  $y$ . Otherwise, microdefault and microdefault mode also produce a trace which is in  $\Sigma$ . This concludes the proof of Theorem 7.

## 6 Examples

*Example 4 (Finite untraceable T1 subshift).* No CA traces subshift  $\Sigma = \{0^\omega, (01)^\omega, (10)^\omega\}$ , even though it is T1.

*Example 5 (T1, T3, non-SFT, non-T2 subshift).* The subshift  $\Sigma = (0^*1 + 1^*)0^\omega$  is neither a SFT nor T2, but it is T1 and T3. Hence, by Theorems 4 and 7 it is traceable.

*Example 6 (Traceable non-T3 subshift).* Let  $f$  be the local rule of anchor 3 and diameter 7 such that  $f(u_{-3}000111) = 1$ ,  $f(000111u_3) = 0$ ,  $f(u_{-3}001011) = 0$ ,  $f(001011u_3) = 1$ , and  $f(u) = u_0$  otherwise. The trace subshift of the corresponding CA is  $\tau(F) = \{0^\omega, (10)^\omega, (01)^\omega, 1^\omega\}$ . In this case,  $\tau(F)$  is finite but not T3.

*Example 7 (Traceable non-sofic subshift).* Let  $F$  be the CA on alphabet  $A = \{b, r, l, w\}$  (the white, the right, the left and the wall particles, respectively) defined by the following local rule  $f$  of anchor 1 and diameter 3:

$x_{-1}x_0x_1$	$rl?$	$?rl$	$r?l$	$?w?$	$?rw$	$wl?$	$r??$	$??l$	$???$
$f(x_{-1}x_0x_1)$	$w$	$w$	$w$	$w$	$l$	$r$	$r$	$l$	$b$

where ? stands for any letter in  $A$  and the first applicable rule is used (left to right). Then,  $\tau(F)$  is not sofic.

## 7 Putting things together

In this paper, we have given sufficient conditions for a subshift to be the trace of a CA.

T1  $\Rightarrow$   $k$ -traceable (Theorem 4)

SFT  $\Rightarrow$  T1 (Theorem 5)

T2  $\Rightarrow$  T1 (Theorem 6)

T3 and  $k$ -traceable  $\Rightarrow$  traceable (Theorem 7)

The following summarizes all these results:

**Theorem 8.** *Any T3+T1, T3 SFT, or T3+T2 subshift is traceable.*

The present result follows other works on the structure that the trace of a CA can have. Here, we take the problem the other way around: we construct a CA that traces a particular kind of subshifts. Though we do not have a necessary and sufficient condition, Conditions T0, T1, T2 and T3 are a first step toward a better understanding of what makes a sofic subshift traceable or not.

Moreover, we expect our construction to be generalizable to weaker conditions. Nevertheless, the non-sofic case is still obscure. The general feeling is that it needs a completely different approach.

In [7], it is proved that every factor subshift of a CA  $F$  is a factor of some  $\beta$ -trace, where  $\beta$  is a partition of  $A^{\mathbb{Z}}$ , and every  $\beta$ -trace is a factor of some *column factor* (i.e. a subshift  $\{(F^j(x)_{[0,q-1]})_{j \in \mathbb{N}} \mid x \in A^{\mathbb{Z}}\}$ , for some  $q \in \mathbb{N}$ ). Hence we can wonder now whether this kind of result can be generalized, in particular to the *canonical factor*  $\{(F^j(x)_{[0,d-1]})_{j \in \mathbb{N}} \mid x \in A^{\mathbb{Z}}\}$  of the CA.

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## A Proofs of Section 4

*Proof (of Proposition 3).* Assume that  $\Sigma$  is a T1 subshift; then there is a 2-SFT  $\Gamma$  such that  $\pi_0(\Gamma) = \Sigma$ . As  $\pi_0$  is a factorization,  $\Sigma$  is a factor of the SFT  $\Gamma$ .

