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On the propagation of an optical wave in a photorefractive medium

B. BIDÉGARAY-FESQUET
Laboratoire Jean Kuntzmann,
CNRS UMR 5224 et Grenoble Universités,
B.P. 53, 38041 Grenoble Cedex 9, France
Brigitte.Bidegaray@imag.fr

J.-C. SAUT
UMR de Mathématiques, Université Paris-Sud,
91405 Orsay, France
Jean-Claude.Saut@math.u-psud.fr

Abstract

The aim of this paper is first to review the derivation of a model describing the propagation of an optical wave in a photorefractive medium and to present various mathematical results on this model: Cauchy problem, solitary waves.

Keywords: Photorefractive media; Cauchy problem; solitary waves.
AMS Subject Classification: 35A05, 35A07, 35Q55, 35Q60, 78A60

1 Introduction

A modification of the refraction index in $LiNbO_3$ or $LiTaO_3$ crystals has been observed in the 1960s and first considered as a drawback. This photo-induced variation of the index is called the photorefractive effect and occurs in any electro-optical or photoconductive crystal. Applications have been found in the 1970s–1980s to real-time signal processing, phase conjugation, or amplification of beams or images.

In this paper we are interested in deriving a not-too-simple but tractable mathematical model for the propagation of light in such materials. Solitonic propagation is one of our concern but we focus here on initial value problems.

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A very complete review of solitonic propagation in photorefractive media may be found in [6]. Our derivation follows the same guidelines as theirs but point out the different approximations made for future mathematical studies.

The outline of the paper is the following. In Sec. 2 we first derive the Kukhtarev model for the material and then couple it to a wave propagation model for light to obtain a complete set of equations. A 1D model is obtained keeping only one of the two transverse space variables. This is a saturated nonlinear Schrödinger equation, the mathematical theory of which is addressed in Sec. 3 in arbitrary dimension. Section 4 is devoted to the study of the full 2D model with emphasis on the Cauchy problem and the solitary wave solutions.

2 Derivation of the Model

2.1 The photorefractive effect

The propagation of an optical wave in insulating or semi-insulating electro-optical crystals induces a charge transfer. The new distribution of charges induces in turn an electric field which produces a variation of the refraction index. The main characteristics of this effect are the following: (1) Sensibility to energy (and not to the electric field), (2) Nonlocal effect (charge distributions and the electric field are not located at the same position), (3) Inertia (charges need a certain time to move), (4) Memory and reversibility (in the dark the space charge, and therefore the index variation, is persistent but an uniform light redistributes uniformly all charges — this yields applications to holography).

The sensibility to energy reminds us of Kerr media yielding the classical cubic nonlinear Schrödinger (NLS) equation. The nonlocal effects will of course complicate the mathematical analysis compared to NLS equations, but the general ideas will be the same. In our final model, inertial effects will be neglected since time is removed from the material equations. Memory and reversibility effects involve ion displacement in materials like Bi_2TeO_5, which we will not take into account in the present study.

2.2 The Kukhtarev model

The physical modeling of the photorefractive effect assumes that charges are trapped in impurities or defaults of the crystal mesh. We chose here to derive the model only in the case when charges are electrons. Some materials like semi-conductors necessitate to model both electrons and holes. Therefore we restrict our study to insulating media.

2.2.1 Charge equation

Electrons come from donor sites with density N_{D}. This density is supposed to be much greater than that of the acceptor sites (impurities) which we denote by N_{A}. The density of donor sites which are indeed ionized is N_{D}^{+} and we of course have $N_{\mathrm{D}}^{+} \leq N_{\mathrm{A}} \ll N_{\mathrm{D}}$. Local neutrality, i.e. no electrons in the conduction

band, corresponds to the relation $N_{\mathrm{D}}^{+} = N_{\mathrm{A}}$. The total charge is given by

$$\rho = e(N_{\mathrm{D}}^{+} - N_{\mathrm{A}} - n_{\mathrm{e}}), \tag{2.1}$$

where e is the electron charge and n_{e} the electron density.

2.2.2 Evolution of ionized donor sites

Photoionization and recombination affect the density of ionized donor sites. Photoionization is proportional to the density of not ionized donor sites ($N_{\mathrm{D}} - N_{\mathrm{D}}^{+}$). In the dark it is proportional to a thermal excitation rate β but is also sensitive to light intensity I_{em} with a photoexcitation coefficient s. Recombination is proportional to the density of electrons and occurs over a time scale $\tau = 1/(\gamma_{\mathrm{r}} N_{\mathrm{D}}^{+})$ which does not depend on n_{e} if the excitation rate is low, therefore the total evolution of ionized donor sites is

$$\partial_t N_{\mathrm{D}}^{+} = (\beta + sI_{\mathrm{em}})(N_{\mathrm{D}} - N_{\mathrm{D}}^{+}) - \gamma_{\mathrm{r}} n_{\mathrm{e}} N_{\mathrm{D}}^{+}. \tag{2.2}$$

2.2.3 Charge transport

Now the main point is to describe the three phenomena which contribute to the charge transport or current density. The first phenomenon is isotropic and is due to thermal diffusion. It is proportional to the gradient of the electron density. The electron mobility is denoted by μ, T is the temperature and k_{B} the Boltzmann constant. The second phenomenon is drift and is collinear to the electric field $\mathbf{E}_{\mathrm{tot}}$. Finallly, the photovoltaic effect is collinear to the optical axis $\mathbf{c}$ and proportional to the non-ionized donor density and the field intensity with a photovoltaic coefficient β_{ph}. The total current density is therefore

$$\mathbf{J} = e\mu n_{\mathrm{e}} \mathbf{E}_{\mathrm{tot}} + \mu k_{\mathrm{B}} T \nabla n_{\mathrm{e}} + \beta_{\mathrm{ph}} (N_{\mathrm{D}} - N_{\mathrm{D}}^{+}) \mathbf{c} I_{\mathrm{em}}. \tag{2.3}$$

2.2.4 Closure of the model

The closure of the model is first based on charge conservation and the Poisson equation:

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0, \tag{2.4}$$

$$\nabla \cdot (\varepsilon_0 \widehat{\varepsilon} \mathbf{E}_{\mathrm{sc}}) = \rho. \tag{2.5}$$

The crystal is anisotropic and this is accounted for in the relative permittivity $\widehat{\varepsilon}$ which is a tensor. A careful analysis of the different electric fields has to be done. In Poisson equation (2.5), $\mathbf{E}_{\mathrm{sc}}$ is the space charge field which is induced by the charge density. The total field $\mathbf{E}_{\mathrm{tot}}$ only occurs in the equations through its gradient (Eqs. (2.4) and (2.3)). Two fields are constant and disappear in the final equations: the photovoltaic field $\mathbf{E}_{\mathrm{ph}} = \beta_{\mathrm{ph}} \gamma_{\mathrm{r}} N_{\mathrm{A}} \mathbf{c} / e\mu s = E_{\mathrm{ph}} \mathbf{c}$, and an external field $\mathbf{E}_{\mathrm{ext}}$ which is often applied in one transverse direction on the faces of the crystal. A last contribution to the total field is $\mathbf{E}$, connected to the light which propagates in the crystal and its description is given in Sec. 2.3.

The set of five equations (2.1)–(2.5) is called the Kukhtarev model and was first given in [9].

2.3 Propagation of the light wave in the crystal

We have already introduced the relative permittivity tensor $\widehat{\varepsilon}$ which plays a rôle in the description of the propagation of a light wave in the crystal via the wave equation:

$$\partial_t^2(\widehat{\varepsilon}\mathbf{E}) - c^2\nabla^2\mathbf{E} = 0.$$

In a non-centrosymmetric crystal the preponderant nonlinear effect is the Pockels effect which yields the following $\mathbf{E}$-dependence for the permittivity tensor:

$$\widehat{\varepsilon(\mathbf{E})} = \widehat{\varepsilon(0)} - \widehat{\varepsilon}\widehat{(\widehat{\mathbf{r}}\cdot\mathbf{E})}\widehat{\varepsilon} = n^2 - \widehat{\varepsilon}\widehat{(\widehat{\mathbf{r}}\cdot\mathbf{E})}\widehat{\varepsilon},$$

where $\widehat{\mathbf{r}}$ is the linear electro-optic tensor and n the mean refraction index. We now suppose that $\mathbf{E}$ is a space perturbation of a plane wave (paraxial approximation) of frequency ω, wave vector $\mathbf{k}$ and polarization $\mathbf{e}$:

$$\mathbf{E}(t,\mathbf{x}) = A(\mathbf{x})\exp(i(\omega t - \mathbf{k}\cdot\mathbf{x}))\mathbf{e}.$$

Such a wave with polarization $\mathbf{e}$ only "sees" a part of tensor $\widehat{\varepsilon(\mathbf{E})}$, or equivalently a variation δn of the refraction index n:

$$\delta n = -\frac{1}{2n}[\mathbf{e}\widehat{\varepsilon}\,\widehat{\mathbf{r}\mathbf{e}}\,\widehat{\varepsilon}\mathbf{e}^*]\mathbf{E}.$$

Now we can write an equation for the amplitude A which takes into account the dispersion relation $c^2|\mathbf{k}|^2 = n^2\omega^2$ and the slowly varying envelope approximation in the $\mathbf{k}$ direction. We denote by $\nabla_\perp$ the gradient in the perpendicular directions to $\mathbf{k}$ and

$$\left[\nabla_\perp^2 - 2i\mathbf{k}\cdot\nabla + 2|\mathbf{k}|^2\frac{\delta n}{n}\right]A(\mathbf{x})\mathbf{e} = 0. \tag{2.6}$$

Of course, we can consider the superposition of such waves to describe for example pump and probe experiments.