For the converse implication, assume that  $\Sigma$  is a subshift on alphabet  $A$ ,  $\Gamma$  a SFT on alphabet  $B \subset A^k$ , and  $\phi : \Gamma \rightarrow \Sigma$  a factorization, where  $k \in \mathbb{N}$  and  $\pi(\Gamma) \subset \Sigma$ . We can suppose without loss of generality that  $\pi(\Gamma) = \pi_0(\Gamma) = \Sigma$ , should we replace it by its conjugate  $\{(a_j w_j)_{j \in \mathbb{N}} \in (AB)^{\mathbb{N}} \mid (w_j)_{j \in \mathbb{N}} \in \Gamma \text{ and } (a_j)_{j \in \mathbb{N}} = \phi((w_j)_{j \in \mathbb{N}})\}$ , which is still a SFT. From Theorem 5,  $\Gamma$  is T1, i.e. there exists a 2-SFT  $\Gamma'$  on alphabet  $C \subset B^l$  for some  $l \in \mathbb{N}$  such that  $\pi_0(\Gamma') = \pi(\Gamma') = \Gamma$ . We have just used the projections with respect to alphabet  $B$  (recall that  $C \subset B^l \subset (A^k)^l$ ), which projections with respect to  $A$  itself are  $\pi_0(\Gamma) = \pi(\Gamma) = \Sigma$ . Hence, with respect to  $A$ , we get  $\pi_0(\Gamma') = \pi(\Gamma') = \Sigma$ , and  $\Sigma$  is T1.  $\square$

*Proof (of Lemma 1).* If  $|B| \geq 2$ , then  $B^\omega$  is infinite (since all words of  $B$  have the same length). Conversely, if  $\Sigma$  is an infinite transitive sofic subshift, and  $(A, Q, \delta)$  a deterministic finite automaton which recognizes the regular language  $\mathcal{L}(\Sigma)$ , then there is a state  $q \in Q$  and two letters  $a, b \in A$  such that  $\delta(a, q)$  and  $\delta(b, q)$  are defined (otherwise there would only be one possibility to extend the word from each state, and there would be at most as much infinite words in  $\Sigma$  as initial states in  $Q$ ). By transitivity, there is a word  $\tilde{u}$  and a word  $\tilde{v}$  such that  $\delta(\tilde{u}, \delta(a, q)) = \delta(\tilde{v}, \delta(b, q)) = q$ . If  $n = |a\tilde{u}| \cdot |b\tilde{v}|$ ,  $u = (a\tilde{u})^{|b\tilde{v}|}$ ,  $v = (b\tilde{v})^{|a\tilde{u}|}$ , and  $l$  is such that  $2^l \geq n$ , then  $B = \{u, v\}^l \subset (A^n)^l$  is appropriate.  $\square$

### A.1 Proof of Lemma 2

The function  $H$  defined in the following lemma allows us to define a clock of period  $2n$ : the word  $H(i)$  represents the  $i$ -th clock tick.

**Lemma 4.** *Consider an alphabet  $B \subset A^n$ , with  $|B| \geq 2$  and  $n \in \mathbb{N}^*$ . Then there is an injection  $H : [0, 2n - 1] \rightarrow A^{3n}$  such that the subshift  $\Gamma = \{(H((q + j) \bmod 2n))_{j \in \mathbb{N}} \mid q \in \mathbb{N}\}$  verifies  $\pi(\Gamma) \subset B^\omega$ .*

*Proof.* Let  $u, v \in B$ , suppose  $u_0 \neq v_0$  (other cases are obtained by rotation), and define:

$$H : \begin{array}{l} [0, 2n - 1] \rightarrow A^{3n} \\ h \mapsto \gamma^h(u)\gamma^h(uv) \end{array} .$$

Assume  $H(h_1) = H(h_2)$  for some distinct  $h_1, h_2 \in [0, n - 1]$ . Let  $h = \min(h_1 - h_2 \bmod 2n, h_2 - h_1 \bmod 2n) \in [1, n]$ . Then, by the definition of  $H$ ,  $\gamma^{h_1}(u)\gamma^{h_1}(uv) = \gamma^{h_2}(u)\gamma^{h_2}(uv)$ . Hence,  $\gamma^{h_1}(u) = \gamma^{h_2}(u)$  and  $\gamma^{h_1}(uv) = \gamma^{h_2}(uv)$ , giving  $u = \gamma^h(u)$  and  $uv = \gamma^h(uv)$ . Finally,

$$u_0 = \gamma^h(u)_{n-h} = u_{n-h} = (uv)_{n-h} = \gamma^h(uv)_{n-h} = (uv)_n = v_0 .$$

This is a contradiction, hence  $H$  is injective.