The system is now closed but it is impossible to solve Eqs. (2.1)–(2.6). We have to simplify them taking into account characteristic scales. Our description follows (or more precisely makes explicit the assumptions in [16]) and is purely formal. The rigorous justification is certainly difficult and should include the approximations made in Sec. 2.3.

2.4 Characteristic values

We first want to define a characteristic electron intensity n_0 by considering uniform solutions in space and time. Equations (2.1), (2.2) and (2.5) yield

$$\rho = e(N_\mathrm{D}^+ - N_\mathrm{A} - n_\mathrm{e}) = 0 \text{ and } 0 = (\beta + sI)(N_\mathrm{D} - N_\mathrm{D}^+) - \gamma_\mathrm{r} n_\mathrm{e} N_\mathrm{D}^+.$$

With a characteristic intensity I_0, neglecting β and assuming $n_\mathrm{e} \ll N_\mathrm{A}$, we have $n_0 = sI_0(N_\mathrm{D} - N_\mathrm{A})/\gamma_\mathrm{r} N_\mathrm{A}$.

There are three characteristic times: (1) the characteristic lifetime of an electron (in the dark) $\tau_\mathrm{e} = 1/\gamma_\mathrm{r} N_\mathrm{A}$, (2) the characteristic evolution time of ionized donors $\tau_\mathrm{d} = 1/\gamma_\mathrm{r} n_0$ (and a consequence of $n_0 \ll N_\mathrm{A}$ is $\tau_\mathrm{e} \ll \tau_\mathrm{d}$), (3)

the characteristic relaxation time of the electric field $t_0 = \varepsilon_0\varepsilon_c/e\mu n_0$, where ε_c is the characteristic value of $\widehat{\varepsilon}$ along the $\mathbf{c}$ direction. It is obtained combining Eqs. (2.3) and (2.5) assuming there is only drift. If a timescale has to be kept, it is t_0, but we do not detail this point since we neglect time-dependence in the final equations.

The Debye length L_D is the characteristic value of the field space variation. It is determined together with the characteristic field E_0. The Poisson equation (2.5) yields $L_D = \varepsilon_0\varepsilon_c E_0/eN_A$. If drift and isotropic diffusion have the same order, $E_0 = k_B T/eL_D$ and therefore

$$L_D = \left(\frac{k_B T\varepsilon_0\varepsilon_c}{e^2 N_A}\right)^{1/2}, \quad E_0 = \left(\frac{k_B T N_A}{\varepsilon_0\varepsilon_c}\right)^{1/2}, \quad \text{and} \quad I_0 = \frac{k_B T N_A}{\varepsilon_0\varepsilon_c}.$$

2.5 The Zozulya–Anderson model

Zozulya–Anderson model[16] is obtained using the above characteristic values and for a specific material ($LiNbO_3$) which imposes certain symmetries. The adiabatic assumption allows to get rid of the time-dependence and an asymptotic formal analysis which accounts for the very large donors density N_D ends the derivation.

Dimensionless equations are obtained using n_0 and N_A for electron and ion densities respectively, I_0 for intensities, E_0 for fields and ε_c for the permittivity tensor. Coefficient β is normalized as a dark intensity $I_d = \beta/sI_0$. We keep all the other notations but they now denote the normalized variables. The total intensity is $I = I_{em} + I_d$. We assume that the space charge field $\mathbf{E}_{sc}$ derives from a potential: $L_D\nabla\varphi = -\mathbf{E}_{sc}$. In the adiabatic assumption matter equations reduce to

$$I\frac{1 - N_D^+ N_A/N_D}{1 - N_A/N_D} = n_e N_D^+,$$

$$L_D\nabla\cdot\left\{n_e\mathbf{E}_{tot} + L_D\nabla n_e + \mathbf{E}_{ph} I_{em}\frac{1 - N_D^+ N_A/N_D}{1 - N_A/N_D}\right\} = 0,$$

$$-L_D^2\nabla\cdot(\widehat{\varepsilon}\nabla\varphi) = N_D^+ - 1 - \frac{n_0}{N_A}n_e.$$

In $LiNbO_3$, $N_A/N_D \sim 10^{-3}$ and $n_0/N_D \sim 10^{-6}$ and we neglect them. Finally, we make different assumptions on the fields: first the beam is not too thin, the photogalvanic and the external applied fields are not too large and therefore we may neglect $-L_D^2\nabla\cdot(\widehat{\varepsilon}\nabla\varphi)$; second the propagation field amplitude is relatively small and we assimilate $\mathbf{E}_{tot}$ and $\mathbf{E}_{sc}$. This implies $n_e = I$ and $N_D^+ = 1$ and we have only one matter equation, namely

$$\nabla I\cdot\nabla\varphi + I\nabla^2\varphi - \nabla^2 I - k_D E_{ph}\mathbf{c}\cdot\nabla I = 0,$$

where $k_D = 1/L_D$. To obtain a "simpler" equation, in physics papers the variable $U = \varphi - \ln I$ is often used. This variable seems however to lack physical meaning.

The final matter equation is

$$\nabla U \cdot \nabla \varphi + \nabla^2 U - k_{\mathrm{D}} E_{\mathrm{ph}} \mathbf{c} \cdot \nabla I = 0. \tag{2.7}$$

We now fix different space directions. Propagation is supposed to take place in the z-direction and $\mathbf{k} = k\mathbf{e}_z$. The two transverse directions are therefore x and y. The $\mathbf{e}_x$ direction is chosen as both $\mathbf{c}$ and $\mathbf{e}$. If an external field is applied, it will be along $\mathbf{e}_x$ as well. In the matter equation (2.7), the quantity $\mathbf{c} \cdot \nabla I$ simply reads $\partial_x I$. In $LiNbO_3$, $r = r_{xxx}$ is responsible for the change of refractive index (it is r_{xxy} in some other materials) and we approximate $\widehat{\varepsilon}$ by n^2 in the expression for δn which becomes $\delta n = \frac{1}{2} n^3 r E_0 L_{\mathrm{D}} \partial_x \varphi$. Together with Eq. (2.6) the envelope equation now reads

$$\left[\partial_z + \frac{i}{2k}\nabla_\perp^2\right] A(\mathbf{x}) = -i\frac{k}{2} n^2 r E_0 L_{\mathrm{D}} \partial_x \varphi A(\mathbf{x}).$$

The last step is to have dimensionless space variables. We set $\alpha = \frac{k}{2} n^2 r E_0$ which has the dimension of the inverse of a space variable. We denote $z' = |\alpha| z$, $(x', y') = \sqrt{k|\alpha|}(x, y)$, $A' = A/\sqrt{I_0 I_{\mathrm{d}}}$, $\varphi' = \sqrt{k|\alpha|}\varphi/k_{\mathrm{D}}$ and $U' = \sqrt{k|\alpha|}U/k_{\mathrm{D}}$. The last approximations are now $U' = \varphi'$, $k \gg |\alpha|$ and $E_{\mathrm{ph}} \sim E_0$, and omitting primes:

$$\begin{aligned} \left[\partial_z - \frac{i}{2}\nabla_\perp^2\right] A &= -iA\partial_x \varphi, \\ \nabla_\perp^2 \varphi + \nabla_\perp \ln(1 + |A|^2) \cdot \nabla_\perp \varphi &= \partial_x \ln(1 + |A|^2). \end{aligned}$$

These equations are usually referred to as a model derived in [16] but only seeds of these equations are derived there usually including many other terms and especially time derivatives.

In the wide literature devoted to photorefractive media, many equations are written which resemble those above but with different choices of asymptotic approximations. In particular numerical results are very often obtained keeping the time in the matter equations (see [13] or [15]).

2.6 Mathematical setting

If we look at a wider class of materials we may have different signs for the nonlinearity (in reference to the cubic nonlinear Schrödinger equation, the case $a = 1$ is classically called the focusing case, and $a = -1$ the defocusing case). Besides mathematicians are more accustomed to use t as the evolution variable. We will therefore consider the system

$$\begin{cases} i\partial_t A + \Delta A = -aA\partial_x \varphi, & a = \pm 1, \\ \Delta\varphi + \nabla \ln(1 + |A|^2) \cdot \nabla\varphi = \partial_x \ln(1 + |A|^2), \end{cases} \tag{2.8}$$

where $\Delta = \partial_x^2 + \partial_y^2$ or $\Delta = \partial_x^2$.

These expressions with logarithms are widely used in the physics literature, maybe because they are the starting point of solitonic studies and logarithms

appear naturally in the expression of solitary waves (see Sec. 3.2). This form is however cumbersome to handle for the mathematical analysis, and it is much more convenient to cast (2.8) as

$$\begin{cases} i\partial_t A + \Delta A = -aA\partial_x\varphi, \\ \operatorname{div}\big((1+|A|^2)\nabla\varphi\big) = \partial_x(|A|^2), \end{cases} \tag{2.9}$$

which is closer to the original Kukhtarev equations.

We have seen that the main effects take place in the t- (propagation) and the x-directions (drift, anisotropic diffusion, external field, polarization). It is therefore natural to study the equations with no dependence in the y variable. In the one-dimensional case, we infer immediately from the last equation in System (2.9) that $(1+|A|^2)\partial_x\varphi = |A|^2 - C(t)$ where the constant $C(t)$ is given by the boundary conditions. If no external field is applied $C(t) \equiv 0$. This is the case for bright solitary waves (see [11]). In the case of dark solitary waves $C(t) = \lim_{x\to\pm\infty}|A|^2$ (see [12]), which does not depend on t either. In both cases, System (2.9) reduces to the saturated NLS equation

$$i\partial_t A + \partial_x^2 A = -a\frac{|A|^2 - |A_\infty|^2}{1+|A|^2}A. \tag{2.10}$$

In the sequel we will mainly consider the case when $A_\infty = 0$ and show that, in some sense, the dynamics of (2.9) is similar to that of (2.10) which we will recall in Sec. 3.

In the two-dimensional case (2.9) can be viewed as a saturated version of a Davey–Stewartson system. Namely, replacing $1+|A|^2$ by 1 in the L.H.S. of (2.9) yields

$$\begin{cases} i\partial_t A + \Delta A &= -aA\partial_x\varphi, \\ \Delta\varphi &= \partial_x(|A|^2), \end{cases}$$

which is the Davey–Stewartson system of the elliptic–elliptic type (see [7]).

3 The Saturated NLS Equation

We review here some mathematical facts, more or less known, on the saturated NLS equation

$$\begin{cases} i\partial_t A + \Delta A = -a\dfrac{|A|^2 A}{1+|A|^2}, \quad a = \pm 1, \\ A(\mathbf{x},0) = A_0(\mathbf{x}), \end{cases} \tag{3.11}$$

where $A = A(\mathbf{x},t)$ and $\mathbf{x} \in \mathbb{R}^d$. We have derived this equation for $d = 1$, but give here results for a general d. This equation is also derived in other contexts, for example the propagation of a laser beam in gas vapors [14].

3.1 The Cauchy problem

The Cauchy problem (3.11) can be solved in L^2 and in the energy space H^1.

Theorem 1 (i) *Let $A_0 \in L^2(\mathbb{R}^d)$. Then there exists a unique solution $A \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$ of* (3.11) *which satisfies furthermore*

$$\int_{\mathbb{R}^d} |A(\mathbf{x},t)|^2 d\mathbf{x} = \int_{\mathbb{R}^d} |A_0(\mathbf{x})|^2 d\mathbf{x}, \qquad t \in \mathbb{R}. \tag{3.12}$$

(ii) *Let $A_0 \in H^1(\mathbb{R}^d)$. Then the solution above satisfies $A \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ and*

$$\int_{\mathbb{R}^d} \left[|\nabla A(\mathbf{x},t)|^2 d\mathbf{x} + a\ln(1+|A(\mathbf{x},t)|^2)\right] d\mathbf{x}$$
$$= \int_{\mathbb{R}^d} \left[|\nabla A_0(\mathbf{x})|^2 d\mathbf{x} + a\ln(1+|A_0(\mathbf{x})|^2)\right] d\mathbf{x}, \qquad t \in \mathbb{R} \tag{3.13}$$

Proof. The norm conservations (3.12) and (3.13) result from multiplying (3.11) by $\bar{A}$ and $\partial_t \bar{A}$ respectively and integrating the complex and real parts respectively. This formal proof is justified by the standard truncation process.

Let $S(t)$ be the group operator associated to the linear Schrödinger equation $i\partial_t A + \Delta A = 0$. Then the Duhamel formula for (3.11) reads

$$A(\mathbf{x},t) = S(t)A_0(\mathbf{x}) - a\int_0^t S(t-s)\frac{|A(\mathbf{x},s)|^2}{1+|A(\mathbf{x},s)|^2}A(\mathbf{x},s)\ ds. \tag{3.14}$$

Since $x \mapsto x/(1+x)$ is Lipschitz, we easily infer that the R.H.S. of (3.14) defines a contraction on a suitable ball of $\mathcal{C}([0,T]; L^2(\mathbb{R}^d))$ for some $T > 0$. The local well-posedness in $L^2(\mathbb{R}^d))$ follows. Global well-posedness is derived from the conservation law (3.12).

The H^1 theory follows the same argument, noticing that

$$\left|\nabla\left(\frac{|A|^2}{1+|A|^2}A\right)\right| = \left|\frac{A^2}{(1+|A|^2)^2}\nabla\bar{A} + \frac{|A|^2}{(1+|A|^2)^2}\nabla A\right| \leq \frac{1}{2}|\nabla A|.$$

□

Remark. As a consequence of (3.12), (3.13) and $\ln(1+|A|^2) \leq |A|^2$, we obtain the uniform bound

$$\int_{\mathbb{R}^d} |\nabla A(\mathbf{x},t)|^2 d\mathbf{x} \leq \int_{\mathbb{R}^d} |A_0(\mathbf{x})|^2 d\mathbf{x} + \int_{\mathbb{R}^d} |\nabla A_0(\mathbf{x})|^2 d\mathbf{x}, \qquad t \in \mathbb{R}. \tag{3.15}$$

Contrarily to the context of the usual nonlinear cubic Schrödinger equation, this bound does not depend on the sign of a and in particular saturation means that no blow-up is occurs.

3.2 Solitary waves — one-dimensional results

In the one-dimensional case, it is possible to compute first integral formulations of the solitary waves.

Bright solitary waves are sought for in the form $A(x,t) = e^{i\omega t}u(x)$ (see [11]), where A is a solution to (2.10) with $A_\infty = 0$. The function u is supposed to have a maximum at $x = 0$ ($u(0) = u_m > 0$ and $u'(0) = 0$), therefore

$$[u'(x)]^2 = (\omega - a)[u^2(x) - u_m^2] + a[\ln(1+u^2) - \ln(1+u_m^2)].$$

We furthermore want that for $x \to \infty$, $u(x) \to 0$ and $u'(x) \to 0$. This yields a unique possible frequency for the solitary wave, namely

$$\omega = a\left(1 - \frac{\ln(1+u_m^2)}{u_m^2}\right)$$

and

$$[u'(x)]^2 = a\left(-\frac{u^2(x)}{u_m^2}\ln(1+u_m^2) + \ln(1+u^2)\right).$$

Since u_m is supposed to be the maximum of u, this quantity is positive only if $a = 1$ (focusing case) and the bright soliton is solution to the first order equation:

$$u'(x) = -\mathrm{sign}(x)\sqrt{\ln(1+u^2) - \frac{u^2}{u_m^2}\ln(1+u_m^2)} \text{ with } \omega = 1 - \frac{\ln(1+u_m^2)}{u_m^2}.$$

Dark solitary waves are sought for in the form $A(x,t) = u(x)$ (see [12]) where A is solution to (2.10). There is no time-dependence. We assume that $\lim_{x\to\pm\infty} u'(x) = 0$ and consistently with $A_\infty \neq 0$,

$$\lim_{x\to+\infty} u(x) = -\lim_{x\to-\infty} u(x) = u_\infty.$$

Then

$$[u'(x)]^2 = a\left(-(u^2 - u_\infty^2) + (1+u_\infty^2)\ln\left(\frac{1+u^2}{1+u_\infty^2}\right)\right).$$

At the origin $u(0) = 0$ and we want more generally that $|u(x)| \leq |u_\infty|$. Therefore, dark solitary waves only exist if $a = -1$ (defocusing case). In this context $u(x)$ is a monotonous function and is solution to the first order equation:

$$u'(x) = \mathrm{sign}(u_\infty)\sqrt{u^2 - u_\infty^2 - (1+u_\infty^2)\ln\left(\frac{1+u^2}{1+u_\infty^2}\right)}.$$

For both bright and dark solitary waves, no explicit solution is known.

3.3 Solitary waves — *a priori* estimates and non existence

Consider now the solitary wave solutions of (3.11) in any dimension d, that is solutions of the type $A(\mathbf{x},t) = e^{i\omega t}U(\mathbf{x})$, where $U \in H^1(\mathbb{R}^d)$ (we thus are only concerned with "bright" solitary waves). A solitary wave is a solution of the elliptic equation

$$-\Delta U + \omega U = a\frac{|U|^2 U}{1+|U|^2}, \qquad U \in H^1(\mathbb{R}^d). \tag{3.16}$$

A trivial solution is $U \equiv 0$. We seek other nontrivial solutions.