Let  $q \in [0, 2n - 1]$  and  $p \in [0, n - 1]$ . Then,

$$\begin{aligned}\pi_p((H((q + j) \bmod 2n))_{j \in \mathbb{N}}) &= (\gamma^{(q+j) \bmod 2n}(u))_{j \in \mathbb{N}} = (u_{(q+j) \bmod n})_{j \in \mathbb{N}} \\ &= \sigma^q(u^\omega) \in B^\omega .\end{aligned}$$

Similarly, taking the projection  $p$  in  $[n, 3n - 1]$  we find

$$\begin{aligned}\pi_p((H(q + j \bmod 2n))_{j \in \mathbb{N}}) &= (\gamma^{q+j \bmod 2n}(uv))_{j \in \mathbb{N}} = ((uv)_{q+j \bmod 2n})_{j \in \mathbb{N}} \\ &= \sigma^q((uv)^\omega) \in B^\omega .\end{aligned}$$

□

**Lemma 5.** *A full shift  $(B^\mathbb{N}, \sigma)$ , where  $B \subset A^n$  and  $n \in \mathbb{N}^*$ , is a factor of a SFT  $\Psi$  on alphabet  $A^{4n}$  such that  $\pi(\Psi) \subset \mathcal{O}_\sigma(B^\omega)$ , seen as a subshift on  $A$ .*

*Proof.* For  $0 \leq q < n$ , we define the set of *time- $q$  encodings* as

$$Psi_q = \{ ((w_0)_j, \dots, (w_{n-1})_j)_{j \in \mathbb{N}} \in (A^n)^\mathbb{N} \mid \forall p \in [0, n - 1], w_p \in \sigma^{q-p \bmod n}(B^\omega) \}$$

and the *decoding function* as:

$$\phi_q : ((w_0)_j \dots (w_{n-1})_j)_{j \in \mathbb{N}} \in \mathbb{N} \mapsto ((w_{q+j \bmod n})_{[0, n-1]})_{j \in \mathbb{N}} .$$

$\phi_q$  is surjective. If we apply  $\sigma$ , we pass from a time- $q$  encoding to a time- $(q + 1)$  encoding:  $\sigma(\Psi_q) = \Psi_{q+1 \bmod n}$ . In order to know which step we are in (which of the first  $n$  columns to look at), we use a “clock” represented by the last  $3n$  columns as described in Lemma 4. We use the set:

$$\Psi = \bigcup_{0 \leq q < 2n} \{ (w_j, H(q + j \bmod 2n))_{j \in \mathbb{N}} \in (A^{4n})^\mathbb{N} \mid (w_j)_{j \in \mathbb{N}} \in \Psi_{q \bmod n} \} ,$$

which is a disjoint union from Lemma 4, and has the advantage of being a subshift, since  $\sigma(\Psi_{q \bmod n}) = \Psi_{q+1 \bmod n}$  and  $\sigma((H(q + j \bmod 2n))_{j \in \mathbb{N}}) = (H(q + 1 + j \bmod 2n))_{j \in \mathbb{N}}$ . The decoding

$$\phi : (w_j, H(q + j \bmod 2n))_{j \in \mathbb{N}} \mapsto \phi_{H(q \bmod n)}((w_j)_{j \in \mathbb{N}})$$

is a factorization deriving from the  $n$ -block map  $((w_0)_j \dots (w_{n-1})_j, H(q + j \bmod 2n))_{0 \leq j < n} \mapsto (w_{H(q \bmod n)})_{[0, n-1]}$ , since  $\phi(\Psi) \supset \phi_0(\Psi_0) = B^\mathbb{N}$ . By definition of  $\Psi_q$  and Lemma 4, the projections are in  $B^\omega$ . Last point, we can see that  $\Psi$  is an  $n$ -SFT, since:

$$\begin{aligned}& ((w_0)_j \dots (w_{4n-1})_j)_{j \in \mathbb{N}} \in \Psi \\ \Leftrightarrow & \exists q \in [0, 2n - 1], \begin{cases} ((w_0)_j \dots (w_{n-1})_j)_{j \in \mathbb{N}} \in \Psi_{q \bmod n} \\ ((w_n)_j \dots (w_{4n-1})_j)_{j \in \mathbb{N}} = (H(q + j \bmod 2n))_{j \in \mathbb{N}} \end{cases} \\ \Leftrightarrow & \begin{cases} \forall p \in [0, n - 1], w_p \in \sigma^{q-p \bmod n}(B^\omega) \\ \forall j \in \mathbb{N}, (w_n)_j \dots (w_{4n-1})_j = H(q + j \bmod 2n) \end{cases}\end{aligned}$$