Lemma 2 *Any $H^1(\mathbb{R}^d)$ solitary wave satisfies*

$$\int_{\mathbb{R}^d} \left[|\nabla U|^2 + \left(\omega - a \frac{|U|^2}{1+|U|^2} \right) |U|^2 \right] d\mathbf{x} = 0 \tag{3.17}$$

(energy identity)

$$(d-2) \int_{\mathbb{R}^d} |\nabla U|^2 d\mathbf{x} + d\omega \int_{\mathbb{R}^d} |U|^2 d\mathbf{x} - ad \int_{\mathbb{R}^d} \left[|U|^2 - \ln(1+|U|^2) \right] d\mathbf{x} = 0 \tag{3.18}$$

(Pohozaev identity).

Proof. As for Theorem 1, (3.17) results from multiplying (3.16) by $\bar{U}$ and integrating. To get (3.18), one multiplies (3.16) by $x_k \partial_{x_k} \bar{U}_k$, integrates the real part, and sums from 1 to d. This is justified by a standard truncation argument. □

Corollary 3 *No nontrivial solitary wave (solution of* (3.16)*) exists when*

(i) $a = -1$ (*defocusing case*), *for* $\omega \geq 0$.

(ii) $a = 1$ (*focusing case*) *and* $\omega \geq 1$.

(iii) $a = \pm 1$ *if* $\omega < 0$ *provided* $|U|^2/(1+|U|^2) = O(1/|\mathbf{x}|^{1+\varepsilon})$, $\varepsilon > 0$ *as* $|\mathbf{x}| \to +\infty$.

Proof. Identity (3.17) implies that no solitary wave may exist when $a = -1$ and $\omega \geq 0$ or $a = 1$ and $\omega \geq 1$. When $d = 1, 2$, Eq. (3.18) implies that no solitary wave exist when $\omega \leq 0$ and $a = 1$. Recall $d = 2$ is the physical case. The remaining cases ($\omega < 0$, $a = -1$ or $a = 1$, $d \geq 3$) follow from the classical result of Kato[8] on the absence of embedded eigenvalues. Indeed, we can write (3.16) as

$$\Delta U + (-\omega + V(\mathbf{x}))U = 0, \qquad V(\mathbf{x}) = a \frac{|U|^2}{1+|U|^2},$$

assuming furthermore that $V(\mathbf{x}) = O(1/|\mathbf{x}|^{1+\varepsilon})$, $\varepsilon > 0$, as $|\mathbf{x}| \to \infty$. A proof for $d = 3, 4$ or $d \geq 5$, $\omega \leq -(d-2)/2$ with no decaying assumption is given in Appendix. □

Corollary 4 *Solitary waves may exist only when* $a = 1$ *and* $0 < \omega < 1$.

Corollary 4 is consistent with the one-dimensional "explicit" result. We first have a classical regularity and decay result.

Proposition 5 *Let* $a = 1$ *and* $0 < \omega < 1$. *Then any* $U \in H^1(\mathbb{R}^d)$ *solution of* (3.16) *satisfies*

$$U \in H^\infty(\mathbb{R}^d), \tag{3.19}$$

$$e^{\delta|\mathbf{x}|} U \in L^\infty(\mathbb{R}^d) \text{ for any } \delta < \omega/2. \tag{3.20}$$

Proof. $U \in H^\infty(\mathbb{R}^d)$ results trivially from a bootstrapping argument using $|U|^2/(1+|U|^2) < 1$. To prove (3.20), we first derive the estimate

$$\int_{\mathbb{R}^d} e^{\omega|\mathbf{x}|} \left[|\nabla U|^2 + |U|^2\right] d\mathbf{x} < +\infty. \tag{3.21}$$

In fact, as in Cazenave[4, 5], we multiply (3.16) by $e^{\omega|\mathbf{x}|}\bar{U}$ and integrate the real part (this formal argument is made rigorous by replacing $e^{\omega|\mathbf{x}|}$ by $e^{\omega|\mathbf{x}|/(1+\varepsilon|\mathbf{x}|)}$, $\varepsilon > 0$, $\varepsilon \to 0$) to get

$$\int_{\mathbb{R}^d} e^{\omega|\mathbf{x}|} \left[|\nabla U|^2 + \omega|U|^2\right] d\mathbf{x} \leq \int_{\mathbb{R}^d} e^{\omega|\mathbf{x}|} \frac{|U|^4}{1+|U|^2} d\mathbf{x} + \omega \int_{\mathbb{R}^d} e^{\omega|\mathbf{x}|} |U||\nabla U| d\mathbf{x}. \tag{3.22}$$

By (3.19) there exists $R > 0$ such that $|U|^2/(1+|U|^2) < \omega/4$ on $\mathbb{R}^d \setminus B_R$. Thus we infer from (3.22) that

$$\begin{aligned} \int_{\mathbb{R}^d} e^{\omega|\mathbf{x}|} \left[|\nabla U|^2 + \omega|U|^2\right] d\mathbf{x} \quad \leq \quad & \int_{B_R} e^{\omega|\mathbf{x}|} \frac{|U|^4}{1+|U|^2} d\mathbf{x} + \frac{\omega}{4} \int_{\mathbb{R}^d} e^{\omega|\mathbf{x}|} |U|^2 d\mathbf{x} \\ & + \frac{\omega}{2} \left(\int_{\mathbb{R}^d} e^{\omega|\mathbf{x}|} |U|^2 d\mathbf{x} + \int_{\mathbb{R}^d} e^{\omega|\mathbf{x}|} |\nabla U|^2 d\mathbf{x} \right) \end{aligned}$$

which implies (3.21).

Now we write U as a convolution

$$U(\mathbf{x}) = H_\omega \star \frac{|U|^2 U}{1+|U|^2}, \qquad \text{where } H_\omega = \mathcal{F}^{-1}\left(\frac{1}{\omega + |\xi|^2}\right). \tag{3.23}$$

As it is well known ([1]), $H_\omega(\mathbf{x}) = \omega^{(d-2)/2} G_1(\omega^{1/2}\mathbf{x})$ where

$$G_1(z) = \begin{cases} C|z|^{(2-d)/2} K_{(d-2)/2}(|z|), & d \geq 3, \\ K_0(|z|), & d = 2, \end{cases}$$

where K_ν is the modified Bessel function of order ν. Furthermore (see [1]), one has the asymptotic behavior:

$$\begin{cases} K_\nu(z) & \sim & \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}|z|\right)^{-\nu}, & \text{for } \nu > 0, \text{ as } |z| \to 0, \\ K_0(z) & \sim & -\ln(|z|), & \text{as } |z| \to 0, \\ K_\nu(z) & \sim & C|z|^{-1/2} e^{-|z|}, & \text{as } |z| \to \infty. \end{cases} \tag{3.24}$$

We infer from (3.23) that

$$e^{\delta|\mathbf{x}|}|U(\mathbf{x})| \leq \int_{\mathbb{R}^d} e^{\delta|\mathbf{x}-\mathbf{x}'|} H_\omega(\mathbf{x}-\mathbf{x}') e^{\delta|\mathbf{x}'|} \frac{|U|^3}{1+|U|^2}(\mathbf{x}') d\mathbf{x}'. \tag{3.25}$$

Since by (3.24) $e^{\delta|\mathbf{x}|} H_\omega(\mathbf{x}) \in L^2(\mathbb{R}^d)$ for $0 < \delta < \omega^{1/2}$, and by (3.21) $e^{\delta|\mathbf{x}|}|U|^3/(1+|U|^2) \in L^2(\mathbb{R}^d)$ for $\delta \leq \omega/2 < \omega^{1/2}$, we deduce from (3.25) that $e^{\delta|x|}U \in L^\infty(\mathbb{R}^d)$ for $0 < \delta < \omega/2$. $\square$

Remark. Actually, the saturated cubic NLS equation should involve a small parameter $\varepsilon > 0$, namely, in the focusing case, we should consider instead of (3.11)

$$i\partial_t A^\varepsilon + \Delta A^\varepsilon = -\frac{|A^\varepsilon|^2 A^\varepsilon}{1+\varepsilon|A^\varepsilon|^2}. \tag{3.26}$$

Theorem 1 is of course still valid for a fixed $\varepsilon > 0$, but (3.13) and (3.15) should be replaced by

$$\begin{aligned}\int_{\mathbb{R}^d} \left[|\nabla A(\mathbf{x},t)|^2 d\mathbf{x} + \frac{1}{\varepsilon^2}\ln(1+|A(\mathbf{x},t)|^2\right] d\mathbf{x} \\ = \int_{\mathbb{R}^d} \left[|\nabla A_0(\mathbf{x})|^2 d\mathbf{x} + \frac{1}{\varepsilon^2}\ln(1+|A_0(\mathbf{x})|^2\right] d\mathbf{x},\end{aligned}$$

$$\int_{\mathbb{R}^d} |\nabla A(\mathbf{x},t)|^2 d\mathbf{x} \le \frac{1}{\varepsilon}\int_{\mathbb{R}^d} |A_0(\mathbf{x})|^2 d\mathbf{x} + \int_{\mathbb{R}^d} |\nabla A_0(\mathbf{x})|^2 d\mathbf{x}.$$

For solitary waves $A^\varepsilon(\mathbf{x},t) = e^{i\omega t}U(\mathbf{x})$, (3.26) reduces to the elliptic equation

$$-\Delta U + \omega U = \frac{|U|^2 U}{1+\varepsilon|U|^2}.$$

Setting $V = \varepsilon^{1/2}U$, one obtains

$$-\Delta V + \omega V = \frac{1}{\varepsilon}\,\frac{|V|^2 V}{1+|V|^2}.$$

The only possible range for the existence of nontrivial H^1 solitary waves is $\omega \in]0, 1/\varepsilon[$. Proposition 5 is still valid for ω in this range.