$$\begin{aligned}
& \text{where } q = H^{-1}((w_n)_0 \dots (w_{4n-1})_0) \\
\Leftrightarrow \forall j \in \mathbb{N}, & \begin{cases} \forall p \in [0, n-1], (w_p)_{[0, n-1]} \in \sigma^{q-p \bmod n}(B^\omega)_{[0, n-1]} \\ (w_n)_1 \dots (w_{4n-1})_1 = H(q+1 \bmod 2n) \end{cases} \\
& \text{where } q = H^{-1}((w_n)_0 \dots (w_{4n-1})_0)
\end{aligned}$$

□

*Proof (of Lemma 2).* Let  $\Sigma$  be a factor subshift of a SFT  $\Gamma$  on alphabet  $B$  such that  $B^\omega \subset \Sigma$ . By Lemma 5, the full shift  $B^\mathbb{N}$ , where  $B \subset A^n$ , is a factor of a SFT  $\Psi$  on alphabet  $A^{4n}$  such that  $\pi(\Psi) = \mathcal{O}_\sigma(B^\omega)$ . Let  $\phi : \Psi \rightarrow B^\mathbb{N}$  the corresponding factorization.  $\Gamma$  is a factor of the subshift  $\Gamma' = \phi^{-1}(\Gamma)$ . Of course,  $\pi(\Gamma') \subset \pi(\Psi) = \mathcal{O}(B^\omega)$ . Moreover, it is a SFT too, since  $w \in \Gamma' \Leftrightarrow w \in \Psi$  and  $\phi(w) \in \Gamma$ . To sum up,  $\Sigma$  is a factor of the SFT  $\Gamma'$  such that  $\pi(\Gamma') \subset \Sigma$ . Hence, by Proposition 3,  $\Sigma$  is T1. □

## B Proofs of Section 5

*Remark 1.* From the definition of  $w$ , we immediately notice that border words have at least one letter that is not in  $\phi(A)$ :  $[\mathcal{T}] \cap [\phi(A)^{|w|}]_{|w|} = [\mathcal{T}] \cap [\phi(A)^{|w|}]_{2|w|} = \emptyset$ .

Here is a formalization of the property that border words cannot overlap each other too much:

**Definition 8 (Freezingness).** A language  $W \subset A^h$  is  $k$ -freezing, for some integers  $k, h \in \mathbb{N}$ , if cylinders  $[W]$  and  $[W]_i$  do not intersect for any  $i \in [1, k]$ .

*Remark 2.*  $W$  is  $k$ -freezing if and only if  $[W]_i \cap [W]_j = \emptyset$  if  $|j - i| \in [1, k]$  (we will not have overlapping border words sharing more than  $k$  letters).

**Lemma 6.** The set  $\mathcal{Y}$  of borders is  $(k + 3|w|)$ -freezing.

*Proof.*

- If  $|w| \leq i \leq k + 3|w|$ , Remark 1 gives us  $[\mathcal{Y}] \cap [\mathcal{Y}]_i \subset [\phi(A)^{k+3|w|}]_{3|w|} \cap [\mathcal{Y}]_i \subset [\phi(A)^{|w|}]_{i+|w|} \cap [\mathcal{Y}]_i = \emptyset$ .
- Suppose there are an integer  $i \in [0, |w| - 1]$ , words  $u, v \in \mathcal{O}_\gamma(w)$  and a configuration  $x \in [\phi(A)^{|w|}u\bar{u}\phi(A)^{k+|w|}] \cap [\phi(A)^{|w|}v\bar{v}\phi(A)^{k+|w|}]_i$ . Here we use the symmetry of words  $u\bar{u}$  and  $v\bar{v}$ . Let  $p = \min \{ j \in [0, l-1] \mid x_{j+|v|} \notin \phi(A) \}$ . On the one hand:

$$\begin{aligned}
p &= \min \{ j \in [0, |w| - 1] \mid u_j \notin \phi(A) \} \\
&= |w| - \max \{ j \in [0, |w| - 1] \mid \bar{u}_j \notin \phi(A) \} \\
&= 2|w| - \max \{ j \in [0, l-1] \mid x_j \notin \phi(A) \}.
\end{aligned} \tag{1}$$

On the other hand:

$$\begin{aligned}
p &= i + \min \{ j \in [0, |w| - 1] \mid w_j \notin \phi(A) \} \\
&= i + |w| - \max \{ j \in [0, |w| - 1] \mid \bar{w}_j \notin \phi(A) \} \\
&= i + 2|w| - \max \{ j \in [0, l-1] \mid x_j \notin \phi(A) \}.
\end{aligned} \tag{2}$$

Combining Equalities (1) and (2), we get  $i = 0$ .