3.4 Solitary waves — existence results

We now turn to the existence of non-trivial H^2 solutions of

$$-\Delta U + \omega U = \frac{|U|^2 U}{1+|U|^2}$$

when $0 < \omega < 1$. We will look for real radial solutions $U(\mathbf{x}) = u(|\mathbf{x}|) \equiv u(r)$ and thus consider the ODE problem

$$\begin{cases} -u'' - \dfrac{d-1}{r}u' + \omega u = \dfrac{u^3}{1+u^2}, \\ u \in H^2(]0,\infty[), \qquad u'(0) = 0. \end{cases} \tag{3.27}$$

We recall a classical result of Berestycki *et al.*[2]

Theorem 6 ([2], p. 143) *Let g be a locally Lipschitz continuous function on $\mathbb{R}_+ = [0, +\infty[$ such that $g(0) = 0$, satisfying the following hypotheses.*

(H1) $\alpha = \inf\{\zeta > 0,\ g(\zeta) \ge 0\}$ *exists and* $\alpha > 0$.

(H2) *Let* $G(t) = \int_0^t g(s)ds$. *There exists* $\zeta > 0$ *such that* $G(\zeta) > 0$.
Let $\zeta_0 = \inf\{\zeta > 0,\ G(\zeta) \geq 0\}$. *In view of* (H1) *and* (H2), ζ_0 *exists and* $\zeta_0 > \alpha$.

(H3) $\lim_{s\searrow\alpha} g(s)/(s-\alpha) > 0$

(H4) $g(s) > 0$ *for* $s \in]\alpha, \zeta_0]$.
Let $\beta = \inf\{\zeta > \zeta_0,\ G(\zeta) \geq 0\}$. *In view of* (H4), $\zeta_0 < \beta \leq +\infty$.

(H5) *If* $\beta = +\infty$, *then* $\lim_{s\to+\infty} g(s)/s^l = 0$ *for some* $l < (d+2)/(d-2)$ *(if* $d = 2$, *we may choose for* l *any finite real number).*

Let us consider the Cauchy problem

$$\begin{cases} -u'' - \dfrac{d-1}{r}u' = g(u), & r > 0, \\ u(0) = \zeta, & u'(0) = 0. \end{cases} \tag{3.28}$$

Then there exists $\zeta \in]\zeta_0, \beta[$ *such that* (3.28) *has a unique solution satisfying* $u(r) > 0$ *for* $r \in]0, +\infty[$, $u'(r) < 0$ *for* $r \in]0, +\infty[$ *and* $\lim_{r\to+\infty} u(r) = 0$. *If in addition* $\limsup_{s\searrow 0} g(s)/s < 0$, *then there exists* $C > 0$ *and* $\delta > 0$ *such that* $0 < u(r) \leq Ce^{-\delta r}$, *for* $0 \leq r < +\infty$.

Theorem 7 *If* $a = 1$ *and* $0 < \omega < 1$, *there exists a nontrivial positive solution of* (3.27).

Proof. The case $d = 1$ has been addressed in Sec. 3.2. Consider now $d \geq 2$. We apply Theorem 6 with

$$g(u) = -\omega u + \frac{u^3}{1+u^2},$$

which graph is displayed in Fig. 1.

Note that $\alpha = \sqrt{\omega/(1-\omega)}$ which yields (H1). Setting $u = \sqrt{\omega/(1-\omega)} + \varepsilon$, one easily checks that

$$\frac{g(u)}{u-\alpha} = 2\omega(1-\omega) + O(\varepsilon),$$

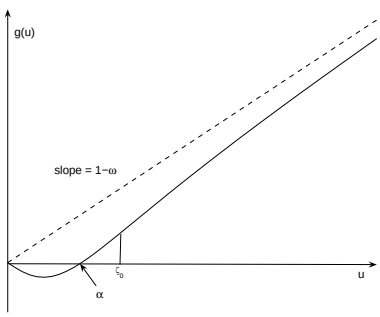

Figure 1: Graph of function g

and (H3) is satisfied. One computes $G(u) = (1-\omega)u^2 - \frac{1}{2}\ln(1+u^2)$, which obviously satisfies (H2) and (H4) with $\beta = +\infty$. Last (H5) holds true (for $l > 1$). $\square$

Remark. u satisfies the decay rate of Proposition 5.

4 The Zozulya–Anderson System

4.1 Estimate on the potential

We now restrict to the space-dimension $d = 2$ which is the context of the derivation. To mimic the proof for the Cauchy problem in the one-dimensional case, we would like to express φ in terms of A for say $A \in L^2(\mathbb{R}^2)$. With such a data A, we indeed have a unique φ in some convenient space but no Lipschitz regularity for the mapping $A \mapsto \varphi$, which is required to perform some fixed point procedure. To ensure this we will have to assume $A \in H^2(\mathbb{R}^2)$.

To derive the first estimates, we consider time as a parameter and do not express it. We therefore introduce the weighted homogeneous Sobolev space

$$H = \{\varphi \in \mathcal{S}'(\mathbb{R}^d),\ (1+|A|^2)^{1/2}\nabla\varphi \in L^2(\mathbb{R}^d)\}/\mathbb{R}$$

together with its natural Hilbertian structure.

Lemma 8 (i) *Let $A \in L^2(\mathbb{R}^2)$. There exists a unique $\varphi \in H$ solution of*

$$\mathrm{div}((1+|A|^2)\nabla\varphi) = \partial_x(|A|^2) \text{ in } \mathcal{D}'(\mathbb{R}^2) \tag{4.29}$$

such that

$$\int_{\mathbb{R}^2}(1+\frac{1}{2}|A|^2)|\nabla\varphi|^2 d\mathbf{x} \leq \frac{1}{2}\int_{\mathbb{R}^2}|A|^2 d\mathbf{x}. \tag{4.30}$$

(ii) *If furthermore $A \in H^2(\mathbb{R}^2)$, then $\nabla\varphi \in H^2(\mathbb{R}^2)$ and there exists a polynomial P vanishing at 0 such that*

$$\|\nabla\varphi\|_{H^2(\mathbb{R}^2)} \leq P(\|A\|_{H^2(\mathbb{R}^2)}). \tag{4.31}$$

Proof. (i) We define a smoothing sequence $(\theta_\varepsilon)_{\varepsilon>0}$ with $\int_{\mathbb{R}^2}\theta_\varepsilon d\mathbf{x} = 1$ and $A_\varepsilon = A \star \theta_\varepsilon$ is such that $A_\varepsilon \to A \in L^2(\mathbb{R}^2)$. In particular

$$\|A_\varepsilon\|_{L^2(\mathbb{R}^2)} \leq \|A\|_{L^2(\mathbb{R}^2)}. \tag{4.32}$$

By Riesz theorem there exists a unique solution to

$$\mathrm{div}((1+|A_\varepsilon|^2)\nabla\varphi_\varepsilon) = \partial_x(|A_\varepsilon|^2), \tag{4.33}$$

i.e.

$$-\Delta\varphi_\varepsilon - \mathrm{div}(|A_\varepsilon|^2\nabla\varphi_\varepsilon) = \partial_x(|A_\varepsilon|^2) \tag{4.34}$$

after noticing that the R.H.S. of Eq. (4.33) defines a linear continuous form on H given by

$$\langle\partial_x(|A_\varepsilon|^2), \psi\rangle = \int_{\mathbb{R}^2}|A_\varepsilon|^2\partial_x\psi d\mathbf{x}.$$

Now we get from (4.33)

$$\begin{aligned}\int_{\mathbb{R}^2}(1+|A_\varepsilon|^2)|\nabla\varphi_\varepsilon|^2 d\mathbf{x} &= -\int_{\mathbb{R}^2}|A_\varepsilon|^2\partial_x\varphi_\varepsilon d\mathbf{x}\\ &\leq \frac{1}{2}\int_{\mathbb{R}^2}|A_\varepsilon|^2 d\mathbf{x}+\frac{1}{2}\int_{\mathbb{R}^2}|A_\varepsilon|^2(\partial_x\varphi_\varepsilon)^2 d\mathbf{x},\end{aligned}$$

which yields (together with (4.32))

$$\int_{\mathbb{R}^2}(1+\frac{1}{2}|A_\varepsilon|^2)|\nabla\varphi_\varepsilon|^2 d\mathbf{x}\leq\frac{1}{2}\int_{\mathbb{R}^2}|A_\varepsilon|^2 d\mathbf{x}\leq\frac{1}{2}\int_{\mathbb{R}^2}|A|^2 d\mathbf{x}. \tag{4.35}$$

Up to the extraction of a sub-sequence, we have $\nabla\varphi_\varepsilon\to\nabla\varphi$ and $\partial_x(|A_\varepsilon|^2)\to\partial_x(|A|^2)$ in $\mathcal{D}'(\mathbb{R}^2)$. From Eq. (4.35), $A_\varepsilon\nabla\varphi_\varepsilon\rightharpoonup B$ weakly in L^2 and for all $\psi\in\mathcal{D}$,

$$\int_{\mathbb{R}^2}A_\varepsilon\nabla\varphi_\varepsilon\cdot\nabla\psi d\mathbf{x}\to\int_{\mathbb{R}^2}A\nabla\varphi\cdot\nabla\psi d\mathbf{x},$$

therefore $B=A\nabla\varphi$. Since $||A_\varepsilon|^2\nabla\varphi_\varepsilon|=|A_\varepsilon||A_\varepsilon\nabla\varphi_\varepsilon|$, $|A_\varepsilon|^2\nabla\varphi_\varepsilon\to|A|^2\nabla\varphi$ in $\mathcal{D}'$. We can pass to the limit in Eq. (4.34) and obtain

$$-\Delta\varphi-\mathrm{div}(|A|^2\nabla\varphi)=\partial_x(|A|^2)\text{ in }\mathcal{D}'(\mathbb{R}^2),$$

i.e. $\mathrm{div}((1+|A|^2)\nabla\varphi)=\partial_x(|A|^2)$ and deduce estimate (4.30) from (4.35). This yields the existence of $\varphi\in H$. The uniqueness is straightforward: two solutions φ_1 and φ_2 would satisfy

$$\int_{\mathbb{R}^2}(1+\frac{1}{2}|A|^2)|\nabla(\varphi_1-\varphi_2)|^2 d\mathbf{x}=0,\text{ i.e. }\nabla(\varphi_1-\varphi_2)=0\text{ a.e.}$$

and hence be equal in H.

(ii) We first notice that $|A|^2\Delta\varphi$ is meaningful in $H^{-1}(\mathbb{R}^2)$. Actually, for any $\psi\in H^1(\mathbb{R}^2)$, one defines

$$\langle|A|^2\Delta\varphi,\psi\rangle_{H^{-1}(\mathbb{R}^2),H^1(\mathbb{R}^2)}=\langle\Delta\varphi,|A|^2\psi\rangle_{H^{-1}(\mathbb{R}^2),H^1(\mathbb{R}^2)},$$

which makes sense since $|A|^2\psi\in H^1(\mathbb{R}^2)$ for $A\in H^2(\mathbb{R}^2)$, $\psi\in H^1(\mathbb{R}^2)$. Thus we can write (4.29) as

$$(1+|A|^2)\Delta\varphi=-\nabla|A|^2\cdot\nabla\varphi+\partial_x(|A|^2),$$

and

$$\Delta\varphi=-\frac{\nabla|A|^2}{1+|A|^2}\cdot\nabla\varphi+\frac{\partial_x(|A|^2)}{1+|A|^2}=:F.$$

We claim that $F\in L^r(\mathbb{R}^2)$, for any $r\in(1,2)$, with

$$\|F\|_{L^r(\mathbb{R}^2)}\leq C\|A\|_{L^2(\mathbb{R}^2)}\|A\|_{H^2(\mathbb{R}^2)}.$$

First, $|\nabla|A|^2\cdot\nabla\varphi|\leq 2|\nabla A||A\nabla\varphi|$ and by Hölder

$$\|\nabla|A|^2\cdot\nabla\varphi\|_{L^r(\mathbb{R}^2)}\leq 2\|\nabla A\|_{L^p(\mathbb{R}^2)}\|A\nabla\varphi\|_{L^2(\mathbb{R}^2)}$$

for any $1 < r < 2$ and $p = 2r/(2-r) \in (2, \infty)$. Since $\|A\nabla\varphi\|_{L^2(\mathbb{R}^2)} \leq \|A\|_{L^2(\mathbb{R}^2)}$ and $H^1(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$ for all $q > 2$, we obtain that

$$\left\| \frac{\nabla|A|^2}{1+|A|^2} \cdot \nabla\varphi \right\|_{L^r(\mathbb{R}^2)} \leq C\|A\|_{H^2(\mathbb{R}^2)}\|A\|_{L^2(\mathbb{R}^2)}, \qquad 1 < r < 2.$$

Similarly

$$\begin{aligned} \left\| \frac{\partial_x|A|^2}{1+|A|^2} \right\|_{L^r(\mathbb{R}^2)} &\leq 2\|A\partial_x A\|_{L^r(\mathbb{R}^2)} \leq 2\|A\|_{L^2(\mathbb{R}^2)}\|\partial_x A\|_{L^p(\mathbb{R}^2)} \\ &\leq C\|A\|_{H^2(\mathbb{R}^2)}\|A\|_{L^2(\mathbb{R}^2)}, \qquad 1 < r < 2. \end{aligned}$$

By elliptic regularity, we infer thus that for any r, $1 < r < 2$,

$$\|\nabla\varphi\|_{W^{1,r}(\mathbb{R}^2)} \leq C\|A\|_{H^2(\mathbb{R}^2)}\|A\|_{L^2(\mathbb{R}^2)}.$$

By Sobolev embedding,

$$\|\nabla\varphi\|_{L^q(\mathbb{R}^2)} \leq C\|\nabla\varphi\|_{W^{1,r}(\mathbb{R}^2)} \leq C\|A\|_{H^2(\mathbb{R}^2)}\|A\|_{L^2(\mathbb{R}^2)}.$$

for $\frac{1}{q} = \frac{1}{r} - \frac{1}{2}$, i.e. $q = 2r/(2-r)$ for all r, $1 < r < 2$. Thus for any $p > 2$

$$\begin{aligned} \left\| \frac{\nabla|A|^2}{1+|A|^2} \cdot \nabla\varphi \right\|_{L^p(\mathbb{R}^2)} &\leq \|\nabla\varphi\|_{L^{2p}(\mathbb{R}^2)}\|\nabla|A|^2\|_{L^{2p}(\mathbb{R}^2)} \\ &\leq C\|A\|_{H^2(\mathbb{R}^2)}\|A\|_{L^2(\mathbb{R}^2)}\|A\|^2_{H^2(\mathbb{R}^2)} \\ &= C\|A\|_{L^2(\mathbb{R}^2)}\|A\|^3_{H^2(\mathbb{R}^2)} \end{aligned}$$

(we have used the fact that $H^2(\mathbb{R}^2)$ is an algebra and the embedding $H^1(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$ for all $q > 2$).

Similarly, for any $p > 2$

$$\left\| \frac{\partial_x(|A|^2)}{1+|A|^2} \right\|_{L^p(\mathbb{R}^2)} \leq 2\|A\|_{L^{2p}(\mathbb{R}^2)}\|\partial_x A\|_{L^{2p}(\mathbb{R}^2)} \leq C\|A\|_{H^1(\mathbb{R}^2)}\|A\|_{H^2(\mathbb{R}^2)}.$$

Finally for any $p > 2$

$$\|F\|_{L^p(\mathbb{R}^2)} \leq C\|A\|^2_{H^2(\mathbb{R}^2)}(1 + \|A\|_{L^2(\mathbb{R}^2)}\|A\|_{H^2(\mathbb{R}^2)}).$$

and by elliptic regularity

$$\|\nabla\varphi\|_{W^{1,p}(\mathbb{R}^2)} \leq C\|A\|^2_{H^2(\mathbb{R}^2)}(1 + \|A\|_{L^2(\mathbb{R}^2)}\|A\|_{H^2(\mathbb{R}^2)}), \qquad \forall p > 2.$$

We now check that $\nabla\varphi \cdot \nabla|A|^2/(1+|A|^2) \in H^1(\mathbb{R}^2)$. This easily reduces to showing that $\nabla(\nabla\varphi \cdot \nabla|A|^2) \in L^2(\mathbb{R}^2)$. For $(i,j) \in \{1,2\}$, $\partial_{x_i}\partial_{x_j}\varphi \in L^p(\mathbb{R}^2)$ since

$$\widehat{\partial_{x_i}\partial_{x_j}\varphi} = \frac{\xi_i\xi_j}{|\xi|^2}\widehat{\Delta\varphi} \text{ and } \Delta\varphi \in L^p(\mathbb{R}^2),\ p > 2.$$

Thus $\partial_{x_i}\partial_{x_j}\varphi\nabla|A|^2 \in L^2(\mathbb{R})$ ($\nabla|A|^2 \in H^1(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$, $\forall q > 2$).

On the other hand, taking $p > 2$ we see that $\nabla\varphi \in L^\infty(\mathbb{R}^2)$ and thus $\nabla\varphi\partial_{x_i}\partial_{x_j}|A|^2 \in L^2(\mathbb{R}^2)$.