□

The following lemma grants the columns that it produces are in  $\Sigma$ .

**Lemma 7.** *For every border  $b \in \mathcal{Y}$ , and every column  $i \in [0, l - 1]$ , the infinite word  $((\Delta_{\mathcal{Y}}^j(b))_i)_{j \in \mathbb{N}}$  is in subshift  $\Sigma$ .*

*Proof.* Consider a border word  $b = a^{|w|} \gamma^j(w) \overline{\gamma^j(w)} a^{k+3|w|}$ .

- If  $i \in [0, |w| - 1] \cup [3|w|, l - 1]$ , then  $((\Delta_{\mathcal{Y}}^j(b))_i)_{j \in \mathbb{N}} = (\phi^j(a))_{j \in \mathbb{N}} \in \Sigma$ .
- If  $i \in [|w|, 3|w| - 1]$ , then by a direct induction one finds that:

$$((\Delta_{\mathcal{Y}}^j(b))_i)_{j \in \mathbb{N}} = \sigma^{|w| - |2|w| - i + j}(w^\omega) \in \Sigma .$$

□

*Remark 3.*  $\Theta$  is  $(h - 1)$ -freezing.

The following lemma grants that in particular we will be able to apply function  $\Delta$  if there are two macrocells in the neighborhood.

**Lemma 8.**  $B\mathcal{Y}B\mathcal{Y} \subset \Theta$ .

*Proof.* First, if  $0 < i \leq k + 3|w|$ , then  $[B\mathcal{Y}B\mathcal{Y} \cap A^i B\mathcal{Y}A^{h-i}] \subset [\mathcal{Y}]_k \cap [\mathcal{Y}]_{k+i} = \emptyset$  (from Lemma 6).

Similarly, if  $k + 3|w| \leq i < h$ , then  $[B\mathcal{Y}B\mathcal{Y} \cap A^i B\mathcal{Y}A^{h-i}] \subset [\mathcal{Y}]_{h+k} \cap [\mathcal{Y}]_{i+k} = \emptyset$ . □

*Proof (of Lemma 3).* From the definition of  $f$  in execution mode and of  $\Delta$ , we notice that:

$$F([\Theta]) \subset [B\mathcal{Y}] \tag{3}$$

The apparition of a letter which is not in  $\phi(A)$  would mean we are in the left part of an execution mode around that cell: if  $x \in A^{\mathbb{Z}}$  is such that  $F(x)_0 \notin \phi(A)$ , then there is a  $j \in [0, k + 3|w| - 1]$  such that  $x \in [\Theta]_{-j}$ .

$$F^{-1}([\phi(A)]^C) \subset \bigcup_{0 \leq j < k+3|w|} [\Theta]_{-j} \tag{4}$$

Now if a configuration  $x$  has an image in cylinder  $[B\mathcal{Y}]$ , then, from Remark 1, there is a cell  $g \in [k + |w|, k + 2|w| - 1]$  such that  $F(x)_g \notin \phi(A)$ . Hence, combining with Equation (4), with  $i = g - j$ :

$$F^{-1}([B\mathcal{Y}]) \subset \bigcup_{-2|w| < i < k+2|w|} [\Theta]_i \tag{5}$$

Let  $x \in F^{-1}([B\mathcal{Y}])$ . Then  $x \in [\Theta]_i$  for some  $i \in [-2|w| + 1, k + 2|w| - 1]$  (from Equation (5)). Then  $F(x) \in [B\mathcal{Y}] \cap F([\Theta]_i) \subset [\mathcal{Y}]_k \cap [\mathcal{Y}]_{k+i}$  (from Equation (3)).  $\mathcal{Y}$  being  $(k + 3|w|)$ -freezing (Lemma 6), we can conclude that  $i = 0$ .