It is also easy to check that $\partial_x(|A|^2)/(1+|A|^2) \in H^1(\mathbb{R}^2)$.
Finally, $\Delta\varphi = F \in H^1(\mathbb{R}^2)$, proving that $\nabla\varphi \in H^2(\mathbb{R}^2)$ with an estimate of the form

$$\|\nabla\varphi\|_{H^2(\mathbb{R}^2)} \leq P(\|A\|_{H^2(\mathbb{R}^2)}),$$

where P is a polynomial vanishing at 0, which proves (4.31). □
Remark. All above estimates are therefore uniform in time, and if $A \in \mathcal{C}([0,T]; H^2(\mathbb{R}^2))$ for some $T > 0$, one has

$$\|\nabla\varphi\|_{\mathcal{C}([0,T];H^2(\mathbb{R}^2))} \leq P(\|A\|_{\mathcal{C}([0,T];H^2(\mathbb{R}^2))}).$$

4.2 Solitary waves — non existence results

We now look for solitary wave solutions of (2.9), that is solutions of the form $(e^{i\omega t}U(x), \phi(x))$ with $x \in \mathbb{R}^d$, $\omega \in \mathbb{R}$, $U \in H^1(\mathbb{R}^d)$, and $\phi \in H$. Thus (U, ϕ) should satisfy the system

$$\begin{cases} -\Delta U + \omega U = aU\partial_x\phi, \\ \operatorname{div}((1+|U|^2)\nabla\phi) = \partial_x(|U|^2). \end{cases} \tag{4.36}$$

The existence of nontrivial solutions of (4.36) is an open problem. Note that (4.36) does not seem to be the Euler–Lagrange equation associated to a variational problem. We have however:

Proposition 9 (i) *Let $a = -1$ (defocusing case). Then no nontrivial solution of* (4.36) *exists for $\omega \geq 0$.*
(ii) *Let $a = 1$ (focusing case). No nontrivial solution of* (4.36) *exists for $\omega \geq 1$.*
(iii) *Let $a = \pm 1$. No nontrivial solution of* (4.36) *exists if $\omega < 0$ provided $\partial_x\phi = O(1/|\mathbf{x}|^{1+\varepsilon})$, $\varepsilon > 0$ as $|\mathbf{x}| \to +\infty$.*

Proof. From (4.36) we have

$$\int_{\mathbb{R}^d} |\nabla U|^2 d\mathbf{x} + \omega \int_{\mathbb{R}^d} |U|^2 d\mathbf{x} = a \int_{\mathbb{R}^d} |U|^2 \partial_x\phi d\mathbf{x},$$

$$\int_{\mathbb{R}^d} (1+|U|^2)|\nabla\phi|^2 d\mathbf{x} = -\int_{\mathbb{R}^d} |U|^2 \partial_x\phi d\mathbf{x},$$

and

$$\int_{\mathbb{R}^d} |\nabla U|^2 d\mathbf{x} + \omega \int_{\mathbb{R}^d} |U|^2 d\mathbf{x} - a \int_{\mathbb{R}^d} (1+|U|^2)|\nabla\phi|^2 d\mathbf{x} = 0, \tag{4.37}$$

which proves (i). Now independent of the sign of a, and from (4.30) and (4.37),

$$\int_{\mathbb{R}^d} |\nabla U|^2 d\mathbf{x} + \omega \int_{\mathbb{R}^d} |U|^2 d\mathbf{x} \leq \int_{\mathbb{R}^d} |U|^2 d\mathbf{x}.$$

Thus

$$\int_{\mathbb{R}^d} |\nabla U|^2 d\mathbf{x} + (\omega - 1) \int_{\mathbb{R}^d} |U|^2 d\mathbf{x} \leq 0,$$

which proves (ii). Part (iii) results from [8]. □

4.3 The Cauchy problem

We consider the system

$$\begin{cases} i\partial_t A + \Delta A = -aA\partial_x\varphi, \\ \operatorname{div}\big((1+|A|^2)\nabla\varphi\big) = \partial_x(|A|^2), \\ A(\cdot,0) = A_0. \end{cases} \tag{4.38}$$

Theorem 10 *Let $A_0 \in H^2(\mathbb{R}^2)$.*
Then there exists $T_0 > 0$ and a unique solution $(A, \nabla\varphi)$ of (4.38) such that $A \in \mathcal{C}([0,T_0]; H^2(\mathbb{R}^2))$ and $\nabla\varphi \in \mathcal{C}([0,T_0]; H^2(\mathbb{R}^2))$. Moreover

$$\|A(\cdot,t)\|_{L^2(\mathbb{R}^2)} = \|A_0\|_{L^2(\mathbb{R}^2)}, \qquad 0 \le t \le T_0$$

and

$$\int_{\mathbb{R}^2} (1 + \frac{1}{2}|A|^2)|\nabla\varphi|^2 d\mathbf{x} \le \frac{1}{2}\int_{\mathbb{R}^2} |A_0|^2 d\mathbf{x}, \qquad 0 \le t \le T_0.$$

Proof. *Uniqueness.* Let $(A, \nabla\varphi) \in L^\infty(0,T; H^2(\mathbb{R}^2))$ and $(B, \nabla\psi) \in L^\infty(0,T; H^2(\mathbb{R}^2))$ two solutions of (4.38) with $A(\cdot,0) = B(\cdot,0)$. Then from $(4.38)_2$ one gets

$$\Delta(\varphi - \psi) + \operatorname{div}(|A|^2\nabla\varphi - |B|^2\nabla\psi) = \partial_x(|A|^2) - \partial_x(|B|^2),$$

yielding

$$\begin{aligned} &\int_{\mathbb{R}^2} |\nabla(\varphi-\psi)|^2 d\mathbf{x} + \int_{\mathbb{R}^2} |A|^2|\nabla(\varphi-\psi)|^2 d\mathbf{x} \\ &\quad = \int_{\mathbb{R}^2} (|A|^2 - |B|^2)\partial_x(\varphi-\psi) d\mathbf{x} - \int_{\mathbb{R}^2} (|A|^2 - |B|^2)\nabla\psi\cdot\nabla(\varphi-\psi) d\mathbf{x}. \end{aligned} \tag{4.39}$$

Observing that $|A|^2 - |B|^2 = A(A-\bar{B}) + \bar{B}(A-B)$, the R.H.S. of (4.39) is majorized by

$$\begin{aligned} &\frac{1}{4}\int_{\mathbb{R}^2} |\partial_x(\varphi-\psi)|^2 d\mathbf{x} + (\|A\|_{L^\infty(\mathbb{R}^2)} + \|B\|_{L^\infty(\mathbb{R}^2)})\int_{\mathbb{R}^2} |A-B|^2 d\mathbf{x} \\ &\qquad + \frac{1}{4}\int_{\mathbb{R}^2} |\nabla(\varphi-\psi)|^2 d\mathbf{x} \\ &\qquad + (\|A\|_{L^\infty(\mathbb{R}^2)} + \|B\|_{L^\infty(\mathbb{R}^2)})\|\nabla\psi\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2} |A-B|^2 d\mathbf{x} \end{aligned}$$

and by Sobolev embedding

$$\|\nabla(\varphi-\psi)\|_{L^2(\mathbb{R}^2)} \le C(\|A\|_{H^2(\mathbb{R}^2)}, \|B\|_{H^2(\mathbb{R}^2)}, \|\nabla\psi\|_{H^2(\mathbb{R}^2)})\|A-B\|_{L^2(\mathbb{R}^2)}. \tag{4.40}$$

On the other hand, we obtain readily from $(4.38)_1$ that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |A-B|^2 d\mathbf{x} \le \int_{\mathbb{R}^2} |A-B|^2|\partial_x\varphi| d\mathbf{x} + \int_{\mathbb{R}^2} |B||\partial_x(\varphi-\psi)||A-B| d\mathbf{x}$$

which together with (4.40) and the Cauchy-Schwarz lemma yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|A-B|^2d\mathbf{x}\\&\qquad\leq C(\|A\|_{H^2(\mathbb{R}^2)},\|B\|_{H^2(\mathbb{R}^2)},\|\nabla\varphi\|_{H^2(\mathbb{R}^2)},\|\nabla\psi\|_{H^2(\mathbb{R}^2)})\|A-B\|_{L^2(\mathbb{R}^2)}\end{aligned}$$

and $A=B$ by Gronwall lemma.

H^2 a priori estimate. We derive a (formal) H^2 *a priori* estimate on the solution of (4.38). Since $H^2(\mathbb{R}^2)$ is an algebra, we deduce from Lemma 8 that

$$\|A\partial_x\varphi\|_{\mathcal{C}([0,T];H^2(\mathbb{R}^2))}\leq\|A\|_{\mathcal{C}([0,T];H^2(\mathbb{R}^2))}P(\|A\|_{\mathcal{C}([0,T];H^2(\mathbb{R}^2))}),\tag{4.41}$$

where P was introduced in (4.31). From the energy estimate

$$\frac{1}{2}\frac{d}{dt}\|A(\cdot,t)\|^2_{H^2(\mathbb{R}^2)}\leq C\|A\partial_x\varphi(\cdot,t)\|_{H^2(\mathbb{R}^2)}\|A(\cdot,t)\|_{H^2(\mathbb{R}^2)},$$

we infer with (4.41) the local H^2 bound

$$\|A(\cdot,t)\|_{H^2(\mathbb{R}^2)}\leq C\left(\|A_0\|_{H^2(\mathbb{R}^2)}\right)\ \text{for } 0<t<T_0,\tag{4.42}$$

$T_0<T$ sufficiently small.

Approximation of (4.38). The strategy is now to implement a compactness method using the (justified) *a priori* estimate (4.42). For $\varepsilon>0$, we consider the system

$$i\partial_tA^\varepsilon+\Delta A^\varepsilon = -aA^\varepsilon\partial_x\varphi^\varepsilon,\tag{4.43}$$

$$\operatorname{div}\left((1+\varepsilon\Delta^2+|A^\varepsilon|^2)\nabla\varphi^\varepsilon\right) = \partial_x(|A^\varepsilon|^2),\tag{4.44}$$

$$A^\varepsilon(\cdot,0) = A_0.\tag{4.45}$$

Solving $\nabla\varphi^\varepsilon$ in terms of A^ε, we obtain from (4.44) that $\nabla\varphi^\varepsilon$ satisfies

$$\varepsilon\int_{\mathbb{R}^2}|\Delta\nabla\varphi^\varepsilon|^2d\mathbf{x}+\int_{\mathbb{R}^2}(1+\frac{1}{2}|A^\varepsilon|^2)|\nabla\varphi^\varepsilon|^2d\mathbf{x}\leq\frac{1}{2}\int_{\mathbb{R}^2}|A^\varepsilon|^2d\mathbf{x}.\tag{4.46}$$

Well-posedness of approximate system. We now check that the Cauchy problem (4.43)–(4.45) is globally well-posed in $H^2(\mathbb{R}^2)$. Let first A^ε, $B^\varepsilon\in H^2(\mathbb{R}^2)$ and φ^ε, ψ^ε the corresponding solutions of (4.44). Proceeding as in the uniqueness proof above, one gets

$$\begin{aligned}&\varepsilon\int_{\mathbb{R}^2}|\nabla\Delta(\varphi^\varepsilon-\psi^\varepsilon)|^2d\mathbf{x}+\int_{\mathbb{R}^2}|\nabla(\varphi^\varepsilon-\psi^\varepsilon)|^2d\mathbf{x}\\&\qquad\leq C(\|A^\varepsilon\|_{H^2(\mathbb{R}^2)},\|B^\varepsilon\|_{H^2(\mathbb{R}^2)},\|\nabla\varphi^\varepsilon\|_{H^2(\mathbb{R}^2)},\|\nabla\psi^\varepsilon\|_{H^2(\mathbb{R}^2)})\|A^\varepsilon-B^\varepsilon\|_{L^2(\mathbb{R}^2)}.\end{aligned}\tag{4.47}$$

Denoting $\partial_x\varphi^\varepsilon$ by $F^\varepsilon(A^\varepsilon)$ we write (4.43) on the Duhamel form with $S(t)=\exp(it\Delta)$,

$$A^\varepsilon(t)=S(t)A_0-a\int_0^tS(t-s)A^\varepsilon F^\varepsilon(A^\varepsilon)ds.\tag{4.48}$$

Using (4.47) and the unitarity of $S(t)$ in $H^s(\mathbb{R}^2)$, we deduce that the R.H.S. of (4.48) defines a contraction in $\mathcal{C}([0,T_\varepsilon];H^2(\mathbb{R}^2))$ for some $T_\varepsilon > 0$.

This implies the local well-posedness of (4.43)–(4.45) in $H^2(\mathbb{R}^2)$. Using the H^2 bound (4.47) on $\nabla\varphi^\varepsilon$, we infer from (4.43) an *a priori* bound in $\mathcal{C}([0,T];H^2(\mathbb{R}^2))$ for A^ε and for all $T > 0$. This proves that the Cauchy problem (4.43)–(4.45) is globally well-posed, for any fixed $\varepsilon > 0$.

Limit $\varepsilon \to 0$. Now we have the bounds (4.46) and

$$\|\nabla\varphi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^2)} + \sqrt{\varepsilon}\|\Delta\nabla\varphi^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C, \qquad 0 \le t \le T, \tag{4.49}$$

where C and T do not depend on ε. Moreover, from (4.43) and (4.49) we have a bound on $\partial_t A^\varepsilon$ which is independent of ε:

$$\|\partial_t A^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C, \qquad 0 \le t \le T.$$

It is now standard to pass to the limit as $\varepsilon \to 0$ (see [10]). By the Aubin–Lions compactness lemma, we obtain a subsequence $(A^\varepsilon, \nabla\varphi^\varepsilon)$ such that $A^\varepsilon \to A$ in $L^\infty(0,T;H^2(\mathbb{R}^2))$ weak-star and $L^2(0,T;H^1_{\mathrm{loc}}(\mathbb{R}^2))$ strongly, $\nabla\varphi^\varepsilon \to \nabla\varphi$ in $L^\infty(0,T;H^2(\mathbb{R}^2))$ weak-star and $L^2([0,T]\times\mathbb{R}^2)$ weakly. The limit $(A,\nabla\varphi)$ belongs to $(L^\infty(0,T;H^2(\mathbb{R}^2)))^2$ and satisfies (4.38). In fact $(4.38)_1$ is satisfied in $L^2(\mathbb{R}^2)$ and $(4.38)_2$ is satisfied in $H^1(\mathbb{R}^2)$.

The fact that $(A,\nabla\varphi) \in (\mathcal{C}(0,T;H^2(\mathbb{R}^2)))^2$ results from the Bona–Smith approximation (see [3]). $\square$

Remark. We do not know whether the local solution obtained in Theorem 10 is global or not.

5 Conclusion

We have given a full description of how to derive from the Kukhtarev equations an asymptotic model for the propagation of light in a photorefractive medium. This derivation is only heuristic insofar as asymptotics are not justified, which would be out of reach now. Some properties of photorefractive media such as memory have also been neglected.

The 1D asymptotic model is a saturated nonlinear Schrödinger equation the Cauchy problem of which is studied (in any space dimension) in L^2 and H^1. We also prove the existence of solitary waves in one and higher dimensions. An interesting and open issue would be to study the transverse stability of the 1D solitary waves in the framework of the asymptotic model.

For the 2D asymptotic model (the Zozulya–Anderson model) we also have studied the Cauchy problem and the non-existence of solitary waves. The question of imposing other boundary conditions, not vanishing in one space direction, can also be addressed to treat a wider range of experimental applications.

A Non-Existence of Solitary Waves in Non-Physical Cases

The goal is here to complete the results of Corollary 3 for $\omega < 0$ with no decaying assumption. We have already seen that Eq. (3.18) implies that no solitary wave may exist for $d = 1, 2$ and $a = 1$ (focusing case).

To go further, let us use both Eqs. (3.17) and (3.18) to obtain

$$\int_{\mathbb{R}^d} \left(2\omega + \frac{(d-2)a|U|^2}{1+|U|^2}\right) |U|^2 d\mathbf{x} - ad \int_{\mathbb{R}^d} \left[|U|^2 - \ln(1+|U|^2)\right] d\mathbf{x} = 0.$$

We set

$$F(X) = \left(2\omega + \frac{(d-2)aX}{1+X}\right) X - ad(X - \ln(1+X)),$$

and we know that $\int_{\mathbb{R}^d} F(|U|^2) d\mathbf{x} = 0$. Now $F(0) = 0$ and

$$F'(X) = \frac{2X^2(\omega - a) + X(4\omega - (4-d)a) + 2\omega}{(1+X)^2} < 0,$$

if $\omega < 0$, $a = 1$ and $d = 3, 4$. Therefore $F(|U|^2) = 0$ a.e. By a bootstrapping argument, we notice that any H^1 solution to Eq. (3.16) is indeed in H^k for all k and therefore continuous. Hence $F(|U|^2) = 0$ on $\mathbb{R}^d$. Since $F'(X) < 0$ the only possible value for U is $U = 0$ on $\mathbb{R}^d$.

We can refine this result, finding other parameter ranges for which $2X^2(\omega - a) + X(4\omega - (4-d)a) + 2\omega < 0$. If $d \geq 5$ and $a = 1$, this holds for $\omega \leq -(d-4)/4$. Moreover,

$$\begin{aligned} &2X^2(\omega - a) + X(4\omega - (4-d)a) + 2\omega \\ &\qquad = 2\left[(\omega - a)(X-1)^2 + X(4\omega - 4a + \frac{d}{2}a) + \omega + a\right]. \end{aligned}$$

No solitary wave can exist for $a = -1$ and $\omega \leq -1$. Hence we complete Corollary 3 with

Corollary 11 *No non-trivial solitary wave (solution of* (3.16)*) of the saturated NLS equation exists when*

(i) $a = -1$ (*defocusing case*), *for* $\omega \leq -1$.

(ii) $a = 1$ (*focusing case*), *for* $\omega \leq 0$, *if* $d = 3, 4$ *and* $\omega \leq -(d-4)/4$ *if* $d \geq 5$.

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