$$F^{-1}([B\mathcal{Y}]) \subset [\Theta] \tag{6}$$

Combining with Equation (3), we get:

$$F^{-1}([B\mathcal{Y}]) = [\Theta] \quad (7)$$

Another consequence of Equation (6) is the stability:

$$F([\Theta]^C) \subset [B\mathcal{Y}]^C \subset [\Theta]^C \quad (8)$$

Let  $x \in [\Theta]$ . In particular  $x \notin [\Theta]_i$  for  $0 < i < h$ . Hence,  $F(x) \notin [B\mathcal{Y}]_i$  (from Equation (6)), and from Equation (3),  $F(x) \in [B\mathcal{Y}]$ . Finally,  $F(x) \in [\Theta]$ .

$$F([\Theta]) \subset [\Theta] \quad (9)$$

□

**Lemma 9.** *Let  $y \in B^{\mathbb{Z}}$ , and  $x \in A^{\mathbb{Z}}$  such that  $\forall i \in \mathbb{Z}, x_{[ih, ih+k-1]} = y_i \in B$  and  $x_{[ih+k, (i+1)h-1]} \in \mathcal{Y}$ . Then for  $0 \leq q < k$ ,  $T_F^q(x) = \pi_q(\tau_G(y))$ .*

*Proof.* We can prove by induction on the generation  $j \in \mathbb{N}$ , that for any  $i \in \mathbb{N}$ ,  $F^j(x) \in [G^j(y)\mathcal{Y}]_{ih}$ . This property holds for  $j = 0$ . Now suppose it is true at generation  $j \in \mathbb{N}$ , and let us prove it for time  $j + 1$ . Let  $i \in \mathbb{N}$ .  $F^j(x) \in [B\mathcal{Y}B\mathcal{Y}]_{ih} \subset \Theta$  from the induction hypothesis and Lemma 8. Therefore we are in execution mode between cells  $ih$  and  $(i + 1)h$ :

$$\begin{aligned} F^{j+1}(x)_{[0, h-1]} &= \Delta(F^j(x)_{[ih, (i+1)h-1]}, F^j(x)_{[(i+1)h, (i+2)h-1]}) \\ &= \Delta(F^j(y)_i, F^j(y)_{i+1}) \\ &= F^{j+1}(y)_i. \end{aligned}$$

In particular,  $T_F(x) = (F^j(y)_0)_{j \in \mathbb{N}} = \pi_0(\tau_F(y))$ . □

**Lemma 10.** *The trace of every configuration  $x \in A^{\mathbb{Z}}$  is in  $\Sigma$ .*

*Proof.*

- First, if  $x \notin [\Theta]_{-i}$  for any  $i \in [0, h - 1]$ , then we will always (see Lemma 3) apply the default mode in cell 0:  $\forall i \in [0, h - 1], \forall j \in \mathbb{N}, F^j(x) \notin [\Theta]_{-i}$ , and by a trivial recurrence,  $T_F(x) = (F^j(x)_0)_{j \in \mathbb{N}} = (\phi^j(x_0))_{j \in \mathbb{N}} \in \Sigma$ .
- If  $x \in [\Theta]_{-i}$  for some  $i \in [k, h - 1]$ , then from Lemma 9,  $\forall j \in \mathbb{N}, F^j(x) \in [\Theta]_{-i} \subset [\mathcal{Y}]_{k-i}$ , and by recurrence,  $F^j(x)_0 = \Delta_{\mathcal{Y}}^j(x_{[k-i, h-i-1]})_{i-m}$ , so  $T_F(x) = (F^j(x)_0)_{j \in \mathbb{N}} = (\Delta_{\mathcal{Y}}^j(x_{[k-i, h-i-1]})_{i-m})_{j \in \mathbb{N}} \in \Sigma$  by Lemma 7.
- If  $x \in [\Theta]_{qh-i}$  for some  $i \in [0, k - 1]$  and every  $q \in \mathbb{N}$ , then from Lemma 9,  $T_F(x) \in \overset{\circ}{\tau}(G)$ .
- Otherwise, there is some  $i \in [0, k - 1]$  and some  $q \in \mathbb{N}^*$  such that  $x \in [\Theta^q \Theta^C]_{-i}$ , and thanks to Lemma 3,  $\forall j \in \mathbb{N}, F^j(x) \in [\Theta^q \Theta^C]_{-i}$ . Let  $y \in B^{\mathbb{N}}$  a configuration such that for  $0 \leq p < q$ ,  $x_{[ph-i, ph+k-i-1]} = y_p$  and for  $p \geq q$ ,  $y_p y_{q-1}$ . Then we can show by induction on  $j \in \mathbb{Z}$  that for  $0 \leq p < q$ ,  $F^j(x)_{[ph-i, ph+k-i-1]} = G^k(y)_i$ , from the definition of  $f$ . Hence,  $T_F(x) = \pi_i(T_G(y))$ .

□



## C Proofs of examples

*Proof (of Example 4).* By contradiction, assume that such a CA  $F$  exists. Let  $f$  be the corresponding local rule,  $m$  its anchor and  $d$  its diameter. Being surjective (we can immediately check that for any configuration  $x \in A^{\mathbb{Z}}$ ,  $F^2(x) = x$ ), it is balanced i.e.  $\forall a \in A, |f^{-1}(a)| = |A|^{d-1}$ . In particular,  $|f^{-1}(1)| = |A^m 0 A^{d-m-1}|$ . Moreover, from the definition of  $\Sigma$ ,  $f^{-1}(1) \subset A^m 0 A^{d-m-1}$ . Equality of cardinals gives  $A^m 0 A^{d-m-1} = f^{-1}(1)$ . Hence  $0^\omega \notin \Sigma$ , which is a contradiction.  $\square$

*Proof (of Example 5).*  $10^*$  and  $0^*1$  are included in the language of  $(0^*1 + 1^*)0^\omega$ , but not  $10^*1$ ; hence  $\Sigma$  is not a SFT.  $\Sigma$  contains two distinct transitive subshifts, namely  $0^\omega$  and  $1^\omega$ , both of which are finite; hence it is not T2. If  $\phi$  is defined by  $\forall a \in \{0, 1\}, \phi(a) = 0$ , then  $0^\omega, 1^\omega \in \Sigma$  and  $w = 1 \in \{0, 1\}^* \setminus 0^\omega$ . Hence  $\Sigma$  is T3. Finally, build a 2-SFT  $\Gamma = ((0, 1)^*(1, 1) + (1, 1)^*)(0, 0)^\omega \subset (\{0, 1\}^2)^\omega$ , and remark that  $\pi_0(\Gamma) = \Sigma$ ; therefore  $\Sigma$  is T1.  $\square$

In order to prove Example 7, we first need to prove the following lemma.

**Lemma 11.** *Let  $A$  and  $F$  be as in Example 7. Then*

$$\mathcal{L}(\tau(F)) \cap lb^*rb^*lb^*r = \bigcup_{p,q \in \mathbb{N}} (lb^{2p}rb^{2q}lb^{2p}r).$$

*Proof.* Remark that the the wall  $w$  is invariant and blocking. Now we can prove that the set of configurations whose trace has a prefix in  $lb^*rb^*lb^*r$  is

$$T_F^{-1}(lb^*rb^*lb^*rA^\omega) = \bigcup_{p,q \in \mathbb{N}} [wb^p lb^q w]_{-p-1}.$$

Intuitively, if there were no wall on the left, no particle would come from the left, since right particles cannot cross left particles. Similarly, there must be a wall on the right.

Fix  $p, q \in \mathbb{N}$ . For any  $j \in \mathbb{N}$ , let  $i = j - p - 1 \bmod 2(p + q + 1)$  and  $S_j = F^j([wb^p lb^q w]_{-p-1})$ . By induction on  $j$  one can easily prove that  $S_j \subseteq [wb^i lb^{p+q-i} w]_{-p-1}$  if  $i \leq p+q$  and  $S_j \subseteq [wb^{2(p+q)+1-i} lb^{i-p-q-1} w]_{-p-1}$  otherwise. Hence, the trace of a configuration  $x \in [wb^p lb^q w]_{-p-1}$  is  $T_F(x) = (lb^{2p}rb^{2q})^\omega$ .  $\square$

*Proof (of Example 7).* By Lemma 11 we have that

$$L = \mathcal{L}(\tau(F)) \cap lb^*rb^*lb^*r = \bigcup_{p,q \in \mathbb{N}} (lb^{2p}rb^{2q}lb^{2p}r).$$

Since  $lb^*rb^*lb^*r$  is regular and  $L$  is not, we conclude that  $\mathcal{L}(\tau(F))$  is irregular.  $\square$